Supersymmetric pairing of kinks for polynomial nonlinearities

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We show how one can obtain kink solutions of ordinary differential equations with polynomial nonlinearities by an efficient factorization procedure directly related to the factorization of their nonlinear polynomial part. We focus on reaction-diffusion equations in the traveling frame and damped-anharmonic-oscillator equations. We also report an interesting pairing of the kink solutions, a result obtained by reversing the factorization brackets in the supersymmetric quantum-mechanical style. In this way, one gets ordinary differential equations with a different polynomial nonlinearity possessing kink solutions of different width but propagating at the same velocity as the kinks of the original equation. This pairing of kinks could have many applications. We illustrate the mathematical procedure with several important cases, among which are the generalized Fisher equation, the FitzHugh-Nagumo equation, and the polymerization fronts of microtubules.

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I. INTRODUCTION

Factorization of second-order linear differential equations, such as the Schrödinger equation, is a well established method to get solutions in an algebraic manner [1]. Here we are interested in factorizations of ordinary differential equations (ODE) of the type

$$u'' + \gamma u' + F(u) = 0, \tag{1}$$

where $F(u)$ is a given polynomial in u. If the independent variable is the time, then γ is a damping constant and we are in the case of nonlinear damped oscillator equations. Many examples of this type are collected in the Appendix of a paper of Tuszyński *et al.* [2]. However, the coefficient γ can also play the role of the constant velocity of a traveling front if the independent variable is a traveling coordinate used to reduce a reaction-diffusion (RD) equation to the ordinary differential form as briefly sketched in the following. These RD traveling fronts or kinks are important objects in low-dimensional nonlinear phenomenology describing topologically switched configurations in many areas of biology, ecology, chemistry, and physics.

Consider a scalar RD equation for $u(x,t)$,

$$\frac{\partial u}{\partial t} = \mathcal{D}\frac{\partial^2 u}{\partial x^2} + sF(u), \tag{2}$$

where $\mathcal{D}$ is the diffusion constant and s is the strength of the reaction process. Equation (2) can be rewritten as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u), \tag{3}$$

where the coefficients have been eliminated by the rescalings $\tilde{t}=st$ and $\tilde{x}=(s/\mathcal{D})^{1/2}x$, and dropping the tilde. Usually, the scalar RD equation possesses traveling-wave solutions $u(\xi)$ with $\xi=x-vt$, propagating at speed v. For this type of solutions, the RD equation turns into the ODE,

$$u'' + vu' + F(u) = 0, \tag{4}$$

where the prime denotes $D=d/d\xi$. The latter equation has the same form as nonlinear damped oscillator equations with the velocity playing the role of the friction constant.

For applications in physical optics and acoustics, it is convenient to write the traveling coordinate in the form $\xi=kx-\omega t=k(x-vt)$ with $kv=\omega$. This is a simple scaling by k of the previous coordinate turning Eq. (4) into the form

$$u'' + \frac{v}{k}u' + \frac{1}{k^2}F(u) = 0, \tag{5}$$

which can be changed back to the form of Eq. (1) by redefining $\tilde{\gamma}=v/k$ and $\tilde{F}(u)=(1/k^2)F(u)$.

In general, performing the factorization of Eq. (1) means the following:

$$[D - f_2(u)][D - f_1(u)]u = 0. \tag{6}$$

This leads to the equation

$$u'' - \frac{df_1}{du}uu' - f_1u' - f_2u' + f_1f_2u = 0. \tag{7}$$

The following groupings of terms are possible related to different factorizations.

(a) Berkovich grouping. In 1992, Berkovich [3] proposed to group the terms as follows:

$$u'' - (f_1 + f_2)u' + \left(f_1f_2 - \frac{df_1}{du}u'\right)u = 0, \tag{8}$$

and furthermore discussed a theorem according to which any factorization of an ODE of the form given in Eq. (6) allows us to find a class of solutions that can be obtained from

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solving the first-order differential equation $u'=f_1u$. Substituting the latter expression in the Berkovich grouping, one gets

$$u'' - (f_{1b}+f_{2b})u' + \left(f_{1b}f_{2b} - \frac{df_{1b}}{du}f_{1b}u\right)u = 0, \tag{9}$$

where we redefined $f_1=f_{1b}$ and $f_2=f_{2b}$ to distinguish this case from our proposal following next. For the specific form of the ODEs we consider here, Berkovich's conditions read

$$f_{1b}\left(-\gamma - f_{1b} - \frac{df_{1b}}{du}u\right) = \frac{F(u)}{u}, \tag{10}$$

$$f_{1b}+f_{2b} = -\gamma. \tag{11}$$

(b) Grouping of this work. We propose here the different grouping of terms

$$u'' - \left(\frac{d\phi_1}{du}u + \phi_1 + \phi_2\right)u' + \phi_1\phi_2 u = 0 \tag{12}$$

that can be considered the result of changing the Berkovich factorization by setting $f_{1b}=\phi_1$ and $f_{2b}\rightarrow\phi_2$ under the conditions

$$\phi_1\phi_2 = \frac{F(u)}{u}, \tag{13}$$

$$\phi_1 + \phi_2 + \frac{d\phi_1}{du}u = -\gamma. \tag{14}$$

The following simple relationship exists between the factoring functions:

$$\phi_2 = f_{2b} - \frac{df_{1b}}{du}u$$

and further (third, and so forth) factorizations can be obtained through linear combinations of the functions f_{1b}, f_{2b}, and ϕ_2.

Based on our experience, we think that the grouping we propose is more convenient than that of Berkovich and also of other people employing more difficult procedures. The main advantage resides in the fact that whereas in Berkovich's scheme Eq. (10) is still a differential equation to be solved, in our scheme we make a choice of the factorization functions by merely factoring polynomial expressions according to Eq. (13), and then imposing Eq. (14) leads easily to an n-dependent γ coefficient for which the factorization works. This fact makes our approach extremely efficient in finding particular solutions of the kink type, as one can see in the following.

We will show next in the explicit case of the generalized Fisher equation all the mathematical constructions related to the factorization brackets and their supersymmetric quantum-mechanical-like reverse factorization. In addition, in less detail, we treat within the same approach damped nonlinear oscillators of Dixon-Tuszyński-Otwinowski type and the FitzHugh-Nagumo equation.

II. GENERALIZED FISHER EQUATION

Let us consider the generalized Fisher equation given by

$$u'' + \gamma u' + u(1-u^n) = 0. \tag{15}$$

The case $n=1$ refers to the common Fisher equation and it will be briefly discussed as a subcase. Equation (13) allows us to factorize the polynomial function

$$\phi_1\phi_2 = \frac{F(u)}{u} = (1-u^n) = (1-u^{n/2})(1+u^{n/2}). \tag{16}$$

Now, by choosing

$$\phi_1 = a_1(1-u^{n/2}), \quad \phi_2 = \frac{1}{a_1}(1+u^{n/2}), \quad a_1 \neq 0, \tag{17}$$

the explicit forms of a_1 and γ can be obtained from Eq. (14),

$$\begin{aligned}\frac{d\phi_1}{du}u + \phi_1 + \phi_2 &= -\frac{n}{2}a_1u^{n/2} + a_1(1-u^{n/2}) + (1/a_1)(1+u^{n/2}) \\ &= -\gamma.\end{aligned} \tag{18}$$

Introducing the notation $h_n=[(n/2)+1]^{1/2}$, one gets

$$a_1 = \pm h_n^{-1}, \quad \gamma = \mp(h_n + h_n^{-1}). \tag{19}$$

Then Eq. (15) becomes

$$u'' \pm (h_n + h_n^{-1})u' + u(1-u^n) = 0 \tag{20}$$

and the corresponding factorization is

$$[D \pm h_n(u^{n/2}+1)][D \mp h_n^{-1}(u^{n/2}-1)]u = 0. \tag{21}$$

It follows that Eq. (20) is compatible with the first-order differential equation

$$u' \mp h_n^{-1}(u^{n/2}-1)u = 0. \tag{22}$$

Integration of Eq. (22) gives for $\gamma > 0$

$$u_>^{\pm} = \{1 \pm \exp[(h_n - h_n^{-1})(\xi - \xi_0)]\}^{-2/n}. \tag{23}$$

Rewritten in the hyperbolic form, we get

$$u_>^{+} = \left(\frac{1}{2} - \frac{1}{2}\tanh\left[\frac{1}{2}(h_n - h_n^{-1})(\xi - \xi_0)\right]\right)^{2/n},$$

$$u_>^{-} = \left(\frac{1}{2} - \frac{1}{2}\coth\left[\frac{1}{2}(h_n - h_n^{-1})(\xi - \xi_0)\right]\right)^{2/n}. \tag{24}$$

The tanh form is precisely the solution obtained long ago by Wang [4] and Hereman and Takaoka [5] by more complicated means.

Moreover, a different solution is possible for $\gamma < 0$,

$$u_<^{\pm} = \{1 \pm \exp[-(h_n - h_n^{-1})(\xi - \xi_0)]\}^{-2/n}, \tag{25}$$

or

$$u_<^+ = \left(\frac{1}{2} + \frac{1}{2}\tanh\left[-\frac{1}{2}(h_n - h_n^{-1})(\xi - \xi_0)\right]\right)^{2/n},$$

$$u_<^- = \left(\frac{1}{2} + \frac{1}{2}\coth\left[-\frac{1}{2}(h_n - h_n^{-1})(\xi - \xi_0)\right]\right)^{2/n}, \qquad (26)$$

respectively.

A. Reversion of factorization brackets without the change of the scaling factors

Choosing now $\phi_1 = a_1(1+u^{n/2})$ and $\phi_2 = (1/a_1)(u^{n/2}-1)$ leads to the same Eq. (20) but now with the factorization

$$[D \mp h_n(u^{n/2}-1)][D \pm h_n^{-1}(u^{n/2}+1)]u = 0, \qquad (27)$$

and therefore the compatibility is with the different first-order equation,

$$u' \pm h_n^{-1}(u^{n/2}+1)u = 0. \qquad (28)$$

However, the direct integration gives the solution (for $\gamma > 0$)

$$u = \left(-\frac{1}{1 \pm \exp[(h_n - h_n^{-1})(\xi - \xi_0)]}\right)^{2/n}$$
$$= (-1)^{2/n}\{1 \pm \exp[(h_n - h_n^{-1})(\xi - \xi_0)]\}^{-2/n}, \qquad (29)$$

which is similar to the known solution Eq. (23). For $\gamma < 0$, solutions of the type given by Eq. (25) are obtained.

B. Direct reversion of factorization brackets

Let us perform now a direct inversion of the factorization brackets in Eq. (21) similar to what is done in supersymmetric quantum mechanics in order to enlarge the class of exactly solvable quantum Hamiltonians,

$$[D \mp h_n^{-1}(u^{n/2}-1)][D \pm h_n(u^{n/2}+1)]u = 0. \qquad (30)$$

Doing the product of differential operators, the following RD equation is obtained:

$$u'' \pm (h_n + h_n^{-1})u' + u[1+u^{n/2}][1 - h_n^4 u^{n/2}] = 0. \qquad (31)$$

Equation (31) is compatible with the equation

$$u' \pm h_n(u^{n/2}+1)u = 0, \qquad (32)$$

and integration of the latter gives the kink solution of Eq. (31),

$$u_>^{\pm} = \left(-\frac{1}{1 \pm \exp[(h_n^3 - h_n)(\xi - \xi_0)]}\right)^{2/n}$$
$$= \{1 \pm \exp[(h_n^3 - h_n)(\xi - \xi_0)]\}^{-2/n} \qquad (33)$$

for $\gamma > 0$. On the other hand, for $\gamma < 0$ the exponent is the same but of opposite sign. Hyperbolic forms of the latter solutions are easy to write down and are similar up to widths to Eqs. (24) and (26), respectively.

Thus, a different RD equation given by Eq. (31) with modified polynomial terms and its solution have been found by reverting the factorization terms of Eq. (20). Although the reaction polynomial is different, the velocity parameter remains the same. This is the main result of this work: *At the velocity corresponding to the traveling kink of a given RD equation, there is another propagating kink corresponding to a different RD equation that is related to the original one by reverse factorization*. We can call this kink the supersymmetric (SUSY) kink because of the mathematical construction.

Finally, one can ask if the process of reverse factorization can be continued with Eq. (31). It can be shown that this is not the case because Eq. (31) has already a discretized (polynomial-order-dependent) γ and this fact prevents further solutions of this type. Suppose we consider the following factorization functions:

$$\tilde{\phi}_1 = \tilde{a}_1^{-1}[1 - h_n^4 u^{n/2}], \quad \tilde{\phi}_2 = \tilde{a}_1(1+u^{n/2}). \qquad (34)$$

Then, one gets $\tilde{a}_1 = \pm h_n^3$ and solves $\tilde{a}_1^{-1} + \tilde{a}_1 = h_n^{-1} + h_n$. The solutions are $n=0$, which implies linearity, and $n=-4$, which leads to a Milne-Pinney equation. On the other hand, Eq. (31) with an arbitrary γ can be treated by the inverse factorization procedure to get the SUSY partner RD equation and its SUSY kink.

C. Subcase $n=1$

This subcase is the original Fisher equation describing the propagation of mutant genes,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u). \qquad (35)$$

In the traveling frame, the Fisher equation has the form

$$u'' + \alpha u' + u(1-u) = 0. \qquad (36)$$

When the γ parameter takes the value $\gamma_1 = \frac{5}{6}\sqrt{6}$ (i.e., $h_1 = \sqrt{6}/2$), one can factor Fisher's equation, and employing our method leads easily to the known kink solution,

$$u_{\mathrm{F}} = \frac{1}{4}\left(1 - \tanh\left[\frac{\sqrt{6}}{12}(\xi - \xi_0)\right]\right)^2, \qquad (37)$$

that was first obtained by Ablowitz and Zeppetella [7] with a series solution method. On the other hand, the SUSY kink for this case reads

$$u_{\mathrm{F,SUSY}} = \frac{1}{4}\left(1 - \tanh\left[\frac{\sqrt{6}}{8}(\xi - \xi_0)\right]\right)^2, \qquad (38)$$

i.e., it has a width one and a half times greater than the common Fisher kink and is a solution of the partner equation

$$u'' + \frac{5\sqrt{6}}{6}u' + u\left(1 - \frac{5}{4}u^{1/2} - \frac{9}{4}u\right) = 0. \qquad (39)$$

A plot of the kinks u_{F} and $u_{\mathrm{F,SUSY}}$ is displayed in Fig. 1.

D. Subcase $n=6$

This subcase is of interest in light of experiments on polymerization patterns of microtubules in centrifuges. It has been discovered that the polymerization of the tubulin dimers proceeds in a kink-switching fashion propagating with a constant velocity within the sample. Portet, Tuszynski,

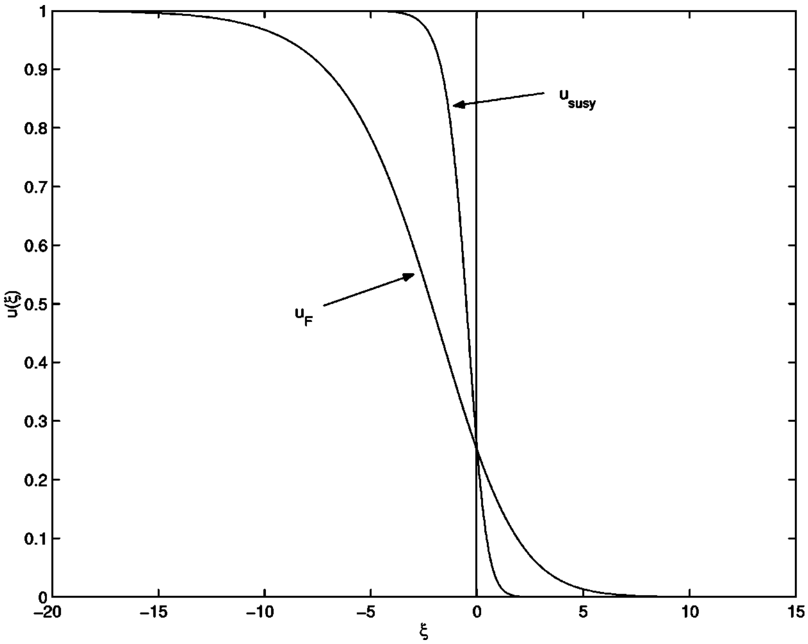

FIG. 1. The front of mutant genes (Fisher's wave of advance) in a population and the partner SUSY kink propagating with the same velocity. The axes are in arbitrary units.

and Dixon [6] used RD equations to discuss the modification of self-organization patterns of MTs as well as the tubulin polymerization under the influence of reduced gravitational fields. They used the value $n=6$ for the mean critical number of tubulin dimers at which the polymerization process starts and showed that the same nucleation number enters the polynomial term of the RD process for the number concentration c of tubulin dimers,

$$c'' + \tfrac{5}{2}c' + c(1 - c^6) = 0. \tag{40}$$

The polymerization kink in their work reads

$$c_{\rm PTD} = 2^{-1/3}\left(1 - \tanh\left[\frac{3}{4}(\xi - \xi_0)\right]\right)^{1/3}. \tag{41}$$

On the other hand, the SUSY polymerization kink (see Fig. 2) of the form

$$c_{\rm SUSY} = 2^{-1/3}\{1 - \tanh[3(\xi - \xi_0)]\}^{1/3} \tag{42}$$

can be taken into account according to the hyperbolic form of Eq. (33). It propagates with the same speed and corresponds to the equation

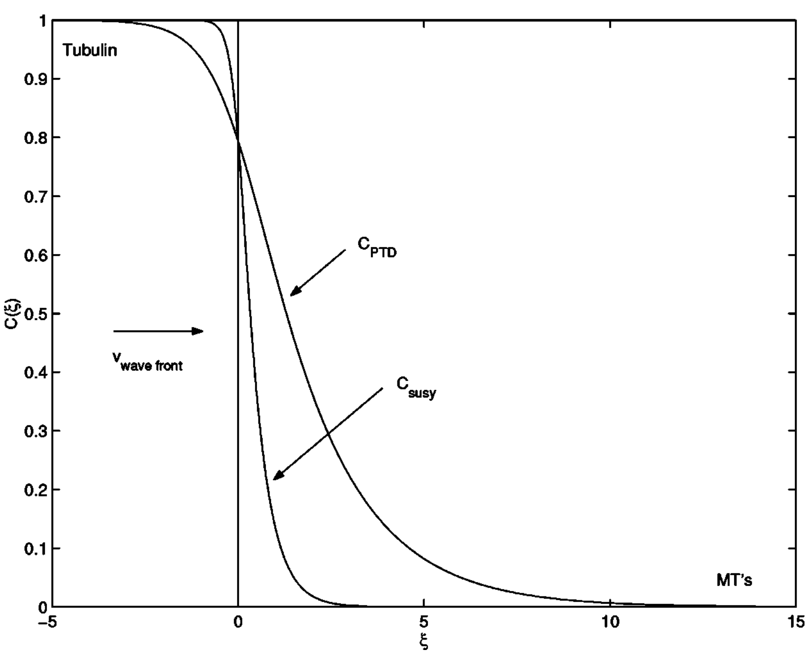

FIG. 2. The polymerization kink of Portet, Tuszyński, and Dixon [6] and the SUSY kink propagating with the same velocity (axes in arbitrary units).

$$c'' \pm \tfrac{5}{2}c' + c(1 - 15c^3 - 16c^6) = 0. \tag{43}$$

In principle, this equation could be obtained as a consequence of modifying the kinetics steps in the microtubule polymerization process.

III. EQUATIONS OF THE DIXON-TUSZYŃSKI-OTWINOWSKI TYPE

In the context of damped anharmonic oscillators, Dixon *et al.* [8] studied equations of the type (in this section, the prime denotes $D_\tau = d/d\tau$)

$$u'' + u' + Au - u^{n-1} \equiv u'' + u' + u(\sqrt{A} - u^{(n/2)-1})(\sqrt{A} + u^{(n/2)-1}) = 0 \tag{44}$$

and gave solutions for the cases $A=\frac{2}{9}$ and $A=\frac{3}{16}$, with $n=4$ and $n=6$, respectively. For this case, time is the independent variable. The factorization method works nicely if one uses $g_n=\sqrt{n/2}$ and dealing with the more general equation

$$u'' \pm \sqrt{A}(g_n + g_n^{-1})u' + u(A - u^{n-2}) = 0, \tag{45}$$

for which we can employ either the factorization functions

$$\phi_1 = \mp g_n^{-1}(\sqrt{A} - u^{(n/2)-1}),$$

$$\phi_2 = \mp g_n(\sqrt{A} + u^{(n/2)-1}) \tag{46}$$

or

$$\phi_1 = \mp g_n^{-1}(\sqrt{A} + u^{(n/2)-1}),$$

$$\phi_2 = \mp g_n(\sqrt{A} - u^{(n/2)-1}). \tag{47}$$

Then, Eq. (45) can be factored in the forms

$$[D_\tau \pm g_n(u^{(n/2)-1} + \sqrt{A})][D_\tau \mp g_n^{-1}(u^{(n/2)-1} - \sqrt{A})]u = 0 \tag{48}$$

and

$$[D_\tau \mp g_n(u^{(n/2)-1} - \sqrt{A})][D_\tau \pm g_n^{-1}(u^{(n/2)-1} + \sqrt{A})]u = 0. \tag{49}$$

Thus, Eq. (45) is compatible with the equations

$$u' \mp g_n^{-1}(u^{(n/2)-1} - \sqrt{A})u = 0, \tag{50}$$

$$u' \pm g_n^{-1}(u^{(n/2)-1} + \sqrt{A})u = 0 \tag{51}$$

that follow from Eq. (48) and Eq. (49). Integration of Eqs. (50) and (51) gives the solution of Eq. (45),

$$u_> = \left(\frac{\sqrt{A}}{1 \pm \exp[\sqrt{A}(g_n - g_n^{-1})(\tau - \tau_0)]}\right)^{2/(n-2)}, \quad \gamma > 0 \tag{52}$$

and

$$u_< = \left(\frac{\sqrt{A}}{1 \pm \exp[-\sqrt{A}(g_n - g_n^{-1})(\tau - \tau_0)]}\right)^{2/(n-2)}, \quad \gamma < 0. \tag{53}$$

The solutions obtained by Dixon *et al.* are particular cases of the latter formulas.

Reversing now the factorization brackets in Eq. (48),

$$[D_\tau \mp g_n^{-1}(u^{(n/2)-1} - \sqrt{A})][D_\tau \pm g_n(u^{(n/2)-1} + \sqrt{A})]u = 0, \tag{54}$$

leads to the following equation:

$$u'' \pm \sqrt{A}(g_n + g_n^{-1})u' + u(\sqrt{A} + u^{(n/2)-1})\left(\sqrt{A} - \frac{n^2}{4}u^{(n/2)-1}\right) = 0, \tag{55}$$

which is compatible with the equation

$$u' \pm g_n(u^{(n/2)-1} + \sqrt{A})u = 0 \tag{56}$$

whose integration gives the solution of Eq. (55),

$$u_> = \left(\frac{\sqrt{A}}{1 \pm \exp[\sqrt{A}g_n(\tau - \tau_0)]}\right)^{2/(n-2)}, \quad \gamma > 0 \tag{57}$$

and

$$u_< = \left(\frac{\sqrt{A}}{1 \pm \exp[-\sqrt{A}g_n(\tau - \tau_0)]}\right)^{2/(n-2)}, \quad \gamma < 0. \tag{58}$$

IV. FITZHUGH-NAGUMO EQUATION

Let us consider the FitzHugh-Nagumo equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u(1-u)(a-u) = 0, \tag{59}$$

where a is a real constant. If $a=-1$, one gets the real Newell-Whitehead equation describing the dynamical behavior near the bifurcation point for the Rayleigh-Bénard convection of binary fluid mixtures. The traveling frame form of Eq. (59) has been discussed in detail by Hereman and Takaoka [5],

$$u'' + \gamma u' + u(u-1)(a-u) = 0. \tag{60}$$

The FitzHugh-Nagumo polynomial function allows the following factorizations:

$$\phi_1 = \pm(\sqrt{2})^{-1}(u-1), \quad \phi_2 = \pm\sqrt{2}(a-u) \tag{61}$$

when the γ parameter is equal to $\gamma_{a1} = \pm(-2a+1)/\sqrt{2}$ that we also write as $\gamma_{a,1} = \pm\sqrt{a}(g_{a1} - g_{a1}^{-1})$, where $g_{a1} = -\sqrt{2a}$.

In addition, we can employ the factorization functions

$$\phi_1 = \pm(\sqrt{2})^{-1}(a-u), \quad \phi_2 = \pm\sqrt{2}(u-1) \tag{62}$$

when $\gamma_{a,2} = \pm(-a+2)/\sqrt{2}$, or written again in the more symmetric form $\gamma_{a,2} = \pm\sqrt{a}(g_{a2} - g_{a2}^{-1})$, where $g_{a2} = -\sqrt{a/2}$. Thus, Eq. (60) can be factored in the two cases

$$u'' \pm \gamma_{a,1}u' + u(u-1)(a-u) = 0 \tag{63}$$

and

$$u'' \pm \gamma_{a,2} u' + u(u-1)(a-u) = 0. \tag{64}$$

In passing, we notice that for the Newell-Whitehead case $a = -1$, the two equations coincide and are the same as the generalized Fisher equation for $n=2$.

In factorization bracket forms, Eqs. (63) and (64) are written as follows:

$$[D \mp \sqrt{2}(a-u)][D \pm (\sqrt{2})^{-1}(1-u)]u = 0 \tag{65}$$

and

$$[D \mp \sqrt{2}(u-1)][D \mp (\sqrt{2})^{-1}(a-u)]u = 0, \tag{66}$$

and are compatible with the first-order differential equations

$$u' \pm (\sqrt{2})^{-1}(1-u)u = 0 \text{ for } \gamma_{a,1}, \tag{67}$$

$$u' \mp (\sqrt{2})^{-1}(a-u)u = 0 \text{ for } \gamma_{a,2}. \tag{68}$$

Integration of the latter equations gives the solution of Eq. (60) for the two different values of the wave front velocity $\gamma_{a,1}$ and $\gamma_{a,2}$.

For Eq. (63), we get

$$u_> = \frac{1}{1 \pm \exp[(\sqrt{2})^{-1}(\xi - \xi_0)]},$$

$$u_< = \frac{1}{1 \pm \exp[-(\sqrt{2})^{-1}(\xi - \xi_0)]} \tag{69}$$

for $\gamma_{a,1}$ positive and negative, respectively.

As for Eq. (64), the solutions are

$$u_> = \frac{a}{1 \pm \exp[-(\sqrt{2})^{-1}a(\xi - \xi_0)]},$$

$$u_< = \frac{a}{1 \pm \exp[(\sqrt{2})^{-1}a(\xi - \xi_0)]} \tag{70}$$

for $\gamma_{a,2}$ positive and negative, respectively.

Considering now the factorizations (65) and (66), the change of order of the factorization brackets gives

$$[D \pm (\sqrt{2})^{-1}(1-u)][D \mp \sqrt{2}(a-u)]u = 0 \tag{71}$$

and

$$[D \mp (\sqrt{2})^{-1}(a-u)][D \mp \sqrt{2}(u-1)]u = 0. \tag{72}$$

Doing the product of differential operators (and considering the factorization term $u' - \phi_2 u = 0$) gives the following RD equations:

$$u'' \pm \gamma_{a1} u' + u(4u-1)(a-u) = 0 \tag{73}$$

and

$$u'' \pm \gamma_{a2} u' + u(u-1)(a-u-3u^2) = 0. \tag{74}$$

Equations (73) and (74) are compatible with the equations

$$u' \mp \sqrt{2}(a-u)u = 0 \tag{75}$$

and

$$u' \mp \sqrt{2}(u-1)u = 0, \tag{76}$$

respectively. Integrations of Eqs. (75) and (76) give the solutions of Eqs. (73) and (74), respectively. The explicit forms are the following:

(i) For Eq. (73),

$$u_> = \frac{a}{1 \pm \exp[-\sqrt{2}a(\xi - \xi_0)]},$$

$$u_< = \frac{a}{1 \pm \exp[\sqrt{2}a(\xi - \xi_0)]}. \tag{77}$$

(ii) For Eq. (74),

$$u_> = \frac{1}{1 \pm \exp[\sqrt{2}(\xi - \xi_0)]},$$

$$u_< = \frac{1}{1 \pm \exp[-\sqrt{2}(\xi - \xi_0)]}. \tag{78}$$

V. CONCLUSION

This paper has been concerned with stating an efficient factorization scheme of ordinary differential equations with polynomial nonlinearities that leads to an easy finding of analytical solutions of the kink type that previously have been obtained by far more cumbersome procedures. The main result is an interesting pairing between equations with different polynomial nonlinearities, which is obtained by applying the SUSY quantum-mechanical reverse factorization. The kinks of the two nonlinear equations are of different widths but they propagate at the same velocity, or if we deal with damped polynomial nonlinear oscillators the two kink solutions correspond to the same friction coefficient. Several important cases, such as the generalized Fisher and the FitzHugh-Nagumo equations, have been shown to be simple mathematical exercises for this factorization technique. The physical prediction is that for commonly occurring propagating fronts, there are two kink fronts of different widths at a given propagating velocity. Moreover, the reverse factorization procedure can also be applied to the Berkovich scheme with similar results. It will be interesting to apply the approach of this work to the discrete case in which various exact results have been obtained in recent years [9]. More general cases in which the coefficient γ is an arbitrary function could also be of much interest because of possible applications.

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