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A polynomial approach for generating a monoparametric family of chaotic attractors via switched linear systems

B. Aguirre-Hernández^a, E. Campos-Cantón^{b,*}, J. A. López-Rentería^c, E. C.
Díaz González^a

^a DEPARTAMENTO DE MATEMÁTICAS,
*División de Ciencias Básicas e Ingeniería, Universidad Autónoma
Metropolitana-Iztapalapa,*
AV. SAN RAFAEL ATLIXCO 186, COL. VICENTINA, 09340,
MÉXICO, D. F., MÉXICO

^b DIVISIÓN DE MATEMÁTICAS APLICADAS,
Instituto Potosino de Investigación Científica y Tecnológica A.C.
CAMINO A LA PRESA SAN JOSÉ 2055 COL. LOMAS 4A SECCIÓN, 78216,
SAN LUIS POTOSÍ, SLP, MÉXICO

^c DEPARTAMENTO DE MATEMÁTICAS,
División de Ciencias Exactas y Naturales, Universidad de Sonora,
BOULEVARD ROSALES Y LUIS ENCINAS S/N, COL. CENTRO, 83000,
HERMOSILLO, SONORA, MÉXICO

Abstract

In this paper, we consider characteristic polynomials of n -dimensional systems that determine a segment of polynomials. One parameter is used to characterize this segment of polynomials in order to determine the maximal interval of dissipativity and unstability. Then we apply this result to the generation of a family of attractors based on a class of *unstable dissi-*

*Corresponding author

Email addresses: bahe@xanum.uam.mx (B. Aguirre-Hernández),
eric.campos@ipicyt.edu.mx (E. Campos-Cantón), jalr@mat.uson.mx (J. A.
López-Rentería), edgardazgonzalez@yahoo.com.mx (E. C. Díaz González)

pative systems (UDS) of type affine linear systems. This class of systems is comprised of switched linear systems yielding strange attractors. A family of these chaotic switched systems is determined by the maximal interval of perturbation of the matrix that governs the dynamics for still having scroll attractors.

Keywords: unstable dissipative systems; switched systems; multi-scroll attractors; chaotic systems, segment of polynomial.

1. Introduction

Let us to consider a linear system capable of displaying multiscroll attractors with chaotic behavior when is driven by the suitable control law:

$$\dot{x} = Ax \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector and $A \in \mathbb{R}^{n \times n}$ determines the dynamics of the system. A great deal of qualitative information about the behaviors of its solutions is determined by the linear operator A of (1) and specifically its eigenvalues $\Lambda = \{\lambda_1, \dots, \lambda_n\}$. An equilibrium point x^* of (1) is called saddle if it is a hyperbolic equilibrium point in the sense that A has at least one eigenvalue with positive real part and at least one with negative real part but no eigenvalues having zero real parts. Saddle equilibria, which connect the stable and unstable manifolds, W^s and W^u , respectively, are responsible for successive stretching and folding therefore play an important role in generating chaos, particularly saddle-focus equilibria. The stretching causes the system trajectories to exhibit sensitive dependence on initial conditions whereas the folding creates the complicated microstructure [1]. The saddle-focus equilibria of a chaotic systems in \mathbb{R}^3 can be characterized into two types

according to their eigenvalues $\Lambda = \{\lambda_i, \lambda_j, \lambda_k\} \subset \mathbb{C}$: (i) The saddle-focus equilibria that is stable in one of its components but unstable and oscillatory in the other two ones [2]. That is, the stable component corresponds to a negative real eigenvalue ($\text{Re}\{\lambda_i\} < 0$ and $\text{Im}\{\lambda_i\} = 0$), whereas the unstable components are related to two complex conjugate eigenvalues ($\text{Re}\{\lambda_k\} > 0$ and $\text{Im}\{\lambda_k\} \neq 0$). (ii) The saddle-focus equilibria that are stable in two of its components but unstable in the other one [3]. That is, the dissipative components are oscillatory: $\text{Im}\{\lambda_k\} \neq 0$ and $\text{Re}\{\lambda_k\} < 0$, while the unstable component corresponds to the real positive eigenvalue $\text{Re}\{\lambda_k\} > 0$ and $\text{Im}\{\lambda_k\} = 0$.

If system (1) has a saddle-focus equilibrium which is responsible for stable and unstable manifolds and the sum of its eigenvalues is negative, then the system is called an unstable dissipative system (UDS). According to the above discussion, it is possible to define two types of UDS, and two types of corresponding equilibria in \mathbb{R}^3 .

Definition 1.1. Consider the system (1) in \mathbb{R}^3 with eigenvalues λ_i , $i = 1, 2, 3$ such that $\sum_{i=1}^3 \lambda_i < 0$. Then the system is said to be an UDS of *type I* (UDS-I) if one of its eigenvalues is negative real and the other two are complex conjugate with positive real part; and it is said to be of *type II* (UDS-II) if one of its eigenvalues is positive real and the other two are complex conjugate with negative real part.

For the corresponding equilibria, their two types are defined accordingly. The above definitions imply that the UDS-I is dissipative in one of its components but unstable in the other two, which are oscillatory. The converse is the UDS-II, which are dissipative and oscillatory in two of its components

but unstable in the other one. Some hyperbolic chaotic dynamical systems in \mathbb{R}^3 may be related to these two types of UDS around equilibria; for instance, Chua's system [4] and another piecewise linear system [5] and nonlinear systems [6, 7, 8, 9].

Switched systems have been widely used in many different areas in science. Some recent analysis have been made regarding their stability (See [10, 11, 12], and the references therein). There is interest in generating chaotic or hyperchaotic attractors with multiple scroll based on UDS-I and synchronization phenomena have been studied between them, for example, in [13] has been studied preservation of scrolls via chaotic synchronization. Since the work reported by [14] about n-Double scroll from the Chua's system [15, 16], there have been many different approaches to yield multi-scroll attractors in the last coupled decades. These approaches may be ranged from modifying the Chua's system by replacing the nonlinear part with different nonlinear functions [14, 17, 18, 19], to the use of nonsmooth nonlinear functions such as, hysteresis [20, 21], saturation [22, 23], saw-tooth [24], threshold and step functions [25, 26, 27, 28]. Multiscroll attractors based on the Jerk equation are given in [29]. Recently, it has been reported the generation of multi-scroll attractors in fractional order chaotic systems [30, 31], and based on nonlinear systems [32].

On the other hand, even though there are a lot of approaches to generate multiscroll chaotic attractors, there are few works to generate families of multiscroll chaotic attractors. For example in [27] a family of multiscroll attractors was presented based on a system given by (1) and applying different controls signals. Therein families into three called 1D-, 2D- and 3D-grid

multiscroll attractors were generated and the families consist on the possibility to increase the number of scrolls in all state variable directions. In [5] a family of multiscroll attractors was presented as a monoparametric family due to with only one parameter the system generates a single, double or tripled scrolls attractors. Recently in [33, 34] a hyperchaotic family of multiscroll attractors was introduced defining a multiscroll system in \mathbb{R}^3 and this is taken as a subsystem for multiscroll systems in \mathbb{R}^n , with $4 \leq n \in \mathbb{N}$. Other interesting works have developed a systematic methodology for constructing continuous-time autonomous hyperchaotic systems with multiple positive Lyapunov exponents [35, 36]. Now in this work, we are introducing a family of multiscroll attractors based on a family of polynomials. That is, consider a system as (1) with characteristic polynomial p_0 and UDS-I, so the interest is to generate a polynomial family $\mathbb{P}(t, k) = p_0 + kp_1$ such that support a new system given by (1) but remaining UDS-I. The background of the theory of polynomial families $\mathbb{P}(t, k)$ can be studied in [37, 38, 39] and references therein. The aim is to find the maximal instability interval, i.e., *the maximal robust dynamics interval* (MDI) that ensures saddle equilibrium of system (1) in order to connect the stable and unstable manifolds, W^s and W^u , respectively. Thus these manifolds are responsible of successive stretching and folding to generate chaos. From the application point of view, chaos has been successfully employed towards encryption algorithms [40, 41] and communication systems [42]. Communication systems based on parametric modulation of chaotic systems which consists of modulating the information signal in a parameter of the chaotic system without destroying the chaotic dynamics. This kind of communication systems needs an interval of values

for the parameter to be modulated with the information. Thus, one of the applications of a monoparametric family of chaotic attractors via switched linear systems may be in the area of communication systems. Also several kinds of chaotic synchronization phenomena have been reported [13, 43, 44]. For example, in [45] a time-varying complex dynamical network model is introduced and synchronization phenomenon is investigated.

In this work, we contribute to the generation of a family of UDS by constructing a monoparametric family of chaotic attractors based on a class of dynamical systems. This class of dynamical systems results of the combination of two unstable “one-spiral” trajectories. The article is organized as follows: Section 2 contains the preliminaries of families of chaotic attractors generated by switched linear systems. Section 3 describes the theory to generate families of polynomials in order to find the *maximal robust dynamics interval* (MDI). Section 4 contains the mechanism to generate a monoparametric family of multiscrolls attractors based on unstable dissipative systems (UDS); finally conclusions are drawn in Section 5.

2. A family of chaotic attractors generated by switched linear systems

It is possible to address the study of generating strange attractors from point of view of the control design theory by considering the following control system:

$$\dot{x} = Ax + Bu, \tag{2}$$

where $B \in \mathbb{R}^{n \times m}$ and the pair (A, B) is in canonical controllable form and close the loop with the very used linear feedback control $u(k, x) \in \mathbb{R}^m$. It

is well known that characteristic polynomial of the closed loop system is $\mathbb{P}(t, k) = p_0(t) + kp_1(t)$, where k is a real parameter, $p_0(t)$ is the characteristic polynomial of A and $p_1(t) = c_n t^n + c_{n-1} t^{n-1} + \dots + c_1$ is an arbitrary polynomial. Therefore, the dynamics of the closed-loop system is governed by the polynomial family $\mathbb{P}(t, k)$. If $k = 0$ then the behavior of the solutions for the system is given by its characteristic polynomial $p_0(t)$. Thus, we can perturb the variable k around zero to keep the same stability and unstability that the original dynamics. Following this concept, a question arises about how can perturb the k parameter to preserve the dynamics of the system? i.e., we are interested in knowing the maximum range of the k parameter such that the system remains UDS-I. The case for $p_0(t)$ stable is called the problem of finding the maximal stability interval and was studied by Bialas around 1985 [37, 46] and recently by López-Rentería *et al.* in [47]. Using these ideas, we generate and characterize the dynamics of systems according to their equilibria. The description of the characterization is given as follows: Suppose that the monoparametric family of systems

$$\dot{x} = \mathfrak{F}_k(A, B, p_1, x, u) \quad (3)$$

for $k = k_0$ has n_1 eigenvalue in \mathbb{C}^- and $n - n_1$ eigenvalues in \mathbb{C}^+ . By continuity of the coefficients of the polynomial $p_0(t)$, the perturbation of k around k_0 gives us the following definition.

Definition 2.1. We shall call a *robust dynamics interval* of the monoparametric family (3) to an interval $[a, b]$ around k_0 if the characteristic polynomial $p_k(t)$ has n_1 roots in \mathbb{C}^- and $n - n_1$ roots in \mathbb{C}^+ for all $k \in [a, b]$.

Thus, the following task is the finding of the maximal robust dynamics

interval, that is, find an interval for k around some k_0 where the system keeps its dynamics.

Under the above scheme, another important related problem is the control design in order to generate chaotic multiscroll attractors by unstable dissipative switching systems of type linear affine with multiple equilibria $x^* = -A^{-1}Bu \in \Omega \subset \mathbb{R}^n$ which they are saddle hyperbolic points with stable and unstable manifolds $W_{x^*}^s$ and $W_{x^*}^u$, respectively. The dynamics of the linear system (2) is characterized by the set of eigenvalues $\Lambda : spec(A)$. Thus, the characteristic polynomial $p_0(t)$ of A is restrained to take values that satisfies hypothesis for systems of either type I or II given in definition 1.1.

We are interested in to provide a family of piecewise affine continuous system in the following way: We shall design a mono parametric family of systems $\dot{x} = A(k)x + Bu$, $k \in \mathbb{R}$, governed by a switching law

$$u = \begin{cases} u_1 & \text{in } \mathcal{D}_{x_1^*}; \\ \vdots & \\ u_j & \text{in } \mathcal{D}_{x_j^*}; \\ \vdots & \\ u_r & \text{in } \mathcal{D}_{x_r^*}. \end{cases} \quad (4)$$

that generate multi-scroll attractors for $k = 0$, with u_i on a domain $\mathcal{D}_i \subset \mathbb{R}^n$, $i = 1, 2, \dots, r$, with $\cap_{i=1}^r \mathcal{D}_i = \emptyset$ and $\cup_{i=1}^r \mathcal{D}_i = \mathbb{R}^n$. For a specific equilibria x_1^*, \dots, x_r^* in any convenient place in \mathbb{R}^n . Thus each x_i^* determines its respective $u_i = B^{-1}Ax_i^*$. Moreover, we shall determine the maximal interval of dissipativity and robust dynamics (\underline{k}, \bar{k}) around $k = 0$, that is, the maximal

interval of perturbation of the matrix $A(k)$ by mean of the parameter k for still having scrolls attractors. To achieve it we assume that the system (2):

DS1: Is dissipative and unstable at an equilibrium $x^* = -A^{-1}Bu \in \Omega \subset \mathbb{R}^n$ which is a saddle hyperbolic point.

DS2: Is completely controllable.

Respect to the hyperbolicity, it is required to be such that A has n_1 eigenvalues in \mathbb{C}^- and $n - n_1$ eigenvalues in \mathbb{C}^+ with non pure imaginary eigenvalues.

3. The Maximal UDS Interval

The aim is to establish a result for a family of polynomials $\mathbb{P}(t, k) = p_0(t) + kp_1(t)$ for which $p_0(t)$ has n_1 roots in \mathbb{C}^- and $n - n_1$ roots in \mathbb{C}^+ for all k in the maximal robust dynamics interval, (k_{\min}^-, k_{\max}^+) . Suppose that the family $\mathbb{P}(t, k)$ satisfies the following assumptions.

Assumption 3.1. $\mathbb{P}(t, k)$ is a family of polynomials of

1. fixed degree n ,
2. coefficients are continuous with respect to $k \in I = [a, b]$.

Therein we compute the maximal unstable and dissipative intervals and consequently, the maximal UDS interval (\underline{k}, \bar{k}) . Finally, we study a family of attractors based on a class of unstable dissipative systems. Thus we start by giving some important theorems that help us to establish the maximal unstable dissipative interval.

THEOREM (3.2) (BOUNDARY CROSSING THEOREM). *Under Assumption 3.1, if $\mathbb{P}(t, a)$ has all its roots in $\mathfrak{S} \subset \mathbb{C}$ whereas $\mathbb{P}(t, b)$ has at least one root in $\mathfrak{U} = \mathbb{C} - \mathfrak{S}$. Then there exists at least one ρ in $(a, b]$ such that*

- $\mathbb{P}(t, \rho)$ has all its roots in $\mathfrak{S} \cup \partial\mathfrak{S}$,
- $\mathbb{P}(t, \rho)$ has at least one root in $\partial\mathfrak{S}$.

Where $\partial\mathfrak{S}$ denotes the boundary of \mathfrak{S} . For a specific polynomial $\mathbb{P}(t, k_0)$, with $k_0 \in [a, b]$, its coefficients depend continuously on the parameter vector $\hat{c}(k_0) \in \mathbb{R}^n$, then the parameter vectors $\hat{c}(k)$ associated to the family of polynomials $\mathbb{P}(t, k)$ vary in a set $\Omega \in \mathbb{R}^n$. If $\mathbb{P}(t^*, k_0) = 0$ then t^* is a root, thus the set of roots $\Lambda_{k_0}(t) := \{t^* \mid \mathbb{P}(t, k_0) = 0\}$ are the eigenvalues of the system.

THEOREM (3.3) (ZERO EXCLUSION PRINCIPLE). . *Under Assumption 3.1, the polynomial family $\mathbb{P}(t, k)$ contains at least one stable polynomial, and Ω is piecewise connected. Then the entire family is stable if and only if*

$$0 \notin \mathbb{P}(t^*, k), \forall t^* \in \partial\mathfrak{S}. \quad (5)$$

As well as its generalization presented in [47].

THEOREM (3.4) (GENERALIZATION OF ZERO EXCLUSION PRINCIPLE). *Consider the polynomial family $P(\lambda, t)$ with constant degree where $\lambda \in \Omega$ and $\Omega \subset \mathbb{R}^l$ is a pathwise connected set. Suppose there exists an element of the family with n_1 roots in \mathbb{C}^- and $n - n_1$ root in \mathbb{C}^+ . Then the entire family still having n_1 roots in \mathbb{C}^- and $n - n_1$ roots in \mathbb{C}^+ if and only if $P(\lambda, i\omega) \neq 0$ for all $\lambda \in \Omega$ and for all $\omega \in \mathbb{R}^l$.*

In [47] the maximal robust stability interval is computed by using the before generalized zero exclusion principle, which can be seen as a particular case in our new context of robust dynamics (definition 2.1 for $n_1 = n$).

Now, in a more general case, we consider the problem of finding the maximal interval of robust dynamics with a polynomial approach. So we are interested in the maximum interval (k_{\min}^-, k_{\max}^+) such that $\mathbb{P}(t, k)$ has n_1 roots in \mathbb{C}^- and $n - n_1$ roots in \mathbb{C}^+ for all k in (k_{\min}^-, k_{\max}^+) and $n > \deg p_1(t)$. In the same spirit that [47], we calculate k_{\min}^- and k_{\max}^+ . If $p_0(-i\omega) = P(\omega^2) - i\omega Q(\omega^2)$ and $p_1(i\omega) = p(\omega^2) + i\omega q(\omega^2)$, then

$$\mathbb{P}(i\omega, k)p_0(-i\omega) = G(\omega) + kF(\omega) + ik\omega H(\omega)$$

where

$$\begin{aligned} F(\omega) &= p(\omega^2)P(\omega^2) + \omega^2 q(\omega^2)Q(\omega^2), \\ G(\omega) &= P^2(\omega^2) + \omega^2 Q^2(\omega^2), \\ H(\omega) &= q(\omega^2)P(\omega^2) - p(\omega^2)Q(\omega^2). \end{aligned}$$

Now let us to define, for an arbitrary polynomial $f(t)$, the set

$$R(f) = \{\xi \in \mathbb{C} \mid f(\xi) = 0\}$$

Let $R(f)_{\mathbb{R}^+}$ denotes the set of positive real elements of $R(f)$ and now we define the sets

$$\begin{aligned} K^+ &= \{F(\omega_l) : \omega_l \in R(H)_{\mathbb{R}^+} \cup \{0\}, F(\omega_l) > 0\} \\ K^- &= \{F(\omega_l) : \omega_l \in R(H)_{\mathbb{R}^+} \cup \{0\}, F(\omega_l) < 0\} \end{aligned}$$

Then, with the aforementioned we give the following result for a family of polynomials.

THEOREM (3.5) (MAXIMAL ROBUST DYNAMICS INTERVAL). *Consider the polynomial family $\mathbb{P}(t, k) = p_0(t) + kp_1(t)$, where $p_0(t)$ is a n -degree polynomial with n_1 roots in \mathbb{C}^- and $n - n_1$ roots in \mathbb{C}^+ . Suppose the $n > \deg p_1(t)$*

and let $F(\omega)$, $G(\omega)$ and $H(\omega)$ be the polynomials defined above. Then $\mathbb{P}(t, k)$ has n_1 roots in \mathbb{C}^- and $n - n_1$ roots in \mathbb{C}^+ for all $k \in (k_{\min}^-, k_{\max}^+)$, where

$$k_{\min}^- = \max \left\{ -\frac{G(\omega_l)}{F(\omega_l)} : F(\omega_l) \in K^+ \right\}$$

$$k_{\max}^+ = \min \left\{ -\frac{G(\omega_l)}{F(\omega_l)} : F(\omega_l) \in K^- \right\}$$

Proof. For $k = 0$ we have that $\mathbb{P}(t, 0) = p_0(t)$ has n_1 roots in \mathbb{C}^- and $n - n_1$ roots in \mathbb{C}^+ . In order to apply the generalized zero exclusion principle we need that if for all $k \in [k^-, k^+]$, $\mathbb{P}(i\omega, k)p_0(i\omega) \neq 0$ for all $\omega \in \mathbb{R}$, then $\mathbb{P}(t, k)p_0(-t)$ has n roots in \mathbb{C}^- and n roots in \mathbb{C}^+ . It is clear that if $k = 0$ then $\mathbb{P}(i\omega, 0)p_0(-i\omega) = p_0(i\omega)p_0(-i\omega)$ has n roots in \mathbb{C}^- and n roots in \mathbb{C}^+ for all $\omega \in \mathbb{R}$. Thus, we are interested in the maximum and the minimum values (around $k = 0$) of k where it occurs a change of stability in the system. That is, the maximum k^- and the minimum k^+ for which occurs $\mathbb{P}(i\omega, k)p_0(-i\omega) = G(\omega) + kF(\omega) + ik\omega H(\omega) = 0$. To achieve it, we must solve the system of equations

$$G(\omega) + kF(\omega) = 0 \tag{6}$$

$$\omega H(\omega) = 0. \tag{7}$$

The aforementioned system is simultaneously satisfied if $G(\omega_l) + kF(\omega_l) = 0$, where $\omega_l \in R(H)_{\mathbb{R}^+} \cup \{0\}$. Thence we get $k = -\frac{G(\omega_l)}{F(\omega_l)}$. Therefore, since

$G(\omega) > 0$ for all $\omega \in \mathbb{R}$, the desired values are given by

$$\begin{aligned} k_{\min}^- &= \max \left\{ -\frac{G(\omega_l)}{F(\omega_l)} : F(\omega_l) > 0, \omega_l \in R(H)_{\mathbb{R}^+} \cup \{0\} \right\} \\ &= \max \left\{ -\frac{G(\omega_l)}{F(\omega_l)} : F(\omega_l) \in K^+ \right\} \\ k_{\max}^+ &= \min \left\{ -\frac{G(\omega_l)}{F(\omega_l)} : F(\omega_l) < 0, \omega_l \in R(H)_{\mathbb{R}^+} \cup \{0\} \right\} \\ &= \min \left\{ -\frac{G(\omega_l)}{F(\omega_l)} : F(\omega_l) \in K^- \right\} \end{aligned}$$

as we claim. □

Remark 3.6. In the proof we just consider positive real $\omega_l \in R(H)_{\mathbb{R}^+}$ due to the symmetry of $H(\omega)$ and the realness of values of k . Moreover, it is not necessary to consider cases when $F(\omega_l) = 0$, since otherwise the system (6)-(7) we would have $G(\omega_l) = 0$, but

$$\begin{aligned} G(\omega_l) &= |p_0(i\omega_l)|^2 \\ &= p_0(i\omega_l)p_0(-i\omega_l) \\ &= 0, \end{aligned}$$

which is impossible to happen because $p_0(t)$ does not have roots in $i\mathbb{R}$. However, if it occurs either $K^+ = \{0^+\}$ or $K^- = \{0^-\}$, then we evaluate in $\omega = 0$ and depending on the sign of $F(0)$ we will get either

$$k_{\min}^- = \lim_{r \rightarrow 0^+} -\frac{G(0)}{r} = -\infty$$

or

$$k_{\max}^+ = \lim_{r \rightarrow 0^-} -\frac{G(0)}{r} = +\infty.$$

Similar results but for segments and rays of Hurwitz polynomials can be seen in [48] and [49].

Now, we are closer of getting what we desire. If $n = 3$ and $n_1 = 1$, we can know about the maximal interval of “saddleness” (MDI) and we do just need the condition for negative sum of roots (dissipativity condition).

LEMMA (3.7) (DISSIPATIVITY LEMMA). *The sum of the roots of the polynomial $p(t) = a_0t^n + a_1t^{n-1} + \dots + a_{n-1}t + a_n$ is negative if and only if $\frac{a_1}{a_0} > 0$.*

Proof. Let z_1, \dots, z_n be the roots of $p(t)$ and

$$p(t) = a_0 \prod_{j=1}^n (t - z_j)$$

its factorization. By the Viète’s formulae (see in [50]) we have that

$$p(t) = a_0(t^n - s_1t^{n-1} + \dots + (-1)^n s_n)$$

where $s_j = \sum_{1 \leq i_1 < \dots < i_j \leq n} z_{i_1} \dots z_{i_j}$, for $j = 1, 2, \dots, n$. Thus, $a_j = (-1)^j a_0 s_j$, $j = 1, \dots, n$. It is clear that for $j = 1$, we have that $s_1 = z_1 + \dots + z_n$ and $a_1 = -a_0 s_1$. Therefore s_1 is negative if and only if $\frac{a_1}{a_0} > 0$. \square

By applying theorem 3.5 and lemma 3.7 to the family $P(t, k) = p_0(t) + kp_1(t)$ we can establish a family of multiple scrolls as is shown in the following Section.

4. Generation of a monoparametric family of multiscrolls attractors

The control system (2) can be determined by its pair (A, B) :

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \quad (8)$$

Now let define the matrix D in terms of the coefficients of the polynomial $p_1(t) = c_{n-1}t^{n-1} + c_{n-2}t^{n-2} + \dots + c_0$ as follows:

$$D = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ c_0 & c_1 & \dots & c_{n-1} \end{pmatrix} \quad (9)$$

THEOREM (4.1). *The control system (2) satisfying hypothesis DS1) and DS2) if $\dot{x} = (A + kD)x + Bu$ is a member of the family $\mathfrak{F}_k(A, B, p_1, x, u)$ with characteristic polynomial $\mathbb{P}(t, k)$ for all $k \in (\underline{k}, \bar{k})$ then it is possible to generate a monoparametric family of multiscroll attractors.*

Remark 4.2. If $c_1 = 0$ then we have that $(\underline{k}, \bar{k}) = (k_{\min}^-, k_{\max}^+)$. If $c_1 \neq 0$, note that the system (2) is an UDS-I, then it satisfies that the sum of all of its eigenvalues are negative and this condition is also affected by the perturbation kc_1 in the coefficient a_1 and therefore $(k_{\min}^-, k_{\max}^+) \cap S$ is not empty.

To understand completely the idea, we shall develop an illustrative example.

Example 4.3. Consider the control system given by (2) for

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -30 & -4 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0.15 \\ 0.3 \\ 0.4 \end{bmatrix}$$

then the number of scrolls displayed in the attractor can be determined by the control signal u . For example a four-scroll attractor can be given by the following control signal

$$u = \begin{cases} 3, & \text{for } 0.375 < x_1; \\ 2, & \text{for } 0.225 < x_1 \leq 0.375; \\ 1, & \text{for } 0.075 < x_1 \leq 0.225; \\ 0, & \text{for } x_1 \leq 0.075. \end{cases}$$

Figure 1 shows the projection of the attractor onto the planes: (a) (x_1, x_2) ; (b) (x_1, x_3) ; and (c) (x_2, x_3) . As we are interested in generating a monoparametric family of multiscroll attractors determined as follows:

$$\dot{x} = (A + kD)x + Bu. \tag{10}$$

where k is the parameter of family given a matrix D . Without losing generality, the matrix D can be given as follows: $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$

Note that the characteristic polynomial for $k = 0$ is $p_0(t) = t^3 + t^2 + 4t + 30$ and from the example 4.3 and lemma 3.7, the system (2) is UDS-I for $u = 0$. Moreover, the characteristic polynomial of $A + kD$ is $P(t, k) = t^3 + (1 -$

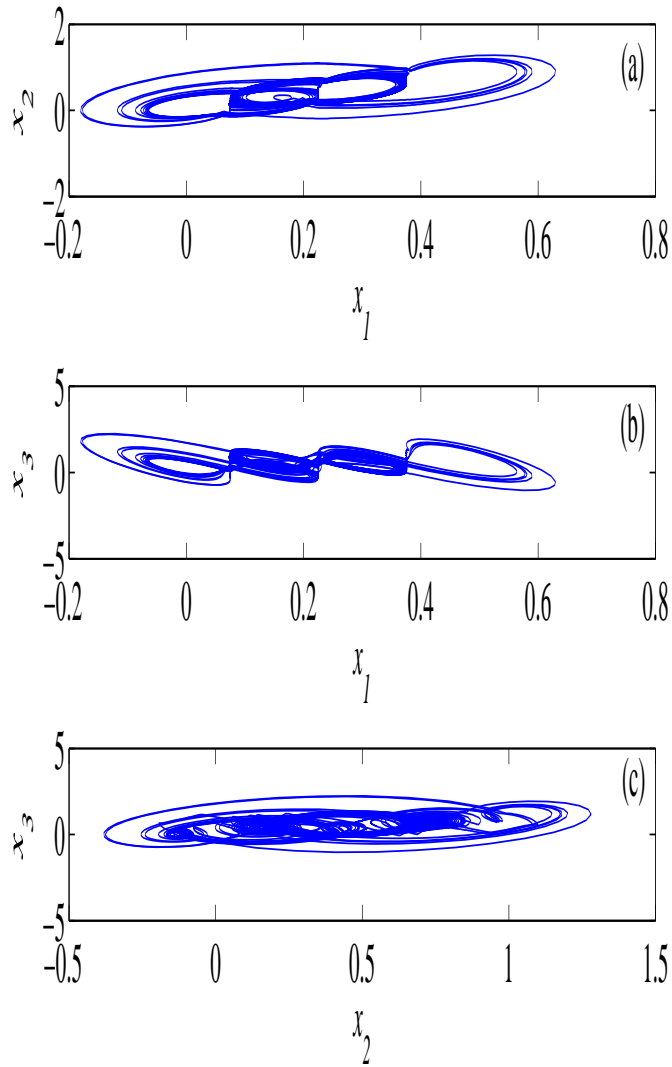


Figure 1: Projections of the attractor for $k = 0$ onto the planes: (a) (x_1, x_2) ; (b) (x_1, x_3) ; and (c) (x_2, x_3) .

$\frac{1}{2}k)t^2 + 4t + 30$, so we have that $p_1(t) = -\frac{1}{2}t^2$. Furthermore,

$$\begin{aligned} p_A(i\omega) &= 30 - \omega^2 + i\omega(4 - \omega^2), \\ p_1(i\omega) &= \frac{1}{2}\omega^2, \end{aligned}$$

thence

$$\begin{aligned} P(\omega^2) &= 30 - \omega^2, \\ Q(\omega^2) &= 4 - \omega^2, \\ p(\omega^2) &= \frac{1}{2}\omega^2, \\ q(\omega^2) &= 0, \end{aligned}$$

and therefore

$$\begin{aligned} F(\omega) &= \frac{1}{2}\omega^2(30 - \omega^2), \\ G(\omega) &= \omega^6 - 7\omega^4 - 44\omega^2 + 900, \\ H(\omega) &= -\frac{1}{2}\omega^2(4 - \omega^2). \end{aligned}$$

Hence, $R(H)_{\mathbb{R}^+} = \{2\}$ and $F(2) = 52 > 0$. Then $K^- = \{0^-\}$ and $K^+ = \{52\}$. So that, $k_{\max}^+ = +\infty$ and $k_{\min}^- = -\frac{G(2)}{F(2)} = -\frac{676}{52} = -13$. Consequently, the maximum interval of hyperbolicity is $(k_{\min}^-, k_{\max}^+) = (-13, +\infty)$ and since $a_1 + kc_1 = 1 - \frac{1}{2}k$, then $S = (-\infty, 2)$. Thus, if $k \in (-13, 2)$ then the system (10) is an UDS-I of which emerges a family of multi-scrolls attractor.

Figure 1 shows the projections of the attractor for $k = 0$ and Figure 2 shows the projections of the attractor for $k = 3$ onto the planes: (a) (x_1, x_2) ;

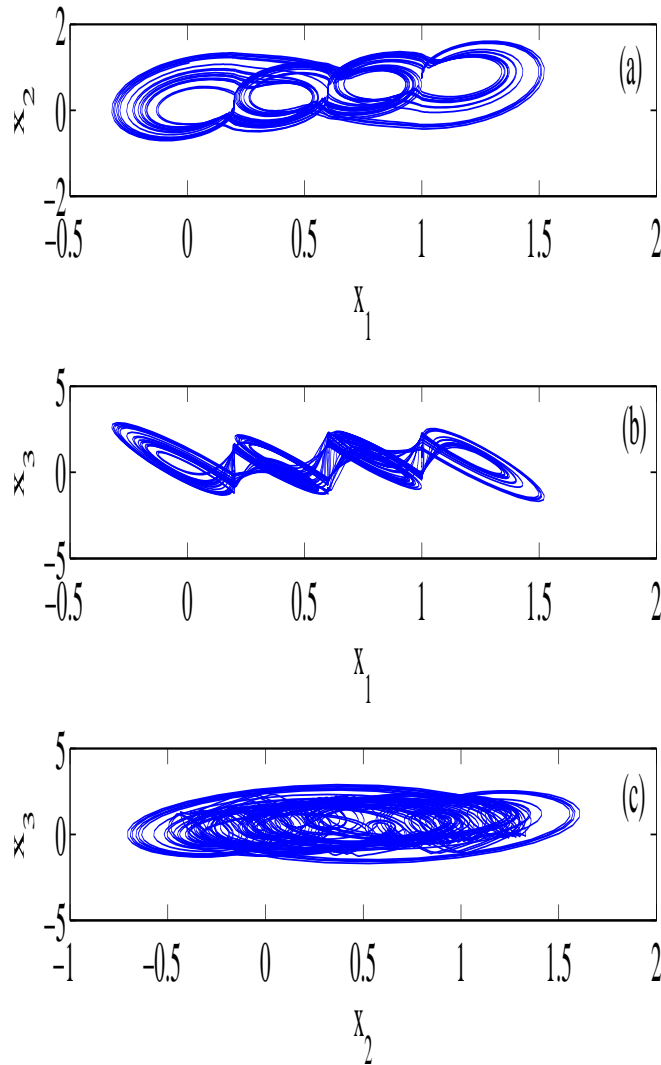


Figure 2: Projections of the attractor for $k = 3$ onto the planes: (a) (x_1, x_2) ; (b) (x_1, x_3) ; and (c) (x_2, x_3) .

(b) (x_1, x_3) ; and (c) (x_2, x_3) . Due to the stable and unstable manifolds of the system given by Eq.10 for $k = 3$ change of direction it is necessary to define B as follows:

$$B = \begin{bmatrix} 0.4 \\ 0.3 \\ 0.4 \end{bmatrix}$$

In order to generate a four-scroll attractor, the control signal can be given as follows:

$$u = \begin{cases} 3, & \text{for } 1 < x_1; \\ 2, & \text{for } 0.6 < x_1 \leq 1; \\ 1, & \text{for } 0.2 < x_1 \leq 0.6; \\ 0, & \text{for } x_1 \leq 0.2. \end{cases}$$

Figure 3 shows the projections of the attractor for $k = -1$ onto the planes: (a) (x_1, x_2) ; (b) (x_1, x_3) ; and (c) (x_2, x_3) . It is necessary to define B as follows:

$$B = \begin{bmatrix} 0.3 \\ 0.25 \\ 0.4 \end{bmatrix}$$

In order to generate a four-scroll attractor, the control signal can be given as follows:

$$u = \begin{cases} 3, & \text{for } 0.75 < x_1; \\ 2, & \text{for } 0.45 < x_1 \leq 0.75; \\ 1, & \text{for } 0.15 < x_1 \leq 0.45; \\ 0, & \text{for } x_1 \leq 0.15. \end{cases}$$

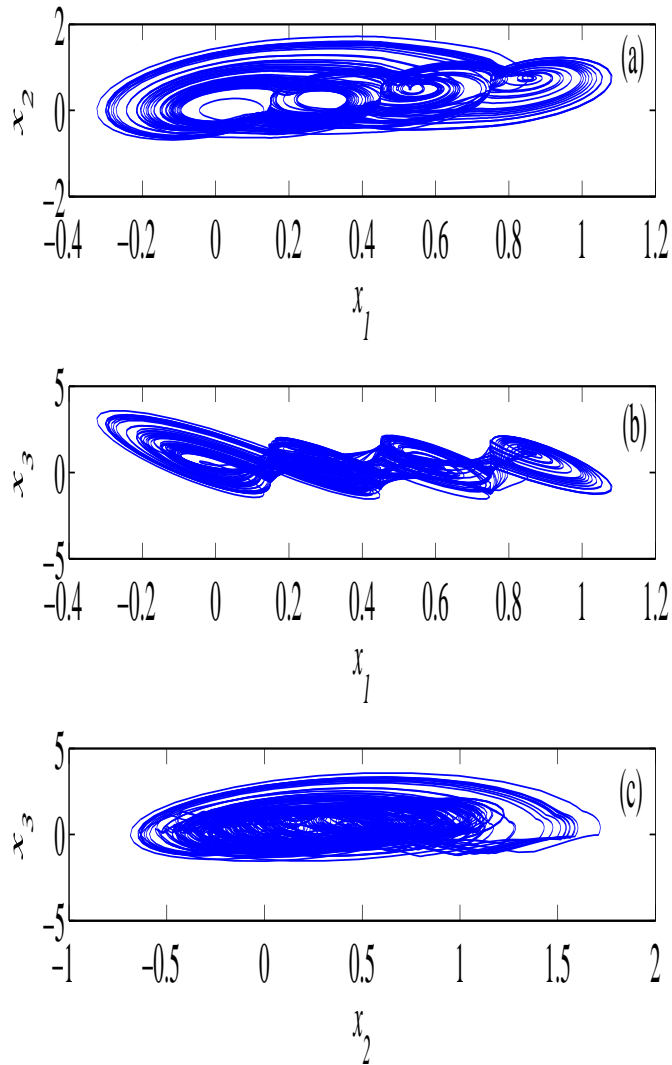


Figure 3: Projections of the attractor for $k = -1$ onto the planes: (a) (x_1, x_2) ; (b) (x_1, x_3) ; and (c) (x_2, x_3) .

5. Conclusions

This paper has studied a monoparametric family of multiscrolls-generating systems based on piecewise linear systems. Particularly, it deals with UDS type I. It derives conditions to generate a family of polynomials $\mathbb{P}(t, k) = p_0(t) + kp_1(t)$ for which $p_0(t)$ has n_1 roots in \mathbb{C}^- and $n - n_1$ roots in \mathbb{C}^+ for all k in the maximal robust dynamics interval, thus emerging monoparametric family of chaotic attractors. So, a way to perturb a system and still having multiscrolls attractors is derived. The attractor arises from a switching system having at least two UDS type I. This approach may be further extended to generate chaotic systems with multistabilities by adding more UDS type II.

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- [1] R. L. DEVANEY, *A First Course in Chaotic Dynamical Systems: Theory and Experiment*, 1st Ed., Westview Press (1992).
- [2] E. CAMPOS-CANTÓN, J. G. BARAJAS-RAMÍREZ, G. SOLÍS-PERALES AND R. FEMAT,, *Multiscroll attractors by switching systems*, *Chaos*, **20**, 013116 (2010).
- [3] E. CAMPOS-CANTÓN, R. FEMAT AND G. CHEN, *Attractors generated from switching unstable dissipative systems*, *Chaos*, **22**, 033121 (2012).

- [4] T. MATSUMOTO, *A chaotic attractor from Chua's circuit*, IEEE Trans. on Circ. Syst., **31**, 12, pp. 1055–1058 (1984).
- [5] E. CAMPOS-CANTÓN, I. CAMPOS-CANTÓN, J. S. GONZÁLEZ SALAS AND F. CRUZ ORDAZ, *A parametrized family of single-double-triple-scroll chaotic oscillations*, Rev. Mex. de Fís., **54**, pp. 411–415 (2008).
- [6] S. JAFARI AND J. C. SPROTT, *Simple chaotic flows with a line equilibrium*, Chaos, Solitons & Fractals, **57**, pp. 79–84 (2013).
- [7] G. CHEN AND T. UETA, *Yet another chaotic attractor*, Int. J. Bifurcation and Chaos, **9**, pp. 1465–1466 (1999).
- [8] G. GRASSI, F. L. SEVERANCE AND D. A. MILLER, *Multi-wing hyperchaotic attractors from coupled Lorenz systems*, Chaos Solitons and Fractals, **41**, pp. 284–291 (2009).
- [9] J. LU AND G. CHEN, *A new chaotic attractor coined*, Int. J. Bifurcation and Chaos, **12**, pp. 659–661 (2002).
- [10] ALEKSANDRONOV A. Y., CHEN Y., PLATONOV A. V. & ZHANG L, *Stability analysis for a class of switched nonlinear systems*, Automatica, **47**, pp. 2286–2291 (2011).
- [11] CHIOU J. S., WANG C. J., CHENG C. M. & WANG C. C., *Analysis and synthesis of switched nonlinear systems using the T-S fuzzy model*, Applied Mathematical Modelling, **34**, pp. 1467–1481 (2010).

- [12] MA R. & ZHAO J., *Backstepping desing for global stabilization of switched nonlinear systems in lower triangular form under arbitrary switchings*, Automatica, 46, 1819–1823 (2010).
- [13] E. JIMÉNEZ-LÓPEZ, J. S. GONZÁLEZ-SALAS, L. J. ONTAÑÓN-GARCÍA, E. CAMPOS-CANTÓN, AND A. N. PISARCHIK, *Generalized multistable structure via chaotic synchronization and preservation of scrolls*, Journal of the Franklin Institute **350**, pp. 2853–2866 (2013).
- [14] SUYKENS J. A. K. & VANDEWALLE J. *Generation of n-double scrolls ($n=1;2;3;4; \dots$)*, IEEE Transactions on Circuits and Systems, Part I, 40(11),861–867 (1993).
- [15] CHUA L. O. , KOMURO M. & MATSUMOTO T., *The double scroll family. IEEE Transactions on Circuits and Systems*, 33(11),1072–1118 (1986).
- [16] MADAN R. N., *Chua’s Circuit: A Paradigm for Chaos*, World Scientific: Singapore. (1993).
- [17] SUYKENS J. A. K., HUANG A. & CHUA L. O. *A family of n-scroll attractors from a generalized Chua’s circuit*, International Journal of Electronics and Communications, 51(3),131–138 (1997).
- [18] TANG K. S., ZHONG G. Q., CHEN G. R. & MAN K. F., *Generation of n-scroll attractors via sine function*, IEEE Transactions on Circuits and Systems, Part I, vol. 48, pp. 11, pp.1369-1372 (2001).

- [19] YALÇIN M.E., SUYKENS J.A.K. & VANDEWALLE J., *Experimental confirmation of 3- and 5-scroll attractors from a generalized Chua's circuit*, IEEE Transactions on Circuits and Systems, 47(3), 425-429 (2000).
- [20] J. LÜ, F. HAN, X. YU AND G. CHEN, *Generating 3 – D multis-croll chaotic attractors: A hysteresis series switching method*, Automatica, **40**, pp. 1677–1687 (2004).
- [21] W. DENG AND J. LÜ, *Generating multi-directional multi-scroll chaotic attractors via fractional differential hysteresis system*, Phys. Lett. A, **369**, pp. 438–443 (2007).
- [22] LÜ J., CHEN G. & YU X., *Design and analysis of multiscroll chaotic attractors from saturated function series*, IEEE Transactions on Circuits and Systems, Part I, 51(12) (2004).
- [23] SÁNCHEZ-LÓPEZ C., TREJO-GUERRA R., MUÑOZ-PACHECO J. M. & TLELO-CUAUTLE E., *N-scroll chaotic attractors from saturated function series employing CCII+s*, Nonlinear Dynamics, Volume 61, Numbers 1-2, pp. 331-341 (2010).
- [24] TREJO-GUERRA R., TLELO-CUAUTLE E., J.M. JIMÉNEZ-FUENTES, C. SÁNCHEZ-LÓPEZ, J.M. MUÑOZ-PACHECO, G. ESPINOSA-FLORES-VERDAD, J.M. ROCHA-PÉREZ , *Integrated circuit generating 3- and 5-scroll attractors*, Commun Nonlinear Sci Numer Simulat, Volume 17, pp. 4328-4335 (2012).

- [25] J. LÜ, X. YU AND G. CHEN, *Generating chaotic attractors with multiple merged basins of attractions: A switching piecewise-linear control approach*, IEEE Trans. on Circ. and Syst. I, **50**, 2, pp. 198-207 (2003).
- [26] LÜ J., MURALI K., SINHA S., LEUNG H. & AZIZ-ALAOIU M. A., *Generating multi-scroll chaotic attractors by thresholding*, Phys. Lett. A **372** 3234–3239 (2008).
- [27] M. E. YALCIN, J. A. K. SUYKENS, J. VANDEWALLE AND S. ÖZOĞUZ, *Families of scroll grid attractors*, Int. J. of Bifurcations and Chaos, **12**, 1, pp. 23–41 (2002).
- [28] Yu S., Lü J., Leung H. & Chen G. *Design and implementation of n-scroll chaotic attractors from a general jerk circuit*, IEEE Transactions on Circuits and Systems I **52**(7) (2005).
- [29] M. E. YALCIN, *Multi-scroll and hypercube attractors from a general jerk circuit using Josephson junctions*, Chaos Solitons and Fractals, **34**, pp. 1659–1666 (2007).
- [30] D. CAFAGNA, G. GRASSI, *Chaos in a new fractional-order system without equilibrium points*, Commun Nonlinear Sci Numer Simulat, **19**, pp. 2919–2927 (2014).
- [31] W. M. AHMAD, *Generation and control of multi-scroll chaotic attractors in fractional order systems*, Chaos Solitons and Fractals, **25**, pp. 727–735 (2005).

- [32] K. BOUALLEGUE, A. CHAARI AND A. TOUMI, *Multi-scroll and multi-wing chaotic attractors generated with Julia process fractal*, Chaos Solitons and Fractals, **44**, pp. 79–85 (2011).
- [33] L.J. ONTANON-GARCIA, E. JIMENEZ-LOPEZ, E. CAMPOS-CANTON, & M. BASIN, *A family of hyperchaotic multi-scroll attractors in R^n* , Applied Mathematics and Computation 233, 522-533. (2014).
- [34] W. M. AHMAD, *A simple multi-scroll hyperchaotic system*, Chaos Solitons and Fractals, **27**, pp. 1213–1219 (2006).
- [35] C. SHEN, S. YU, J. L AND G. CHEN, *A systematic methodology for constructing hyperchaotic systems with multiple positive Lyapunov exponents and circuit implementation*, IEEE Trans. Circuits Syst. I, 61, 3, (2014).
- [36] C. SHEN, S. YU, J. L AND G. CHEN, *Designing hyperchaotic systems with any desired number of positive Lyapunov exponents via a simple model*, IEEE Trans. Circuits Syst. I, 61, 8, (2014).
- [37] BARMISH B. R., *New tools for robustness of linear systems.*, Macmillan. xvi, New York, NY, 1994.
- [38] BHATTACHARYYA S. P., H. CHAPPELLAT AND L. H. KEEL, *Robust control: The parametric approach*, Prentice Hall, 1995.
- [39] HINRICHSSEN D., AND PRITCHARD J., *Mathematical Systems Theory I: Modeling, state space analysis, stability and robustness*, Texts in Applied Mathematics **48**, Springer-Verlag, Berlin, 2005.

- [40] X. HUANG, G. YE, *An efficient self-adaptive model for chaotic image encryption algorithm*, Commun Nonlinear Sci Numer Simulat, **19**, pp. 4094–4104 (2014).
- [41] Y. XU, H. WANG, Y. LI, B. PEI, *Image encryption based on synchronization of fractional chaotic systems*, Commun Nonlinear Sci Numer Simulat, **19**, pp. 3735–3744 (2014).
- [42] A. E.S. KUETCHE MBE, H.B. FOTSIN, J. KENGNE AND P. WOAFO, *Parameters estimation based adaptive Generalized Projective Synchronization (GPS) of chaotic Chua's circuit with application to chaos communication by parametric modulation*, Chaos Solitons and Fractals, **61**, pp. 27–37 (2014).
- [43] XINSONG YANG, ZHICHUN YANG, XIAOBING NIE, *Exponential synchronization of discontinuous chaotic systems via delayed impulsive control and its application to secure communication*, Commun Nonlinear Sci Numer Simulat, **19**, pp. 1529–1543 (2014).
- [44] A. KRAWIECKI, *Chaotic synchronization on complex hypergraphs*, Chaos Solitons and Fractals, **65**, pp. 44–50 (2014).
- [45] J. L AND G. CHEN, *A time-varying complex dynamical network model and its controlled synchronization criteria*, IEEE Trans. Automatic Control, 50, 6, (2005).
- [46] S. BIAŁAS, *A necessary and sufficient condition for the stability of convex combinations of stable polynomials or matrices*, Bull. Polish Acad. Sci. Tech. Sci., **33**, No. 9–10, pp. 473–480 (1985).

- [47] J. A. LÓPEZ-RENTERIA, B. AGUIRRE-HERNÁNDEZ AND F. VERDUZCO, *The bounday crossing theorem and the maximal stability interval*, Math. Prob. in Engineering, doi:10.1155/2011/12340, 13 pages (2011).
- [48] B. AGUIRRE, C. IBARRA, AND R. SUÁREZ, *Sufficient algebraic conditions for stability of cones of polynomials*, Systems Control Lett., **46**, No. 4, pp. 255–263 (2002).
- [49] B. AGUIRRE AND R. SUÁREZ, *Algebraic test for the Hurwitz stability of a given segment of polynomials*, Bol. Soc. Mat. Mexicana (3), **12**, pp. 261–275 (2006).
- [50] J. A. LÓPEZ-RENTERIA, B. AGUIRRE-HERNÁNDEZ AND F. VERDUZCO, *On Hurwitz and Schur connecting-curves and dense trajectories*, AIP conf. Proc., **1368**, 271, doi: 10.1063/1.3663511, 4 pages (2011).