CLOSED-LOOP STABILIZATION OF A CLASS OF LUR'E SYSTEM: APPLICATION TO CHAOS CONTROL

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Abstract: In this paper the problem of closed-loop stabilization of a class of Lur'e system is addressed. Under certain conditions of the static non-linearity, the control problem is proved that can be converted to a simultaneous stabilization condition for a linear controller. The design strategy is considered for chaos control of a type of $C^3-equivalents$ chaotic systems. Experimental results corroborate the analysis presented.

Keywords: Stabilization, Linear Control, Chaos Control, Lur'e Problem.

1. INTRODUCTION

In almost any practical problem, there are nonlinear static characteristics that affect the performance of the control system. This issue could be characterized by actuator saturation or rate-limiter, friction, output-limiter, etc. (Khalil 1996), (Sastry 1999). As a result, an unpredictable behavior could be observed and the closed-loop performance can be deteriorated. A special problem is the Lur'e system where a linear system presents feedback interconnection from a sector bounded nonlinear function of the output. This problem is classical in control theory, and well-known results have been derived to conclude absolute stability, as the circle and Popov criterion (Sastry 1999), (Khalil 1996), (Vidyasagar 1993). Recently, a generalization to the sector condition was also presented by Hu *et al.* (2004).

In regard chaos, the Lur'e problem has been related to chaotic oscillators as the Chua circuit and Sprott's systems (Sprott 2000). In chaos control two problems are addressed: chaos suppression and chaos synchronization. The main goal of the suppression is to remove the erratic and unpredictable behavior that characterize the chaotic systems. The foundational for the chaos suppression problem is scientific and technological. The controlled chaotic systems have allowed to claim and understand that structured disorder and its entropy/information relationship extending the determinism concept (Bricmont 1998), (Hayles 1990). In addition, the chaos suppression impacts biomedical, life and engineering sciences. For example, it can be applied to control pathological rythm in the heart (Christini and Collins 1996). Roughly speaking, the chaos suppression problem can be defined as the stabilization of unstable periodic orbits of a chaotic attractor in equilibrium points or periodic orbits with period n (Aguirre and Billings 1995). Since the seminal paper by Ott *et al.* (1990) was published, several control schemes have been proposed to suppress chaos. Some feedback controllers have been designed for robustness against a noisy environment (Cazelles *et al.* 1995). Others have been proposed as robust approaches for output feedback control (Femat *et al.* 1999a), and few schemes have been designed in the frequency domain. In this sense, integral actions have shown to be capable of stabilizing chaotic systems in equilibrium points and periodic orbits (Puebla *et al.* 2003), (Femat *et al.* 1999b).

In this paper the closed-loop stability of a special type of Lur'e system is analyzed. The application to a chaos control problem is illustrated, and a sub-class of $C^3 - equivalents$ chaotic systems are studied (Malasoma 2002). Such class of chaotic systems involve the Chua's and Sprott's circuits (Sprott 2000). In this contribution, the controller is interconnected with the chaotic system to be tested, by numerical simulation and physically implemented in circuits to explore its performance. The paper is organized as follows. Section 2 contains the problem statement. The equivalence between the closed-loop stabilization and the simultaneous stabilization is introduced in Section 3. The application to chaos control is discussed in Section 4, and finally, Section 5 presents some concluding remarks.

2. PROBLEM STATEMENT

Consider a general form for a n^{th} order SISO system with internal non-linear feedback

$$x^{(n)} + a_{n-1}x^{(n-1)} + \ldots + a_1\dot{x} = \xi(x) \qquad (1)$$

where the coefficients $a_j \geq 0 \; \forall j = 1, \ldots, n-1$ are constant values, and the static nonlinear function $\xi(\cdot)$ is piecewise linear and presents the following properties

- There exist Υ_i disjoint sets $i \in \mathcal{I}$ such that $\Upsilon_1 \cup \cdots \cup \Upsilon_r = R$ and the nonlinear map is linear inside that set, i.e.

$$\forall x \in \Upsilon_i \quad \xi(x) = \alpha_i x + \beta_i \qquad (2)$$

- It is satisfied either

$$\forall i \quad \begin{cases} \text{sign}(\alpha_i) < \text{sign}(\alpha_{i+1}) \\ \text{sign}(\alpha_i) > \text{sign}(\alpha_{i+1}) \end{cases} \qquad (3)$$

  Hence there is a slope sign change from one linear region to the next or the previous one.
- For $\alpha_i \neq 0$, the static nonlinearity satisfies that there exists a unique element $z_i \in \Upsilon_i$ such that $\xi(z_i) = 0$. The slope sign change produces that inside the set Υ_i, there is a crossing through the zero axis.

Using the previous description, define the sets

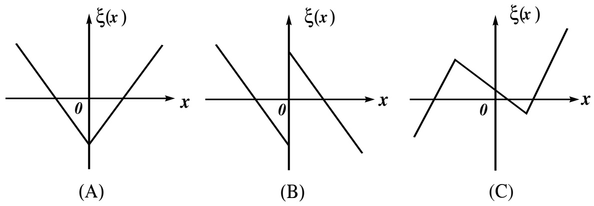

Fig. 1. Examples of Static Nonlinear Terms $\xi(\cdot)$.

$$\Lambda = \bigcup_{\forall i \; s.t. \; \alpha_i \neq 0} \Upsilon_i \qquad (4)$$

$$\mathcal{Z} = \{z_1, z_2, \ldots\} \qquad (5)$$

$$\therefore \mathcal{Z} \subset \Lambda \qquad (6)$$

Thus, Λ is the union of all the intervals that have a zero crossing, and $\mathcal{Z}$ the set of all the zero crossing points. Examples of the static maps $\xi(\cdot)$ are presented in Figure 1. Now, the system (1) can be put into controllable canonical form taking as the output $y = x$, i.e.

$$\dot{x}_1 = x_2 \qquad (7)$$

$$\dot{x}_2 = x_3 \qquad (8)$$

$$\vdots \qquad (9)$$

$$\dot{x}_{n-1} = x_n \qquad (10)$$

$$\dot{x}_n = -a_{n-1}x_n - \ldots - a_1 x_2 + \xi(x_1) \qquad (11)$$

$$y = x_1 \qquad (12)$$

Therefore, the open-loop equilibrium points are

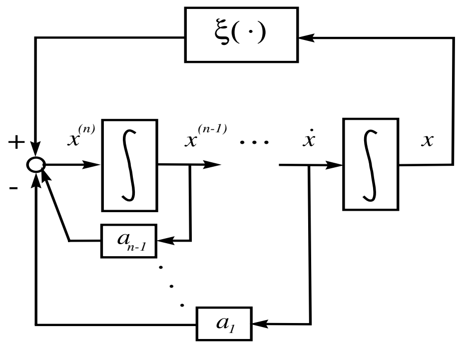

Fig. 2. General Structure of the Systems Studied.

given by

$$x_{2e} = x_{3e} = \ldots = x_{ne} = 0 \quad \& \quad \xi(x_{1e}) = 0 \qquad (13)$$

Hence the equilibriums are defined in terms of the set $\mathcal{Z}$. The control objective is then stated as to design an output feedback controller such that it guarantees local stability for the non-linear closed-loop system at any equilibrium point. In order to analyze the closed-loop stability, the location of the control input into the state-space description (7)-(11) is analyzed, and two possible scenarios are studied:

$$\Pi_1 = \left\{ \frac{s^{n-1}}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s - \alpha_i} \in RL_2 | \forall i \; s.t. \; \alpha_i \neq 0 \right\} \quad (14)$$

$$\Pi_2 = \left\{ \frac{1}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s - \alpha_i} \in RL_2 | \forall i \; s.t. \; \alpha_i \neq 0 \right\} \quad (15)$$

Setup 1 : the control signal enters into the first differential equation $\longrightarrow$ through equation (7),
Setup 2 : the control signal enters into the n^{th} differential equation $\longrightarrow$ through equation (11),

Therefore, associated with each open-loop equilibrium point, a set of the linearized plants can be deduced by defining the position of the control input for Setups 1 and 2 as in (14) and (15), since at each equilibrium point, the static nonlinearity $\xi(\cdot)$ is given by (2). The set RL_2 represents the space of all strictly proper and real rational transfer functions with no poles on the imaginary axis (Zhou and Doyle 1998).

3. CLOSED-LOOP CONTROL TO SIMULTANEOUS STABILIZATION

The two locations for the control signal previously defined will be studied, assuming that the controllers have either a PI structure

$$K(s) = K_p + \frac{K_i}{s} \quad (16)$$

or a general q^{th} order SISO proper structure

$$K(s) = \frac{c_1 s^{q-1} + c_2 s^{q-2} + \ldots + c_q}{s^q + b_1 s^{q-1} + \ldots + b_q} + d \quad (17)$$

where the controller parameters c_j and b_j $\forall j = 1, \ldots, q$ are constant values. The linear controller in (17) can be put into state-space form through a controllable canonical form (Zhou and Doyle 1998)

$$K(s) = \left[\begin{array}{ccccc|c} -b_1 & -b_2 & \cdots & -b_{q-1} & -b_q & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \hline c_1 & c_2 & \cdots & c_{q-1} & c_q & d \end{array} \right] \quad (18)$$

Remark 1. In this contribution, it is assumed negative feedback from the controllers in all the following derivations.

3.1 Analysis for Control Setup 1

The Setup 1 is used for the control signal and conditions for the new closed-loop equilibrium points are derived. Assume that $\mathbf{x} \in R^n$ represents the plant state vector and $\tilde{\mathbf{x}} \in R^q$ the controller state vector.

Lemma 1. For the closed-loop feedback interconnection between the system (1) and

i.) The PI controller, the new single equilibrium point is given by

$$x_{1e} = x_{3e} = \ldots = x_{ne} = 0 \quad (19)$$
$$\tilde{x}_{1e} = -x_{2e} \quad (20)$$
$$x_{2e} = -\frac{1}{a_1}\xi(0) \quad (21)$$

ii.) For the general q^{th} order controller, there are multiple equilibrium points given by

$$\tilde{x}_{1e} = \tilde{x}_{2e} = \ldots = \tilde{x}_{q-1e} = 0 \quad (22)$$
$$\tilde{x}_{qe} = -\frac{1}{b_q} x_{1e} \quad (23)$$
$$x_{3e} = x_{4e} = \ldots = x_{ne} = 0 \quad (24)$$
$$x_{2e} = \left[d + \frac{c_q}{b_q} \right] x_{1e} \quad (25)$$

where the equilibrium conditions for x_1 are obtained by the solutions to

$$\xi(x_{1e}) - \gamma x_{1e} = 0 \quad \gamma \triangleq a_1 \left[d + \frac{c_q}{b_q} \right] \quad (26)$$

Therefore, the new closed-loop equilibrium points are given by the intersection of the static nonlinearity $\xi(\cdot)$ and a line of constant slope γ. Define the set of solution to (26) by

$$\Omega_1 = \{ x \in R \mid \xi(x) - \gamma x = 0 \} \quad (27)$$

The set Ω_1 will have an important role in the stability analysis of the resulting closed-loop system. Note that in Lemma 1, the slope γ related to the new equilibrium points can be defined in terms of the controller dc-gain $K(0)$, i.e.

$$K(0) = d + \frac{c_q}{b_q} \quad (28)$$

From the conclusion of Lemma 1, if the linear controllers are synthesized based on the linearized versions of (1), local closed-loop stability cannot be guaranteed for the PI controller, however it can be concluded for the general q^{th} order controller by the next theorem.

Theorem 1. If the general q^{th} controller $K(s)$ in (17) simultaneously stabilizes the linearized plants Π_1 in (14) then local stability is guaranteed for all the new closed-loop equilibrium points.

proof: First, since the controller $K(s)$ stabilizes all the plants in the set Π_1, then there exist positive definite matrices $P_i > 0$ associated with each open-loop equilibrium point $\mathbf{x_e^i} = [z_i \ 0 \ \dots \ 0]^T$, such that for the closed-loop state vector $\mathbf{X} = [\mathbf{x} - \mathbf{x_e^i} \ \tilde{\mathbf{x}}]^T$, a quadratic Lyapunov functions satisfy

$$V_i(\mathbf{X}) = \mathbf{X}^T P_i \mathbf{X} > 0 \quad and \quad \dot{V}_i < 0 \tag{29}$$

Note that due to the properties of the static nonlinear term $\xi(\cdot) \implies \Omega_1 \subset \Lambda$. Therefore, for each $y \in \Omega_1$, $\exists \Upsilon_i$ such that $y \in \Upsilon_i$. Hence since $\xi(\cdot)$ is piecewise linear, then the linear dynamics of the open loop plant in the sets Υ_i remain unchanged. Therefore, $\forall y \in \Omega_1 \implies \exists P_i > 0$ to construct a quadratic Lyapunov function at the new equilibrium point in order to prove local stability.

□

Remark 2. Note that no conclusion for the PI controller can be derived, since there is only one closed-loop equilibrium point in that case, and it is shifted from the open-loop linear region.

3.2 Analysis for Control Setup 2

The Setup 2 is now used for the control signal. In this section, the conditions for the new closed-loop equilibrium points are derived for both types of controllers.

Lemma 2. For the closed-loop feedback interconnection between the system (1) and

i.) The PI controller, the new single equilibrium point is given by

$$x_{1e} = \ldots = x_{ne} = 0 \tag{30}$$
$$\tilde{x}_{1e} = -\xi(0) \tag{31}$$

ii.) For the general q^{th} order controller, there are multiple equilibrium points given by

$$\tilde{x}_{1e} = \ldots = \tilde{x}_{q-1e} = 0 \tag{32}$$
$$\tilde{x}_{qe} = -\frac{1}{b_q} x_{1e} \tag{33}$$
$$x_{2e} = \ldots = x_{ne} = 0 \tag{34}$$
$$\tag{35}$$

where the equilibrium condition for x_1 is obtained by the solutions to

$$\xi(x_{1e}) - \lambda x_{1e} = 0 \quad \lambda \triangleq d + \frac{c_q}{b_q} \tag{36}$$

As a result, the new closed-loop equilibrium points are given by the intersection of the static nonlinearity $\xi(\cdot)$ and a line of slope $\lambda = K(0)$. Define now the set of solution to (36) by

$$\Omega_2 = \{x \in R \mid \xi(x) - \lambda x = 0\} \tag{37}$$

Theorem 2. If the general q^{th} controller in (17) simultaneously stabilizes the family of linearized plants around the open-loop equilibrium points Π_2 in (15), then local closed-loop stability is guaranteed for all the new equilibrium points.

proof: Similar justification and construction to Theorem 1 since $\Omega_2 \subset \Lambda$.

□

Remark 3. Note that no conclusion for the PI controller can also be derived, since there is only one closed-loop equilibrium point and it is shifted again from the open-loop linear region.

4. APPLICATION TO CHAOS CONTROL

The oscillators studied in this work are a sub-class of $C^3 - equivalents$ chaotic systems (Malasoma 2002) of third order with a nonlinear static term. Examples of these oscillators are the ones described by Sprott (2000) and the popular Chua's circuit. Thus, the systems can be written as ordinary differential equations of third order (Figure 2):

$$\dddot{x} + \alpha\ddot{x} + \beta\dot{x} = \xi(x), \tag{38}$$

where $\xi(x)$ is a nonlinear piecewise function, and α and β are the bifurcation parameters of the oscillator.

Several other chaotic systems can be represented as the integrator chain; for example the Lorenz equation for any point except the origin. These oscillators present different dynamic behavior according with their bifurcation parameters. The main advantage of the systems described by (38) is that they could be implemented experimentally using electronics circuits as operational amplifiers, electric elements as capacitors and resistors, and nonlinear devices as rectifier and zener diodes. The nonlinear static function $\xi(x)$ has different representations that could exhibit chaos; among others, the followings

$$\begin{aligned} \xi_1(x) &= \rho + \sigma|x| \\ \xi_2(x) &= \rho \cdot \mathrm{sgn}(x) + \sigma x \\ \xi_3(x) &= \rho + \sigma \cdot \max(x, 0) \\ \xi_4(x) &= \rho + \sigma f(x) \\ \xi_5(x) &= \rho x + \sigma g(x) + \eta h(x) \end{aligned} \tag{39}$$

where

$$f(x) = \begin{cases} \chi & x < \frac{-\chi}{\epsilon} \\ -\epsilon x & \frac{-\chi}{\epsilon} \leq x \leq \frac{\chi}{\epsilon} \\ -\chi & x > \frac{\chi}{\epsilon} \end{cases}$$

$$g(x) = \begin{cases} 0 & x > \epsilon \\ 1 & -\epsilon \leq x \leq \epsilon \\ 0 & x < -\epsilon \end{cases}$$

$$h(x) = |x + \epsilon| - |x - \epsilon|$$

and $(\rho, \sigma, \eta, \epsilon, \chi)$ are real constant values. ξ_1, ξ_2 and ξ_3 were firstly suggested in (Sprott 2000) and ξ_5 corresponds to the Chua's circuit. In addition, Malasoma (2002) has also reported polynomial approaches for the nonlinear terms $\xi(x)$.

Without loss of generality and due to the topological similarities among the oscillators, the study was focused only on the system with nonlinear term $\xi_1(x)$. The electronic realization is shown in Figure 3, where its mathematical model is represented by

$$\begin{aligned} \dot{x}_1 &= -\frac{x_2}{P_3C_3} - \frac{u}{R_{Al}C_3} \\ \dot{x}_2 &= -\frac{x_3}{R_1C_2} \\ \dot{x}_3 &= \frac{|x_1|}{R_3C_1} + \frac{R_4x_2}{R_2R_5C_1} - \frac{x_3}{P_2C_1} - \frac{15}{P_1C_1} \\ y &= x_1, \end{aligned} \tag{40}$$

where $R_1 = R_2 = R_3 = R_4 = R_5 = R_{AL} = 1k\Omega$, $C_1 = C_2 = C_3 = 100\mu F$, P_1 is a variable resistor that modifies the size of the attractor, and P_2,P_3 are tuned to obtain chaotic behavior in the oscillator (i.e., P_2 and P_3 are bifurcation parameters). R_{Al} is the resistance through that the control authority u is provided into the first derivative equation with negative feedback (**Setup 1**). The

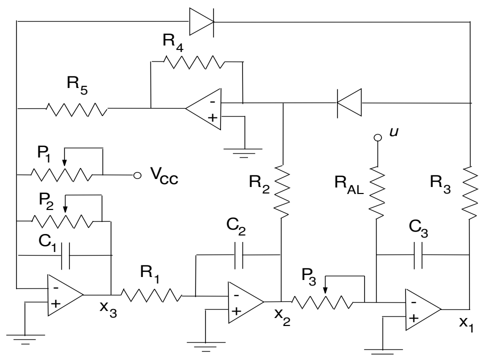

Fig. 3. Electronic Realization of the Chaotic Oscillator with Nonlinear Term $\xi_1(x)$

output of the system is selected by the voltage measurement of the state x_1. The state-space representation of (40) is related to the ordinary differential equation formulation of (38) by the following relations:

$$\begin{aligned} \alpha &= \frac{1}{P_2C_1} = 5.99 \\ \beta &= \frac{R_4}{R_1R_2R_5C_1C_2} = 100 \\ \rho &= \frac{-15}{R_1P_1P_3C_1C_2C_3} = 1044.9 \\ \sigma &= \frac{1}{R_1R_3P_3C_1C_2C_3} = -1751.3. \end{aligned} \tag{41}$$

The chaotic oscillator has two unstable equilib-

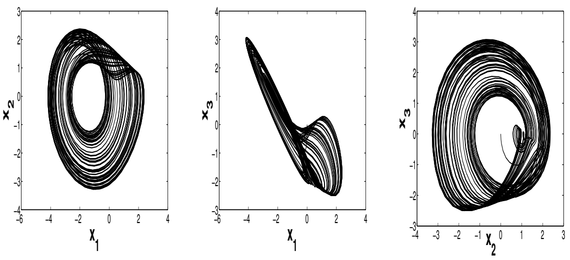

Fig. 4. Projection on canonical planes of the phase portrait for the chaotic circuit in Figure 3.

rium points in $(\pm\frac{15R_3}{P_1}, 0, 0)$, that depend on the variable resistor P_1. In the implementation, the variable resistors in (40) were fixed to the values $P_1 = 8.95k\Omega$, $P_2 = 1.67k\Omega$ and $P_3 = 957\Omega$. The experimental phase portraits of the oscillator states are shown in Figure 4.

A controller that simultaneously stabilizes both equilibrium points was designed following $\mathcal{H}_\infty$ theory (Zhou and Doyle 1998), (Vidyasagar 1985). The controller is a third order proper LTI system:

$$K(s) = 10.8 + \frac{631.2s^2 + 9085s + 6767}{s^3 + 64.15s^2 + 746.7s + 553.2}$$

It presents high gain $K(0) = 23.03 \implies \gamma = 2303.4$. Due to this property, there is only **one closed-loop equilibrium point**, since there exists only one solution to (27). By the analysis presented in the paper, the resulting closed-loop is stable and the new equilibrium condition for the electronic implementation in (38) is listed below

$$\begin{aligned} x_{1e} &= -0.52\ V \\ x_{2e} &= 1.15\ V \\ x_{3e} &= 0\ V \end{aligned} \tag{42}$$

The Figure 5 shows the experimental implementation of the controller by using a dSPACE DS1104 board running at a sampling frequency of 10 kHz. It is observed that the three states of the chaotic oscillator converge to their equilibrium values (42), with some minor error due to the tolerance in the components values (resistors and capacitors ±10%). Note that there is some scaling factor among the state of the original system in (38) and the electronic implementation (40). Besides stabilization, the tracking capabilities of

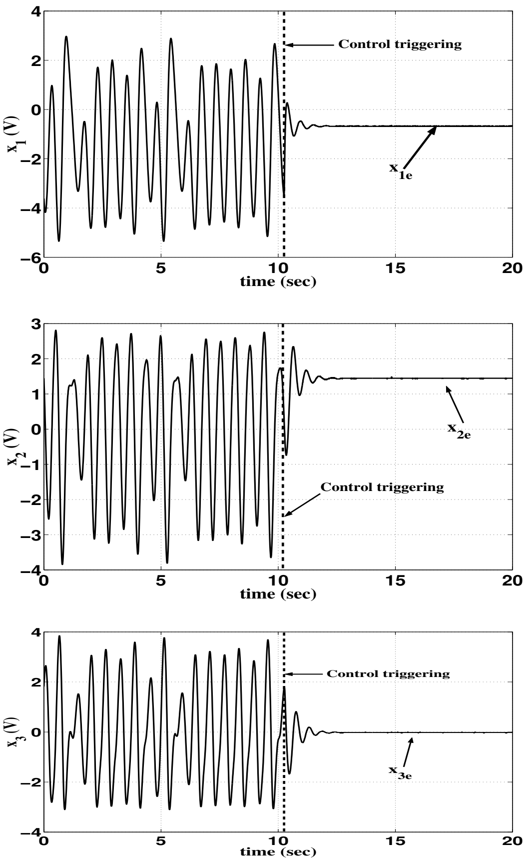

Fig. 5. Experimental Closed-Loop Response for Stabilization.

the resulting closed-loop were tested. For this purpose, an unstable periodic orbit of the chaotic attractor was selected. Figure 6 shows that the state x_1 can follow the periodic reference with almost no error. The remaining states x_2, x_3 follow naturally the corresponding states of the original periodic attractor. Thus, the attractor is fully reconstructed once the control is applied.

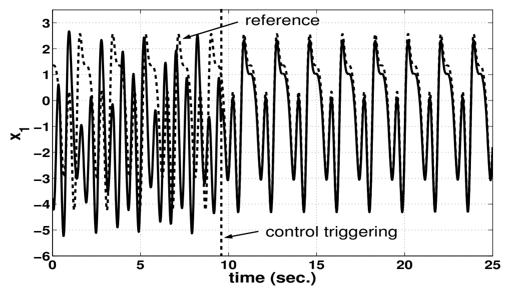

Fig. 6. Experimental Closed-Loop Response for Attractor Tracking.

5. CONCLUSIONS AND FINAL REMARKS

In this paper, the stabilization of a linear system with static nonlinear feedback (Lur'e system) by a linear controller was addressed. Under some conditions in the static nonlinear term, the closed-loop stabilization can be translated to simultaneous stabilization of the open-loop linearized plants. It is shown that the new closed-loop equilibrium points depend on the gain of the controller. The analysis for a PI controller was also presented, and it was observed that only one equilibrium point is obtained. However, the closed-loop stability of the new equilibrium cannot be deduced. The control of a class of chaotic oscillators was used as case study. An experimental implementation was carried out that corroborated the theoretical results presented. In general, the derivations can be extended to the control of linear systems that present internal static nonlinear feedback.

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