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Integrable Abel equations and Vein's Abel equation

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We first reformulate and expand with several novel findings some of the basic results in the integrability of Abel equations. Next, these results are applied to Vein's Abel equation whose solutions are expressed in terms of the third order hyperbolic functions and a phase space analysis of the corresponding nonlinear oscillator is also provided.

Keywords: Abel equation; Appell invariant; normal form; canonical form; third order hyperbolic function.

I. INTRODUCTION

The first order Abel nonlinear equation of the second kind has the form

$$(w + s) \frac{dw}{dx} + p + q_1 w + q_2 w^2 + r w^3 = 0, \quad (1)$$

where p , q_1 , q_2 , r , and s are all some functions of x . The case $r = 0$ has been introduced more than two hundred years ago by Abel [1]. Many solvable equations of this type are collected in [2] and other ones can be found in more recent works [3–9]. The transformation $1/y = w + s$ converts this equation in the form

$$\frac{dy}{dx} = r + (q_2 - 3rs)y + (q_1 - \frac{ds}{dx} - 2q_2s + 3rs^2)y^2 + (p - q_1s + q_2s^2 - rs^3)y^3, \quad (2)$$

which is Abel's equation of the first kind. We see that Abel's original equation of the second kind is actually a homogeneous case of that of the first kind.

One can also write (2) in the canonical form

$$\frac{dy}{dx} = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3 = F(x, y), \quad (3)$$

where only the case $s = 0$ leads to the simple identifications: $f_0 = r$, $f_1 = q_2$, $f_2 = q_1$, and $f_3 = p$.

The integrability features of Abel's equation in the canonical form are extremely important because of its connection with nonlinear second order differential equations that phenomenologically describe a wide class of nonlinear oscillators which in this way can be treated analytically. In particular, Vein [10] introduced an Abel equation with $f_0 = 0$ and rational forms of f_1 , f_2 , and f_3 whose solutions are expressed in terms of third order hyperbolic functions. However, to the best of our knowledge, the paper of Vein went unnoticed for many years and only recently Yamaleev [11, 12] provided a generalization in the framework of third order multicomplex algebra. Needless to say, the corresponding Vein's nonlinear oscillator has not been studied in the literature. This was the main motivation for writing this paper, which is organized as follows. In section II, we present a lemma that encodes the connection of Abel equation with nonlinear oscillator equations and some related simple results. In section III, we reformulate several integrability results for Abel's equation with the purpose to apply them to Vein's Abel equation, which is the subject of section IV. The dynamical systems analysis of Vein's oscillator is developed in section V, and we end up with the conclusions.

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II. CONNECTION WITH SECOND ORDER NONLINEAR ODES

The known importance of Abel's equation in its canonical form (3) stems from the fact that its integrability leads to closed form solutions to a general nonlinear ODE of the form

$$x_\zeta \zeta + f_2(x)x_\zeta + f_3(x) + f_1(x)x_\zeta^2 + f_0(x)x_\zeta^3 = 0 , \quad (4)$$

where the variables $x(\zeta)$ and $y(x(\zeta))$ are some parametric solutions that depend on a generalized coordinate ζ . This can be expressed by the following lemma [9]

Lemma 1. Solutions to a general second-order ODE of type (4) may be obtained via the solutions to Abel's equation (3) and vice versa using the following relationship

$$\frac{dx}{d\zeta} = v(x(\zeta)) . \quad (5)$$

Proof. To show the equivalence, one just needs the chain rule

$$\frac{d^2x}{d\zeta^2} = \frac{dv}{dx} \frac{dx}{d\zeta} = v \frac{dv}{dx} \quad (6)$$

which turns (4) into the Abel equation of the second kind in canonical form

$$v \frac{dv}{dx} + f_3(x) + f_2(x)v + f_1(x)v^2 + f_0(x)v^3 = 0. \quad (7)$$

Via the inverse transformation

$$v(x(\zeta)) = \frac{1}{y(x(\zeta))} . \quad (8)$$

of the dependent variable, equation (7) becomes (3). Moreover, the linear term in (3) can always be eliminated via the transformation

$$y(x) = z(x)e^{\int f_1(x)dx} \quad (9)$$

which gives

$$\frac{dz}{dx} = h_0(x) + h_2(x)z^2 + h_3(x)z^3 = H(x, z) , \quad (10)$$

where

$$\begin{aligned} h_0(x) &= f_0(x)e^{-\int f_1(x)dx} \\ h_2(x) &= f_2(x)e^{\int f_1(x)dx} \\ h_3(x) &= f_3(x)e^{2\int f_1(x)dx} . \end{aligned} \quad (11)$$

The case where $f_0 = 0 \rightarrow r = 0$ is seen from (2) to be the one actually considered by Abel, for in this case the reduced Abel equation

$$(w + s) \frac{dw}{dx} + p + q_1 w + q_2 w^2 = 0 , \quad (12)$$

can be always put in the form

$$\frac{dz}{dx} = h_2(x)z^2 + h_3(x)z^3 \quad (13)$$

where $h_2(x)$ and $h_3(x)$ are reduced functions and are given by the expressions

$$\begin{aligned} h_2(x) &= (q_1 - \frac{ds}{dx} - 2q_2 s)e^{\int q_2(x)dx} \\ h_3(x) &= (p - q_1 s + q_2 s^2)e^{2\int q_2(x)dx} . \end{aligned} \quad (14)$$

Using the lemma, the reduced Abel equation corresponds to a linear ODE without higher-order dissipative terms

$$x_\zeta \zeta + h_2(x)x_\zeta + h_3(x) = 0 . \quad (15)$$

On the other hand, in the case of $f_3 = 0$, the Abel equation of the first-kind (3) becomes a Riccati equation, while the second-kind homogeneous Abel equation (7) is reduced to the Riccati equation

$$\frac{dv}{dx} = -f_2(x) - f_1(x)v - f_0(x)v^2 , \quad (16)$$

which is equivalent to

$$\frac{dy}{dx} = f_0(x) + f_1(x)y + f_2(x)y^2 . \quad (17)$$

This corresponds to the simple fact that an inverse power of a Riccati solution also satisfies a Riccati equation with redistributed coefficients. Now, we eliminate the linear part to obtain the reduced Riccati equation

$$\frac{dz}{dx} = h_0(x) + h_2(x)z^2 \quad (18)$$

which corresponds to a nonlinear ODE with higher-order dissipative terms

$$x_\zeta \zeta + h_2(x)x_\zeta + h_0(x)x_\zeta^3 = 0 , \quad (19)$$

where now the coefficients are

$$\begin{aligned} h_0(x) &= r e^{-\int (q_2 - 3rs) dx} \\ h_2(x) &= (q_1 - s' - 2q_2s + 3rs^2) e^{\int (q_2 - 3rs) dx} . \end{aligned} \quad (20)$$

III. SOME ABEL INTEGRABILITY CASES

A. Abel's equation of constant coefficients

Denote the coefficients $f_i(x) = A_i$ of (3), where $A_i \in \mathbb{R}$ are constants and $A_3 \neq 0$ so that $F(x, y) = F(y)$. It is obvious that the roots of the equation $F(y) = 0$ are themselves solutions of (3). More generally, the general solution of (3) is obtained via factorization of the denominator in the right-hand side of

$$\frac{dy}{y^3 + \frac{A_2}{A_3}y^2 + \frac{A_1}{A_3}y + \frac{A_0}{A_3}} = A_3 dx , \quad (21)$$

which leads to the following cases:

(i) If $y_1 \neq y_2 \neq y_3 \in \mathbb{R}$, so integration of (21) has the form

$$\int \frac{dy}{(y - y_1)(y - y_2)(y - y_3)} = (y - y_1)^{y_2 - y_3} (y - y_2)^{y_3 - y_1} (y - y_3)^{y_1 - y_2} = c_1 e^{A_3 x} , \quad (22)$$

(ii) If $y_1 \neq y_2 = y_3 \in \mathbb{R}$, then the integration of (21) is of the form

$$\int \frac{dy}{(y - y_1)(y - y_2)^2} = \frac{1}{(y_1 - y_2)(y - y_2)} + \frac{1}{(y_1 - y_2)^2} \ln \left| \frac{y - y_1}{y - y_2} \right| = A_3 x + c_2 , \quad (23)$$

(iii) If $y_1 = y_2 = y_3 \in \mathbb{R}$, so (21) becomes

$$\int \frac{dy}{(y - y_1)^3} = \frac{1}{-2(y - y_1)^2} = A_3 x + c_3 , \quad (24)$$

(iv) If $y_1 = \overline{y_2} = \alpha + i\beta \in \mathbb{C}$ and $y_3 \in \mathbb{R}$, the integration of (21) leads to

$$\int \frac{dy}{((y - \alpha)^2 + \beta^2)(y - y_3)} = \ln \left| \sqrt{(y - \alpha)^2 + \beta^2} \right| + \frac{\alpha - y_1}{\beta} \arctan \frac{y - \alpha}{\beta} = A_3 x + c_4 . \quad (25)$$

B. Integrability based on the normal form of Abel's equation

If the following transformations as given in Kamke's book [13]

$$\begin{aligned} y(x) &= \omega(x)\eta(\xi(x)) - \frac{f_2(x)}{3f_3(x)}, & \omega(x) &= e^{\int \left(f_1 - \frac{f_2^2}{3f_3}\right) dx}, \\ \xi(x) &= \int f_3 \omega^2 dx \end{aligned} \quad (26)$$

are applied to equation (3), then one obtains Abel's equation in normal form

$$\frac{d\eta}{d\xi} = \eta^3 + I(x), \quad (27)$$

where the invariant $I(x)$ is given by

$$I(x) = \frac{f_0 + \frac{1}{3} \frac{d}{dx} \left(\frac{f_2}{f_3} \right) - \frac{f_1 f_2}{3f_3} + \frac{2f_2^3}{27f_3^2}}{f_3 \omega^3}. \quad (28)$$

Thus, we conclude that if $I(x) \equiv \text{constant}$, then (27) is integrable since it is separable. If one chooses relations between the functions f_i such that the invariant is null and letting $f_2(x)/f_3(x) = n(x)$, then

$$\frac{dn}{dx} + \frac{2}{9} f_2 n^2 - f_1 n + 3f_0 = 0 \quad (29)$$

is a Riccati equation, which is always integrable because it is obtained from Abel's equation in normal form, which is integrable. Thus, the solution to (27) is

$$\eta = \frac{1}{\sqrt{c - 2\xi}}, \quad (30)$$

and explicitly, the solution to (3) with null invariant $I(x)$ is

$$y = \frac{e^{\int \left(f_1 - \frac{f_2^2}{3f_3}\right) dx}}{\sqrt{c - 2 \int f_3 \left(e^{\int \left(f_1 - \frac{f_2^2}{3f_3}\right) dx} \right)^2}} - \frac{f_2}{3f_3}. \quad (31)$$

C. Integrability of Abel equation with non-constant coefficients and $f_0 = 0$

In this subsection, we will consider the original Abel equation (2) of the first-kind

$$\frac{dy}{dx} = f_1(x)y + f_2(x)y^2 + f_3(x)y^3, \quad (32)$$

which corresponds to (4) without the cubic nonlinearity.

Let us use the transformation

$$y(x) = \frac{e^{\int f_1(x) dx}}{\nu(x)}, \quad (33)$$

which is (9) with $z = \frac{1}{\nu}$ that allows us to put (32) into a differential form

$$\nu d\nu + (P + Q\nu)dx = 0, \quad (34)$$

where

$$\begin{aligned} P(x) &= f_3(x)e^{2\int f_1(x) dx} \\ Q(x) &= f_2(x)e^{\int f_1(x) dx}. \end{aligned} \quad (35)$$

Thus, the original equation (1) is considerably reduced only by having $f_0 = 0$.

For equation (34), if there exists an integrating factor [14]

$$\mu(x, \nu) = e^{-k(\nu + \int Q(x) dx)}, \quad (36)$$

where k is a constant, then

$$(P + Q\nu)\mu dx + \nu\mu d\nu = 0, \quad (37)$$

is integrable, provided that

$$Q\mu + (P + Q\nu)\frac{\partial\mu}{\partial\nu} = \nu\frac{\partial\mu}{\partial x}, \quad (38)$$

which leads to $Q = kP$. Hence, the necessary condition for the integration of (32) is

$$kf_3(x)e^{\int f_1(x) dx} = f_2(x) \rightarrow \frac{d}{dx}\left(\frac{f_2}{f_3}\right) = f_1\frac{f_2}{f_3} \rightarrow \frac{dn}{dx} = f_1n \quad (39)$$

Using $\mu(x, \nu) = e^{-k\nu}e^{-k^2 \int P(x) dx}$, we find that there exists the potential $\Psi(x, \nu) = c_\Psi$, which satisfies

$$\begin{aligned} \frac{\partial\Psi}{\partial x} &= (1 + k\nu)P\mu, \\ \frac{\partial\Psi}{\partial\nu} &= \nu\mu. \end{aligned} \quad (40)$$

Therefore, the solution to (34) is

$$\Psi(x, \nu) = (1 + k\nu)e^{-k(\nu + k \int P dx)} = c_\Psi, \quad (41)$$

or in terms of the integrating factor

$$\Psi(x, \nu) = (1 + k\nu)\mu(x, \nu) = c_\Psi. \quad (42)$$

By substituting the condition (39), the invariant (28) has the form

$$I(x) = \frac{2k^3}{27}e^{\int \frac{f_2^2}{f_3} dx}. \quad (43)$$

D. Canonical form of Abel's equation and the integrating factor

According to the book of Kamke [13], for equations of the type (32) for which there is no condition (39), one should change the variables according to

$$\begin{aligned} y(x) &= \tilde{\omega}(x)\tilde{\eta}(\zeta(x)), & \tilde{\omega} &= e^{\int f_1(x) dx}, \\ \zeta(x) &= \int f_2(x)\tilde{\omega}(x) dx, \end{aligned} \quad (44)$$

which lead to the canonical form

$$\frac{d\tilde{\eta}}{d\zeta} = \tilde{\eta}^2 + g(\zeta)\tilde{\eta}^3, \quad (45)$$

where

$$g(\zeta(x)) = \frac{f_3(x)}{f_2(x)}e^{\int f_1(x) dx} \quad (46)$$

is the Appell invariant. Then, the integrating factor μ can be used to formulate the following interesting result:

Lemma 2. Any Abel equation in the canonical form (45) is integrable as long as the invariant is constant, with $g(\zeta) = \frac{1}{k}$, and solution given by (41).

Proof. In (34), let $\nu(x) = \frac{1}{\tilde{\eta}(\zeta(x))}$, where $\frac{d\zeta(x)}{dx} = Q(x)$, which leads to

$$\frac{d\tilde{\eta}}{d\zeta} = \tilde{\eta}^2 + \frac{P(x)}{Q(x)}\tilde{\eta}^3 = \tilde{\eta}^2 + \frac{f_3(x)e^{\int f_1(x)dx}}{f_2(x)}\tilde{\eta}^3 = \tilde{\eta}^2 + \frac{1}{k}\tilde{\eta}^3, \quad (47)$$

that is separable, with solution

$$\frac{1}{k} \ln \left| \frac{1}{\tilde{\eta}} + \frac{1}{k} \right| = \zeta + c + \frac{1}{\tilde{\eta}}, \quad (48)$$

which after simplification leads back to (41).

IV. VEIN'S ABEL EQUATION

The following Abel equation:

$$\frac{dy}{dx} = \frac{-2b}{bx+a^2}y + \frac{3(ax+b^2)}{bx+a^2}y^2 + \frac{x^3-3abx-a^3-b^3}{bx+a^2}y^3, \quad (49)$$

where $a, b \in \mathbb{R}$, is known from a paper of Vein to be integrable [10]. We first notice that if we define the following determinants,

$$D_3 = \begin{vmatrix} x & -a & -b \\ -b & x & -a \\ -a & -b & x \end{vmatrix}, \quad D_2 = \begin{vmatrix} x & -b \\ -b & -a \end{vmatrix}, \quad D_1 = |-b|, \quad D = \begin{vmatrix} -a & -b \\ x & -a \end{vmatrix},$$

then they allow us to write the coefficients of Vein's Equation (49) as $f_1 = 2D_1/D$, $f_2(x) = -3D_2/D$, and $f_3(x) = D_3/D$. Thus, (49) can be written in the compact form

$$\frac{dy}{dx} = \frac{2D_1}{D}y - \frac{3D_2}{D}y^2 + \frac{D_3}{D}y^3. \quad (50)$$

Next, using the transformation given by (26) with the coefficients from (49), we have

$$\omega = \frac{a^2 + bx}{x^3 - 3abx - a^3 - b^3} \quad (51)$$

and

$$\frac{d\xi}{dx} = f_3(x)\omega^2 = \omega. \quad (52)$$

Using these expressions, the invariant $I(x)$ becomes

$$I(x) = \frac{\frac{d}{dx} \frac{f_2}{3f_3} - \frac{f_1 f_2}{3f_3} + \frac{2f_2^3}{27f_3^2}}{f_3 \omega^3} = \frac{-\omega^2}{\frac{1}{\omega} \omega^3} \equiv -1. \quad (53)$$

Therefore, the normal form of Abel's Equation (45) is

$$\frac{d\eta}{d\xi} = \eta^3 - 1. \quad (54)$$

Using (26) and integrating (54), we obtain

$$\xi(\eta) - c = \frac{1}{6} \ln \frac{(\eta-1)^2}{1+\eta+\eta^2} - \frac{\sqrt{3}}{3} \arctan \frac{\sqrt{3}}{3} (1+2\eta). \quad (55)$$

But because $\xi(x) = \int \omega dx$, we find

$$\xi(x) - c = \frac{1}{6} \ln \frac{(x - (a + b))^2}{x^2 + (a + b)x + a^2 - ab + b^2} - \frac{\sqrt{3}}{3} \arctan \frac{\sqrt{3} 2x + a + b}{a - b}. \quad (56)$$

From the last two equations, one might infer that

$$\eta(x) = \frac{x + b}{a - b} \quad (57)$$

and consequently using (26) would get

$$y(x) = \frac{b}{(x - (a + b))(a - b)}. \quad (58)$$

Unfortunately, because for (57), the argument of arctan is satisfied, but the argument of ln is not satisfied, then we conclude that (58) is not the solution of (49).

To see how the implicit solutions (55), (57) can be untangled and written explicitly, we will proceed as Vein did. Firstly, we show that there are actually three solutions of the type (58), which are the solutions of (49) and can be put in the form [10]:

$$\frac{b}{y_1} = (ax + b^2) - (bx + a^2) \left[\frac{t_3}{t_1} \right], \quad \frac{b}{y_2} = (ax + b^2) - (bx + a^2) \left[\frac{t_1}{t_2} \right], \quad \frac{b}{y_3} = (ax + b^2) - (bx + a^2) \left[\frac{t_2}{t_3} \right], \quad (59)$$

with the t -functions defined cyclically as

$$t_1 = c\phi_1(s) + \phi_2(s), \quad t_2 = c\phi_2(s) + \phi_3(s), \quad t_3 = c\phi_3(s) + \phi_1(s),$$

where c is an arbitrary constant as it does not appear in the differential equation. The ϕ functions are the following triad:

$$\begin{aligned} \phi_1(x) &= \frac{1}{3} \left[e^x + 2e^{-x/2} \cos \frac{x\sqrt{3}}{2} \right], \\ \phi_2(x) &= \frac{1}{3} \left[e^x - 2e^{-x/2} \cos \left(\frac{x\sqrt{3}}{2} + \frac{\pi}{3} \right) \right], \\ \phi_3(x) &= \frac{1}{3} \left[e^x - 2e^{-x/2} \cos \left(\frac{x\sqrt{3}}{2} - \frac{\pi}{3} \right) \right], \end{aligned} \quad (60)$$

also known as the third-order hyperbolic functions, and are plotted in Fig. 1.

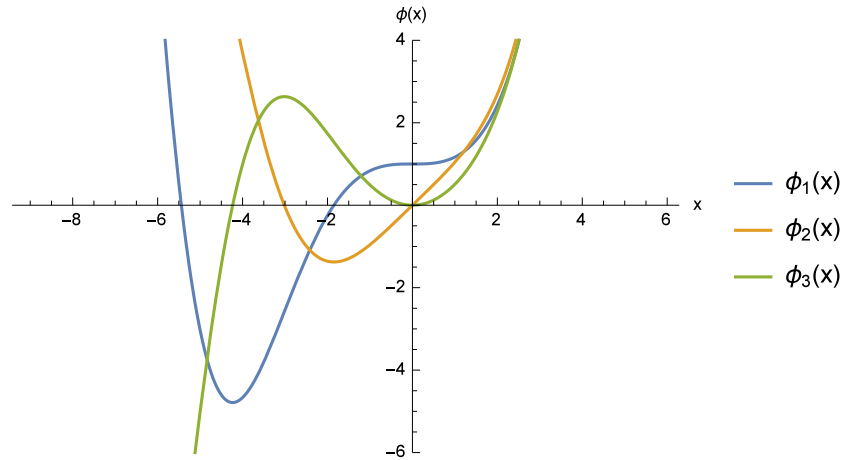


FIG. 1: The third-order hyperbolic functions ϕ_i , $i = 1, 2, 3$ of Equation (60).

They are independent because their Wronskian $W(\phi_1, \phi_2, \phi_3) = 1$, and they satisfy the following relationships

$$\begin{cases} \phi_1 + \phi_2 + \phi_3 = e^x, \\ \phi_1^3 + \phi_2^3 + \phi_3^3 - 3\phi_1\phi_2\phi_3 = 1. \end{cases}$$

They also fulfill the following relationships

$$\begin{aligned} d\phi_1/dx &= \phi_3, & d^2\phi_1/dx^2 &= \phi_2, & d^3\phi_1/dx^3 &= \phi_1, \\ d\phi_2/dx &= \phi_1, & d^2\phi_2/dx^2 &= \phi_3, & d^3\phi_2/dx^3 &= \phi_2, \\ d\phi_3/dx &= \phi_2, & d^2\phi_3/dx^2 &= \phi_1, & d^3\phi_3/dx^3 &= \phi_3, \end{aligned} \quad (61)$$

which are cyclic. That is, the fourth-order derivatives have the same property as the first ones, the fifth-order derivatives as the second ones, and so forth. In particular, they are independent solutions of the differential equation $d^3\phi/dx^3 = \phi$, but also of $d^6\phi/dx^6 = \phi$ and of any $d^{3+3m}\phi/dx^{3+3m} = \phi$, with $m = 0, 1, 2, \dots$

Let

$$t = c\phi_1(s) + \phi_2(s), \quad (62)$$

where c is a constant. Then, t fulfills

$$\begin{aligned} t' &= c\phi_3(s) + \phi_1(s), \\ t'' &= c\phi_2(s) + \phi_3(s), \\ t''' &= t \end{aligned} \quad (63)$$

with $' = d/ds$. Suppose s is defined implicitly as a multi-valued function of x as follows

$$x = a \left(\frac{t'}{t} \right) + b \left(\frac{t''}{t} \right) = \frac{a\phi_1(s) + bc\phi_2(s) + (ac + b)\phi_3(s)}{c\phi_1(s) + \phi_2(s)} = \Phi(s) \quad (64)$$

and denote the inverse relation by the explicit formula

$$s = \Phi^{-1}(x). \quad (65)$$

Then $y = ds/dx = d\Phi^{-1}(x)/dx$ satisfies Abel's equation (49).

Indeed, from (64), differentiating with respect to s and by the usage of (63), one obtains

$$\frac{dx}{ds} = a \left[\frac{t''}{t} - \left(\frac{t'}{t} \right)^2 \right] + b \left[1 - \left(\frac{t'}{t} \right) \left(\frac{t''}{t} \right) \right].$$

The quotient $\frac{t''}{t}$ can be eliminated by means of (64), which leads to

$$b \frac{dx}{ds} = (ax + b^2) - (bx + a^2) \frac{t'}{t}. \quad (66)$$

Differentiating again, this time with respect to x we have,

$$-\frac{b \frac{d^2s}{dx^2}}{\left(\frac{ds}{dx} \right)^2} = a - b \left(\frac{t'}{t} \right) - (bx + a^2) \left[\frac{t''}{t} - \left(\frac{t'}{t} \right)^2 \right] \frac{ds}{dx}.$$

Eliminating the t -quotients by means of (64) and (66), one can find after some reduction the following equation

$$(bx + a^2) \frac{d^2s}{dx^2} = -2b \frac{ds}{dx} + 3(ax + b^2) \left(\frac{ds}{dx} \right)^2 + (x^3 - 3abx - a^3 - b^3) \left(\frac{ds}{dx} \right)^3.$$

Substituting $y = \frac{ds}{dx}$ and dividing by $(bx + a^2)$ lead to Abel's equation (49). Thus, the solution can be written explicitly as

$$\frac{b}{y} = (ax + b^2) - (bx + a^2) \left[\frac{c\phi_3(\Phi^{-1}(x)) + \phi_1(\Phi^{-1}(x))}{c\phi_1(\Phi^{-1}(x)) + \phi_2(\Phi^{-1}(x))} \right], \quad (67)$$

with c an arbitrary constant.

The other two solutions can be obtained by cyclically replacing the ϕ 's in the t function

$$t_2 = c\phi_2(s) + \phi_3(s), \quad t_3 = c\phi_3(s) + \phi_1(s). \quad (68)$$

V. VEIN'S NONLINEAR OSCILLATOR

To see what kind of oscillator corresponds to Vein's Abel equation, we will proceed as follows. First, we will eliminate the linear term from (49) using

$$y = ze^{-2b \int \frac{dx}{bx+a^2}} = \frac{z}{(bx+a^2)^2}, \quad (69)$$

to obtain

$$\frac{dz}{dx} = h_2(x)z^2 + h_3(x)z^3, \quad (70)$$

where

$$\begin{aligned} h_2(x) &= \frac{3(ax+b^2)}{(bx+a^2)^3}, \\ h_3(x) &= \frac{x^3 - 3abx - a^3 - b^3}{(bx+a^2)^5}. \end{aligned} \quad (71)$$

Then using lemma 1 with $x_\zeta = v(x(\zeta)) = \frac{1}{z(x(\zeta))}$, we obtain

$$x_\zeta \dot{\zeta} + h_2(x)x_\zeta + h_3(x) = 0. \quad (72)$$

which represents a nonlinear oscillator with rational friction and nonlinearity that are plotted in Figures 2 and 3, respectively, for the following values (i) $a = 1, b = -2$; (ii) $a = 1, b = -1$; (iii) $a = 2, b = -1$; (iv) $a = -1, b = 1$.

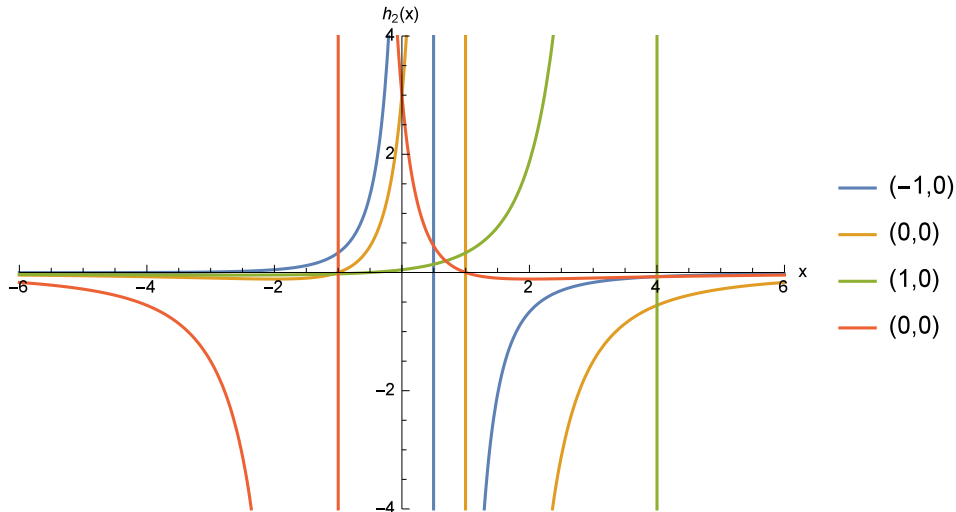


FIG. 2: The friction function $h_2(x)$ of Vein's oscillator at the fixed points $(-1,0)$, $(0,0)$, $(1,0)$, and $(0,0)$.

Writing equation (72) as a dynamical system

$$\begin{aligned} x_\zeta &= v = M(x, v) \\ v_\zeta &= -h_2(x)v - h_3(x) = N(x, v) \end{aligned} \quad (73)$$

and because (73) cannot be put in the form of (40), because $\frac{h_3(x)}{h_2(x)} \neq \text{const.}$, we will be using instead the standard methods of phase-plane analysis and use the linear approximation at the equilibrium points of (73) to classify them instead of solving the equation by finding the potential Ψ .

Also, notice that the system is non-Hamiltonian because there is no potential $\Psi(x, v) = \text{const.}$ such that

$$\begin{aligned} \frac{\partial \Psi}{\partial v} &= v \\ \frac{\partial \Psi}{\partial x} &= h_2(x)v + h_3(x). \end{aligned} \quad (74)$$

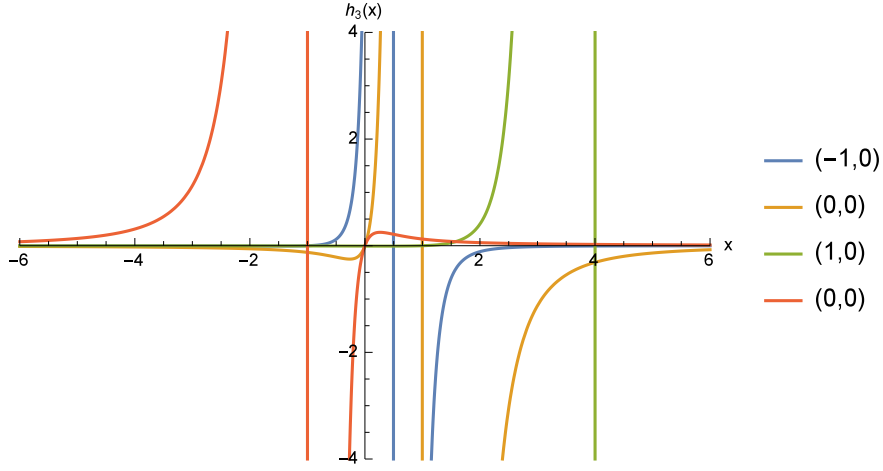


FIG. 3: The nonlinearity function $h_3(x)$ of Vein's oscillator at the fixed points $(-1, 0)$, $(0, 0)$, $(1, 0)$, and $(0, 0)$.

The Jacobian matrix of (73) is

$$J = \begin{bmatrix} \frac{\partial M}{\partial x} & \frac{\partial M}{\partial v} \\ \frac{\partial N}{\partial x} & \frac{\partial N}{\partial v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{dh_2}{dx}v - \frac{dh_3}{dx} & -h_2(x) \end{bmatrix}. \quad (75)$$

We have three equilibrium points of the system (73) from which one is real $(x_0, y_0) = (a + b, 0)$, while the other two are a pair of complex conjugates, $(x_{1,2}, y_0) = (\alpha \pm i\beta, 0)$, with $\alpha = -\frac{a+b}{2}$ and $\beta = -\frac{\sqrt{3}(a-b)}{2}$. The particular case $a = b$ which Vein considered, will reduce the three fixed points to the case of one real fixed point $(2a, 0)$. The characteristic polynomial of the Jacobian matrix is

$$|J - \lambda I_2| = \lambda^2 - \delta_1 \lambda + \delta_2 = 0, \quad (76)$$

while the discriminant is $\Delta = \delta_1^2 - 4\delta_2$. By evaluating the Jacobian at the real fixed point $\tilde{J} = J|_{(C,0)}$, where C is either real or complex, we obtain

$$\tilde{J} = \begin{bmatrix} 0 & 1 \\ -\frac{dh_3}{dx}(C) & -h_2(C) \end{bmatrix}. \quad (77)$$

from which

$$|\tilde{J} - \lambda I_2| = \lambda^2 + h_2(C)\lambda + \frac{dh_3}{dx}(C) = 0. \quad (78)$$

1. *Real fixed point:* In this case we have

$$\begin{aligned} \delta_{1r} &= -\frac{3}{(a^2+ab+b^2)^2} < 0, \\ \delta_{2r} &= \frac{3}{(a^2+ab+b^2)^4} > 0, \\ \Delta_r &= -\delta_{2r} < 0. \end{aligned} \quad (79)$$

then, the real fixed point $(a + b, 0)$ living on the x axis is a stable spiral. The eigenvalues of the Jacobian matrix are complex conjugate pairs $\lambda_{1,2} = \frac{1}{2}(\delta_1 \pm i\sqrt{\delta_2}) = \frac{\sqrt{3}}{2(a^2+ab+b^2)^2}(-\sqrt{3} \pm i)$. By calculating the eigenvectors, the linearized solution around the fixed point is

$$\begin{bmatrix} \tilde{x}(\zeta) \\ \tilde{v}(\zeta) \end{bmatrix} = e^{-\frac{3\zeta}{2(a^2+ab+b^2)^2}} \begin{bmatrix} c_1 \cos \frac{\sqrt{3}\zeta}{2(a^2+ab+b^2)^2} + c_2 \sin \frac{\sqrt{3}\zeta}{2(a^2+ab+b^2)^2} \\ -\frac{\sqrt{3}}{(a^2+ab+b^2)^2} \left\{ c_1 \sin \left(\frac{\sqrt{3}\zeta}{2(a^2+ab+b^2)^2} + \frac{\pi}{3} \right) - c_2 \cos \left(\frac{\sqrt{3}\zeta}{2(a^2+ab+b^2)^2} - \frac{\pi}{3} \right) \right\} \end{bmatrix}, \quad (80)$$

Fixed Points	δ_1	δ_2	Δ	Type
$(x_0, y_0) = (a + b, 0)$	$\tilde{\delta}_{1r}$	$\tilde{\delta}_{2r}$	$-\tilde{\delta}_{2r}$	stable spiral
$(x_{1,2}, y_0) = (-\frac{1}{2}[(a + b) \pm i\sqrt{3}(a - b)], 0)$	$\tilde{\delta}_{1c}$	$\tilde{\delta}_{2c}$	$-\tilde{\delta}_{2c}$	no conclusion

TABLE I: Classification of equilibrium points of (73).

where c_1 and c_2 are arbitrary constants.

2. *Complex conjugated fixed points:* In this case the coefficients of the characteristic polynomial δ_1, δ_2 are complex, and their values are shown in Table I and given in (81).

$$\begin{aligned} \delta_{1c} &= \frac{3}{2(a^3 - b^3)^2} [(a^2 - 2ab - 2b^2) + i\sqrt{3}a(a + b)] , \\ \delta_{2c} &= \frac{3}{2(a^3 - b^3)^4} [(-a^4 - 8a^3b - 6a^2b^2 + 4ab^3 + 2b^4) + i\sqrt{3}a(a^3 - 6ab^2 - 4b^3)] , \\ \Delta_c &= -\tilde{\delta}_{2c} . \end{aligned} \quad (81)$$

Because both coefficients δ_{1c} and δ_{2c} are not real nothing can be said about these complex fixed points.

The phase-plane portrait of the system (73) is shown in Fig. 4. The real fixed point is identified by the red dot, and the isoclines by the dotted curves.

VI. CONCLUSION

This paper recalls several integrability properties of Abel equations together with some simple consequences, followed by a discussion of Vein's Abel equation related to third-order hyperbolic functions. The corresponding nonlinear oscillator system is introduced here, and its dynamical systems analysis is provided. Finally, it is worth noticing that the nonlinear oscillators corresponding to Abel equations are in fact a large class of generalized Liénard equations of the form $\ddot{x} + f(\dot{x})g(x) + h(x) = 0$, where g , and h are arbitrary functions and $f(\dot{x}) = \alpha_1\dot{x}^3 + \alpha_2\dot{x}^2 + \alpha_3\dot{x} + \alpha_4$, where α_i are constants [15] and, as such, have many applications in physics, biology, and engineering [16, 17].

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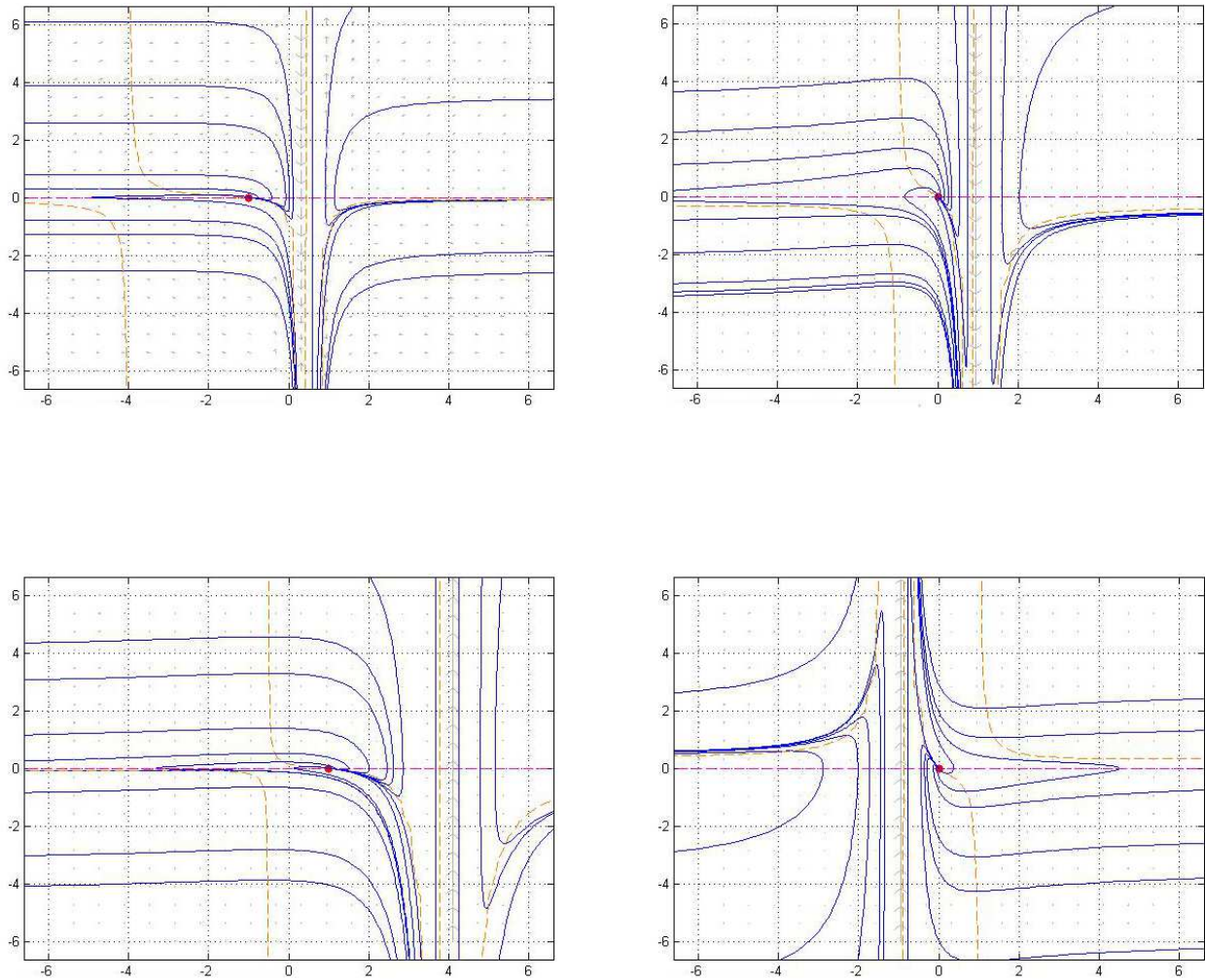


FIG. 4: Phase-plane portrait at the fixed points for $a = 1, b = -2$ (top left); $a = 1, b = -1$ (top right); $a = 2, b = -1$ (bottom left); $a = -1, b = 1$ (bottom right).

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