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On the transitions between dynamical behaviors for small-world and scale-free networks

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Abstract. The effect of network topology on the transition between dynamical behaviors is investigated for two basic networks: the small-world and scale-free models. In our analysis, we consider a network of identical dynamical systems where each node, prior to being coupled into the network, has a stable equilibrium point. In a recent publication we established that, as the number of nodes in the network increases, the collective dynamics of a repellently coupled network transit from stable to bounded and ultimately to unbounded behavior. In this contribution we investigate how these transitions are affected by the choice of network model. We show that for small-world networks, the stable equilibrium point behavior transits to bounded complex behavior for an interval of network sizes, before becoming unbounded for a sufficiently large number of nodes. While for scale-free networks, the transition from stable fixed point to unbounded behavior occurs directly as the number of nodes increases. To illustrate these results, we use numerical simulations of well known chaotic benchmark systems.
Key Words: Networks, Synchronization, Transitions, Emergence.

1 Introduction

Networks are ensembles of nodes connected by links. As such, representing functional units as nodes and their interactions as links, a network can serve as simplified models for many technological, social and even biological real-world systems [1,2]. Recently, significant research efforts has been focus on analyzing the structural characteristics of such representations. This has lead to the discovery of the small-world [3] and scale-free [4] effects as common features of real-world complex networks. Different construction algorithms have been postulated to capture these key topological aspects. One of the earliest network models that successfully capture some aspects related to the small-world effect was proposed by Erdös and Rènyi (ER) in the 1960´s, in the ER random graph model the structure of complex networks is consider to be the result of a stochastic process of assignment of links between a fixed population of nodes [5]. Watts and Strogatz (WS) proposed in 1998 a network model designed to improve on

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ER model [3]. In particular, as to the inability of the ER model to capture the density characteristics of real-world networks. To achieve this, the WS network model proposes the combination of a regular nearest neighbors graph with a stochastic process of rewiring or add a small number of nodes. Since this model is a much better representation of the small-world effect, the WS construction algorithm is also refer to as the small-world network model. The ER and WS network models share a common feature, in both cases the degree distribution follows a Poisson distribution, that is, their connectivity is basically homogeneous, with all nodes having about the same number of nodes. In 1999, Barabási and Albert (BA) proposed an alternative construction algorithm that consists on two processes [4]: *growth* and *preferential attachment*. In the BA algorithm starts with a small number of nodes, at each iteration a new node is added and is coupled to a number of the nodes already existing on the network with a probability directly related to its node degree. The main characteristics of the BA model is that the number of links per node is not fixed to the average for the whole network. In fact, the BA model has a heterogeneous connectivity that follows a Power-law distribution [2], for these reasons the BA construction algorithm is usually refer to as the scale-free network model.Recent works have consider the effect of topological complexity on dynamics for different specific scenarios, from uniformly coupled small-world networks [6], to adaptively weighted scale-free networks [7]; among many others [1,2,8,9]. Unlike most of the previously cited works, we are concern with the transitions of the collective behavior of the network from a synchronized common stable equilibrium point to the emergence of bounded complex behavior, and ultimately, unbounded trajectories. In a recent publication, condition for these transitions, on a network with regular topology, were established in terms of the coupling strength and the number of nodes [10]. In this contribution, we extend those results to consider the effect of complex connectivity. In particular, we investigated how the criteria for transition is changed by choosing small-world or scale-free complex networks.

Throughout the paper, we will consider a network of identical dynamical systems, linearly and diffusively coupled, where each node prior to being coupled into the network has a stable equilibrium point. Under these conditions, the synchronized solution for the entire network, which describes the synchronization manifold, coincides with the dynamical evolution of a single node in isolation [11,6], that is, when synchronization is achieved every node follows a common stable equilibrium point. Then, expressing the dynamics of the network in terms of its deviation from the synchronized solution, a direct relationship can be found between the transverse Lyapunov exponents (*tLe*) [9], the Lyapunov exponents of an isolated node, and the eigenvalues of the coupling matrix. Furthermore, the network connectivity can chosen such that some of the *tLe*s become positive, resulting on transitions from the synchronized equilibrium point solution to bounded complex oscillations, or even unbounded trajectories. In [10], the criteria for these transitions is express as limit values of the coupling strength and the number of nodes in the network. Using this criteria, we show the effect of complex topology on the conditions for transition between dynamical behaviors, In Section 2, the conditions for transition are derive as a relation between the *tLes* of the entire network, the Lyapunov exponents of an isolated node, and the network connectivity, described by the eigenvalue spectrum of the coupling matrix. In Section 3, the small-world and scale-free models are described and their effect on the transition between collective behaviors is establish. These results are illustrated with numerical simulations in Section 4. Finally, in Section 5 this contribution is concluded with comments and remarks.

2 Transitions Between Dynamical Behaviors

Consider a network of N linearly and diffusively coupled identical nodes, where each node is a m-dimensional dynamical system with a stable equilibrium point. The state equations describing the dynamics of the entire network are given by [10]:

$$\dot{x}_i(t) = f(x_i(t)) - c\sum_{j=1}^{N} a_{ij}x_j(t), \qquad i = 1, 2, ..., N \tag{1}$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), ..., x_{im}(t)]^{\top} \in \mathbf{R}^m$ are state variables of the ith node; $f(\cdot) : \mathbf{R}^m \to \mathbf{R}^m$ describes the dynamics of an isolated node; the constant $c > 0$ is the coupling strength between any two nodes in the network. The network connectivity is describe by its coupling matrix $\mathcal{A} = \{a_{ij}\} \in \mathbf{R}^{N\times N}$, where $a_{ij} \in [0, 1]$ is set to one if there is a link between ith and jth nodes and to zero if they are not connected. Diffusive coupling refers to:
(I) Symmetric entries ($a_{ij} = a_{ji}$, $\forall i, j, j \neq i$);
(II) Null sum by rows and columns ($\sum_{j=1}^{N} a_{ij} = \sum_{j=1}^{N} a_{ji} = 0$, $\forall i$), with the diagonal elements are given by

$$a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij} = -\sum_{j=1, j\neq i}^{N} a_{ji}, \qquad i = 1, 2, ..., N \tag{2}$$

For a diffusively coupled network with no isolated nodes, the eigenvalues of $\mathcal{A}$ are [12]:

$$0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_N \tag{3}$$

An isolated node, described by $\dot{x} = f(x)$, can be characterized in terms of its Lyapunov exponents, h_i, given by [13]:

$$h_i = \lim_{t\to+\infty} \frac{1}{t}|J(t, x_0)u_i|, \qquad i = 1, 2, ..., m \tag{4}$$

where $J(t, x_0)$ is the Jacobian matrix of $f()$ evaluated at a randomly selected initial condition x_0 with $\{u_1, u_2, ..., u_m\}$ is a set of orthonormal vectors in the tangent space of the system. Notice that the m Lyapunov exponents of a node, prior to being coupled into the network, are strictly negative and can be ordered as:

$$0 > h_1 \geq h_2 \geq ... \geq h_m \tag{5}$$

A network is said to be synchronized if the trajectories of every node move at unison, in the sense that:

$$x_i(t) = x_j(t) = \bar{x}(t), \quad \forall\, i, j \text{ as } t \to \infty. \tag{6}$$

where $\bar{x}(t)$ is the synchronized state, which coincides with the dynamics of a node in isolation ($\dot{\bar{x}}(t) = f(\bar{x})$ and by construction is a solution for the entire network.

In the mN-dimensional state space the synchronized solution describes a diagonal m-dimensional manifold usually called the synchronization manifold [6,11]. The collective dynamics of the network can be characterized in terms of the divergence of the trajectories of its nodes from the synchronized solution $\bar{x}$. Furthermore, the stability of the synchronized behavior can be determine directly from the transverse Lyapunov exponents [9,14,10], which are obtained as follows:

Defining the synchronization error as $\xi_i(t) = x_i(t) - \bar{x}(t)$. A variational equation is obtained linearizing around the synchronized state

$$\dot{\xi}_i(t) = J(x(t))\xi_i(t) - c\sum_{j=1}^{N} a_{ij}\xi_j(t), \quad i = 1, 2, ..., N \tag{7}$$

which in vector form becomes

$$\dot{\mathcal{X}}(t) = J(x(t))\mathcal{X}(t) - c\mathcal{X}(t)\mathcal{A}^\top \tag{8}$$

where $\mathcal{X}(t) = [\xi_1(t), \xi_2(t), ..., \xi_N(t)] \in \mathbf{R}^{m\times N}$. Notice that by construction the coupling matrix satisfies:

$$\mathcal{A} = \Gamma\Lambda\Gamma^{-1} \tag{9}$$

where $\Gamma = [\gamma_1, \gamma_2, ..., \gamma_N] \in \mathbf{R}^{N\times N}$; and $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_N) \in \mathbf{R}^{N\times N}$; with γ_i the i-th eigenvector of $\mathcal{A}$ and λ_i its corresponding eigenvalue.

In terms of (9), the variational equation (8) becomes:

$$\dot{\nu}_k(t) = J(x(t))\nu_k(t) - c\lambda_k\nu_k(t), \quad k = 1, 2, ..., N \tag{10}$$

where $\nu_k(t) = \mathcal{X}(t)\gamma_k \in \mathbf{R}^m$. Then, applying the definition of Lyapunov exponent in (4) to the variational equation (10). One gets the entire spectrum of *tLes* for the network as:

$$\mu_i(\lambda_k) = h_i - c\lambda_k, \tag{11}$$

for $i = 1, 2, ..., m$, and $k = 1, 2, ..., N$.

The spectrum of *tLe*s inherit, from (3) and (5), the following order:

$$\begin{array}{c} \mu_i(\lambda_N) \geq \mu_i(\lambda_{N-1}) \geq ... \geq \mu_i(\lambda_2) \geq \mu_i(\lambda_1) \\ \mu_1(\lambda_k) \geq \mu_2(\lambda_k) \geq ... \geq \mu_{m-1}(\lambda_k) \geq \mu_m(\lambda_k) \end{array} \tag{12}$$

for $i = 1, 2, ..., m$ and $k = 1, 2, ..., N$.

The stability of the collective behavior around the synchronization manifold can be determine from the sign of the *tLe*s. From (12), the largest *tLe* is $\mu_1(\lambda_N)$;

Then, if $\mu_1(\lambda_N) < 0$, all the nodes synchronize to a common stable equilibrium point. However, if the connectivity is such the eigenvalues of $\mathcal{A}$ make some of the *tLes* positive, then, some of the trajectories move away from the synchronized solution, but if the synchronization manifold remains attractive the trajectories remain bounded. In [10] this situation is refer to as the emergence of bounded behavior. Alternatively, if the eigenvalues of the coupling matrix make a large enough number of *tLe* positive, then the trajectories become unbounded at least for one node.

These conditions for transition between behaviors can be expressed as follows [10]:
(I) The network will synchronize to a common stable equilibrium point if $\mu_1(\lambda_N) < 0$, or equivalently

$$|h_1| > c|\lambda_N| \tag{13}$$

(II) Bounded behavior will emerge in the network if:
(II.1) A small number (τ T with $1 < \tau < m$ and $1 < T < N$) of *tLes* are positive with the remaining *tLes* are negative ($\mu_\tau(\lambda_T) < 0$), or equivalently

$$|h_1| < c|\lambda_N| \text{ and } c|\lambda_T| < |h_\tau| \tag{14}$$

(II.2) Additionally, the overall sum of *tLes* for each node must be negative for every node; that is,

$$\sigma_i = \sum_{i=1}^{m} h_i - c\lambda_k < 0 \text{ for every } k \tag{15}$$

It should be noted that in this case the network is not synchronized, the trajectories move away from the equilibrium point, yet they remain bounded since the equilibrium solution remains attractive. In this situation, the folding and stretching mechanisms may allow for the network to exhibit complex chaotic oscillations.
(III) Ultimately, unbounded trajectories result if the positive *tLes* are such that the sum of any of the nodes is positive, that is

$$\sigma_i = \sum_{i=1}^{m} h_i - c\lambda_k > 0 \text{ for any } k \tag{16}$$

These conditions for the transition from one behavior to another depend directly on the eigenvalues of the connectivity matrix. Since different topologies have different eigenvalue spectrums, in the following section we investigate the effect of two conventional network models on the transitions between collective behaviors.

3 Models of Network Topology

In order to investigate the effect of network topology on the transitions between behaviors, we look at two benchmark models: small-world and scale-free networks. From the criteria for transition in (13)-(16) one can see that the most important aspect to consider is the eigenvalue spectrum of the coupling matrix obtained using each of these models. In particular, two aspects must be considered: (1) the value of the smallest (most negative) eigenvalue, λ_N, since it determines the stability of synchronization towards the equilibrium point. And (2) the dispersion of the eigenvalues, as can be seen from (15) and (16) a larger dispersion makes it more likely to have a positive overall sum even with only a few positive *tLes*. In this contribution, we measure the dispersion of the eigenvalues of $\mathcal{A}$ calculating the following indicator:

$$\rho = \frac{|\lambda_N| - |\lambda_2|}{|\lambda_N|} \tag{17}$$

Then, the dispersion has its minimum ($\rho = 0$) when the eigenvalues are equal, which is the case for a regular globally coupled network; and its maximum ($\rho \approx 1$) when their difference is largest, this happens for a regular star coupled network.

3.1 Small-World Network Model

The coupling matrix of a network with small-world coupling structure can be constructed using the following two-step process [11]:
1. **Regular**: Construct a regular nearest-neighbors connectivity matrix $\mathcal{A}_{nn}$ of size N, by symmetrically letting the two neighboring entries forwards and backwards of each node be turn to one.
2. **Random**: With probability $0 < p_{sw} < 1$, add connections to the network by symmetrically making the corresponding zero entries of $\mathcal{A}_{nn}$ into ones ($a_{ij} = a_{ji} = 1$) and adjusting the diagonal elements of the resulting matrix $\mathcal{A}_{sw}$, such that the diffusive requirements (2) and (3) are satisfied.

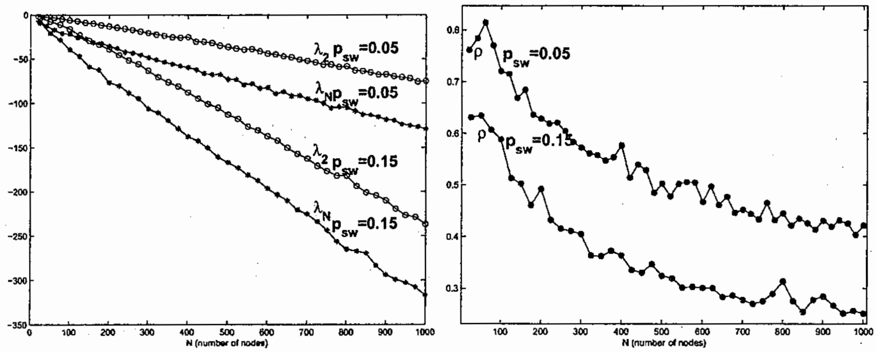

Fig. 1. Left The largest non zero ($\lambda_{sw,2}$) and smallest ($\lambda_{sw,N}$) eigenvalues of $\mathcal{A}_{sw}$. **Right** The dispersion of the eigenvalue spectrum of $\mathcal{A}_{sw}$ for different number of nodes. (Average over twenty realizations for each probability p_{sw})

In Figure 1-Left, the largest ($\lambda_{sw,2}$) and smallest ($\lambda_{sw,N}$) nonzero eigenvalues of $\mathcal{A}_{sw}$ are presented as a function of the number of nodes in the network for a given probability of adding connections. Examining the nonzero eigenvalues of $\mathcal{A}_{sw}$ it can be seen that all of them growth along with the size of the network. In Figure 1-Right, the dispersion of the eigenvalues of $\mathcal{A}_{sw}$ is shown as a function of the number of nodes in the network. It can be seen that as the network growths the eigenvalues become more compacted.

From the observations above, the following can be established for a network with a small-world coupling configuration ($\mathcal{A}_{sw}$):
(I) There will be a critical value of N after which positive *tLe*s will be generated for any given coupling strength. (II) The number of positive *tLe*s will growth larger as the number of nodes in the network increases.
(III) Since the dispersion of the eigenvalues tends to be more compacted, there will be a set of values for which τT positive *tLe*s will be such that at the same time the sum (15) remains negative. In this situation, complex bounded trajectories can be generated in the overall behavior of the network (See Figure 3 in the following Section).
Ultimately, for a large enough number of nodes, the trajectories will become unbounded. (See Figure 3 in the following Section)

3.2 Scale-Free Networks

The connectivity matrix of a network with a scale-free topology can be constructed using the following algorithm [4]:
1. **Growth**: Initially there are m_0 nodes, then every time step a new node is added and connected to m_1 of the existing nodes by setting the entries of the connectivity matrix $\mathcal{A}_{sf}$ to one ($a_{ij} = a_{ji} = 1$) and adjusting the diagonal elements ($a_{ii} = -\sum_{j \neq i} a_{ij}$) such that the sum by rows remains zero.
2. **Preferential Attachment**: The nodes in the network, which are connected to the new node, are chosen with a probability $0 < p_{sfi} < 1$. The probability p_{sf} is a function of the node degree (k_i), which is defined as the number of edges connecting to a node. In this way, the probability of connecting the new node j to the existing node i, is given by $p_{sf}(j \to i) = \frac{k_i}{\sum_l k_l}$

In Figure 2-Left, the largest non-zero ($\lambda_{sf,2}$) and the smallest ($\lambda_{sf,N}$) eigenvalues of the connectivity matrix for a scale-free network ($\mathcal{A}_{sf}$) are shown for different number of nodes. In this case one can observe that as the number of nodes increases, the largest non zero eigenvalue remains basically constant ($\lambda_{sf,2} \approx -1$), while the smallest eigenvalue growths proportionally with the network size ($\lambda_{sf,N} \approx -N$). In Figure 2-Right, the dispersion of the eigenvalues spectrum of $\mathcal{A}_{sf}$ is shown as a function of the number of nodes in the network. One can see that as the number of nodes in the network growths the dispersion tends towards its maximum at one ($\rho_{sf} \approx 1$).

Taking into account this results, for a network with scale-free coupling structure ($\mathcal{A}_{sf}$) the following can be established:
(A) There will be a critical value of N after which positive *tLe*s will be generated

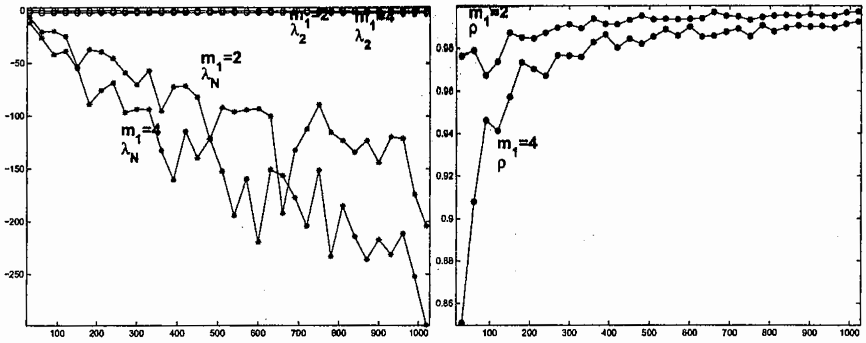

Fig. 2. **Left** The largest non zero ($\lambda_{sf,2}$) and smallest ($\lambda_{sf,N}$) eigenvalues of $\mathcal{A}_{sf}$. **Right** The dispersion of the eigenvalue spectrum of $\mathcal{A}_{sf}$ for different number of nodes. (The numerical values are average over twenty realizations with $m_0 = 4$ $m_1 = 4, 2$)

for any given coupling strength, yet this will be the case only for some of the eigenvalues of $\mathcal{A}_{sf}$ (the ones near λ_N) with the remaining eigenvalues remaining constant (the ones near λ_2).
(B) The number of positive *tLe*s will remain basically the same as the number of nodes increases, yet the value for the smallest eigenvalues will growth with the number of nodes.
(C) The large dispersion between the eigenvalues will allow for a large enough number of nodes that there will be positive *tLe*s with a negative sum for (15), in this situation the trajectories remain bounded but no complex oscillations can be observed.
Finally, for a sufficiently large N the positive *tLe*s will be large enough to make the sum (16) positive. In this case, the network will transit from a stable equilibrium point to a unbounded trajectories for the hub nodes (See Figure 4 in the Section 5).

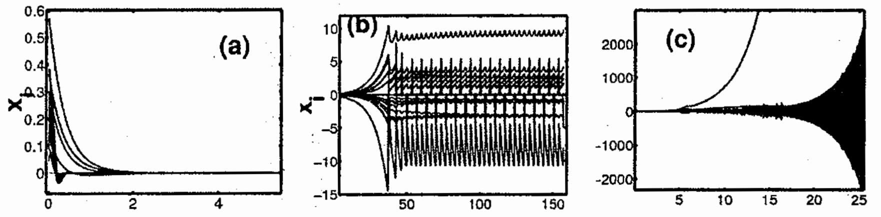

Fig. 3. Illustrative example of the effect of topology on the transitions between behavior for a small-world network of Lorenz systems with a stable equilibrium point. (a) four (b) fifteen and (c) Twenty five nodes.

4 Illustrative Example

For illustrative purposes consider a network with each node is a Chen system [15] with a stable equilibrium point. We assume that the nodes are coupled linearly and diffusively according to the small-world algorithm presented in the previous Section, and further consider that the coupling strength is fixed at $c = 0.2$. As discussed in the above, under these conditions for a small number of nodes, all nodes will have a common equilibrium point (See Figure 3(a) $N = 4$), as the number of nodes increases the conditions on (14) and (15) are satisfied and bounded complex oscillations can be observed (See Figure 3(b) $N = 15$); finally for a large number of nodes, the trajectories become unbounded (See Figure 3(c) $N = 25$)

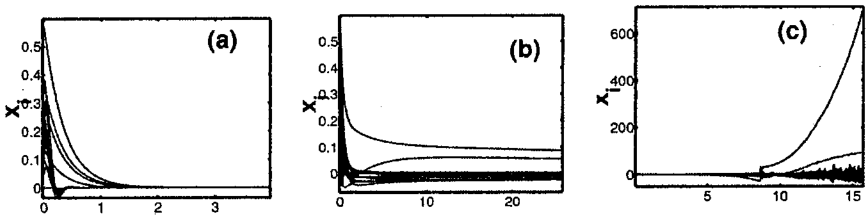

Fig. 4. Illustrative example of the effect of topology on the transitions between behavior for a scale-free network of Lorenz systems with a stable equilibrium point. (a) Four (b) Fifteen and (c) Twenty nodes.

Then, consider that the nodes are coupled linearly and diffusively according to the scale-free algorithm presented above for a fixed coupling strength $c = 0.2$. In this case, for a small number again all nodes will have a common equilibrium point (See Figure 4(a) $N = 4$), as the number of nodes increases where both conditions (14) and (15) are satisfied, the trajectories of the network remain bounded but only the hub node is moves away from the synchronized solution (See Figure 4(b) $N = 15$); finally for a large number of nodes, the trajectories of the hub node become unbounded long before the rest of the nodes move away from the common equilibrium point (See Figure 4(c) $N = 20$)

5 Concluding Remarks

We extended the results on the transition between behaviors for networks of coupled dynamical systems presented in [10] to include the cases of networks with small-world and scale-free topology. Analyzing differences,in terms of values and dispersion of the eigenvalue spectrums generated by each of these network models, we established that network with small-world topology tend to have an interval of values for the number of nodes (N) such that complex oscillations can be observed. On the other hand, network with scale-free topology will tend to transit directly from a stable bounded behavior to a state where at least the hub nodes become unbounded. Therefore, scale-free networks do not have a window of values where complex oscillations may occur. This difference in the transitions results from a fundamental difference in the connectivity of these models and may have significant implications for the design of real-world networks. These issues require further investigation, which will be reported elsewhere.

Acknowledgments

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