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# A dynamic parameter estimator to control chaos with distinct feedback schemes

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## Abstract

A dynamic strategy is proposed to estimate parameters of chaotic systems. The dynamic estimator of parameters can be used with diverse control functions; for example those based on: (i) Lie algebra, (ii) backstepping or (iii) variable feedback structure (sliding-mode). The proposal has adaptive structure because of interaction between dynamic estimation of parameters and a feedback control function. Without loss of generality, a class of dynamical systems with chaotic behavior is considered as benchmark. The proposed scheme is compared with a previous low-parameterized robust adaptive feedback in terms of execution and performance. The comparison is motivated to ask: *What is the suitable adaptive scheme to suppress chaos in an specific implementation?* Experimental results of proposed scheme are discussed in terms of control execution and performance and are relevant in specific implementations; for example, in order to induce synchrony in complex networks.

**Keywords:** Chaos control, Adaptive observers, Controlled nonlinear dynamics.

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# 1 Introduction

Chaos control comprises suppression and synchronization of chaotic systems and can be potentially exploited, among others, to deal with current engineering problems like chaos suppression on dc-dc converters; multimode laser in surgery of carbon nanotubes; regulation of fluid dynamics; design of systems for secure communications via internet; and some problems in biomedical sciences as arrhythmias or epilepsy. Particularly, chaos suppression problem has been a studied topic to address these problems (see reviews and seminal papers [1, 2, 3, 4]). Complementarily, synchronization problem has been exploited to deal with secure communication, synchrony on multimode lasers, biological systems and, more recently, complex networks (see [5, 6]). Diverse schemes have been proposed to understand mechanisms of the chaos control [2] or to design control devices [4, 7].

Two equivalent issues are important on chaos control: (i) to exploit the chaotic solutions of systems [1] or (ii) to use the vector fields of systems [6]. Latter is the major concern from the control theory viewpoint because it means the possibility of driving a desired chaotic behavior along time. For instance, several control schemes have been widely studied in last two decades to induce synchronous behavior (see [5, 4, 8]). As matter of fact, a practical implementation is often limited by the information available for feeding back. Then, a control practitioner has only partial information available (i.e., only measured states or nominal parameters values). This has served as motivation to diverse adaptive control techniques like, among others, feedback output linearization, backstepping, sliding-mode, and observer-based. These approaches lead us to the following problem of selecting a specific control scheme to reach a desired control performance. This problem is a challenge if we consider the trade off among simplicity in implementation, control execution and convergence rate. This particular problem takes major relevance in light of recent scientific questions as the stabilization of complex networks [6]. Moreover, due to chaotic dynamic is highly sensitive to initial conditions and parameter values, chaos control has been oriented on robust feedback approaches [4, 5]. Thus, selection of a specific technique can depend on: performance of control, structural simplicity, and closed-loop stability, etcetera.

The purpose of this paper is to propose a dynamic parameter estimator that can

be used under different control strategies. In order to reach this purpose, a class of chaotic systems is considered as benchmark to design and the posteriori experimental comparison. The considered class includes the most important chaotic systems in the form  $\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})u$  with  $\mathbf{x} \in \mathbb{R}^3$  and vector fields  $f, g : \mathbb{D} \rightarrow \mathbb{R}^3$  have, at least, first derivative, where  $\mathbb{D} \subseteq \mathbb{R}^3$  stands for the system domain. The performance study is carried out for experimental comparison. The comparison is presented in terms of the quantitative execution and performance. The paper is organized as follows. A class of chaotic systems is presented in Section 2. Section 3 describes three controllers for chaos suppression: state feedback linearization, backstepping techniques, sliding-mode. These three controllers are based on an adaptive observer designed to estimate unmeasurable state and unknown parameters. Although these schemes were reported in [4, 5, 13], in seek of completeness, closed-loop stability analysis is included. Section 4 shows the fourth scheme, which is a low-order parametrized controller. In this contribution, the main point is performance comparison for the four controllers under noise and parametric uncertainties. Such controllers are designed in Section 5 for the P-class presented in Section 2. The performance comparison is carried out experimentally on the realization of chaotic system in an electronic circuit. Finally, Concluding remarks are given in Section 6.

## 2 Chaotic systems used as benchmark

A class of chaotic systems is introduced in this section. This class contains dissipative chaotic terms and can be transformed into the equation  $\ddot{x} = J_P(x, \dot{x}, \ddot{x})$ , which is defined by a polynomial jerk function. It is know that jerk equation exhibits chaotic behavior for a set of parameter values, represents different nature or man-made systems and is a sub-class of Lur'e systems when bursting in velocity is involved (see for instance [10] and references therein). Without control, jerk equation has the homogeneous form  $\ddot{x} + \alpha\dot{x} + x - \phi_Q(x, \dot{x}) = 0$ , where  $\phi_Q(x, \dot{x})$  includes polynomial terms; e.g.,  $\phi_Q(x, \dot{x}) = \dot{x}^2$  or  $\phi_Q(x, \dot{x}) = x\dot{x}$  [9]. In what follows, the jerk equation is represented in state space form:  $\dot{x}_1 = x_2, \dot{x}_2 = x_3, \dot{x}_3 = -x_1 - \alpha x_3 + \phi_Q(x_1, x_2)$ . Thus, jerk equation constitutes a family of systems with only one parameter related to damping  $\alpha > 0$ . Among many others, the

following collection of dynamical systems is included in such a family [9]:

$$\begin{aligned}
\Sigma_1 : & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -\alpha x_3 - x_1 + x_2^2 \end{cases} & \Sigma_2 : & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -\alpha x_3 - x_1 + x_1 x_2 \end{cases} \\
\Sigma_3 : & \begin{cases} \dot{x}_1 = x_2 + 1 \\ \dot{x}_2 = -\alpha x_2 + x_3 \\ \dot{x}_3 = x_1 x_2 \end{cases} & \Sigma_4 : & \begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_1 + 1 \\ \dot{x}_3 = -\alpha x_3 + x_1 x_2 \end{cases} \\
\Sigma_5 : & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\alpha x_2 + x_3 \\ \dot{x}_3 = -x_1 + x_2^2 \end{cases} & \Sigma_6 : & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\alpha x_2 + x_3 \\ \dot{x}_3 = -x_1 + x_1 x_2 \end{cases}
\end{aligned} \tag{1}$$

All systems in collection (1) is grouped in a class by using  $C^k$ -equivalence of vector fields [9, 11]. Noted that the family can be found for a large variety of systems by constructiong a diffeomorphic transformation. As matter of fact, there exists conditions such that the Rössler, Lorenz, and Chen systems can be written as the form (1). Additionally, dynamical properties can be studied through transformations of systems; as, for example, its stability can be analyzed via preservation [12].

### 3 An observer adaptive for distinct control approaches

We can mention state feedback linearization, backstepping and sliding-mode methods among distinct control techniques [13]. However, physical implementation is usually limited due to only partial or imprecise information is available in regard parameters of measurement. In order to overcome this difficulty, estimation of state and identification of parameter values is often required. An alternative is to use adaptive observer-based schemes. An adaptive observer is interpreted as a virtual (software) sensor for simultaneous estimation and identification. An observer is designed as dynamic parameter estimator. Convergence and closed-loop stability proofs are shown [15, 16]. We show how is possible to use our proposal under three distinct control techniques, which are described in what follows.

**State Feedback Linearization:** This technique exploits Lie algebra and is very popular in control practice. In order to design a control, P-class chaotic systems is written in the following control affine form:

$$\begin{cases} \dot{w} = f_l(w; \alpha) + g_l(w; \alpha)u_L \\ y_c = h_l(w) \end{cases} \quad (2)$$

where  $w \in \mathbb{R}^3$ ,  $u_L \in \mathbb{R}$ ,  $y_c \in \mathbb{R}$  are state vector, control input and output, respectively. Functions  $f_l$  and  $g_l$  are smooth vector fields. For a given continuous function  $h_l : \mathbb{R}^3 \rightarrow \mathbb{R}$ , Lie derivative is given by  $L_{f_l} h_l(w) = \frac{\partial h_l}{\partial w} f_l(w; \alpha)$ . Then, state feedback control

$$u_L(w; \alpha) = \frac{(-L_{f_l}^p h_l(w) + v_L)}{L_{g_l} L_{f_l}^{p-1} h_l(w)} \quad (3)$$

induces a linear behavior, where  $v_L = (-k_1 y_c - k_2 y_c^{(1)} - \dots - k_p y_c^{(p-1)})$ . The constants  $k_i$  ( $i = \{1, 2, \dots, p\}$ ) are such that  $s^p + k_p s^{p-1} + \dots + k_2 s + k_1$  is a Hurwitz polynomial.

**Backstepping Control:** This technique is based on solving a sequence of first-order systems succeeding is backward configuration. The method starts by writing the P-class system at the form:

$$\dot{\varpi} = f_{\varpi}(\varpi; \alpha) + g_{\varpi}(\varpi; \alpha)\varsigma \quad (4)$$

$$\dot{\varsigma} = f_{\varsigma}(\varpi, \varsigma; \alpha) + g_{\varsigma}(\varpi, \varsigma; \alpha)u_B \quad (5)$$

where  $(\varpi, \varsigma) \in \mathbb{R}^2 \times \mathbb{R}$  is state vector and  $u_B \in \mathbb{R}$  stands for control. Functions  $f_{\varpi} : \mathbb{D} \rightarrow \mathbb{R}^2$  and  $g_{\varpi} : \mathbb{D} \rightarrow \mathbb{R}^2$  are smooth in  $\mathbb{D} \subset \mathbb{R}^2$  that contains  $\varpi = 0$  and  $f_{\varpi}(0; \alpha) = 0$ . System (4)-(5) is a cascade connection of two components. First component (4) with  $\varsigma$  as the virtual control. Second component is the integrator (5). Assuming that (4) can be stabilized by a smooth state feedback control  $\varsigma = \phi_B(\varpi)$ , with  $\phi_B(0) = 0$ , the origin of  $\dot{\varpi} = f_{\varpi}(\varpi; \alpha) + g_{\varpi}(\varpi; \alpha)\phi_B(\varpi)$  is asymptotically stable [13]. Thus, by means of the change of variables  $z_B = \varsigma - \phi_B(\varpi)$ , it follows that system

$$\begin{cases} \dot{\varpi} = [f_{\varpi}(\varpi; \alpha) + g_{\varpi}(\varpi; \alpha)\phi_B(\varpi)] + g_{\varpi}(\varpi; \alpha)z_B \\ \dot{z}_B = u_B - \dot{\phi}_B \end{cases} \quad (6)$$

can be derived. Hence, by computing  $\dot{\phi}_B$ , and using the Lyapunov function  $V_B(\varpi)$ , the state feedback control law can be derived

$$u_B(\varpi, \varsigma; \alpha) = \frac{\partial \phi_B}{\partial \varpi} [f_{\varpi}(\varpi; \alpha) + g_{\varpi}(\varpi; \alpha)\varsigma] - \frac{\partial V_B}{\partial \varpi} g_{\varpi}(\varpi; \alpha) - k[\varsigma - \phi_B(\varpi)] \quad (7)$$

where  $k$  is a positive real constant.

**Sliding-mode Control:** This technique has been used due to its robustness properties. An important feature is that control leads system trajectories at finite time towards a manifold and holds them on it. The manifold is constructed in terms of desired reference. Then, once the trajectories reach the sliding manifold, they tend to desired reference. P-class is written in the form to design controller

$$\dot{\eta} = f_{\eta}(\eta, \xi; \alpha) + \delta_{\eta}(\eta, \xi; \alpha) \quad (8)$$

$$\dot{\xi} = f_{\xi}(\eta, \xi; \alpha) + G_{\xi}(\eta, \xi; \alpha)[u_S + \delta_{\xi}(\eta, \xi, u_S; \alpha)] \quad (9)$$

where  $\eta \in \mathbb{R}^2$ ,  $\xi \in \mathbb{R}$ ,  $u_S \in \mathbb{R}$  and  $\delta_{\eta}$  and  $\delta_{\xi}$  denote uncertainties. Let us consider the subsystem (8), where  $\xi$  is interpreted as a secondary control. By defining  $\xi = \phi_S(\eta)$ , where  $\phi_S(\eta)$  is a smooth function satisfying  $\phi_S(0) = 0$ , the origin of (8) is asymptotically stable. Now, let us take  $z_S = \xi - \phi_S(\eta)$ . Control  $u$  entering into (9) leads  $z_S(t)$  to zero, in finite time, holding it along time. Note dynamics of  $z_S$  is  $\dot{z}_S = f_{\xi}(\eta, \xi; \alpha) + G_{\xi}(\eta, \xi; \alpha)[u_S + \delta_{\xi}(\eta, \xi, u; \alpha)] - \frac{\partial \phi_S}{\partial \eta}[f_{\eta}(\eta, \xi; \alpha) + \delta_{\eta}(\eta, \xi; \alpha)]$ . Hence, the controller becomes

$$u_S(\eta, \xi; \alpha) = u_{eq}(\eta, \xi; \alpha) + G_{\xi}^{-1}(\eta, \xi; \alpha)v(\eta, \xi) \quad (10)$$

where  $u_{eq}(\eta, \xi; \alpha) = G_{\xi}^{-1}(\eta, \xi; \alpha)[-f_{\xi}(\eta, \xi; \alpha) + \frac{\partial \phi_S}{\partial \eta}f_{\eta}(\eta, \xi; \alpha)]$  is chosen to cancel terms in  $G_{\xi}(\eta, \xi; \alpha)$ .  $v(\eta, \xi)$  is determined by substituting  $u_S(\cdot)$  into  $\dot{z}_S$ -equation to have

$$\dot{z}_S = v(\eta, \xi) + \Delta(\eta, \xi, v; \alpha) \quad (11)$$

Assuming  $\Delta$  satisfies  $\|\Delta(\eta, \xi, v; \alpha)\|_{\infty} \leq \rho_S(\eta, \xi; \alpha) + k\|v\|_{\infty}$ , where  $\rho_S(\eta, \xi; \alpha) \geq 0$  and  $k \in [0, 1)$  are known. Then,  $v(\eta, \xi)$  is designed such that  $z_S(t)$  is forced towards the manifold  $z_S = 0$ . In particular,  $v(\eta, \xi)$  is chosen to be  $v(\eta, \xi) = -\frac{\beta(\eta, \xi)}{1-k} \text{sgn}(z_S)$ , where  $\beta(\eta, \xi) \geq \rho_S(\eta, \xi; \alpha) + b_o$  for any constant  $b_o > 0$ , and  $\text{sgn}(\cdot)$  is the signum function. Closed-loop stability of (11) can be ensured by taking the following Lyapunov function  $V_S = \frac{1}{2}z_S^2$ . As chattering is exhibited, as higher order sliding-mode can be designed to improve robustness and to diminish chattering.

### 3.1 Adaptive Observer and closed-loop stability

P-class system has only one unknown parameter and a single output  $y_m \in \mathbb{R}$  available for feedback. Under its features, this P-class can be represented as a state affine system with

unknown parameters as in [14, 15]:

$$\begin{cases} \dot{z} = A(u, y_m)z + \varphi(u, y_m) + \Phi(u, y_m)\theta \\ y_m = Cz \end{cases} \quad (12)$$

where entries of  $A(u, y_m)$ ,  $\varphi(u, y_m)$  and  $\Phi(u, y_m)$  are continuous functions depending on  $u$  and  $y_m$  uniformly bounded and  $\theta$  is a vector of unknown parameters. The following assumptions are introduced [16]:

**Assumption 1** *There exists a bounded time-varying matrix  $K(t)$  such that the following system  $\dot{\Lambda}(t) = (A(t) - K(t)C(t))\Lambda(t)$  is exponentially stable.*

**Assumption 2** *The solution  $\Lambda(t)$  of  $\dot{\Lambda}(t) = [A(t) - K(t)C(t)]\Lambda(t) + \Phi(t)$  is persistently exciting in the sense that exist  $\alpha_1, \beta_1, T_1$  such that*

$$\alpha_1 I \leq \int_t^{t+T_1} \Lambda(\tau)^T C^T \Sigma C(\tau) \Lambda(\tau) d\tau \leq \beta_1 I \quad (13)$$

for some bounded positive definite matrix  $\Sigma$ .

**Assumption 3** *Control  $u$  is persistently exciting in the sense that there exist  $\alpha_2, \beta_2, T_2 > 0$  and  $t_0 \geq 0$  such that:*

$$\alpha_2 I \leq \int_t^{t+T_2} \Psi_u(\tau, t)^T C^T \Sigma C(\tau) \Psi_u(\tau, t) d\tau \leq \beta_2 I \quad (14)$$

$\forall t \geq t_0$ , where  $\Psi_u$  denotes the transition matrix for the system  $\dot{z} = A(u, y_m)z$ ,  $y_m = Cz$ , and  $\Sigma$  some positive definite bounded matrix.

From (14) and (13) with  $K = S^{-1}C^T$  and  $S$  as solution of  $\dot{S} = -\rho S - A(u, y_m)^T S - SA(u, y_m) + C^T \Sigma C$ , adaptive observer is given by

$$\begin{cases} \dot{\hat{z}} = A(u, y_m)\hat{z} + \varphi(u, y_m) + \Phi(u, y_m)\hat{\theta} + \{\Lambda S_\theta^{-1} \Lambda^T C^T + S_z^{-1} C^T\} \Sigma (y_m - C\hat{z}) \\ \dot{\hat{\theta}} = S_\theta^{-1} \Lambda^T C^T \Sigma (y_m - C\hat{z}) \\ \dot{\Lambda} = \{A(u, y_m) - S_z^{-1} C^T C\} \Lambda + \Phi(u, y_m) \\ \dot{S}_z = -\rho_z S_z - A(u, y_m)^T S_z - S_z A(u, y_m) + C^T \Sigma C \\ \dot{S}_\theta = -\rho_\theta S_\theta + \Lambda^T C^T \Sigma C \Lambda \end{cases} \quad (15)$$

where  $S_z(0) > 0$  and  $S_\theta(0) > 0$ , and  $\rho_z$  and  $\rho_\theta$  are positive constants sufficiently large and  $\Sigma$  some bounded positive definite matrix. Moreover, it is remarkable that if Assumptions 2 and 3 are verified, they ensures that the matrices  $S_z$  and  $S_\theta$  are invertible and symmetric positive definite. Now, we can establish the following result.



**Lemma 1** *Let us consider system (12). Suppose 1, 2 and 3 hold. Then, system (15) is an adaptive observer for system (12). That is, estimation error vector ( $e_z := \hat{z} - z$ ,  $\epsilon_\theta := \hat{\theta} - \theta$ ) converges exponentially to zero with a rate given by  $\rho = \min(\rho_z, \rho_\theta)$ .*

**Sketch of the Proof.** Let  $e_z := \hat{z} - z$  and  $\epsilon_\theta := \hat{\theta} - \theta$  be the convergence errors for states and parameter, respectively. Defining  $\epsilon_z = e_z - \Lambda\epsilon_\theta$ , it follows that

$$\dot{\epsilon}_z = \{A(u, y_m) - \Lambda S_\theta^{-1} \Lambda^T C^T \Sigma C - S_z^{-1} C^T \Sigma C\} \epsilon_z + \Phi(u, y_m) \epsilon_\theta - \dot{\Lambda} \epsilon_\theta - \Lambda \dot{\epsilon}_\theta$$

Replacing suitable expressions in above equation, we get:

$$\begin{cases} \dot{\epsilon}_z &= \{A(u, y_m) - S_x^{-1} C^T \Sigma C\} \epsilon_z \\ \dot{\epsilon}_\theta &= -S_\theta^{-1} \Lambda^T C^T \Sigma C (\epsilon_z + \Lambda \epsilon_\theta) \end{cases}$$

We consider  $V(\epsilon_z, \epsilon_\theta) = \epsilon_z^T S_z \epsilon_z + \epsilon_\theta^T S_\theta \epsilon_\theta$  as a Lyapunov function to prove observer convergence. Then, taking  $\dot{V}$  and replacing appropriated expressions, we obtain  $\dot{V}(\epsilon_z, \epsilon_\theta) \leq -\rho_z \epsilon_z^T S_z \epsilon_z - \rho_\theta \epsilon_\theta^T S_\theta \epsilon_\theta$ . Taking  $\rho = \min(\rho_z, \rho_\theta)$ , we have  $\dot{V}(\epsilon_z, \epsilon_\theta) \leq -\rho V(\epsilon_z, \epsilon_\theta)$ . Finally,  $\epsilon_z$  and  $\epsilon_\theta$  converge to zero exponentially with a rate given by  $\rho$ . This ends the proof. ■

Note that the chaotic system can be represented in the following general form

$$\dot{\mathbf{x}} = f(\mathbf{x}; \alpha), \quad y_m = C\mathbf{x} \quad (18)$$

where  $\mathbf{x}$  is state vector,  $y_m$  is measured output and constant  $\alpha$  is a real parameter whose exact value is unknown. By means of a change of coordinates ( $z = T(\mathbf{x})$ ), (18) can be transformed into a state-affine system (12), for which is possible to design adaptive observer (15). Now, closed-loop system is analyzed with the three first controllers (3), (7), and (10) based on adaptive observer (15). From system (12), we have that by extending the state vector by parameters vector  $\theta$ , into  $Z := (z \ \theta)^T$ , state affine structure is preserved as follows:

$$\begin{cases} \dot{Z} &= F(\vartheta)Z + G(\vartheta) \\ y_m &= HZ \end{cases} \quad (19)$$

where  $\vartheta := (u \ y_m)$ ,  $H = (C \ 0)$ ,  $F(\vartheta) = \begin{pmatrix} A(\vartheta) & \Phi(\vartheta) \\ 0 & 0 \end{pmatrix}$ , and  $G(\vartheta) = \begin{pmatrix} \varphi(\vartheta) \\ 0 \end{pmatrix}$ .

Thus, extended system is given by

$$\begin{cases} \dot{Z} = F(\vartheta(\hat{Z}))Z + G(\vartheta(\hat{Z})) \\ \dot{\hat{Z}} = F(\vartheta(\hat{Z}))\hat{Z} + G(\vartheta(\hat{Z})) + S^{-1}H^T(y_m - H\hat{Z}) \\ \dot{S} = -\rho S - F^T(\vartheta(\hat{Z}))S - SF(\vartheta(\hat{Z})) + H^T H \end{cases} \quad (20)$$

where  $S = \begin{pmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{pmatrix}$ , and  $S_z, S_\theta, \Lambda$  of (15) are related to solution  $S$  through:

$$\begin{cases} S_z = S_1 \\ S_\theta = S_3 - S_2^T S_1^{-1} S_2 \\ \Lambda = -S_1^{-1} S_2 \end{cases} \quad (21)$$

Let us define  $e := \hat{Z} - Z$  be the estimation error. Then, the dynamics of resulting observer-based controller can be rewritten as:

$$\begin{cases} \dot{e} = \{F(\vartheta(\hat{Z})) - S^{-1}H^T H\}e \\ \dot{Z} = F(\vartheta(\hat{Z}))Z + G(\vartheta(\hat{Z})) \\ \dot{S} = -\theta S - F^T(\vartheta(\hat{Z}))S - SF(\vartheta(\hat{Z})) + H^T H \end{cases} \quad (22)$$

where  $u(\hat{Z})$  is the controller given by (3), (7), or (10) for each case. Next lemma is stated to prove closed-loop stability:

**Lemma 2** [16] *Assume that  $\vartheta$  is regularly persistent for (19) and let be*

$$\dot{S}(t) = -\theta S(t) - F^T(\vartheta(\hat{Z}))S(t) - S(t)F(\vartheta(\hat{Z})) + H^T H$$

*a Lyapunov differential equation, with  $S(0) > 0$ . Then,  $\exists \theta_0 > 0$  such that for any symmetric positive definite matrix  $S(0); \forall \theta \geq \theta_0$*

$$\exists \bar{\alpha} > 0, \bar{\beta} > 0, t_0 > 0 : \forall t > t_0$$

$\bar{\alpha}I \leq S(t) \leq \bar{\beta}I$  where  $S(t)$  satisfies  $\dot{S}(t)$  ■.

Although the formal proof of Lemma 2 can be found in [16], we show the rationale behind such a formalism to seek completeness in presentation. Consider two continuous functions  $f_1, f_2$ ; therefore  $\exists \alpha_i, \beta_i T_i$  such that  $\beta_i I \geq \int_t^{t+T_i} f_i(\tau, t)^* f_i(\tau, t) d\tau \geq \alpha_i I \Leftrightarrow \exists \alpha_j, \beta_j T_j$  such that

$$\beta_j I \geq \int_t^{t+T_j} \begin{pmatrix} f_1(\tau, t)^* \\ f_2(\tau, t)^* \end{pmatrix} (f_1(\tau, t) \ f_2(\tau, t)) d\tau \geq \alpha_j I$$

for  $i, j = 1, 2 \wedge t \geq t_0$ . The implication  $\Rightarrow$  can be verified directly by developing the product inside the integral. Now, for  $\Leftarrow$ , let us consider any vector  $z_1 \neq 0$  and  $z_2 \neq 0$  of same dimension as  $x, \theta$ , respectively, and define  $F_i = f_i z_i$ , with  $i = 1, 2$ . Note  $F_1, F_2$  are both of same dimension as output  $y_m$ . Then, let us take a  $T \triangleq \max(T_1, T_2)$  and notice that Cauchy-Schwartz inequality yields

$$\left( \int_t^{t+T} F_1(\tau, t) F_2(\tau, t) d\tau \right)^2 < \left( \int_t^{t+T} F_1(\tau, t)^* F_1(\tau, t) d\tau \right) \left( \int_t^{t+T} F_2(\tau, t)^* F_2(\tau, t) d\tau \right)$$

Next, let us define  $\gamma(\mu) \triangleq \left( \int_t^{t+T} F_1(\tau, t)^* F_1(\tau, t) d\tau \right) \left( \int_t^{t+T} F_2(\tau, t)^* F_2(\tau, t) d\tau \right) - \mu \int_t^{t+T} F_1(\tau, t)^* F_1(\tau, t) d\tau \|z_2\|^2 - \mu \int_t^{t+T} F_2(\tau, t)^* F_2(\tau, t) d\tau \|z_1\|^2 + \mu^2 \|z_1\|^2 \|z_2\|^2 - \left( \int_t^{t+T} F_1(\tau, t) F_2(\tau, t) d\tau \right)^2$ . Thus,  $\gamma$  is continuous in  $\mu$ , and, from Cauchy-Schwartz inequality,  $\gamma(\mu) > 0$ . Hence, there exists  $\mu_0 > 0$  such that  $\gamma(\mu) > 0$  for any  $-\mu_0 < \mu < \mu_0$ . Moreover,  $\exists \mu_1 : 0 < \mu_1 \leq \mu_0$  such that  $\int_t^{t+T} F_1(\tau, t)^* F_1(\tau, t) d\tau \geq \mu_1 \|z_1\|^2 \wedge \int_t^{t+T} F_2(\tau, t)^* F_2(\tau, t) d\tau \geq \mu_1 \|z_2\|^2$ , which yields

$$\begin{pmatrix} \int_t^{t+T} F_1(\tau, t)^* F_1(\tau, t) d\tau - \mu_1 \|z_1\|^2 & \int_t^{t+T} F_1(\tau, t)^* F_2(\tau, t) d\tau \\ \int_t^{t+T} F_2(\tau, t)^* F_1(\tau, t) d\tau & \int_t^{t+T} F_2(\tau, t)^* F_2(\tau, t) d\tau - \mu_1 \|z_2\|^2 \end{pmatrix} \geq 0$$

Conclusion on lower bound of  $S(t)$ , in Lemma 2, can be derived by following this rationale. The upper bound conclusion is quite clear from definition of  $f_1, f_2$ . Therefore, the following result can be established:

**Theorem 1** *Under assumptions that nominal controller is globally asymptotically stable and that state  $Z$  in (20) remains for all  $t > t_0 > 0$  in a compact set  $\Omega$  (containing equilibrium point of nominal controller)  $\forall Z(0) \in \Omega$ , extended system (20) is globally asymptotically stable on  $\Omega \times \mathbb{R}^n \times S_n^+$  (i.e.,  $\forall Z(0) \in \Omega, \forall \hat{Z}(0) \in \mathbb{R}^n; \forall S(0) > 0$ ).*

**Proof.** Since observer (15) is such that estimation error goes to zero, it is bounded and matrix  $S$  is solution of the Lyapunov differential equation in (20), is bounded from above and from below in set of positive definite matrices, see Lemma 2. Hence, whole state  $e = (\hat{Z} - Z, \hat{Z}, S)$  of (20) remains in a compact set along any trajectory.

Let  $\Lambda = \{(e(t), \hat{Z}(t), S(t)), t \geq 0\}$  be a semitrajectory of observer-based controller given by (20). This semitrajectory, lying in a compact set, has a nonempty  $\omega$ -limit set (i.e.,  $\omega$ -limit set of a trajectory is set of its accumulation points). Let  $[\bar{e}, \bar{Z}, \bar{S}]$  be an element of  $\omega$ -limit set of  $\Lambda$ . It is clear that when  $e \rightarrow 0$  implies that  $\bar{e} = 0$ . Let

$\{(0, \hat{Z}(t), S(t)), t \geq 0\}$  be a semitrajectory starting at time  $t = 0$  from  $[0, \bar{Z}, \bar{S}]$ . Since the  $\omega$ -limit set is positively invariant, it follows that the semitrajectory  $\{(0, \hat{Z}(t), S(t)), t \geq 0\}$  belongs to  $\omega$ -limit set of  $\Lambda$ . The estimation error is here equal to zero for this semitrajectory, and using our closed-loop stability assumption,  $\hat{Z}$  is globally asymptotically stable, i.e.,  $\hat{Z}(t) \rightarrow Z^* = \psi(Z^*)$ . So, there are points at which  $e = 0$  and  $\hat{Z} = Z^*$  in  $\omega$ -limit set of  $\Lambda$ , since it is a closed set. Letting  $[0, Z^*, \bar{S}(t)]$  be an element of  $\omega$ -limit set of  $\Lambda$  and following same reasoning: let  $\{(0, Z^*, S(t)), t \geq 0\}$  be a semitrajectory starting at  $t = 0$  from  $[0, Z^*, \bar{S}]$ . This semitrajectory belongs to  $\omega$ -limit set of  $\Lambda$ . The dynamics of  $S(t)$  are given by Lyapunov differential equation and using the observability of constant linear system  $(F^* + G^*, H)$ , that  $S(t)$  tends to  $S^*$ , unique positive definite solution of Lyapunov algebraic equation. So,  $[0, Z^*, S^*]$  belongs to  $\omega$ -limit set of  $\Lambda$ . It follows that, under the assumption of (local) asymptotic stability of (20),  $\Lambda$  enters in a finite time into the basin of attraction of  $[0, Z^*, S^*]$ . Hence (20) is globally asymptotically stable on  $\Omega \times \mathbb{R}^n \times S_n^+$  ■.

## 4 An adaptive low-order parametrization scheme to compare execution

Above controllers are compared with other adaptive scheme previously reported [5]. This last adaptive scheme compensates uncertainties in parameters and unmeasured states with a low order equation. Let us re-write P-class system (1) in the canonical form (named strict-feedback form):  $\dot{\chi}_i = \chi_{i+1}$  with  $1 \leq i \leq p - 1$ ,  $\dot{\chi}_p = f_p(\chi, \nu; \alpha) + \Gamma(\chi, \nu; \alpha)u_A$  and  $\dot{\nu} = \zeta(\chi, \nu)$ . Then, by exploiting Lie algebra of vectors fields, P-class system can be stabilized as  $p$  is a integer constant such that  $\Gamma(\chi, \nu; \alpha) \neq 0$  at any point belonging to domain (see details in [5]). Functions  $f_p(\chi, \nu; \alpha)$  and  $\Gamma(\chi, \nu; \alpha)$  are assumed uncertain and unavailable for feedback. Thus, the  $\gamma(\chi, \nu; \alpha) = \Gamma(\chi, \nu; \alpha) - \Gamma_{nom}(\chi)$  and  $\Theta(\chi, \nu, u_A; \alpha) = f_p(\chi, \nu; \alpha) + \gamma(\chi, \nu; \alpha)u_A$  can be defined. After algebraic manipulations, we have:

$$\begin{cases} \dot{\chi}_i = \chi_{i+1}, & 1 \leq i \leq p - 1 \\ \dot{\chi}_p = \Theta(\chi, \nu, u_A; \alpha) + \Gamma_{nom}(\chi)u_A \\ \dot{\nu} = \zeta(\chi, \nu) \end{cases} \quad (23)$$

where  $\Theta(\chi, \nu, u_A; \alpha)$  is a continuous function that lumps the uncertain terms. Lumping function  $\Theta$  is seen as an augmented state by defining  $\Theta \equiv \Theta(x, \nu, u_A; \alpha)$ . So, system (23)

can be rewritten in the augmented form

$$\begin{cases} \dot{\chi}_i = \chi_{i+1}, & 1 \leq i \leq p-1 \\ \dot{\chi}_p = \Theta + \Gamma_{nom}(\chi)u_A \\ \dot{\Theta} = \Xi(x, \nu, \Theta, u_A; \alpha) \\ \dot{\nu} = \zeta(\chi, \nu) \end{cases} \quad (24)$$

where  $\Xi(x, \nu, \Theta, u_A; \alpha) = \sum_{i=1}^{p-1} \chi_{i+1} \partial_i \Theta(x, \nu, u_A; \alpha) + [\Theta + \Gamma_{nom}(\chi)u_A] \partial_p \Theta(x, \nu, u_A; \alpha) + \gamma(\chi, \nu; \alpha) \dot{u}_A + \partial_t \gamma(\chi, \nu; \alpha) u_A + \partial_\nu \Theta(x, \nu, u_A; \alpha) \zeta(\chi, \nu)$ .

Controller  $u_A(\chi, \Theta) = \frac{1}{\Gamma_{nom}(\chi)}(-\Theta - K^T \chi)$  can be used to stabilize (24), where  $K \in \mathbb{R}^p$  is chosen in such way that  $P_K(s) = s^p + k_p s^{p-1} + \dots + k_2 s + k_1 = 0$  is Hurwitz. Now, problem of estimating  $(\chi, \Theta)$  can be addressed by using a high-gain observer for (24). Thus, dynamics of  $\chi$  and  $\Theta$  is reconstructed from measurements of output  $y_m = \chi_1$  by

$$\begin{cases} \dot{\hat{\chi}}_i = \hat{\chi}_{i+1} + L^i \kappa_i (\chi_1 - \hat{\chi}_1), & 1 \leq i \leq p-1 \\ \dot{\hat{\chi}}_p = \hat{\Theta} + \Gamma_{nom}(\hat{\chi})u_A + L^p \kappa_p (\chi_1 - \hat{\chi}_1) \\ \dot{\hat{\Theta}} = L^{p+1} \kappa_{p+1} (\chi_1 - \hat{\chi}_1) \end{cases} \quad (25)$$

where  $\kappa_j, j = 1, 2, \dots, p+1$ , are such that  $P_\kappa(s) = s^{p+1} + \kappa_1 s^p + \dots + \kappa_p s + \kappa_{p+1} = 0$  is also Hurwitz. Parameter  $L > 0$  stands for estimation rate of uncertainties, being unique tuning parameter. Finally, control becomes

$$u_A(\hat{\chi}, \hat{\Theta}) = \frac{1}{\Gamma_{nom}(\hat{\chi})}(-\hat{\Theta} - K^T \hat{\chi}) \quad (26)$$

## 5 Experimental implementation

Implementation is realized to compare controllers performance. Let us consider only one of chaotic systems in collection  $(\Sigma_2)$ . Only  $y_m = x_2$  is measured and parameter  $\alpha$  is uncertain, the problem is to lead output  $y_c = x_1$  (note  $y_m \neq y_c$ ) when  $u$  is acting on right-side of  $(\Sigma_2)$  as follows

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -\alpha x_3 - x_1 + x_1 x_2 + u \end{cases} \quad (27)$$

## 5.1 Implemented controllers

**State feedback linearizing control:** For (27), we have that  $f_l(w; \alpha) = [x_2, x_3, -\alpha x_3 - x_1 + x_1 x_2]^T$  and  $g_l(w; \alpha) = [0, 0, 1]^T$ . Then, designed controller is given by

$$u_L(w; \alpha) = \alpha x_3 + x_1 - x_1 x_2 - k_1 x_1 - k_2 x_2 - k_3 x_3 \quad (28)$$

**Backstepping control:** Let us write the subsystem

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \end{cases} \quad (29)$$

which is in form (4), where  $\bar{w} = x_1$ ,  $\bar{\varsigma} = x_2$  and  $\bar{u}_B = x_3$ . By proposing  $x_2 = \bar{\phi}_B(\bar{w}) = -\mu x_1$ , where  $\mu$  is a positive real constant, and  $\bar{V}_B(\bar{w}) = \frac{1}{2} x_1^2$  is a Lyapunov function, then the controller (7) for (29) is given by

$$\bar{u}_B(\bar{w}, \bar{\varsigma}) = x_3 = -\mu x_2 - x_1 - \bar{k}(x_2 + \mu x_1) = \phi_B(x_1, x_2) \quad (30)$$

with  $\bar{k} > 0$ . Now, let us consider the complete system (27) re-written as

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_3 \\ \dot{x}_3 = (-\alpha x_3 - x_1 + x_1 x_2) + u_B \end{cases} \quad (31)$$

which is in form (4)-(5), where  $\varpi = (x_1 \ x_2)^T$  and  $\varsigma = x_3$ . Taking (30),  $k > 0$ , and  $V_B(\varpi) = \frac{1}{2} x_1^2 + \frac{1}{2} [x_2 + \mu x_1]^2$  as Lyapunov function, then controller (7) for system (27) is given by:

$$\begin{cases} u_B(\varpi, \varsigma; \alpha) = u_{Ba} - (1 + \bar{k}\mu)x_2 - (\mu + \bar{k})x_3 - (x_2 + \mu x_1) \\ \quad \quad \quad -k[x_3 + (1 + \bar{k}\mu)x_1 + (\mu + \bar{k})x_2] \\ u_{Ba}(\varpi, \varsigma; \alpha) = \alpha x_3 + x_1 - x_1 x_2 \end{cases} \quad (32)$$

**Sliding-mode control:** By defining  $\eta = (x_1 \ x_2)^T$  and  $\xi = x_3$ , (27) can be written as

$$\dot{x}_1 = x_2 + \delta_{\eta_1} \quad (33)$$

$$\dot{x}_2 = x_3 + \delta_{\eta_2} \quad (34)$$

$$\dot{x}_3 = [-\alpha_0 x_3 - x_1 + x_1 x_2] + [u_S + \delta_\xi] \quad (35)$$

where  $\delta_{\eta_1}$ ,  $\delta_{\eta_2}$  and  $\delta_\xi$  are the uncertainties. Sliding surface is given by  $z_S = x_3 + \sigma_1 x_1 + \sigma_2 x_2$ , and from (35) and controller takes the form

$$\begin{cases} u_S(\eta, \xi) = u_{eq}(\eta, \xi) - \frac{\beta(\eta, \xi)}{(1-k)} \text{sgn}(x_3 + \sigma_1 x_1 + \sigma_2 x_2) \\ u_{eq}(\eta, \xi) = [\alpha_0 x_3 + x_1 - x_1 x_2 - \sigma_1 x_2 - \sigma_2 x_3] \\ \beta(\eta, \xi) = k_1 |x_1| |x_2| + k_2 |x_1| + k_3 |x_2| + k_4 |x_3| + b_o \end{cases} \quad (36)$$

Specifically, implementation of above controllers requires state estimation and parameter identification provided by adaptive observer given in Section 4. Then, system structure is transformed into

$$\begin{cases} \dot{z}_1 = -\alpha z_1 + z_2 \\ \dot{z}_2 = -z_3 + z_1 z_3 + u \\ \dot{z}_3 = z_1 \end{cases} \quad (37)$$

By defining  $\theta = \alpha$  and  $y_m = z_1$ , adaptive observer (15) is derived. Then, states and parameter are replaced in controllers (28), (32), and (36) becomes

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & z_1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} -z_1 \\ 0 \\ 0 \end{pmatrix} \alpha + \begin{pmatrix} 0 \\ u \\ z_1 \end{pmatrix}$$

**Scheme based on low-order parametrization:** By defining  $\chi = (x_1 \ x_2 \ x_3)^T$ ,  $\Theta = -\alpha x_3 - x_1 + x_1 x_2$  and  $\Gamma_{nom}(\chi) = 1$ ; system (27) is written as (24) to have:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = \Theta + u_A \\ \dot{\Theta} = \Xi(x, \Theta, u_A; \alpha) \end{cases} \quad (39)$$

Hence, (25) and (26) becomes

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + L\kappa_1(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 = \hat{x}_3 + L^2\kappa_2(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_3 = \hat{\Theta} + u_A + L^3\kappa_3(x_1 - \hat{x}_1) \\ \dot{\hat{\Theta}} = L^4\kappa_4(x_1 - \hat{x}_1) \end{cases} \quad (40)$$

$$u_A(\hat{\chi}, \hat{\Theta}) = -\hat{\Theta} - k_1 \hat{x}_1 - k_2 \hat{x}_2 - k_3 \hat{x}_3 \quad (41)$$

where  $L, \kappa_j$  ( $j = 1, 2, 3, 4$ ) are taken as (25). and  $k_i$  ( $i = 1, 2, 3$ ) are as in (26).

## 5.2 Performance index definition and details for implementation

Control execution can be evaluated via performance indexes. The underlying idea is to provide a quantitative measure of the control execution by emphasizing closed-loop specifications. Because our problem aims the chaos suppression, we consider the reference is given and constant. Thus, our attention can be paid onto time convergence indexes. Three different issues are relevant in chaos suppression: steady-state error, overshooting of control action, and average control effort. Steady-state error index discriminates between overdamped and underdamped behavior of the closed-loop system. This index is very convenient for analytical purposes due to it involves system trajectories providing a measure of "distance" to the reference. Overshooting of control action and average control effort provide measures about of the energy required to reach chaos suppression. Former measures the maximum energy needed by the controller for suppression while latter considers the long-term control requirements. Both overshooting control action and average control effort are relevant when control energy for chaos suppression is restricted. Next, three indexes are defined with time weight to normalize the time duration of the experiment:

- The first one is a measure of stabilization error, equivalent to an index of chaos suppression, during the interval  $[t_0, t_f]$ . This index is defined by  $J_s = \frac{1}{t_f - t_0} \int_{t_0}^{t_f} \{x(t)^T Q(t)x(t)\} dt$ , where  $Q(t)$  is a positive semi-definite symmetric matrix for all  $t \in [t_0, t_f]$ .  $Q(t) = I(1 - e^{-\lambda(t-t_0)})^q$  was chosen to assign major weight to steady-state error, where  $I$  is the identity matrix,  $\lambda > 0$ , and  $q$  is a positive integer.
- In second index, control effort is also measured by its overshoot; which is defined by infinity norm  $u_{max} = \max_{[t_0, t_f]} \text{abs}(u(t))$ .
- Finally, the last index is used to measure average control effort at implementation interval  $[t_0, t_f]$ . This index is defined by  $J_{ue} = \frac{1}{t_f - t_0} \int_{t_0}^{t_f} \{u(t)^T Q(t)u(t)\} dt$ .

System (27) was electronically realized by means of operational amplifiers TL084CN, an analog multiplier AD633JN and passive components. A DSpace 1104 acquisition data



board was used to measure control action, capture system state values along implementation time and estimate states and parameter values from adaptive observer. Figure 1 shows a photo of experimental setup including details of oscilloscopes depicting time series during experiments. Schematic of used circuit is in Figure 2. In this way, control schemes were implemented selecting the following parameters:

- *State feedback linearization* (28):  $k_1 = 27$ ,  $k_2 = 27$  and  $k_3 = 9$ .
- *Backstepping* (32):  $\mu = 1$ ,  $k = 1$  and  $\bar{k} = 1$ .
- *Sliding-mode* (36):  $\sigma_1 = 1$ ,  $\sigma_2 = 1$ ,  $k = 0.04$ ,  $k_1 = 0.34$ ,  $k_2 = 0.1$ ,  $k_3 = 0.08$ ,  $k_4 = 0.3224$ ,  $b_o = 0.01$  and  $\alpha_0 = 2.02$ .
- *Adaptive Observer* (15):  $\rho_z = 50$ ,  $\rho_\theta = 2$ ,  $S_z(0) = I$ ,  $S_\theta(0) = I$  and  $\Lambda(0) = [10, 10, 10]^T$ .
- *Robust control with low-order parametrization* (40)-(41):  $k_1 = 27$ ,  $k_2 = 27$ ,  $k_3 = 9$ ,  $L = 10$ ,  $\kappa_1 = 4$ ,  $\kappa_2 = 6$ ,  $\kappa_3 = 4$  and  $\kappa_4 = 1$ .

Experimental results are shown in the Figures 3, 4, 5, and 6. Now, a comparative study is presented for the four controllers. Performance indexes are evaluated. Table 1 shows indexes values from evaluation of each controller.

Table 1: Performance indexes from experimental implementation.

Control strategy	$J_s$	$u_{max}$	$J_{ue}$
State feedback linearization	0.026381	43.7965	0.0002980
Backstepping	0.030910	4.0440	0.0000097
Sliding-mode	0.199010	3.9620	7.771480
Low-order parametrization	0.818650	2.9216	0.601240

As a summary of experimental implementation, note the following remarks from Table 1: (i) State feedback linearization shows a small stabilization error  $J_s$  which is lightly smaller than Backstepping but has the largest control action overshooting. Although this controller shows the fastest response (see Figure 3), the largest overshoot can induce

saturation in actuator in specific implementations inducing closed-loop will be broken as consequence. (ii) The sliding-mode strategy exhibits acceptable convergence rate without overshooting but the largest control effort (see Figure 5). Actually, the control action includes high frequency signal demanding largest average control effort  $J_{ue}$ , which could be unsuitable for systems because of chattering phenomena. (iii) Robust control with low parametrization shows the lowest overshoot but the largest value for  $J_s$  as consequence of slowest response. This fact could be undesirable for implementations requiring faster suppression.

Since the lowest average control effort controller with fast response and moderated overshoot was found for Backstepping, therefore it seems as best candidate for chaos suppression implementations.

## 6 Conclusions

In this paper, a study of control performance has been shown in regard to chaos suppression. Three control strategies, based on an adaptive observer, have been presented to evaluate them: State feedback linearization, Backstepping and Sliding-mode. Furthermore, the convergence of the adaptive observer has been shown, where sufficient conditions have been given. Then, a stability analysis of the closed loop system has been presented. Additionally, a robust control with low-order parametrization has been also considered in this comparative study.

The comparative study of these schemes has been done considering three performance indexes, which have been taken into account to measure stabilization error, control effort, and average control effort. Experimental results were measured to evaluate the performance of each scheme. As a summary, state feedback with adaptive observer yields the lowest stabilization error but largest overshoot. The lowest average control effort was obtained by backstepping strategy. Lowest overshoot was exhibited by robust control with low-order parametrization, which had the largest stabilization error. It follows that backstepping method showed best performance.

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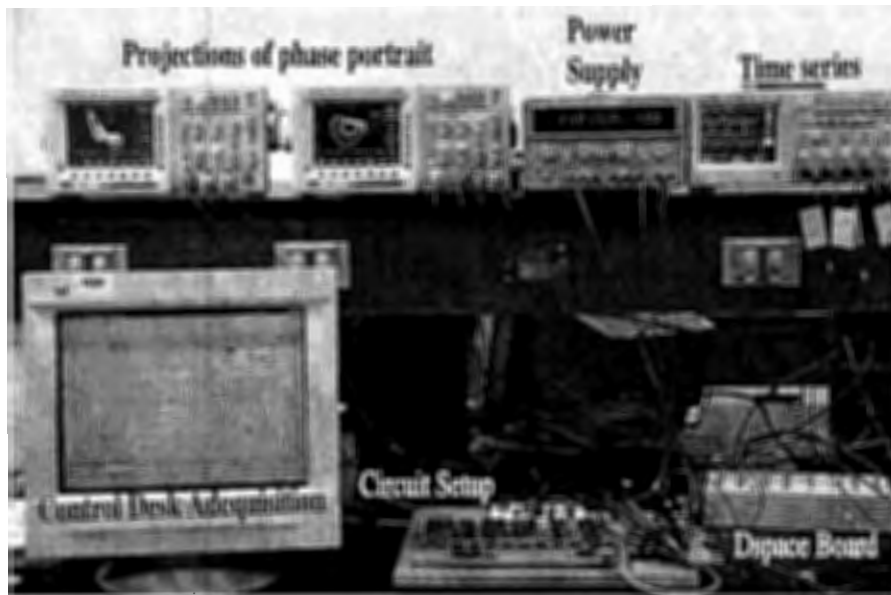


Figure 1: Photo of experimental setup.

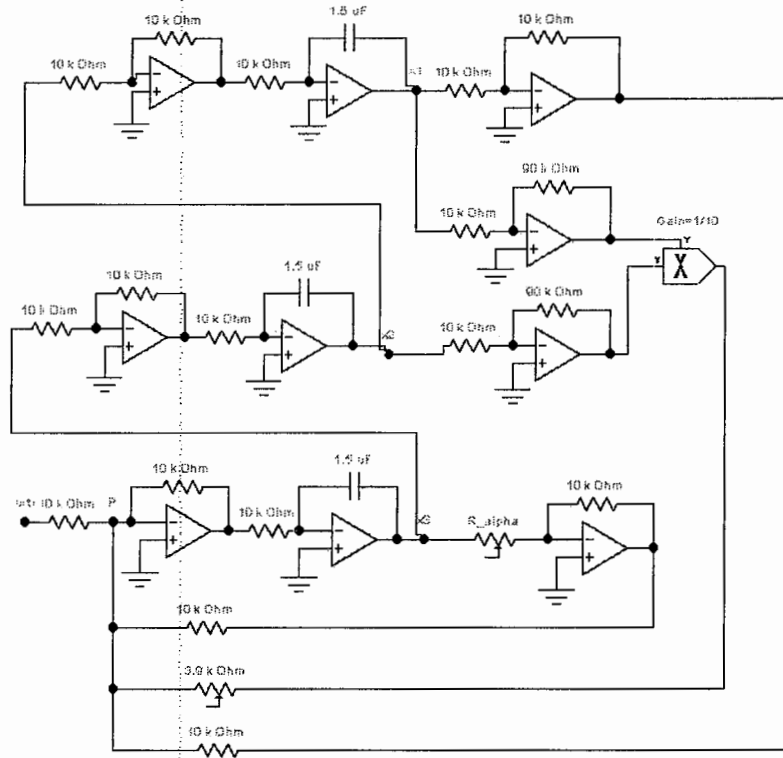


Figure 2: Schematic diagram for physical realization of the system (27).

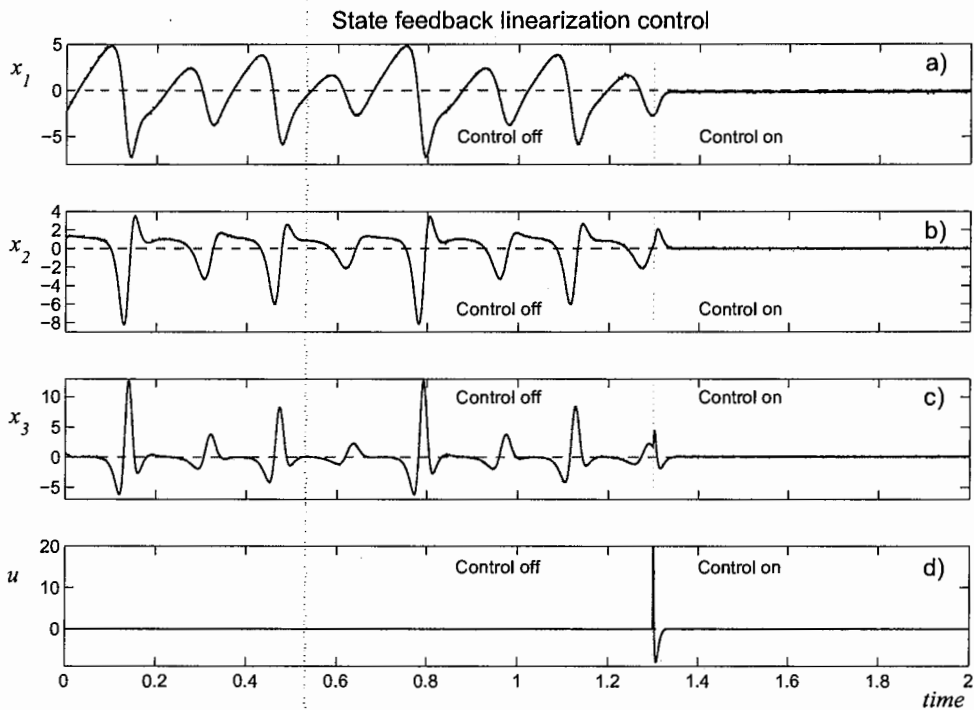


Figure 3: Responses with state feedback linearization control  $u_L$ .

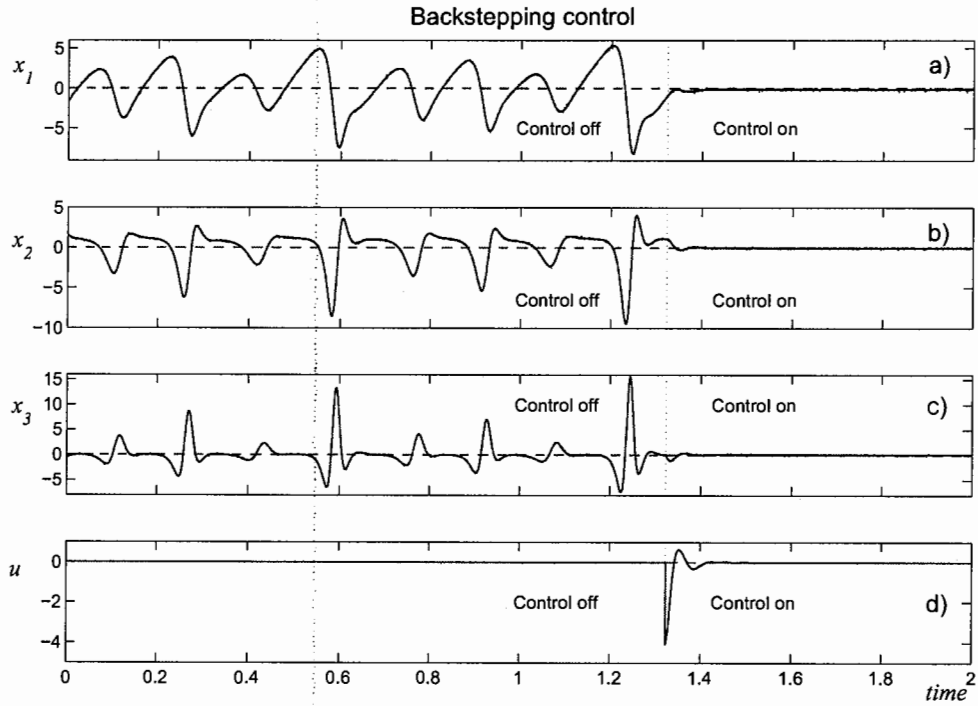


Figure 4: Responses with backstepping control  $u_B$ .

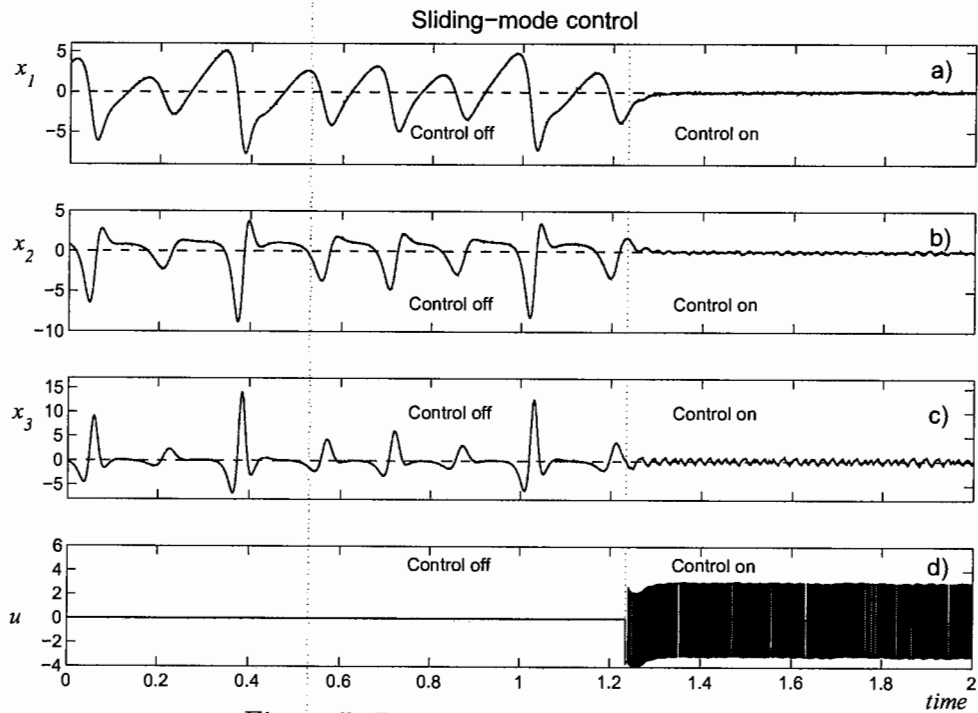


Figure 5: Responses with sliding-mode control  $u_S$ .

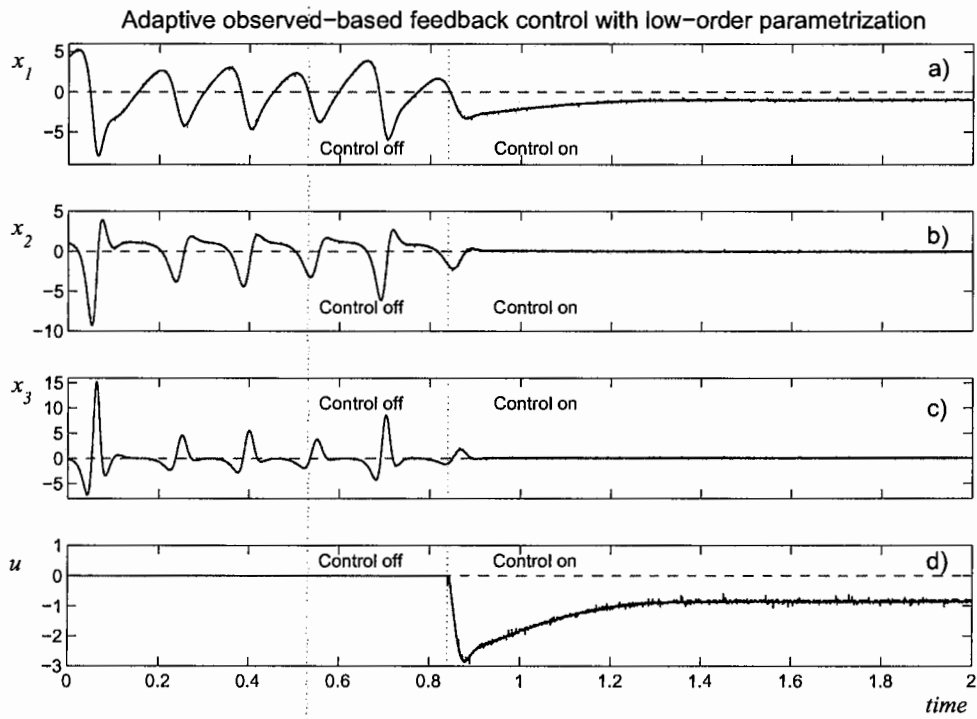


Figure 6: Responses with adaptive observed-based feedback control with low-order parametrization  $u_A$ .