On multistability behavior of Unstable Dissipative Systems

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Abstract

We present an unstable dissipative system capable of generate an multistable dynamical behavior *i.e.* depending on its initial condition, the trajectory of the system converge to a given basin of attraction. The proposed system is based on the a type of piecewise linear system called unstable dissipative systems whose main attribute is to generate trajectories with multiple wings or scrolls. From this system we propose an structure where both the linear part and the switching function depends on two parameters. We shown the range of values of such parameters where the system present a multistable behavior and where the system present trajectories with multiscrolls.

Keywords: Multistability, piecewise linear systems, chaos, multi-scroll.

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1. Introduction

In the evolution of a complex system exists various possible (coexisting) basins of attractions or sinks whose realization depends on its initial state. This phenomena is usually called multistability and it is present in a wide variety complex systems [1, 2]. What is the exact interpretation of a sink depends on the complex system that is studied. Take as an example a social system, where one possible meaning are the distinct forms of government (Democracy, Oligarchy, Autocracy). How a social system transit form each one of these can be seen as a multistable behavior. In the context of biology, there are many examples of systems that manifest multistability phenomena. A worth mentioned example is the cellular differentiation, which is important to understand human development and the distinct forms of diseases. Here, multistability is understood as a processes in which a gene regulation network alternate along several possible cell types [3]. Another example comes from the nonlinear chemical dynamics, where multistability is understood as the different possible final chemical states [4]. In this context, the archetype system is the Oregonator system, where concentrations of the reacting species oscillate between two stable final states (bistability). Several examples can be cited ranging from medicine [5], electronic [6], visual perception [7], superconducting [8], etc. All of these examples motivate the current research works to address the challenger despite by R. Vilela Mendes in [9] of -identifying the universal mechanism that leads to multistability and to prove rigorously under what circumstances the phenomenon may occur. One feasible mode to address this challenge is through the formalism of dynamical systems where the concepts of basin of attraction, stability, convergence among others have a mathematical definition and let us to use some tools from stability theory to analyze its behavior. It is worth to note that this situation is similar with the research works some year ago where chaotic system were modeled and interpreted from the point of view of dynamical systems. Since them various dynamical systems with a chaotic behavior have beed proposed (some examples are the Lorenz, Chua and Rössler systems, to name a few)

In the context of dynamical systems, an attractor is defined as a subset of the phase space toward the trajectories of the dynamical systems tends to evolves (and attractors can be fixed points, limit cycles or periodic, quasiperiodic, chaotic or hyper-chaotic orbits). The basin of attraction or sink is defined as the set of all the initial conditions in the phase space whose corresponding trajectories go to that attractor [10, 11]. The concepts of convergent trajectories and attractor stability are usually associated with a energy-like term called Lyapunov function. Then, with the above concepts it can say that a multistable dynamical system is a dynamical system that, depending on its initial condition, its trajectories solution can alternate between two or more mutually exclusive Lyapunov stable and convergent states [5] .

Some formal definitions of multistage behavior have been proposed by D. Angeli in [2] and Q. Hui in [12] for discontinuous dynamical systems. In this sense, a methodology to induce a multistable behavior is by coupling two o more dynamical systems. For example, E. Jiménez-López *et al* induce a multistable in two Jerk-type dynamical systems coupled in a master-slave system. In this direction, C.R. Hens *et.al.* shown that two coupled Rössler oscillators can achieve a type of multistability called extreme, where the number of coexisting attractors is infinite [13]. It is also been observed that by an appropriate modification of the equations, some classical chaotic systems can exhibit also a multistable behavior. For example, [14, 10] propose a varied of the Duffing-Holmes system and Chua's oscillator; shown that in a given range of its parameter's values this system exhibit coexisting attractors. The experimental evidence of multistability for the Rössler oscillator have ben reported by M. Patel *et al* in [6]. On the other hand, C. Li in [15] and D.Z.T. Njitacke *et al* in [16], have been observed that multistable behavior is also present in the Butterfly Flow and memristive diode bidge-based Jerk circuit, respectively. It is worth to mention that for some discrete-time chaotic systems have been also proved that are able to present multistable behavior [17, 18]

In this paper we propose two methodologies to change the dynamics of an unstable dissipative system of Type I and Type II in such form that both types of systems generate a multistable behavior. The first methodology consist in introduce a bifurcation parameter in the linear operator of the UDS of Type I. With such parameter, we can change the location of the stable and unstable manifold until the trajectories are trapped in a specific switching surface. Once the trajectories are inside such surfaces, it can not scape since the manifold of the neighborhood surfaces do not rise to trapped it. In regard to our second methodology, we consider a UDS Type II and modify the switching law without change the linear operator. With both methodologies we can design a priori the number of multistable regions by introducing another switching surfaces.

We organize this paper as follows: in Section 2 we propose a definition of a multi stable dynamical systems. In section 3, we define and describe the main features of an UDS. Even we present in this section the conditions under which a dynamical system is an UDS Type I or Type II system. In section 4 we present in detail our proposed methodology to induce multistability in a UDS Type I system and in section 5 the corresponding methodology for UDS Type II. In section 6 we present some concluding remarks.

2. Multistable dynamical system

The prototypical system that we consider in this paper is an autonomous piece-wise lineal ordinary differential equation of the form:

$$\dot{\chi} = f(\chi) = A\chi + g(\chi), \quad \chi(0) = \chi_o, \tag{1}$$

where $\chi \in \mathbf{R}^n$ the state variables vector, $A = \{a_{ij}\}_{i,j=1}^{n} \in \mathbf{R}^{n\times n}$ is the linear operator with $a_{ij} \in \mathbf{R}$ and; $g : \mathbf{R}^n \to \mathbf{R}^n$ is a vector-valued function of the form:

$$g(\chi) = \begin{cases} B_1 & if \quad \chi \in S_1 = \{\chi \in \mathbf{R}^n : \delta_1 \le G_1(\chi) < \delta_2\}; \\ B_2 & if \quad \chi \in S_2 = \{\chi \in \mathbf{R}^n : \delta_2 \le G_2(\chi) < \delta_3\}; \\ \vdots & \quad\quad \vdots \\ B_m & if \quad \chi \in S_m = \{\chi \in \mathbf{R}^n : \delta_{m-1} \le G_m(\chi) < \delta_m\}; \end{cases} \tag{2}$$

where $B_i = [b_{i1}, \ldots, b_{in}] \in \mathbf{R}^n$ for $i = 1, \ldots, m$ is a set of vectors with real entries; and $S = \{S_1, S_2, \ldots, S_m\}$ is a finite partition of the phase space called the switching domains, which satisfy $\mathbf{R}^n = \bigcup_{1\le i\le r} S_i$. Each S_i is defined by and surface $G_i(\chi)$ and δ_i (with $1 \le i \le m$) acts as a separatrice (or boundary) between two consecutive switching domain. In what follows we call each δ_i the switching surfaces. Furthermore, we assume that each S_i has at least a saddle equilibria point χ^*. If $\bar{v}_j \in \mathbf{R}^n$ is an eigenvector of the linear operator A and $\lambda_j = \alpha_j + i\beta_j$ its corresponding eigenvalue, then the stable set is $E^s = Span\{\bar{v}_j \in \mathbf{R}^n : \alpha_j < 0\}$ and the unstable set $E^u = Span\{\bar{v}_j \in \mathbf{R}^n : \alpha_j > 0\}$ [19]. Stable sets form boundaries between the basins of attraction of different attractors [20]. We assume that every equilibria point $\chi^* \in \mathbf{R}$ of (1) is an hyperbolic saddle-focus equilibrium.

Let $\phi_t(\chi) \in \mathbf{R}^n$ the trajectory solution of (1):

Definition 2.1. *An attractor (or sink) is a closed invariant set $\mathcal{A}$ which has a shrinking neighborhood i.e. there is an open neighborhood $U \subset A$ such that the trajectory $\phi_t(\chi)$ of any point $\chi \in U$ satisfies $dist(\phi_t(\chi), A) \to 0$ as $t \to \infty$; where $dist(x, A) = inf_{x0 \in A}||\chi, x_0||$. The basin of attraction of $\mathcal{A}$ is the set of initial conditions whose trajectories converge to the attractor, that is $\Omega(\mathcal{A}) = \{\chi_0 \in \mathbf{R}^n : \phi_t(\chi_0) \to \mathcal{A} \quad as \quad t \to \infty\}$.*

It is worth to mention that an attractor can be a fixed point, limit cycles, Quasi-periodic motion or even an chaotic trajectory. Next, based on the above definition, we propose the following definition of a multistable system:

Definition 2.2. *We say that the dynamical system* (1) *is multistable if it satisfy the following requirements:*

1.- There exists a set $\{x^*_\rho\}_{\rho=1}^m$ of saddle equilibria points of (1) *in $\mathbf{R}^n$.*

2.- The phase space can be partitioned in a finite number m of switching domains S_i (for $i = 1, \dots, m$) such that each one has a single equilibria point located at $x^*_i = A^{-1}B_i$, for $i = 1, \dots, m$.

3.- Exist at least two attractors $\mathcal{A}_i$ and $\mathcal{A}_j$, bounded by two switching surfaces such that $\Omega(\mathcal{A}_i) \cap \Omega(\mathcal{A}_j) = \emptyset$

3. Chaotic attractors based on Unstable Dissipative Systems

We consider the following family of affine linear systems:

$$\dot{\chi} = A\chi + B(\chi) \tag{3}$$

where $\chi = (x_1, x_2, x_3)^\top \in \mathbf{R^3}$ is the state vector, the real matrix $A \in \mathbf{R}^{3\times3}$ is the linear operator; and $B : \mathbf{R}^3 \to \mathbf{R}^3$ is a vector-valued function. In particular, we assume that (3) is described by the jerk type equation [21]:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha & -\beta & -\gamma \end{pmatrix}, \quad B(\chi) = \begin{pmatrix} 0 \\ 0 \\ \alpha\sigma(\chi) \end{pmatrix}; \tag{4}$$

where $\alpha, \beta, \gamma \in \mathbf{R}$ and $\sigma(\chi) : \mathbf{R}^3 \to \mathbf{R}$ is the following piecewise-constant and continuous from the right function called the switching law:

$$\sigma(\chi) = \begin{cases} b_1 & if \quad \chi \in S_1 = \{\chi \in \mathbf{R}^3 : \delta_1 \leq \mathbf{v}^\top\chi < \delta_2\} \\ b_2 & if \quad \chi \in S_2 = \{\chi \in \mathbf{R}^3 : \delta_2 \leq \mathbf{v}^\top\chi < \delta_3\} \\ \vdots & \quad\vdots \\ b_m & if \quad \chi \in S_m = \{\chi \in \mathbf{R}^3 : \delta_{m-1} \leq \mathbf{v}^\top\chi < \delta_m\} \end{cases} \tag{5}$$

with $b_i \in \mathbf{R}$ and $S_i = \{\chi \in \mathbf{R}^3 : \delta_{i-1} \leq \mathbf{v}^\top\chi < \delta_i\}$ (for $i = 1, \ldots, m;$) are the switching domains, with $\mathbf{v} \in \mathbf{R}^3$ (with $\mathbf{v} \neq 0$) a constant vector and $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_\eta$ the switching surfaces. Without loss of generality, we assume that the planes $\mathbf{v}^\top\chi = \delta_i$ (for $i = 1, 2, \ldots, m$) are defined with $\mathbf{v} = [1, 0, 0]^\top \in \mathbf{R}^3$. The role of the switching function σ is to specify which system is active at a given switching surface, that is, if $\sigma(\chi) = \beta_k$ for $k \in I = \{1, \ldots, m\}$, then the affine linear system that governs the dynamics in the switching region S_k is: $\dot{\chi} = A\chi + [0, 0, a\beta_k]^\top$.

In particular, we assume that each switching domain contains an single saddle equilibrium point located at $x^*_\rho = A^{-1}B_\rho$, with $\rho \in \{1, \ldots, m\}$. With the above assumptions we ensure that for any initial condition $\chi_0 \in \mathbf{R}^3$, the orbit $\phi(\chi_0)$ of the system (3)-(4) is trapped in the region S_ρ by an one-spiral trajectory called scroll or wing. When the flux reach to the the plane $x_1 = \delta_\tau$ (with $\tau \neq \rho$), it crosses to the region S_τ, where it is again trapped in a new scroll with equilibrium point $x^*_\tau = A^{-1}B_\tau$. In this context, the system (3)-(4) can display various multi-scroll attractors as a result of a combination of several unstable one-spiral trajectories [22], where the switching between regions is governed by the switching function (5).

Definition 3.1. *[22] Let* $\{x^*_\rho\}_{\rho=1}^m$ *a set of equilibrium points of the system* (1)*. We say that the system* (5) *is a multi-scroll system with the minimum of equilibrium points, if each* x^*_ρ *observes oscillations around it and for any initial condition* $\chi_0 \in S_i$*, the orbit* $\phi(\chi_0)$ *generates an attractor between the switching surfaces.*

In what follows, we assume that the eigenspectra $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ of the linear operator $A \in \mathbf{R}^{3\times3}$ has the following features: a) at least one eigenvalue is a real number and; b) at least two eigenvalues are complex numbers. Furthermore, we consider that the sum of the real part of each element of Λ is negative. A dynamical system of the form (3) that satisfy the above requirements is called an Unstable Dissipative System (UDS) [1].

Definition 3.2. *Let* $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ *the eigenspectra of the lineal operator* A*, such that* $\sum_{i=1}^3 Re\{\lambda_i\} < 0$*, with* λ_1 *a real number and* λ_2, λ_3 *two complex numbers. The system* (3) *is said to be an UDS Type I if* $\lambda_1 < 0$*; and it is Type II if* $\lambda_1 > 0$*.*

The above definition imply that the UDS *Type I* is dissipative in one of its components but unstable in the other two, which are oscillatory. The converse is the UDS *Type II*, which are dissipative and oscillatory in two of its components but unstable in the other one. The following result (based on the results in [21]) provide conditions to guaranteed that the system (3) is *UDS Type I* or *Type II* for a general lineal operator $A = \{\alpha_{ij}\} \in \mathbf{R}^3$, with $\alpha_{ij} \in \mathbf{R}$ for $i, j = 1, 2, 3$.

Proposition 3.3. *Consider the family of affine lineal systems* (3)*, the lineal operator* A *given by the jerk system* (4) *with* $\alpha, \beta, \gamma \in \mathbf{R}$ *and let* $\{a, b, c\}$ *a set of non zero real numbers called control parameters. If* $\alpha = c(a^2 + b)$*,* $\beta = a^2 + b + 2ac$ *and* $\gamma = c - 2a$ *with* $b, c > 0$ *and* $a < c/2$*, then the system* (3)-(4) *is UDS Type I; on the other hand, if* $b > 0$ *and* $a, c < 0$ *and* $a > c/2$*, then the system is UDS Type II*

Proof. The characteristic polynomial of the lineal operator (4) is:

$$\begin{aligned} p(\lambda) &= \lambda^3 + \gamma\lambda^2 + \beta\lambda + \alpha \\ &= \lambda^3 + (c - 2a)\lambda^2 + (a^2 + b + 2ac)\lambda + (ca^2 + cb) \\ &= (\lambda + c)(\lambda^2 - 2a\lambda + (a^2 + b)) \end{aligned}$$

The roots of $p(\lambda)$ gives the following expressions for the eigenspectra $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ of (4): $\lambda_1 = -c$ and $\lambda_{2,3} = a \pm i\sqrt{b}$. Note that $\lambda_1 < 0$ and $\sum_{i=1}^{3} Re\{\lambda_i\} = -c + 2a < 0$ if $a < c/2$ and $c > 0$. Then, according to Definition (**??**) the system (3)-(4) is UDS *Type I*. On the other hand, if $a, c < 0$, then $\lambda_1 > 0$ and the above summatory is still negative since $a > c/2$, which implies that the system is UDS *Type II*. □

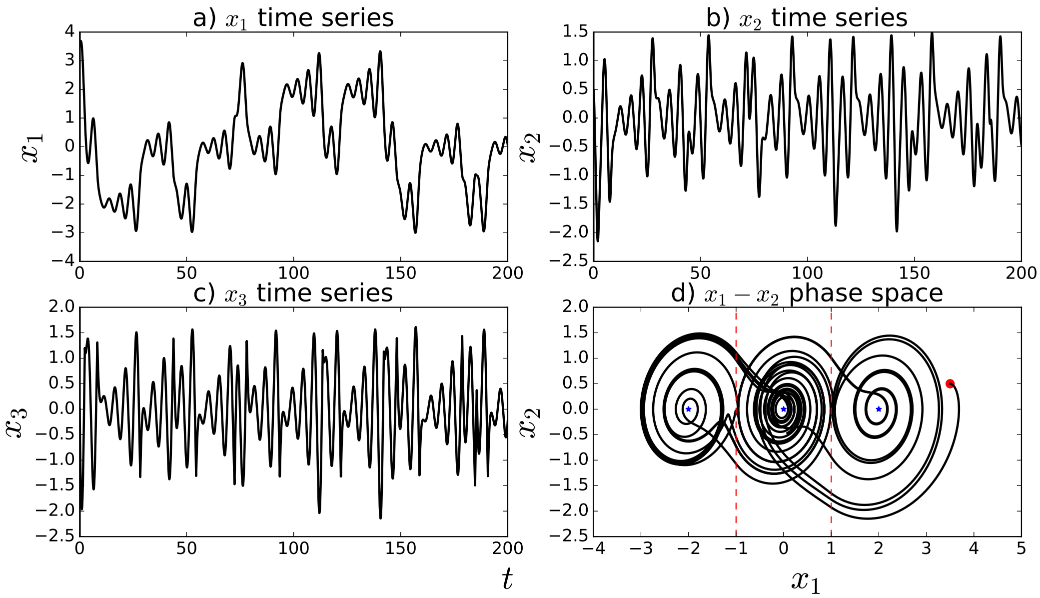

Figure 1: Projection of the UDS *Type I* into the (x_1, x_2) plane with control parameters $a = 0.12501$, $b = 1.5625$ and $c = 1.25$; and switching law (9) . The dashed lines mark the division between the switching surfaces and the red dot indicates the initial position at $\chi_o = (3.5, 0.5, 0)^\top$.

Example 1: In order to illustrate, we consider the dynamical system (3) with control parameters $a = 0.12501$, $b = 1.5625$ and $c = 1.25$. Then, according to the Proposition (3.3), the last row of the lineal operator (4) is defined by the following elements: $\alpha = 1.9727$, $\beta = 1.2656$ and $\gamma = 1$. With this selection of control parameters, the eigenvalues of A are $\lambda_1 = -1.25$ and $\lambda_{2,3} = 0.125 \pm i1.25$, which according to Definition 3.2, the system is an UDS *Type I* . We define the switching law as:

$$\sigma(\chi) = \begin{cases} 2 & if \quad \chi \in S_1 = \{\chi \in \mathbf{R}^3 : x_1 > 1\} \\ 0 & if \quad \chi \in S_2 = \{\chi \in \mathbf{R}^3 : -1 < x_1 \leq 1\} \\ -2 & if \quad \chi \in S_3 = \{\chi \in \mathbf{R}^3 : x_1 \leq -1\} \end{cases} \tag{6}$$

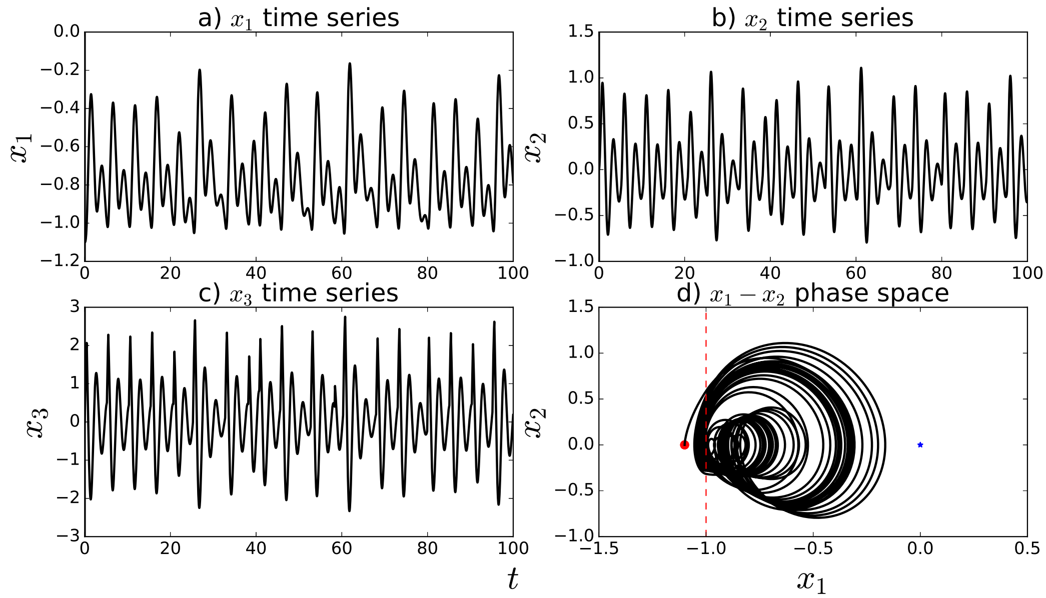

Figure 2: Projection of the UDS *Type II* into the (x_1, x_2) plane with control parameters $a = -0.3494$, $b = 5.9469$ and $c = -0.0988$; and switching law (9) . The dashed lines mark the division between the switching surfaces and the red dot indicates the initial position at $\chi_o = (3.5, 0.5, 0)^\top$.

Then, the equilibrium points for this system are $\chi_1^* = (2, 0, 0), \chi_2^* = (0, 0, 0)$ and $\chi_3^* = (-2, 0, 0)$. In Figure (1) we depicts the time series of the state variables x_1, x_2, x_3 and the projection of the UDS into the phase-space, where we use the switching law (9) and initial condition $\chi_o = (3.5, 0.5, 0)^\top$. It is worth to note that Definition (3.1) is satisfied.

Example 2: As a second example, we consider the following control parameters $a = -0.3494$, $b = 5.9469$ and $c = -0.0988$. Then, according to the results of the Proposition (3.3), $\alpha = -0.6$, $\beta = 6$ and $\gamma = 0.6$. With this selection of parameters the eigenvalues of A are $\lambda_1 = 0.0988$ and $\lambda_{2,3} = -0.3494 \pm 2.4386i$, which according to Definition 3.2, the system is an

UDS *Type II* . In particular, for this second example we define the following switching law:

$$\sigma^*(\chi) = \begin{cases} 0 \quad if \quad \chi \in S_1 = \{\chi \in \mathbf{R}^3 : -1 \leq x_1 \leq 1\} \\ 7 \quad if \quad \chi \in S_2 = \{\chi \in \mathbf{R}^3 : x_1 < -1\} \end{cases} \tag{7}$$

In Figure (2) we illustrate the UDS dynamics for the initial conditions $\chi_0 = (-1.1, 0, 0)^\top$.

4. Induced multistability in a UDS Type I system

Based on the previous description of an UDS, in this section we consider the system (3)-(4) with:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k\alpha & -k\beta & -1 \end{pmatrix}, \quad B(\chi) = \begin{pmatrix} 0 \\ 0 \\ k\alpha\sigma(\chi) \end{pmatrix}; \tag{8}$$

where $k \in \mathbf{R}^+$ is a new parameter introduced in the system. In particular, if $k = 1$, $\alpha = 1.9727$ and $\beta = 1.2656$ (that is, according to Proposition 3.3, the control parameters are $a = 0.12501$, $b = 1.5625$ and $c = 1.25$), the system (3)- (8) satisfy the requirements of Definition (3.2). In Figure (1) we observe that the UDS display varios scrolls in three distinct switching domains. In this section we use the parameter k as a bifurcation parameter whose role is to modify the location of the stable E^s and unstable E^u manifolds. In this sense, k change the dynamical behavior of the UDS *Type I* system from mono-stable to a single scroll dynamics. It is worth to note that by changing k, the switching surfaces and the equilibria points remains unchanged.

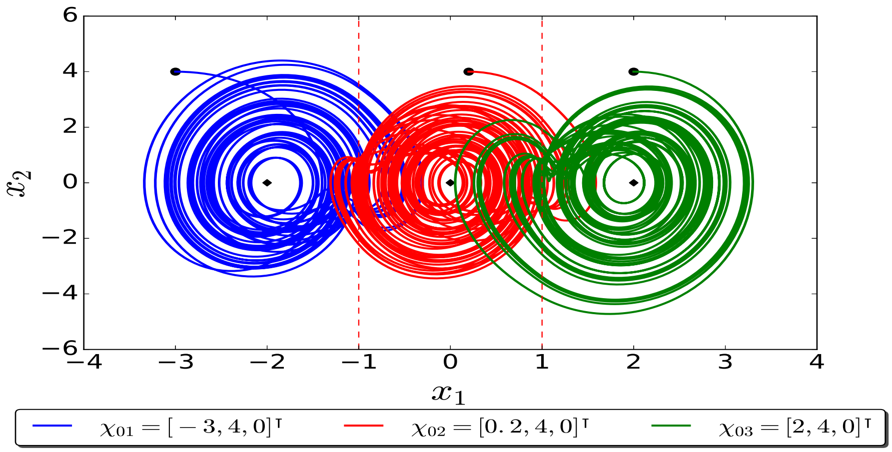

Figure 3: Projection of the UDS *Type I* into the (x_1, x_2) plane with $\alpha = 1.9727$, $\beta = 1.2656$ and $\gamma = 1$, switching law (6) and $k = 7$. The black dots indicates the initial condition at: $\chi_{01} = (-3, 3, 0)^\top$, $\chi_{02} = (0.5, -3.1, 0)^\top$ and $\chi_{03} = (3, -3, 0)^\top$.

Example 3: In order to illustrate, we consider the system given in Example 1. If $k < 7$, the system display three scrolls as we shown in Figure (1). However, if $k > 7$, the manifolds direction change in such a way that for a given initial condition, the trajectories can not display many scrolls as before. Instead, the trajectory remains trapped in a single switching domain $i.e$ becomes multistable as we can see in Figure (3), where we have used the following three distinct initial condition: $\chi_{01} = (-3, 3, 0)^\top \in S_3$, $\chi_{02} = (0.5, -3.1, 0)^\top \in S_2$ and $\chi_{03} = (3, -3, 0)^\top \in S_1$.

Next, we vary the initial condition of the UDS *Type I* (3)-(8) on the (x_1, x_2) plane in order to identify the basin of attraction $\Omega(\mathcal{A}_i)$ of each attractor $\mathcal{A}_i$ (for $i = 1, 2, 3$) . In Figure (4) we shown the three basins of attractions in the region $x_1 \in [-10, 10]$, $x_2 \in [-10, 10]$ and $x_3 = 0$. The color of each dot plotted in such a Figure represent the mean value (over 150

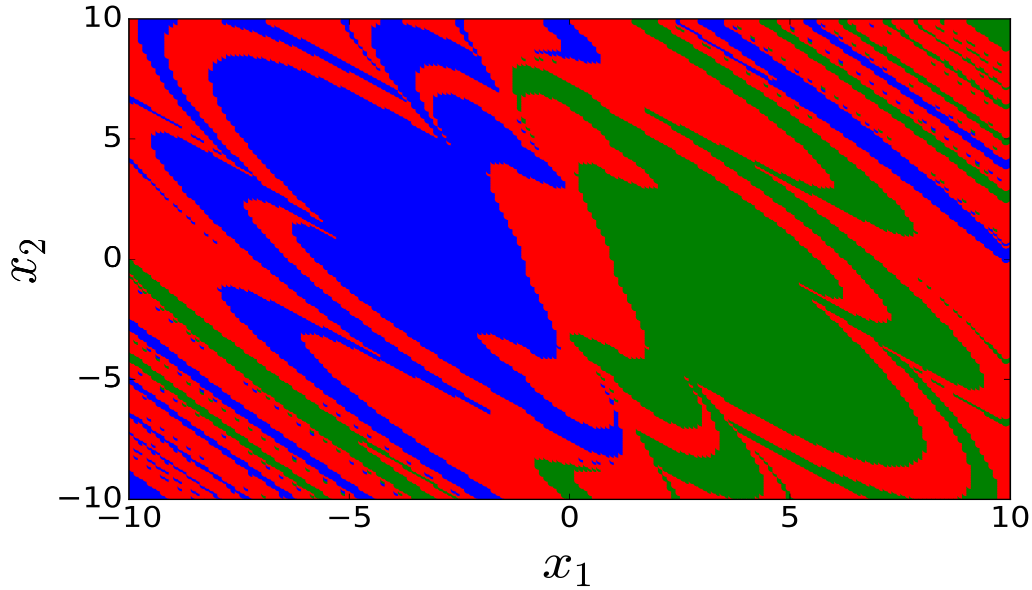

Figure 4: Three basins of attraction $\Omega(\mathcal{A}_i)$ (for $i = 1, 2, 3$) generated with the switching law (6) in $x_1 \in [-10, 10]$, $x_2 \in [-10, 10]$ and $x_3 = 0$. Green dots are used for initial conditions that becomes trapped in S_1, red dots for S_2 and blue dots for S_3.

iterations). A green dot means that for such initial condition, the system is trapped in the switching domain $S_1 = \{\chi \in \mathbf{R}^3 : x_1 > 1\}$. In similar way, the red dot correspond to basins of attraction in the switching domain $S_2 = \{\chi \in \mathbf{R}^3 : -1 < x_1 \leq 1\}$ and blue dots to $S_3 = \{\chi \in \mathbf{R}^3 : x_1 \leq -1\}$.

Example 4: It is worth to mention that it is possible to extend the number of final states of the UDS *Type 1* by increasing the number of switching domains of the systems. The key idea is to design appropriately the switching law by introducing more planes over x_1 [22]. In order to illustrate we change the

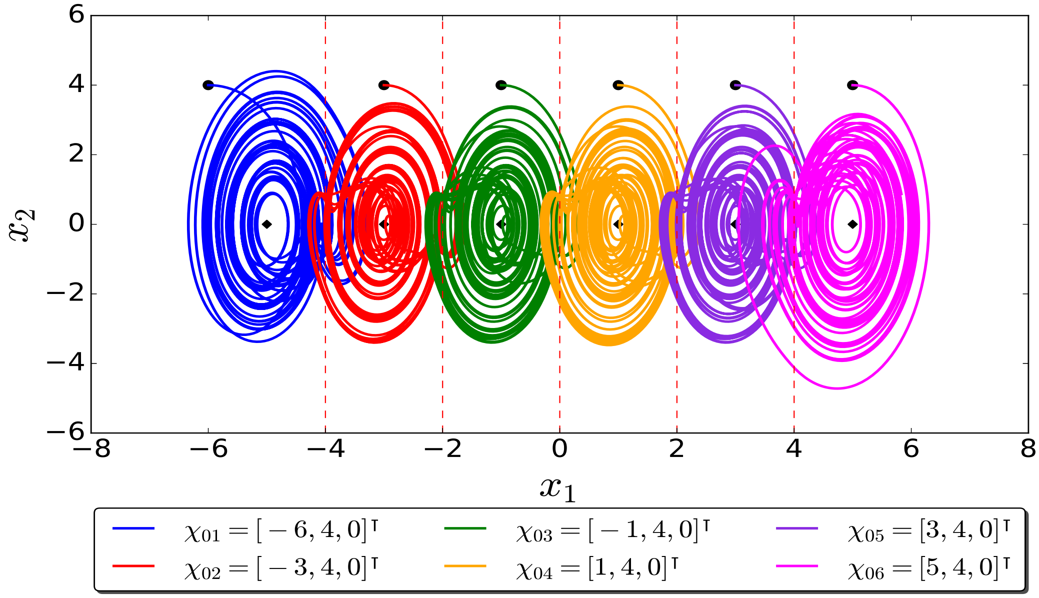

Figure 5: Projection of the UDS *Type I* into the (x_1, x_2) plane with $\alpha = 1.9727$, $\beta = 1.2656$ and $\gamma = 1$, switching law (9) and $k = 7$. The black dots indicates the initial condition.

switching law (6) as follows:

$$\sigma(\chi) = \begin{cases} -5 & if \quad \chi \in \hat{S}_1 = \{\chi \in \mathbf{R}^3 : x_1 \leq -4\} \\ -3 & if \quad \chi \in \hat{S}_2 = \{\chi \in \mathbf{R}^3 : -4 < x_1 \leq -2\} \\ -1 & if \quad \chi \in \hat{S}_3 = \{\chi \in \mathbf{R}^3 : -2 < x_1 \leq 0\} \\ 1 & if \quad \chi \in \hat{S}_4 = \{\chi \in \mathbf{R}^3 : 0 < x_1 \leq 2\} \\ 3 & if \quad \chi \in \hat{S}_5 = \{\chi \in \mathbf{R}^3 : 2 < x_1 \leq 4\} \\ 5 & if \quad \chi \in \hat{S}_6 = \{\chi \in \mathbf{R}^3 : 4 < x_1\} \end{cases} \tag{9}$$

In Figure (5) we shown the behavior of the system (3)-(8) with the same parameter values as in our previous example but with the switching law (9). We shown the dynamics of the system with the following initial conditions: $\chi_{01} = (-6, 4, 0)^\top$, $\chi_{02} = (-3, 4, 0)^\top$, $\chi_{03} = (-1, 4, 0)^\top$, $\chi_{04} = (1, 4, 0)^\top$, $\chi_{05} = (3, 4, 0)^\top$ and $\chi_{06} = (5, 4, 0)^\top$.

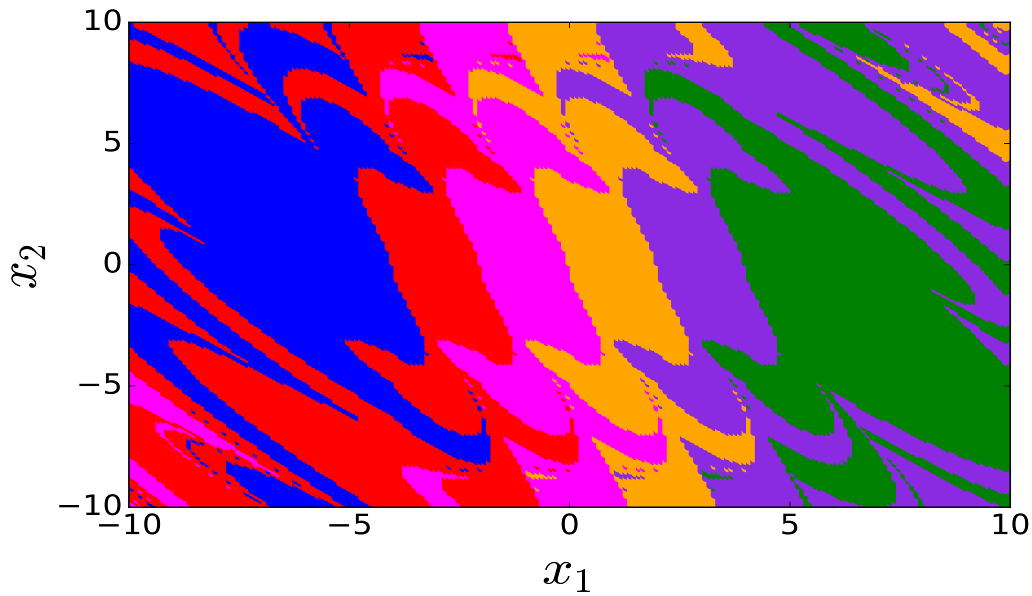

Figure 6: Six basins of attraction $\hat{B}(S_i)$, for $i = 1, \ldots, 6$ of the function (9) in $x_1 \in [-10, 10]$, $x_2 \in [-10, 10]$ and $x_3 = 0$. Blue dots are used for initial conditions in $\hat{B}(S_1)$, red for $\hat{B}(S_2)$, magenta for $\hat{B}(S_3)$, orange for $\hat{B}(S_4)$, violet for $\hat{B}(S_5)$ and green for $\hat{B}(S_6)$.

On the other hand, in Figure (6) we shown the six basin of attraction $\Omega(\mathcal{A}_i)$ (for $i = 1, \ldots, 6$) generated with the switching law (9) and where we vary the initial condition in the range $x_1 \in [-10, 10]$, $x_2 \in [-10, 10]$ and $x_3 = 0$.

5. Induced multistability in UDS Type II

In order to generate multistability behavior in the UDS *Type II*, we first consider the system (3) with:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha & -p\alpha & -\alpha \end{pmatrix}, \quad B(\chi) = \begin{pmatrix} 0 \\ 0 \\ \alpha\sigma^*(\chi) \end{pmatrix}; \tag{10}$$

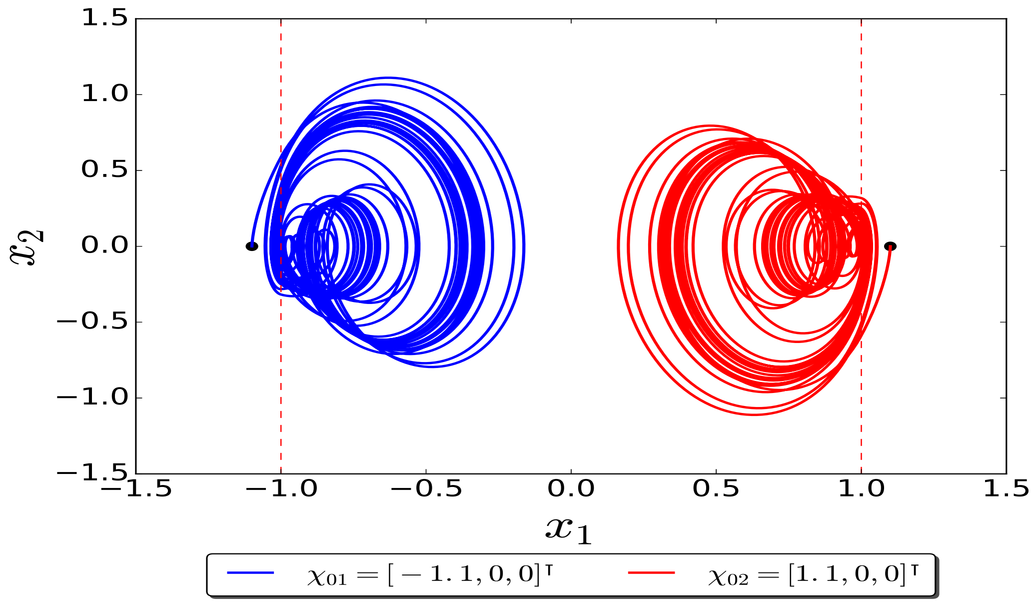

Figure 7: Projection of the UDS *Type II* into the (x_1, x_2) with the switching law (11) and $\alpha = 0.6$ and $p = 10$. The black dots indicates the initial condition at: $\chi_{01} = (-1.1, 0, 0)^\top$ and $\chi_{02} = (1.1, 0, 0)^\top$.

with p is a modular bifurcation parameter and where we modify the switching law (7).

Example 5: by introducing a new switching domain to the function 7 as follows:

$$\sigma^*(\chi) = \begin{cases} -7 & if \quad \chi \in S_1 = \{\chi \in \mathbf{R}^3 : x_1 > 1\} \\ 0 & if \quad \chi \in S_2 = \{\chi \in \mathbf{R}^3 : -1 \leq x_1 \leq 1\} \\ 7 & if \quad \chi \in S_3 = \{\chi \in \mathbf{R}^3 : x_1 < -1\} \end{cases} \tag{11}$$

In Figure (7) we shown the behavior of the system (3)-(8) with $\alpha = 0.6$, $p = 10$ and with the following initial conditions: $\chi_{01} = (-1.1, 0, 0)^\top$ and $\chi_{02} = (1.1, 0, 0)^\top$. On the other hand, in Figure (8) we shown the basin of

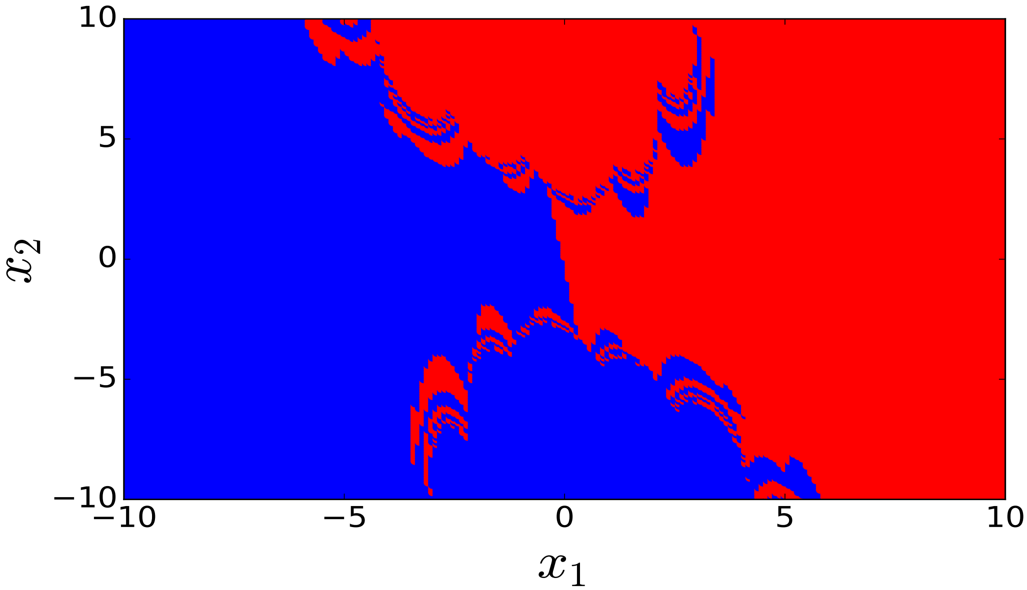

Figure 8: The basins of attraction $\hat{B}(S_i)$, for $i = 1, \ldots, 6$ of (10) generated with the switching function (11) in $x_1 \in [-10, 10]$, $x_2 \in [-10, 10]$ and $x_3 = 0$. Blue dots are used for initial conditions in $\hat{B}(S_1)$ and red for $\hat{B}(S_2)$.

attraction for the the attractors of (10).

Example 6:

In this example we shown how we can generate attractors in $x2$-dimension by modifying the vector-valued function of (10) as follows:

$$B(\chi) = \begin{pmatrix} -f(\chi) \\ 0 \\ \sigma(\chi) + p\alpha f(\chi) \end{pmatrix}; \tag{12}$$

where p is the modular bifurcation parameter used in the lineal operator A, $\sigma(\cdot)$ is a switching law (11) and $f(\chi)$ is the following piecewise constant

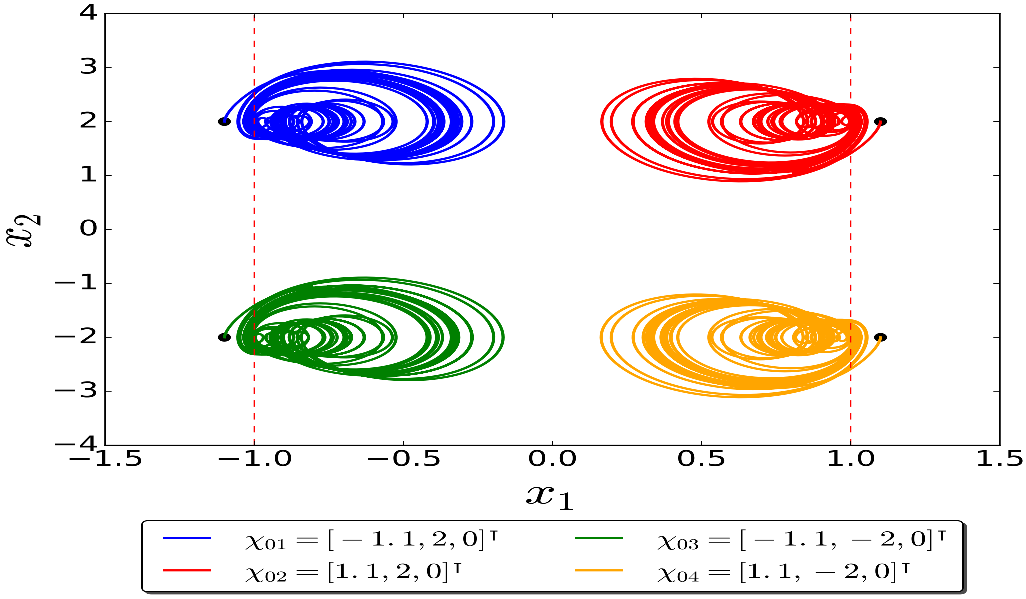

Figure 9: Projection of the UDS *Type II* into the (x_1, x_2) plane with $\alpha = 0.7$; $p = 10$ and, vector-valued function B given in (10).

function:

$$f(\chi) = \begin{cases} -1.4 & if \quad \chi \in S_1 = \{\chi \in \mathbf{R}^3 : x_2 \leq 0\} \\ 1.4 & if \quad \chi \in S_2 = \{\chi \in \mathbf{R}^3 : x_2 > 0\} \end{cases} \tag{13}$$

The role of the function $f(\cdot)$ is to split the direction x_2 to each one of the switching surfaces S_i. In Figure (9) we shown the coexisting attractors generated in each switching surface with the initial conditions: $\chi_{01} = (-1.1, 2, 0)^{\top}$, $\chi_{02} = (1.1, 2, 0)^{\top}$, $\chi_{03} = (-1.1, -2, 0)^{\top}$ and $\chi_{04} = (1.1, -2, 0)^{\top}$. Additionally, in Figure (10) we show the basin of attraction of the UDS *Type II* by varying the initial conditions in the range $x_1 \in [-10, 10]$, $x_2 \in [-10, 10]$ and $x_3 = 0$.

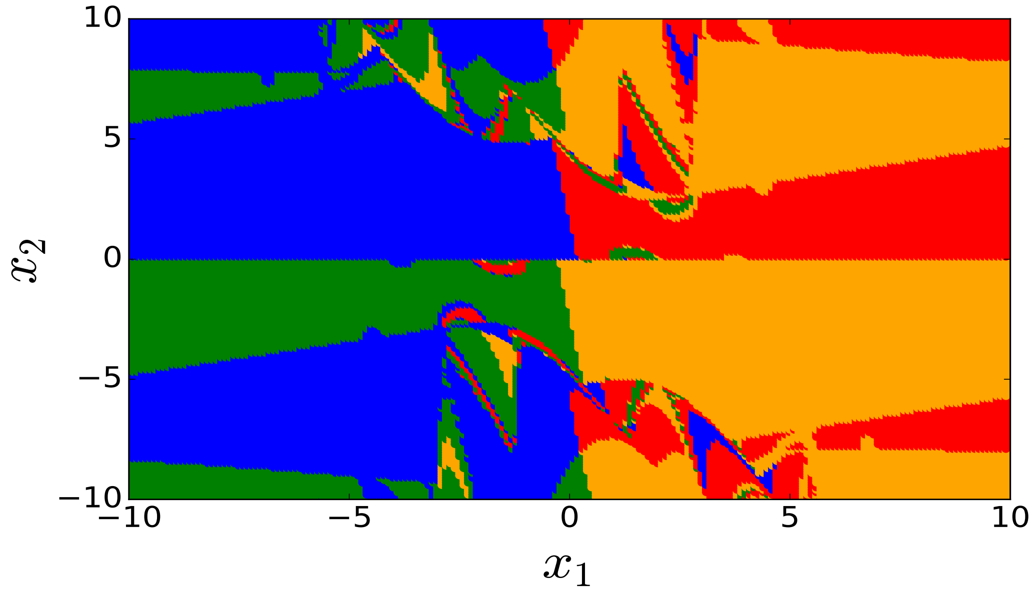

Figure 10: Basins of attraction onto the plane $x_1 \in [-10, 10]$, $x_2 \in [-10, 10]$ and $x_3 = 0$. Each dot represent a given initial condition and its color is the bassin of attractions in which the UDS converge with such initial condition.

6. Concluding remarks

We propose two methodologies to change the dynamics of an unstable dissipative system of Type I and Type II in such form that both types of systems generate a multistable behavior. The first methodology consist in introduce a bifurcation parameter in the linear operator of the UDS of Type I. With such parameter, we can change the location of the stable and unstable manifold until the trajectories are trapped in a specific switching domain. Once the trajectories are inside such domain, it can not scape since the manifold of the neighborhood domains do not rise to trapped it. In regard to our second methodology, we consider a UDS Type II and modify the switching law without change the linear operator. With this methodology we can design a priori the number of attractors by introducing another switching domains.

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