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Further advancements on the output-feedback global continuous control for the finite-time and exponential stabilization of bounded-input mechanical systems: desired conservative-force compensation and experiments

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ABSTRACT
Global Saturating-Proportional Saturating-Derivative (SP-SD) type continuous control for the finite-time or (local) exponential stabilization of mechanical systems with bounded inputs is achieved avoiding velocity variables in the feedback, and further simplified through *desired* conservative-force compensation. The proposed output-feedback controller is not a simple extension of the *on-line* compensation case but it rather proves to entail a closed-loop analysis with considerably higher degree of complexity that gives rise to more involved requirements. Interestingly, the proposal even shows that actuators with higher power-supply capabilities than in the on-line compensation case are required. Other important analytical limitations are further overcome through the developed algorithm. Experimental tests on a multi-degree-of-freedom robot corroborate the efficiency of the proposed approach.

KEYWORDS
Output feedback; finite-time stabilization; mechanical systems; desired conservative-force compensation; bounded inputs

1. Introduction

An output-feedback global continuous control scheme for the finite-time and exponential stabilization of mechanical systems with bounded inputs has been recently proposed and thoroughly motivated in (Zamora-Gómez, Zavala-Río & López-Araujo, 2017). Guaranteeing the corresponding formulated control objective under the explicit consideration of input constraints and the explicit choice on the system trajectory convergence, under the exclusive consideration of position variables in the feedback, are among the main characteristics that distinguish such an approach from continuous finite-time controllers developed for mechanical systems before its appearance: (Hong, Xu & Huang, 2002; Sanyal & Bohn, 2015; Zhao, Li, Zhu & Gao, 2010) (see for instance (Zamora-Gómez et al., 2017, §1) for a brief description of such previ-

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ous works). But there is still an important distinction: while the cited previous works are mainly state-feedback approaches that rely on the *dynamic inversion* technique —or exact compensation of the whole dynamics— (except for one of the two controllers presented in (Hong et al., 2002)), and the only output-feedback extension (formulated in (Hong et al., 2002)) is based on (model-based) finite-time observers, the scheme in (Zamora-Gómez et al., 2017) exploits the inherent passive nature of mechanical systems, avoiding state reconstruction. This is done by keeping a (saturating) Proportional-Derivative type structure with exclusive compensation of the conservative-force (vector) term as a direct way to suitably reshape the closed-loop potential energy so as to set the desired posture as the only equilibrium position on the whole configuration space; damping is further injected through a (model-free) dynamic dissipation subsystem whose output is involved in the feedback as a *damped-derivative* action. Through such a control scheme (which avoids reproduction of any other term of the open-loop dynamics apart from the described *on-line* compensation of the conservative forces), the system model dependence of the designed algorithm is considerably reduced, consequently simplifying the control structure and decreasing the inherent inconveniences of modelling inaccuracies as well as the implied computation burden. But these advantages could still be potentiated by replacing the (unique) on-line compensation term by the conservative-force term exclusively evaluated at the desired position (Kelly, Santibáñez & Loría, 2005, Chapter 8). Such a *desired conservative-force compensation* idea was first developed in an unconstrained-input conventional (infinite-time) stabilization framework by (Takegaki & Arimoto, 1981) and, ever since its introduction in the literature, it has been the subject of diverse studies (Kelly, 1997), been at the core of control design advancements (Zavala-Río and Santibáñez, 2007), and proven to be widely appreciated in view of its simplicity and simplification improvements. This constitutes the main motivation of this work which aims at developing a *desired-conservative-force-compensation* extension of the output-feedback SP-SD-type (Saturating-Proportional Saturating-Derivative) finite-time/exponential stabilization scheme from (Zamora-Gómez et al., 2017). Far from what one could expect, such a design task is not as simple or direct as a simple replacement of the *on-line* compensation term by the *desired* one. Such a replacement turns out to keep the required (desired) closed-loop equilibrium position but not its uniqueness. Contrarily to the on-line compensation case [where the open-loop conservative forces are (ideally) cancelled out], in the desired compensation case further design requirements prove to be needed so as to ensure that the control-induced potential energy component *dominates* the open-loop one (in order to guarantee uniqueness of the desired closed-loop equilibrium configuration). This was already pointed out in the unconstrained-input conventional case (Takegaki & Arimoto, 1981), where such a domination goal was shown to be achieved through a P control (vector) term with a(n absolutely) stronger growing rate than that of the open-loop conservative force term in any direction (at every point) on the configuration space; in particular, under the simple consideration of uncoupled linear P and D control actions, this was shown to be achieved by simply fixing P gains higher than the highest (induced) norm value of the Jacobian matrix of the conservative force term (assuming that such a Jacobian matrix is bounded) (Tomei, 1991). But the solution of the referred uniqueness issue cannot be that simple in the analytical context considered here —under the consideration of input constraints, the contemplated type of trajectory convergence (finite-time or exponential) and the generalized form of the SP-SD controller components— in view of the special functions involved in the SP-SD terms to guarantee the achievement of the formulated stabilization goal. This represents an important analytical challenge to which this work succeeds to give a solution enjoying the technical benefits from desired conservative-force compensation.

It is further worth highlighting that the exhaustive analysis developed here further brings to the fore that actuators with higher power-supply capabilities than in the on-line-compensation case are required. This results from the *worst-case* type design (analytical) procedure followed to guarantee the achievement of the previously described domination feature of the controller-induced conservative force term over the open-loop system one. As a matter of fact, it is the permanence of the open-loop conservative-force term on the system dynamics which is at the origin of the design complication and higher degree of complexity of the closed-loop analysis (with respect to the on-line compensation case where such a term is absent in view of its cancellation). For instance, a further complication to be dealt with —and overcome in this work— is on the support of the controller ability to transit from finite-time to exponential stabilization through a simple control parameter. Indeed, for exponential stability purposes, the counterbalanced (in the desired-compensation sense) open-loop conservative forces turn out to lack of the properties required in the *homogeneity-oriented* analytical framework within which (Zamora-Gómez et al., 2017) and this work are developed. Thus, such a stabilization case has to be treated differently. This work gives a suitable solution to such an additional complication of the closed-loop analysis by supporting the exponential stabilization case through a strict Lyapunov function.

After the publication of (Zamora-Gómez et al., 2017), continuous output-feedback finite-time stabilization of Euler-Lagrange systems was treated in (Cruz-Zavala, Nuño & Moreno, 2017). In that work, the output-feedback version of the *energy shaping plus damping injection* control design methodology from (Loría, Kelly, Ortega & Santibáñez, 1997; Ortega, Loría, Kelly & Praly, 1994) was extended to the finite-time regulation case. Four particular controller cases were presented differing on the type of compensation of the term related to the gradient of the open-loop potential energy function, among desired and on-line (respectively denominated as the cases of *compensation* and *cancellation* of the EL-system potential energy in (Cruz-Zavala et al., 2017)), and on the bounded or unbounded control structure. The bounded controller versions were characterized by the use of specific saturation functions and the application of the control gains to the shaped error correction actions (and not directly to the error variables prior to the shaping). In particular, such *external weighting* leads the control gains to act on the PD-action bounds, generating the need (at every setting or change on the control gain values) for an additional verification and eventual adjustment on the considered saturation function bounds to guarantee the input saturation avoidance requirements. Further, in the desired compensation case, local exponential stability cannot be concluded through the analytical procedure developed therein. Such limitations are surpassed through the proposal developed in this work, characterized by the generalized structuring on (each one of) the SP-SD actions and the dynamic dissipation subsystem, the direct application of the control gains to the error variables —prior to the shaping on the SP-SD actions— which liberates the SP-SD action structures (bounds) from additional verifications and adjustments at every setting or change on the (control gain) tuning, and the more thorough closed-loop analysis and consequent requirement specifications, including the suitable solution given to the proof on its ability to include exponential (in addition to finite-time) stabilization among the control design choices. Experimental corroboration of the developed scheme on a multi-degree-of-freedom (DOF) robotic device is included as a complementary distinction of this work. Both, the desired and on-line compensation versions, were included in the experimental study.

2. Preliminaries

Let $X \in \mathbb{R}^{m\times n}$ and $y \in \mathbb{R}^n$. Throughout this work, X_{ij} denotes the element of X at its i^{th} row and j^{th} column, X_i represents the i^{th} row of X and y_i stands for the i^{th} element of y. With $m = n$, $X > 0$ (conventionally) denotes that X is positive definite while, for a symmetric matrix X, $\lambda_m(X)$ and $\lambda_M(X)$ respectively stand for its minimum and maximum eigenvalues. 0_n represents the origin of $\mathbb{R}^n$ and I_n the $n \times n$ identity matrix. We denote $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ for scalars, and $\mathbb{R}^n_{>0} = \{x \in \mathbb{R}^n : x_i > 0,\ i = 1,\dots,n\}$ and $\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n : x_i \geq 0,\ i = 1,\dots,n\}$ for vectors. $\|\cdot\|$ stands for the standard Euclidean norm for vectors and induced norm for matrices. An $(n-1)$-dimensional sphere of radius $c > 0$ on $\mathbb{R}^n$ is denoted S_c^{n-1}, *i.e.* $S_c^{n-1} = \{x \in \mathbb{R}^n : \|x\| = c\}$. For a continuously differentiable scalar function $f : \mathbb{R}^n \to \mathbb{R}$ and a vector function $g : \mathbb{R}^n \to \mathbb{R}^n$, we denote $D_g f$ the directional derivative of f along g, *i.e.* $D_g f(x) = \frac{\partial f}{\partial x} g(x)$. We will consider the sign function to be zero at zero, *i.e.*

$$\operatorname{sign}(\varsigma) = \begin{cases} \frac{\varsigma}{|\varsigma|} & \text{if } \varsigma \neq 0 \\ 0 & \text{if } \varsigma = 0 \end{cases}$$

and denote $\operatorname{sat}(\cdot)$ the standard (unitary) saturation function, *i.e.* $\operatorname{sat}(\varsigma) = \operatorname{sign}(\varsigma)\min\{|\varsigma|, 1\}$. The contents of the following subsections —except for some complementary properties, assumptions and considerations— were mostly included in (Zamora-Gómez et al., 2017, §2); some of them are recalled here.

2.1. Mechanical systems

Consider the n-DOF fully-actuated frictionless mechanical system dynamics (Brogliato, Lozano, Maschke & Egeland, 2007, §6.1)

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \tag{1}$$

where q, $\dot{q}$, $\ddot{q} \in \mathbb{R}^n$ are the position (generalized coordinates), velocity, and acceleration vectors. $H(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, which is a continuously differentiable positive definite symmetric matrix function, and actually

$$H(q) \geq \mu_m I_n \tag{2}$$

—which implies $\|H(q)\| \geq \mu_m$— $\forall q \in \mathbb{R}^n$, for some $\mu_m > 0$. $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$ is the Coriolis and centrifugal effect matrix defined through the Christoffel symbols of the first kind, which satisfies

$$\dot{H}(q,\dot{q}) = C(q,\dot{q}) + C^T(q,\dot{q}) \tag{3a}$$

$\forall q, \dot{q} \in \mathbb{R}^n$, and consequently

$$z^T\left[\frac{1}{2}\dot{H}(x,y) - C(x,y)\right]z = 0 \tag{3b}$$

$\forall x, y, z \in \mathbb{R}^n$, $\dot{H}$ denoting the rate of change of H, *i.e.* $\dot{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ with $\dot{H}_{ij}(q, \dot{q}) = \frac{\partial H_{ij}}{\partial q}(q)\dot{q}$, $i, j = 1, \ldots, n$; $C(x, y)z = C(x, z)y$, $\forall x, y, z \in \mathbb{R}^n$, whence we have that

$$C(q, a\dot{q})b\dot{q} = C(q, b\dot{q})a\dot{q} = C(q, ab\dot{q})\dot{q} = C(q, \dot{q})ab\dot{q} \tag{4}$$

$\forall q, \dot{q} \in \mathbb{R}^n$, $\forall a, b \in \mathbb{R}$; and

$$\|C(x, y)\| \leq \psi(x)\|y\| \tag{5}$$

$\forall x, y \in \mathbb{R}^n$, for some $\psi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. $g(q) = \nabla\mathcal{U}_{\mathrm{ol}}(q)$, with $\mathcal{U}_{\mathrm{ol}} : \mathbb{R}^n \to \mathbb{R}$ being the potential energy function of the open-loop system, or equivalently[1]

$$\mathcal{U}_{\mathrm{ol}}(q) = \mathcal{U}_{\mathrm{ol}}(q_0) + \int_{q_0}^{q} g^T(z)dz \tag{6}$$

for any $q, q_0 \in \mathbb{R}^n$; and $\tau \in \mathbb{R}^n$ is the external input (generalized) force vector.

In this work, we consider the (realistic) bounded input case, where the absolute value of each input τ_i is constrained to be smaller than a given saturation bound $T_i > 0$, *i.e.* $|\tau_i| \leq T_i$, $i = 1, \ldots, n$. More precisely, letting u_i represent the control variable (controller output) relative to the i^{th} degree of freedom, we have that

$$\tau_i = T_i \mathrm{sat}(u_i/T_i) \tag{7}$$

Further assumptions are stated next.

Assumption 2.1. *The inertia matrix is bounded, i.e.* $\|H(q)\| \leq \mu_M$, $\forall q \in \mathbb{R}^n$, *for some* $\mu_M \geq \mu_m > 0$.

Assumption 2.2. $\psi(\cdot)$ *in* (5) *is bounded and consequently* $\|C(x, y)\| \leq k_C\|y\|$, $\forall x, y \in \mathbb{R}^n$, *for some* $k_C \geq 0$.

Assumption 2.3. *The conservative (generalized) force vector* $g(q)$ *is a continuously differentiable bounded vector function with bounded Jacobian matrix* $\frac{\partial g}{\partial q}$, *or equivalently,*

2.3.1. every element of the conservative force vector, $g_i(q)$, $i = 1, \ldots, n$, *satisfies:* $|g_i(q)| \leq B_{gi}$, $\forall q \in \mathbb{R}^n$, *for some non-negative constant* B_{gi}*;*

2.3.2. $\frac{\partial g}{\partial x}$ *exists and is continuous and such that* $\left\|\frac{\partial g}{\partial q}(q)\right\| \leq k_g$, $\forall q \in \mathbb{R}^n$, *for some non-negative constant* k_g, *and consequently* $\|g(x) - g(y)\| \leq k_g\|x - y\|$, $\forall x, y \in \mathbb{R}^n$.

Assumption 2.4. $T_i > \eta B_{gi}$, $\forall i \in \{1, \ldots, n\}$, *for some scalar* $\eta \geq 1$.

Assumptions 2.1–2.3 apply *e.g.* for robot manipulators having only revolute joints (Kelly et al., 2005, §4.3).

Remark 2.1. By (2), the inverse matrix of $H(q)$, denoted $H^{-1}(q)$, exists and keeps analog analytical properties. More precisely, $H^{-1}(q)$ is a continuously differentiable positive definite matrix function, and actually, under the additional con-

[1] The integration in (6) takes into account the conservative nature of g, as pointed for instance in (Mendoza, Zavala-Río, Santibáñez & Reyes, 2015, Note 1, p. 2009).

sideration of Assumption 2.1: $(1/\mu_M)I_n \leq H^{-1}(q) \leq (1/\mu_m)I_n$ —which implies $1/\mu_M \leq \|H^{-1}(q)\| \leq 1/\mu_m$— $\forall q \in \mathbb{R}^n$, with $\mu_M \geq \mu_m$ being the positive constants characterized through (2) and Assumption 2.1. $\triangle$

2.2. Local homogeneity, finite-time stability and δ-exponential stability

As in (Zamora-Gómez et al., 2017), this work is developed within the analytical framework of *local homogeneity* (Zavala-Río & Fantoni, 2014), which states a formal analytical platform permitting to handle vector fields with bounded components (and consequently, control design under the consideration of input constraints, which would not be formally possible within the conventional coordinate-dependent context of homogeneity (Bhat & Bernstein, 2005)). Definitions and results in such an analytical context are strongly related to *family of dilations* δ_ε^r, defined as $\delta_\varepsilon^r(x) = \left(\varepsilon^{r_1}x_1, \ldots, \varepsilon^{r_n}x_n\right)^T$, $\forall x \in \mathbb{R}^n$, $\forall \varepsilon > 0$, with $r = (r_1, \ldots, r_n)^T$, where the *dilation coefficients* $r_1, \ldots, r_n$ are positive scalars. Other fundamental concepts involved in the analytical context underlying this work are those of *homogeneous norm* —with respect to the family of dilations δ_ε^r, or simply *r-homogeneous norm*: a positive definite continuous function being r-homogeneous of degree 1— (Kawski, 1990; M'Closkey & Murray, 1997; Zavala-Río & Zamora-Gómez, 2017), denoted $\|\cdot\|_r$, and *r-homogeneous $(n-1)$-sphere* of radius $c > 0$: $S_{r,c}^{n-1} = \{x \in \mathbb{R}^n : \|x\|_r = c\}$.

Consider an n-th order autonomous system

$$\dot{x} = f(x) \tag{8}$$

where f is a vector field being continuous on an open neighborhood of the origin $\mathcal{D} \subset \mathbb{R}^n$ and such that $f(0_n) = 0_n$, and let $x(t; x_0)$ represent the system solution with initial condition $x(0; x_0) = x_0$. A fundamental concept underlying this work is that of a (globally) finite-time stable equilibrium, as defined in (Bhat & Bernstein, 2005).

Remark 2.2. The origin is a globally finite-time stable equilibrium of system (8) if and only if it is globally asymptotically stable and finite-time stable. $\triangle$

Theorem 2.1. *(Zavala-Río & Fantoni, 2014) Consider system* (8) *with $\mathcal{D} = \mathbb{R}^n$. Suppose that f is a locally r-homogeneous vector field of degree α with domain of homogeneity $D \subset \mathbb{R}^n$. Then, the origin is a globally finite-time stable equilibrium of system* (8) *if and only if it is globally asymptotically stable and $\alpha < 0$.*

An alternative stability concept proving to be compatible to the framework of (local) homogeneity is that of *δ-exponential stability*, whose definition is found for instance in (Zamora-Gómez et al., 2017; Zavala-Río & Zamora-Gómez, 2017).

Remark 2.3. If f in (8) is locally r-homogeneous of degree $\alpha = 0$ with dilation coefficients $r_i = r_0$, $\forall i \in \{1, \ldots, n\}$, for some $r_0 > 0$, then the origin turns out to be exponentially stable (in the usual or standard sense (Khalil, 2002, Definition 4.5)) if and only if it is δ-exponentially stable (Zavala-Río & Zamora-Gómez, 2017, Remark 2.5). $\triangle$

Consider an n-th order autonomous system of the form

$$\dot{x} = f(x) + \hat{f}(x) \tag{9}$$

where f and $\hat{f}$ are continuous vector fields on $\mathbb{R}^n$ such that $f(0_n) = \hat{f}(0_n) = 0_n$.

Lemma 2.1. *(Zavala-Río & Zamora-Gómez, 2017, Lemma 2.2) Suppose that, for some $r \in \mathbb{R}^n_{>0}$, f in (9) is a locally r-homogeneous vector field of degree $\alpha < 0$, resp. $\alpha = 0$, with domain of homogeneity $D \subset \mathbb{R}^n$, and that 0_n is a globally asymptotically, resp. δ-exponentially, stable equilibrium of $\dot{x} = f(x)$. Then, the origin is a finite-time, resp. δ-exponentially, stable equilibrium of system (9) if*

$$\lim_{\varepsilon \to 0^+} \frac{\hat{f}_i(\delta^r_\varepsilon(x))}{\varepsilon^{\alpha + r_i}} = 0 \tag{10}$$

$i = 1, \dots, n$, $\forall x \in S^{n-1}_c$, resp. $\forall x \in S^{n-1}_{r,c}$, for some $c > 0$ such that $S^{n-1}_c \subset D$, resp. $S^{n-1}_{r,c} \subset D$.

Remark 2.4. Notice that the condition required by Lemma 2.1 may be equivalently verified through the satisfaction of

$$\lim_{\varepsilon \to 0^+} \left\| \varepsilon^{-\alpha} \mathrm{diag}\left[\varepsilon^{-r_1}, \dots, \varepsilon^{-r_n}\right] \hat{f}(\delta^r_\varepsilon(x)) \right\| = 0 \tag{11}$$

$\forall x \in S^{n-1}_c$ (resp. $S^{n-1}_{r,c}$). In other words, (10) is fulfilled for all $i = 1, \dots, n$ and all $x \in S^{n-1}_c$ (resp. $S^{n-1}_{r,c}$) if and only if (11) is satisfied for all $x \in S^{n-1}_c$ (resp. $S^{n-1}_{r,c}$). $\triangle$

2.3. Scalar functions with particular properties

Definition 2.1. *A continuous scalar function $\sigma : \mathbb{R} \to \mathbb{R}$ will be said to be:*

(1) bounded —by M— if $|\sigma(\varsigma)| \leq M$, $\forall \varsigma \in \mathbb{R}$, for some positive constant M;
(2) strictly passive if $\varsigma\sigma(\varsigma) > 0$, $\forall \varsigma \neq 0$;
(3) strongly passive if it is a strictly passive function satisfying $|\sigma(\varsigma)| \geq \kappa\left|a\,\mathrm{sat}(\varsigma/a)\right|^b = \kappa\big(\min\{|\varsigma|, a\}\big)^b$, $\forall \varsigma \in \mathbb{R}$, for some positive constants κ, a and b.

Let us note that a non-decreasing strictly passive function σ is strongly passive (Zavala-Río & Zamora-Gómez, 2017, Remark 2.7).

Remark 2.5. Equivalent characterizations of strictly passive functions are: $\varsigma\sigma(\varsigma) > 0 \iff \mathrm{sign}(\varsigma)\sigma(\varsigma) > 0 \iff \mathrm{sign}(\sigma(\varsigma)) = \mathrm{sign}(\varsigma)$, $\forall \varsigma$. $\triangle$

Lemma 2.2. *(Zavala-Río & Zamora-Gómez, 2017, Lemma 2.3) Let $\sigma : \mathbb{R} \to \mathbb{R}$, $\sigma_0 : \mathbb{R} \to \mathbb{R}$ and $\sigma_1 : \mathbb{R} \to \mathbb{R}$ be strongly passive functions and k be a positive constant. Then:*

(1) $\int_0^\varsigma \sigma(k\nu)d\nu > 0$, $\forall \varsigma \neq 0$;
(2) $\int_0^\varsigma \sigma(k\nu)d\nu \to \infty$ as $|\varsigma| \to \infty$;
(3) $\sigma_0 \circ \sigma_1$ is strongly passive.

3. The proposed control scheme

Consider the following SP-SD type controller with desired conservative-force compensation

$$u(q,\vartheta) = -s_1(K_1\bar{q}) - s_2(K_2\vartheta) + g(q_d) \tag{12}$$

where $\bar{q} = q - q_d$, for any constant —desired equilibrium position— $q_d \in \mathbb{R}^n$; $\vartheta \in \mathbb{R}^n$ is the output vector variable of an auxiliary subsystem defined as

$$\dot{\vartheta}_c = -As_3(\vartheta_c + B\bar{q}) \tag{13a}$$
$$\vartheta = \vartheta_c + B\bar{q} \tag{13b}$$

K_1, K_2, A and B are positive definite diagonal matrices —*i.e.* $K_i = \operatorname{diag}[k_{i1},\ldots,k_{in}]$, $i = 1,2$, $A = \operatorname{diag}[a_1,\ldots,a_n]$, $B = \operatorname{diag}[b_1,\ldots,b_n]$, $k_{ij} > 0$, $a_j > 0$, $b_j > 0$, $\forall j = 1,\ldots,n$— with K_1 involved in an additional requirement stated below (through (15)); for any $x \in \mathbb{R}^n$, $s_i(x) = \big(\sigma_{i1}(x_1),\ldots,\sigma_{in}(x_n)\big)^T$, $i = 1,2,3$, with —for each $j = 1,\ldots,n$— σ_{3j} being a strictly passive function, while σ_{1j} and σ_{2j} are strongly passive functions such that[2]

$$B_j \triangleq \sup_{(\varsigma_1,\varsigma_2)\in\mathbb{R}^2} \big|\sigma_{1j}(\varsigma_1) + \sigma_{2j}(\varsigma_2)\big| < T_j - B_{gj} \tag{14}$$

(recall Assumption 2.3.1), all three being locally Lipschitz-continuous on $\mathbb{R}\setminus\{0\}$; and with —for each $j = 1,\ldots,n$— k_{1j} and σ_{1j} additionally required to be such that

$$|\sigma_{1j}(k_{1j}\varsigma)| > \min\big\{k_g|\varsigma|\,,\,2B_{gj}\big\} \tag{15}$$

$\forall\varsigma \neq 0$ (recall Assumption 2.3.2).

Remark 3.1. Note that, by (14), we have that —for every $j = 1,\ldots,n$— σ_{1j} and σ_{2j} shall both be bounded, while σ_{3j} may be freely chosen to be bounded or not. △

Remark 3.2. The auxiliary subsystem in Eqs. (13) is a version of the *dirty derivative* operator (Zamora-Gómez et al., 2017, Remark 8) (applied to the position error vector variable) with a non-linear structure that will prove to be useful to achieve the control objective with the focused types of trajectory convergence. △

Remark 3.3. From the formulation of the proposed scheme, one can verify that the proper satisfaction of the stated requirements entails that

$$2B_{gj} < |\sigma_{1j}(k_{1j}\varsigma)| \leq \sup_{(\varsigma_1,\varsigma_2)\in\mathbb{R}^2} \big|\sigma_{1j}(\varsigma_1) + \sigma_{2j}(\varsigma_2)\big| < T_j - B_{gj}$$

$\forall|\varsigma| \geq 2B_{gj}/k_g$, whence one sees that Assumption 2.4 with $\eta = 3$ is a necessary condition for the feasibility of the simultaneous fulfilment of (14) and (15). A similar condition on the control input bounds has been required by other approaches where input constraints have been considered (Colbaugh, Barany & Glass, 1997; Mendoza,

[2]Notice that if σ_{1j} and σ_{2j} are (both) chosen to be non-decreasing, then $B_j = \max\Big\{ \lim_{\varsigma\to\infty}\left[\sigma_{1j}(\varsigma) + \sigma_{2j}(\varsigma)\right]\,,\, \lim_{\varsigma\to-\infty} -\left[\sigma_{1j}(\varsigma) + \sigma_{2j}(\varsigma)\right]\Big\}$.

Zavala-Río, Santibáñez & Reyes, 2015a), generally arising from the *worst-case* procedure followed to ensure that the analytical requirements that guarantee the result are fulfilled. $\triangle$

Remark 3.4. Let us note that (15) could have been alternatively stated as requiring $\left|\sigma_{1j}(k_{1j}\varsigma)\right| \geq \min\left\{\hat{k}_{1j}|\varsigma|, b_j\right\}$ for some constants $\hat{k}_{1j} > k_g$ and $b_j > 2B_{gj}$. However, by stating (15), the existence of constants $\hat{k}_{1j} > k_g$ and $b_j > 2B_{gj}$ such that $\left|\sigma_{1j}(k_{1j}\varsigma)\right| \geq \min\left\{\hat{k}_{1j}|\varsigma|, b_j\right\} > \min\left\{k_g|\varsigma|, 2B_{gj}\right\}$, $\forall\varsigma \neq 0$, is implied. $\triangle$

Remark 3.5. Note that the control gains in K_2, A and B are not at all restricted and are consequently free to take any positive value, while those in K_1 are the only ones whose choice remains restricted in accordance to the design requirement stated through (15) (where they are involved in). $\triangle$

Proposition 3.1. *Consider system* (1),(7) *in closed loop with the proposed control law* (12)-(13)*, under Assumptions 2.1–2.3 and 2.4 with $\eta = 3$, and the above stated design specifications. Thus, global asymptotic stability of the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is guaranteed with $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$.*

Proof. Observe that —for every $j = 1, \ldots, n$— by (14), we have that, for any $(q, \vartheta) \in \mathbb{R}^n \times \mathbb{R}^n$ and any $q_d \in \mathbb{R}^n$:

$$\begin{aligned}|u_j(q,\vartheta)| &= \left| -\sigma_{1j}(k_{1j}\bar{q}_j) - \sigma_{2j}(k_{2j}\vartheta_j) + g_j(q_d)\right| \\ &\leq \left|\sigma_{1j}(k_{1j}\bar{q}_j) + \sigma_{2j}(k_{2j}\vartheta_j)\right| + |g_j(q_d)| \\ &\leq B_j + B_{gj} < T_j\end{aligned}$$

From this and (7), one sees that $T_j > |u_j(q,\vartheta)| = |u_j| = |\tau_j|$, $\forall (q,\vartheta) \in \mathbb{R}^n \times \mathbb{R}^n$, which shows that, along the system trajectories, $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$. This proves that, under the proposed scheme, the input saturation values, T_j, are never attained. Hence, the closed-loop dynamics takes the (equivalent) form

$$\begin{aligned}H(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) &= -s_1(K_1\bar{q}) - s_2(K_2\vartheta) + g(q_d) \\ \dot{\vartheta} &= -As_3(\vartheta) + B\dot{q}\end{aligned}$$

By defining $x_1 = \bar{q}$, $x_2 = \dot{q}$ and $x_3 = \vartheta$, the closed-loop dynamics adopts the $3n$-order state-space representation

$$\dot{x}_1 = x_2 \tag{16a}$$
$$\dot{x}_2 = H^{-1}(x_1 + q_d)[-s_1(K_1x_1) - s_2(K_2x_3) - C(x_1 + q_d, x_2)x_2 - g(x_1 + q_d) + g(q_d)] \tag{16b}$$
$$\dot{x}_3 = -As_3(x_3) + Bx_2 \tag{16c}$$

By further defining $x = (x_1^T, x_2^T, x_3^T)^T$, these state equations may be rewritten in the form of system (9) with

$$f(x) = \begin{pmatrix} x_2 \\ -H^{-1}(q_d)[s_1(K_1x_1) + s_2(K_2x_3)] \\ -As_3(x_3) + Bx_2 \end{pmatrix} \tag{17a}$$

$$\hat{f}(x) = \begin{pmatrix} 0_n \\ -H^{-1}(x_1+q_d)[C(x_1+q_d,x_2)x_2 + g(x_1+q_d) - g(q_d)] \\ \qquad - \mathcal{H}(x_1)[s_1(K_1x_1) + s_2(K_2x_3)] \\ 0_n \end{pmatrix} \tag{17b}$$

where

$$\mathcal{H}(x_1) = H^{-1}(x_1+q_d) - H^{-1}(q_d) \tag{18}$$

Thus, the closed-loop stability property stated through Proposition 3.1 is corroborated by showing that $x = 0_{3n}$ is a globally asymptotically stable equilibrium of the state equation $\dot{x} = f(x) + \hat{f}(x)$, which is proven through the following theorem (whose formulation proves to be convenient for subsequent developments and proofs).

Theorem 3.1. *Under the stated specifications, the origin is a globally asymptotically stable equilibrium of* $\dot{x} = f(x) + \ell\hat{f}(x)$, $\forall \ell \in \{0,1\}$, —i.e. *of both the state equation* $\dot{x} = f(x)$ *and the (closed-loop) system* $\dot{x} = f(x) + \hat{f}(x)$,— *with* $f(x)$ *and* $\hat{f}(x)$ *defined through Eqs.* (17).

Proof. For every $\ell \in \{0,1\}$, let us define the continuously differentiable scalar function

$$V_\ell(x_1,x_2,x_3) = \frac{1}{2}x_2^T H(\ell x_1 + q_d)x_2 + \mathcal{U}_\ell(x_1) + \int_{0_n}^{x_3} s_2^T(K_2 z)B^{-1}dz \tag{19}$$

where

$$\int_{0_n}^{x_3} s_2^T(K_2 z)B^{-1}dz = \sum_{j=1}^{n}\int_0^{x_{3j}} \frac{\sigma_{2j}(k_{2j}z_j)}{b_j}dz_j$$

and

$$\mathcal{U}_\ell(x_1) \triangleq \int_{0_n}^{x_1} s_1^T(K_1 z)dz + \ell\,\mathcal{U}(x_1) \tag{20}$$

with

$$\int_{0_n}^{x_1} s_1^T(K_1 z)dz = \sum_{j=1}^{n}\int_0^{x_{1j}} \sigma_{1j}(k_{1j}z_j)dz_j \tag{21}$$

and

$$\mathcal{U}(x_1) \triangleq \mathcal{U}_{\mathrm{ol}}(x_1+q_d) - \mathcal{U}_{\mathrm{ol}}(q_d) - g^T(q_d)x_1 \tag{22a}$$

$$= \int_{0_n}^{x_1} [g(z+q_d) - g(q_d)]^T dz \tag{22b}$$

$$= \int_{0_n}^{x_1} \left[\int_{0_n}^{z} \frac{\partial g}{\partial q}(\bar{z}+q_d)d\bar{z}\right]^T dz \tag{22c}$$

Observe from Eqs. (22) and Assumption 2.3 that

$$\begin{aligned}\mathcal{U}(x_1) &\leq \int_{0_n}^{x_1}\left[\int_{0_n}^{z}\left\|\frac{\partial g}{\partial q}(\bar{z}+q_d)\right\|d\bar{z}\right]^T dz\\ &\leq \int_{0_n}^{x_1}\left[\int_{0_n}^{z}k_g d\bar{z}\right]^T dz = \int_{0_n}^{x_1}k_g z^T dz = \sum_{j=1}^{n}\int_0^{x_{1j}}k_g z_j dz_j \end{aligned} \tag{23}$$

$\forall x_1 \in \mathbb{R}^n$ (more specifically from (22c)), and simultaneously that

$$\mathcal{U}(x_1) \leq \sum_{j=1}^{n}\int_0^{x_{1j}}\mathrm{sign}(z_j)|g_j(z+q_d)-g_j(q_d)|dz_j \leq \sum_{j=1}^{n}\int_0^{x_{1j}}\mathrm{sign}(z_j)2B_{gj}dz_j$$

$\forall x_1 \in \mathbb{R}^n$ (more specifically from (22b)). From these inequalities, Eqs. (20) and (21), the satisfaction of (15), and Remark 3.4, we have that

$$\begin{aligned}\mathcal{U}_\ell(x_1) &\geq \sum_{j=1}^{n}\int_0^{x_{1j}}\mathrm{sign}(z_j)\min\{(\hat{k}_{1j}-\ell k_g)|z_j|\,,\,(b_j-2\ell B_{gj})\}dz_j\\ &\geq \sum_{j=1}^{n}\int_0^{x_{1j}}\mathrm{sign}(z_j)\min\{\bar{k}_{\ell j}|z_j|\,,\,\bar{b}_{\ell j}\}dz_j = \sum_{j=1}^{n}w_{\ell j}(x_{1j}) \triangleq S_\ell(x_1)\end{aligned} \tag{24a}$$

with

$$w_{\ell j}(x_{1j}) = \begin{cases}\frac{\bar{k}_{\ell j}}{2}x_{1j}^2 & \text{if } |x_{1j}| \leq \bar{b}_{\ell j}/\bar{k}_{\ell j}\\ \bar{b}_{\ell j}[|x_{1j}|-\bar{b}_{\ell j}/(2\bar{k}_{\ell j})] & \text{if } |x_{1j}| > \bar{b}_{\ell j}/\bar{k}_{\ell j}\end{cases} \tag{24b}$$

for some $\hat{k}_{1j} > k_g$ and $b_j > 2B_{gj}$, and any positive constants $\bar{k}_{\ell j} \leq \hat{k}_{1j} - \ell k_g$ and $\bar{b}_{\ell j} \leq b_j - 2\ell B_{gj}$.

Remark 3.6. One sees from expressions (24) that S_ℓ, $\ell = 0, 1$, are positive definite radially unbounded functions of x_1. Observe further that (involving previous arguments and Remark 2.5)

$$\begin{aligned}D_{x_1}\mathcal{U}_\ell(x_1) = x_1^T\nabla_{x_1}\mathcal{U}_\ell(x_1) &= x_1^T\Big[s_1(K_1x_1)+\ell(g(x_1+q_d)-g(q_d))\Big]\\ &= \sum_{j=1}^{n}|x_{1j}|\Big[|\sigma_{1j}(k_{1j}x_{1j})|+\ell\,\mathrm{sign}(x_{1j})(g_j(x_1+q_d)-g_j(q_d))\Big]\\ &\geq \sum_{j=1}^{n}|x_{1j}|\Big[|\sigma_{1j}(k_{1j}x_{1j})|-\ell|g_j(x_1+q_d)-g_j(q_d)|\Big]\\ &\geq \sum_{j=1}^{n}|x_{1j}|\min\{(\hat{k}_{1j}-\ell k_g)|x_{1j}|\,,\,(b_j-2\ell B_{gj})\}\\ &\geq \sum_{j=1}^{n}|x_{1j}|\min\{\bar{k}_{\ell j}|x_{1j}|\,,\,\bar{b}_{\ell j}\} > 0\end{aligned} \tag{25}$$

$\forall x_1 \neq 0_n$,[3] whence one sees that, for every $\ell = 0, 1$,

$$\nabla_{x_1} \mathcal{U}_\ell(x_1) = s_1(K_1 x_1) + \ell[g(x_1 + q_d) - g(q_d)] = 0_n \iff x_1 = 0_n \tag{26}$$

$\triangle$

Thus, from (19), (24) and (2), we get that

$$V_\ell(x_1, x_2, x_3) \geq \frac{\mu_m}{2}\|x_2\|^2 + S_\ell(x_1) + \int_{0_n}^{x_3} s_2^T(K_2 z) B^{-1} dz \tag{27}$$

whence, under the additional consideration of Lemma 2.2, positive definiteness and radial unboundedness of V_ℓ, $\ell = 0, 1$, is concluded. Further, for every $\ell \in \{0, 1\}$, the derivative of V_ℓ along the trajectories of $\dot{x} = f(x) + \ell \hat{f}(x)$, is obtained as

$$\begin{aligned}
&\dot{V}_\ell(x_1, x_2, x_3) \\
&\quad = x_2^T H(\ell x_1 + q_d)\dot{x}_2 + \Big[s_1(K_1 x_1) + \ell[g(x_1 + q_d) - g(q_d)]\Big]^T \dot{x}_1 \\
&\qquad + \frac{\ell}{2} x_2^T \dot{H}(x_1 + q_d, x_2) x_2 + s_2^T(K_2 x_3) B^{-1} \dot{x}_3 \\
&\quad = x_2^T \Big[-\ell[C(x_1 + q_d, x_2)x_2 + g(x_1 + q_d) - g(q_d)] - s_1(K_1 x_1) - s_2(K_2 x_3)\Big] \\
&\qquad + \frac{\ell}{2} x_2^T \dot{H}(x_1 + q_d, x_2) x_2 + \Big[s_1(K_1 x_1) + \ell[g(x_1 + q_d) - g(q_d)]\Big]^T x_2 \\
&\qquad + s_2^T(K_2 x_3) B^{-1}[-A s_3(x_3) + B x_2] \\
&\quad = -s_2^T(K_2 x_3) B^{-1} A s_3(x_3) \\
&\quad = -\sum_{j=1}^{n} \frac{a_j}{b_j} \sigma_{2j}(k_{2j} x_{3j}) \sigma_{3j}(x_{3j})
\end{aligned}$$

where, in the case of $\ell = 1$, (3b) has been applied, Note, from the strictly passive character of σ_{2j} and σ_{3j} (recall Definition 2.1 and Remark 2.5), $j = 1, \ldots, n$, that $\dot{V}_\ell(x_1, x_2, x_3) \leq 0$, $\forall (x_1, x_2, x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, with $Z_\ell \triangleq \{(x_1, x_2, x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : \dot{V}_\ell(x_1, x_2, x_3) = 0\} = \{(x_1, x_2, x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : x_3 = 0_n\}$. Further, from the system dynamics $\dot{x} = f(x) + \ell \hat{f}(x)$ —under the consideration of Remark 3.6 (more precisely, of (26))— one sees that $x_3(t) \equiv 0_n \implies \dot{x}_3(t) \equiv 0_n \implies x_2(t) \equiv 0_n \implies \dot{x}_2(t) \equiv 0_n \implies s_1(K_1 x_1(t)) + \ell[g(x_1(t) + q_d) - g(q_d)] \equiv 0_n \iff x_1(t) \equiv 0_n$ (which shows that $(x_1, x_2, x_3)(t) \equiv (0_n, 0_n, 0_n)$ is the only system solution completely remaining in Z_ℓ), and corroborates that at any $(x_1, x_2, x_3) \in Z_\ell \setminus \{(0_n, 0_n, 0_n)\}$, the resulting unbalanced force terms act on the closed-loop dynamics $[\dot{x} = f(x_1, x_2, 0_n) + \ell \hat{f}(x_1, x_2, 0_n)$ with $(x_1, x_2) \neq (0_n, 0_n)]$, forcing the system trajectories to leave Z_ℓ, whence $\{(0_n, 0_n, 0_n)\}$ is concluded to be the only invariant set in Z_ℓ, $\ell = 0, 1$. Therefore, by the invariance theory (Michel, Hou & Liu, 2008, §7.2) (more precisely by (Michel et al., 2008, Corollary 7.2.1)), $x = 0_{3n}$ is concluded to be a globally asymptotically stable equilibrium of both the state equation $\dot{x} = f(x)$ and the (closed-loop) system $\dot{x} = f(x) + \hat{f}(x)$. □

[3]In any radial direction, $\mathcal{U}_\ell(x_1)$ is strictly increasing, and consequently $x_1 = 0_n$ is the unique stationary point of $\mathcal{U}_\ell(x_1)$.

Remark 3.7. As shown in the on-line compensation case developed in (Zamora-Gómez et al., 2017), it is the *dirty-derivative*-based auxiliary subsystem in Eqs. (13) which performs the energy dissipation in the closed-loop system (in the absence of the velocity variables in the feedback). This is analogously visualized —in the desired compensation case developed here— through the feedback-system passivity approach of (Zamora-Gómez et al., 2017, Theorem 2) as follows: under the consideration of the closed-loop system in Eqs. (16), let $e_1 = -y_2 = -s_2(K_2x_3)$, $e_2 = y_1 = x_2$, $\psi(x_3) = s_2^T(K_2x_3)B^{-1}As_3(x_3)$,

$$V_{11}(x_1, x_2) = \frac{1}{2}x_2^T H(x_1 + q_d)x_2 + \mathcal{U}_1(x_1)$$

and

$$V_{12}(x_3) = \int_{0_n}^{x_3} s_2^T(K_2z)B^{-1}dz$$

By previous arguments and developments, V_{11} and V_{12} are radially unbounded positive definite functions in their respective arguments. Following an analysis analog to that of the proof of Theorem 3.1, one obtains

$$\dot{V}_{11} = e_1^T y_1$$

and

$$\dot{V}_{12} = e_2^T y_2 - \psi(x_3)$$

with $\psi(x_3)$ being positive definite (in its argument). Hence, the closed-loop system in Eqs. (16) may be seen as a (negative) feedback system connection among a passive —actually lossless— subsystem Σ_1 with dynamic model

$$\Sigma_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = H^{-1}(x_1 + q_d)[-C(x_1 + q_d, x_2)x_2 - g(x_1 + q_d) + g(q_d) - s_1(K_1x_1) + e_1] \\ y_1 = x_2 \end{cases}$$

and positive definite storage function $V_{11}(x_1, x_2)$, and a strictly passive subsystem Σ_2 with sate model

$$\Sigma_2 : \begin{cases} \dot{x}_3 = -As_3(x_3) + Be_2 \triangleq f_2(x_3, e_2) \\ y_2 = s_2(K_2x_3) \end{cases} \tag{28}$$

and storage function $V_{12}(x_3)$. Moreover, one sees from (28) that $f_2(0_n, e_2) = Be_2 = 0_n \implies e_2 = 0_n$, completing the requirements of (Zamora-Gómez et al., 2017, Theorem 2). $\triangle$

3.1. Finite-time stabilization

Proposition 3.2. *Consider the proposed control scheme under the additional consideration that, for every $j = 1, \ldots, n$, σ_{ij}, $i = 1, 2$, are locally r_i-homogeneous of (com-*

mon) degree $\alpha_i = 2r_2 - r_1$ —i.e. $r_{1j} = r_1$, $r_{2j} = r_2$ and $\alpha_{1j} = \alpha_1 = 2r_2 - r_1 = \alpha_2 = \alpha_{2j}$ for all $j = 1, \ldots, n$— with dilation coefficients such that $2r_2 - r_1 > 0 > r_2 - r_1$ and domain of homogeneity $D_{ij} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{ij} \in (0, \infty]\}$, and σ_{3j} is locally r_1-homogeneous of degree $\alpha_3 = r_2$ —i.e. $r_{3j} = r_3 = r_1$ and $\alpha_{3j} = \alpha_3 = r_2$ for all $j \in \{1, \ldots, n\}$— with domain of homogeneity $D_{3j} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{3j} \in (0, \infty]\}$. Thus, global finite-time stability of the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is guaranteed with $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$.

Proof. Since the proposed control scheme is applied —with all its previously stated specifications— Proposition 3.1 holds and consequently $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$. Then, all that remains to be proven is that the additional considerations give rise to the claimed finite-time stabilization. In this direction, let $\hat{r}_i = (r_{i1}, \ldots, r_{in})^T$, $i = 1, 2, 3$, $r = (\hat{r}_1^T, \hat{r}_2^T, \hat{r}_3^T)^T$, $K_3 = \mathrm{diag}[k_{31}, \ldots, k_{3n}]$ with $k_{3j} = 1$, $\forall j = 1, \ldots, n$, $D \triangleq \{(x_1, x_2, x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : K_i x_i \in D_{i1} \times \cdots \times D_{in}\,,\ i = 1, 2, 3\} = \{(x_1, x_2, x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : |x_{ij}| < L_{ij}/k_{ij}\,,\ i = 1, 2, 3\,,\ j = 1, \ldots, n\}$, and consider the previously defined state (vector) variables and the consequent closed-loop state-space representation $\dot{x} = f(x) + \hat{f}(x)$, with f and $\hat{f}$ as defined through Eqs. (17). Since D defines an open neighborhood of the origin, there exists $\rho > 0$ such that $B_\rho \triangleq \{x \in \mathbb{R}^{3n} : \|x\| < \rho\} \subset D$. Moreover, for every $x \in B_\rho$ and all $\varepsilon \in (0, 1]$, we have that $\delta_\varepsilon^r(x) \in B_\rho$ (since $\|\delta_\varepsilon^r(x)\| < \|x\|$, $\forall \varepsilon \in (0, 1)$), and, for every $j = 1, \ldots, n$,

$$f_j(\delta_\varepsilon^r(x)) = \varepsilon^{r_{2j}} x_{2j} = \varepsilon^{r_2} x_{2j} = \varepsilon^{(r_2 - r_1) + r_1} x_{2j} = \varepsilon^{(r_2 - r_1) + r_{1j}} f_j(x)$$

$$\begin{aligned} f_{n+j}(\delta_\varepsilon^r(x)) &= -H_j^{-1}(q_d)[s_1(K_1 \delta_\varepsilon^{\hat{r}_1}(x_1)) + s_2(K_2 \delta_\varepsilon^{\hat{r}_3}(x_3))] \\ &= -H_j^{-1}(q_d)[s_1(\varepsilon^{r_1} K_1 x_1) + s_2(\varepsilon^{r_3} K_2 x_3)] \\ &= -H_j^{-1}(q_d)[\varepsilon^{\alpha_1} s_1(K_1 x_1) + \varepsilon^{\alpha_2} s_2(K_2 x_3)] \\ &= -H_j^{-1}(q_d)\varepsilon^{2r_2 - r_1}[s_1(K_1 x_1) + s_2(K_2 x_3)] \\ &= -\varepsilon^{(r_2 - r_1) + r_2} H_j^{-1}(q_d)[s_1(K_1 x_1) + s_2(K_2 x_3)] \\ &= \varepsilon^{(r_2 - r_1) + r_{2j}} f_{n+j}(x) \end{aligned} \tag{29a}$$

$$\begin{aligned} f_{2n+j}(\delta_\varepsilon^r(x)) &= -A s_3(\delta_\varepsilon^{\hat{r}_3}(x_3)) + B \delta_\varepsilon^{\hat{r}_2}(x_2) \\ &= -A s_3(\varepsilon^{r_3} x_3) + \varepsilon^{r_2} B x_2 \\ &= -A \varepsilon^{\alpha_3} s_3(x_3) + \varepsilon^{r_2} B x_2 \\ &= \varepsilon^{r_2}[-A s_3(x_3) + B x_2] \\ &= \varepsilon^{(r_2 - r_3) + r_3}[-A s_3(x_3) + B x_2] \\ &= \varepsilon^{(r_2 - r_1) + r_{3j}} f_{2n+j}(x) \end{aligned} \tag{29b}$$

whence one concludes that f is a locally r-homogeneous vector field of degree $\alpha = r_2 - r_1$, with domain of homogeneity B_ρ. Hence, by Theorems 2.1 and 3.1, the origin of the state equation $\dot{x} = f(x)$ is concluded to be a globally finite-time stable equilibrium since $r_2 < r_1$. Thus, by Theorem 3.1, Lemma 2.1, and Remarks 2.2 and 2.4, the origin of the closed-loop system $\dot{x} = f(x) + \hat{f}(x)$ is concluded to be a globally finite-time stable equilibrium provided that $r_2 < r_1$, if

$$\begin{aligned}
\mathcal{L}_0 &\triangleq \lim_{\varepsilon\to 0^+} \left\| \varepsilon^{-\alpha}\mathrm{diag}[\varepsilon^{-r_{11}},\dots,\varepsilon^{-r_{1n}},\varepsilon^{-r_{21}},\dots,\varepsilon^{-r_{2n}},\varepsilon^{-r_{31}},\dots,\varepsilon^{-r_{3n}}]\hat{f}(\delta_\varepsilon^r(x))\right\| \\
&= \lim_{\varepsilon\to 0^+} \left\| \varepsilon^{-\alpha}\mathrm{diag}[\varepsilon^{-r_{21}},\dots,\varepsilon^{-r_{2n}}][\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\dots,\hat{f}_{2n}(\delta_\varepsilon^r(x))]^T\right\| \\
&= \lim_{\varepsilon\to 0^+} \left\| \varepsilon^{-\alpha-r_2}[\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\dots,\hat{f}_{2n}(\delta_\varepsilon^r(x))]^T\right\| \\
&= \lim_{\varepsilon\to 0^+} \varepsilon^{r_1-2r_2}\left\| [\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\dots,\hat{f}_{2n}(\delta_\varepsilon^r(x))]^T\right\| \qquad (30)\\
&= 0
\end{aligned}$$

for all $x \in S_c^{3n-1} = \{x \in \mathbb{R}^{3n} : \|x\| = c\}$, for some $c > 0$ such that $S_c^{3n-1} \subset D$. Hence, from (17b), under the consideration of (4), we have, for all such $x \in S_c^{3n-1}$:

$$\begin{aligned}
&\left\|[\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\dots,\hat{f}_{2n}(\delta_\varepsilon^r(x))]^T\right\| \\
&\quad= \Big\| -H^{-1}(\varepsilon^{r_1}x_1+q_d)[C(\varepsilon^{r_1}x_1+q_d,\varepsilon^{r_2}x_2)\varepsilon^{r_2}x_2+g(x_1+q_d)-g(q_d)] \\
&\qquad\qquad -\mathcal{H}(\varepsilon^{r_1}x_1)[s_1(\varepsilon^{r_1}K_1x_1)+s_2(\varepsilon^{r_3}K_2x_3)]\Big\| \\
&\quad\le \left\| -H^{-1}(\varepsilon^{r_1}x_1+q_d)C(\varepsilon^{r_1}x_1+q_d,x_2)\varepsilon^{2r_2}x_2\right\| \\
&\qquad + \left\|H^{-1}(\varepsilon^{r_1}x_1+q_d)\right\|\left\|g(\varepsilon^{r_1}x_1+q_d)-g(q_d)\right\| \\
&\qquad + \left\|\mathcal{H}(\varepsilon^{r_1}x_1)[\varepsilon^{\alpha_1}s_1(K_1x_1)+\varepsilon^{\alpha_2}s_2(K_2x_3)]\right\|
\end{aligned}$$

whence, through a procedure similar to the one developed to obtain expressions (29), and the consideration of Assumption 2.3.2, we get

$$\begin{aligned}
&\left\|[\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\dots,\hat{f}_{2n}(\delta_\varepsilon^r(x))]^T\right\| \\
&\quad\le \varepsilon^{2r_2}\left\|H^{-1}(\varepsilon^{r_1}x_1+q_d)C(\varepsilon^{r_1}x_1+q_d,x_2)x_2\right\| + \left\|H^{-1}(\varepsilon^{r_1}x_1+q_d)\right\|k_g\varepsilon^{r_1}\|x_1\| \\
&\qquad + \varepsilon^{2r_2-r_1}\left\|\mathcal{H}(\varepsilon^{r_1}x_1)[s_1(K_1x_1)+s_2(K_2x_3)]\right\|
\end{aligned}$$

and consequently, from (30) (recalling that by design specifications: $r_1 > r_2 > 0$), we get

$$
\begin{aligned}
\mathcal{L}_0 \leq\ & \lim_{\varepsilon\to 0^+} \varepsilon^{r_1} \left\| H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2 \right\| \\
& + k_g\|x_1\| \lim_{\varepsilon\to 0^+} \varepsilon^{2(r_1-r_2)} \left\| H^{-1}(\varepsilon^{r_1}x_1 + q_d) \right\| \\
& + \lim_{\varepsilon\to 0^+} \|\mathcal{H}(\varepsilon^{r_1}x_1)[s_1(K_1x_1) + s_2(K_2x_3)]\| \\
\leq\ & \|H^{-1}(q_d)C(q_d, x_2)x_2\| \lim_{\varepsilon\to 0^+} \varepsilon^{r_1} + k_g\|x_1\|\|H^{-1}(q_d)\| \lim_{\varepsilon\to 0^+} \varepsilon^{2(r_1-r_2)} \\
& + \|s_1(K_1x_1) + s_2(K_2x_3)\| \lim_{\varepsilon\to 0^+} \|\mathcal{H}(\varepsilon^{r_1}x_1)\| \\
\leq\ & \|s_1(K_1x_1) + s_2(K_2x_3)\| \cdot \|\mathcal{H}(0_n)\| = 0
\end{aligned}
\tag{31}
$$

(note, from (18), that $\|\mathcal{H}(0_n)\| = \|H^{-1}(q_d) - H^{-1}(q_d)\| = 0$), which completes the proof. $\square$

Corollary 3.1. *Consider the proposed control scheme taking σ_{ij}, $i = 1, 2, 3$, $j = 1, \ldots, n$, such that*

$$\sigma_{ij}(\varsigma) = \mathrm{sign}(\varsigma)|\varsigma|^{\beta_{ij}} \quad \forall|\varsigma| \leq L_{ij} \in (0, \infty) \tag{32}$$

with constants $\beta_{ij} = \beta_i$ such that

$$0 < \beta_1 < 1 \quad , \quad \beta_2 = \beta_1 \quad , \quad \beta_3 = \frac{1+\beta_1}{2} \tag{33}$$

Thus, global finite-time stability of the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is guaranteed with $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$.

Proof. Note that, given any $r_{ij} > 0$, for every $\varsigma \in (-L_{ij}, L_{ij})$: $\varepsilon^{r_{ij}}\varsigma \in (-L_{ij}, L_{ij})$ and $\sigma_{ij}(\varepsilon^{r_{ij}}\varsigma) = \mathrm{sign}(\varepsilon^{r_{ij}}\varsigma)|\varepsilon^{r_{ij}}\varsigma|^{\beta_{ij}} = \varepsilon^{r_{ij}\beta_{ij}}\mathrm{sign}(\varsigma)|\varsigma|^{\beta_{ij}} = \varepsilon^{r_{ij}\beta_{ij}}\sigma_{ij}(\varsigma)$, $\forall \varepsilon \in (0, 1]$. Hence, under the consideration of expressions (33), for every $j = 1, \ldots, n$, we have, for any $r_{1j} = r_1 > 0$, that taking $r_{2j} = r_2 = (1 + \beta_1)r_1/2$ and $r_{3j} = r_3 = r_1$, σ_{ij}, $i = 1, 2$, are locally r_i-homogeneous of degree $\alpha_{1j} = \alpha_1 = r_1\beta_1 = 2r_2 - r_1 = r_3\beta_2 = \alpha_2 = \alpha_{2j}$ with domain of homogeneity $D_{ij} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{ij}\}$, and σ_{3j} is locally r_1-homogeneous of degree $\alpha_{3j} = \alpha_3 = (1 + \beta_1)r_3/2 = (1 + \beta_1)r_1/2 = r_2$ with domain of homogeneity $D_{3j} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{3j}\}$. The requirements of Proposition 3.2 are thus concluded to be satisfied with $0 < \beta_1 < 1 \implies r_2 - r_1 < 0 < 2r_2 - r_1$. $\square$

Remark 3.8. Since the results of this section depart from the application of the proposed control scheme, the cases of Proposition 3.2 with $r_2 \geq r_1$ and Corollary 3.1 with $\beta_1 \geq 1$ are particular cases of Proposition 3.1 where the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is globally asymptotically (but not finite-time) stable. It is further worth pointing out that with $r_2 = r_1$ —or analogously $\beta_1 = 1$ in the case of Corollary 3.1— we have that $\varepsilon^{r_2-r_1} = 1$, $\forall\varepsilon > 0$. Hence, in this case, developments analog to those giving rise to inequalities (31) lead to $\mathcal{L}_0 \leq k_g\|x_1\|\|H^{-1}(q_d)\|$, and consequently, Lemma 2.1 (under the consideration of Remark 2.3) cannot be applied to conclude (local) exponential stability (contrarily to the on-line gravity compensation case of (Zamora-Gómez et al., 2017)). Nevertheless, exponential stability is next proven to be achieved (locally), through an alternative (strict-Lyapunov-function-based) analytical procedure, for the special case obtained under the consideration of (32) with $\beta_{ij} = 1$, $i = 1, 2, 3$, $j = 1, \ldots, n$ (which implies $\beta_1 = 1 \iff r_2 = r_1$). $\triangle$

3.2. Exponential stabilization

Corollary 3.2. *Consider the proposed control scheme taking —for every $i = 1, 2, 3$ and $j = 1, \ldots, n$— σ_{ij} as in (32) with $\beta_{ij} = 1$, i.e. such that*

$$\sigma_{ij}(\varsigma) = \varsigma \quad \forall |\varsigma| \leq L_{ij} \in (0, \infty) \tag{34}$$

Thus: $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$, and the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is globally asymptotically stable and (locally) exponentially stable.

Proof. See Appendix A. $\square$

4. Experimental results

The proposed control scheme was implemented through experimental tests on a Geometric Touch Haptic Device (http://www.geomagic.com). This was used in (Nuño & Ortega, 2018), where a technical description of such a 3-DOF robotic device is presented. As in (Nuño & Ortega, 2018), the gravity (conservative) force vector has been modelled as

$$g(q) = \begin{pmatrix} 0 \\ 105 \sin(q_2 + q_3) + 137 \cos q_2 \\ 105 \sin(q_2 + q_3) \end{pmatrix} \times 10^{-4} \text{ [Nm]}$$

whence the values introduced through Assumption 2.3 are obtained as $B_{g1} = 0$, $B_{g2} = 242 \times 10^{-4}$ Nm, $B_{g3} = 105 \times 10^{-4}$ Nm and $k_g = 299 \times 10^{-4}$ Nm/rad. Input saturation bounds were further valued as $T_j = 1$ Nm, $j = 1, 2, 3$, whence Assumption 2.4 can be taken to be fulfilled with $\eta = 3$. For the sake of simplicity, units will be subsequently omitted.

For the application of the proposed design methodology, let us define the functions

$$\sigma_u(\varsigma; \beta, \bar{\alpha}) = \operatorname{sign}(\varsigma) \max\{|\varsigma|^{\beta}, \bar{\alpha}|\varsigma|\} \tag{35a}$$

$$\sigma_b(\varsigma; \beta, \bar{\alpha}, M) = \operatorname{sign}(\varsigma) \min\{|\sigma_u(\varsigma; \beta, \bar{\alpha})|, M\} \tag{35b}$$

for constants $\beta > 0$, $\bar{\alpha} \in \{0, 1\}$ and $M > 0$. Examples are shown in (Zamora-Gómez et al., 2017, §5).

Based on the functions in Eqs. (35), we define —for every $j = 1, 2$— those involved in the implementations performed in this section as

$$\sigma_{ij}(\varsigma) = \sigma_b(\varsigma; \beta_i, \bar{\alpha}_{ij}, M_{ij}) \qquad i = 1, 2 \tag{36a}$$
$$\sigma_{3j}(\varsigma) = \sigma_u(\varsigma; \beta_3, \bar{\alpha}_{3j}) \tag{36b}$$

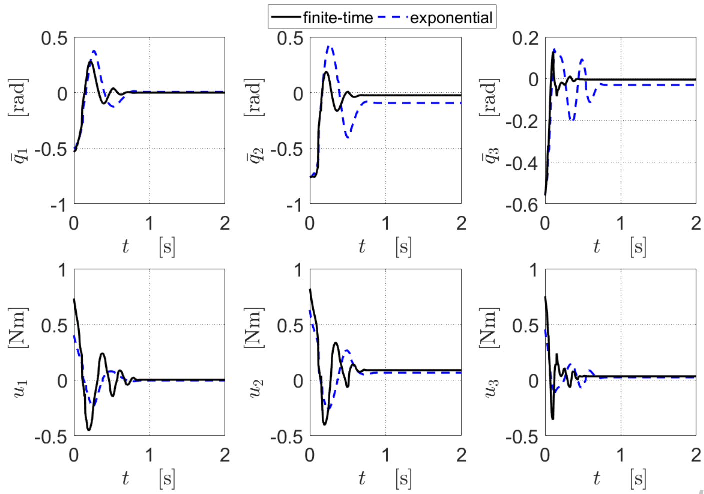

Figure 1. Finite-time *vs* exponential stabilization

with $\bar{\alpha}_{ij} = 0$, $i = 1, 2, 3$, $j = 1, 2, 3$. Conditions on their parameters under which (15) is fulfilled are:

$$k_{1j} > k_g(2B_{gj})^{(1-\beta_1)/\beta_1} \tag{37a}$$

$$M_{1j} > 2B_{gj} \tag{37b}$$

(this is shown in Appendix B). Let us note, from the involved functions, as defined through Eqs. (36), that $B_j = M_{1j} + M_{2j}$, $j = 1, 2$ (see (14) and Footnote 2). Thus, by fixing $M_{ij} = 0.4$, $i = 1, 2$, $j = 1, 2$, the inequalities from expressions (14) and (37b) have been simultaneously satisfied. The rest of the control gain/parameter values were chosen taking care that the design requirements were always satisfied. All the implementations were run taking the desired configuration at $q_d = (\pi/6 \quad \pi/4 \quad \pi/6)^T$ [rad] and initial conditions: $q(0) = (-0.00963 \quad 0.02467 \quad -0.01834)^T$ [rad], $\dot{q}(0) = (0 \quad 0 \quad 0)^T$.

Through experimental tests that show the efficiency of the proposed approach from an actual application whence model inaccuracies constitute an inescapable reality, we further aim at observing a couple of aspects on the closed-loop responses. The first of these focuses on the performance of the finite-time stabilization in contrast to analog exponential regulation implementations. Next, finite-time stabilization tests oriented to conclude on the differences or coincidences among closed-loop responses obtained through the desired and on-line compensation versions of the developed SP-SD-type output-feedback scheme —where the type of the compensation term is the only difference among the implementations— are included.

4.1. Finite-time vs *exponential stabilization*

Figure 1 shows results obtained taking $\beta_1 = \beta_2 = 3/5$ and $\beta_3 = 4/5$, for the finite-time controller (in accordance to (33)), and the remaining control gain/parameters were taken, for both (finite-time and exponential) controllers, as: $K_1 = \mathrm{diag}[0.5, 0.5, 0.5]$ (satisfying (37a)), $K_2 = \mathrm{diag}[0.3, 0.3, 0.3]$ and $A = B = \mathrm{diag}[1, 1, 1]$. One sees that

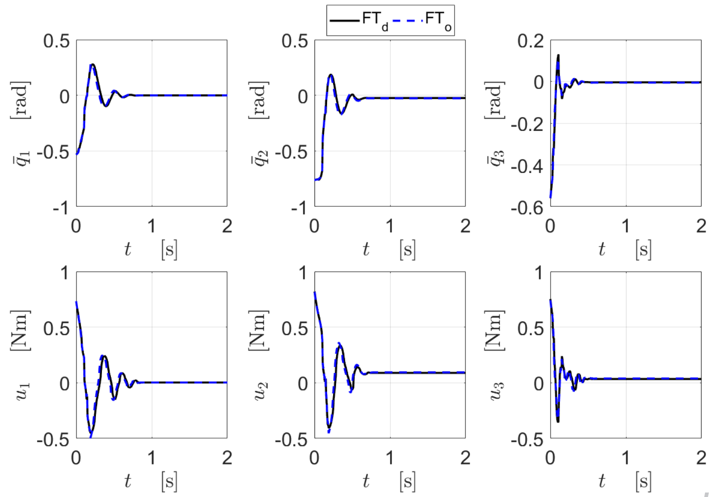

Figure 2. Desired *vs* on-line conservative-force compensation

control signals avoiding input saturation took place in both implementations, while the closed-loop trajectory arising through the exponential stabilizer was observed to present a longer and more important transient. Moreover, while a notorious steady-state error —due to modelling imprecisions such as static friction and biased parameters involved in the gravity vector model— is observed to be obtained with the exponential controller, a considerably smaller one (almost imperceptible) is noticed to take place with the finite-time stabilizer. In particular, this observation is very important since it corroborates the robustness argument frequently given in the literature to motivate finite-time controllers over asymptotical (infinite-time) ones. Further tests repeatedly showed the same result: considerably smaller (always almost imperceptible) steady-state errors arisen with the finite-time controller compared to those obtained with the exponential stabilizer, which were generally notorious.

4.2. Desired vs *on-line conservative-force compensation*

The last test focuses on the comparison among finite-time control implementations involving the desired and on-line conservative-force compensation versions of the SP-SD-type control schemes from this work and that from (Zamora-Gómez et al., 2017), respectively. Of course, for every one of these cases, one can always choose control gain/parameters such that, for the same initial conditions, either of them outperforms the other. Thus, what we really focus on, in this section, is in comparing closed-loop responses when both controllers hold the same control gain/parameter values but differ only on the type of conservative-force (gravity) compensation. With this goal in mind, we repeated exactly the finite-time control test of the precedent subsection simply alternating the referred compensation term. Figure 2 shows the comparison among the tested controllers, with $\mathrm{FT_d}$ and $\mathrm{FT_o}$ denoting the finite-time controllers with desired and on-line compensation term, respectively. No considerable differences among the closed-loop performances can be appreciated. Several alternative tests performing the same comparison but with different control characteristics (*e.g.* different control gain/parameter value combinations) generally gave rise to a similar result, *i.e.* suitable closed-loop responses with no considerable differences among them. We conclude from these results that the cost on the performance for the implementation simplification earned by the desired compensation version of the controller is negligible, in spite of the open-loop conservative-force term that is left acting on the system in this case.

5. Conclusions

Global SP-SD-type continuous control of mechanical systems with input constraints guaranteeing finite-time or exponential stabilization has been made possible avoiding velocity variables in the feedback and further simplified through desired conservative-force compensation. Far from what one could have expected, this output-feedback controller is not a simple extension of the on-line compensation case but it has rather proven to need a closed-loop analysis with considerably higher degree of complexity. Moreover, the proposed approach has overcome the proof on its transition from finite-time to exponential stabilization, which could not be solved keeping the local-homogeneity approach of the former in view of the open-loop conservative force which is kept acting on the closed loop. Experimental tests on a multi-DOF robotic system have shown the actual ability of the proposed approach to guarantee the considered types of convergence avoiding input saturation, corroborating the well-known argument that finite-time control is more robust than asymptotic (infinite-time) stabilization, in the sense that it generally gives rise to more reduced steady-state errors (resulting from unmodelled phenomena, such as static friction). Furthermore, both the on-line and desired conservative-force compensation versions of the developed scheme were tested and actually compared when the only difference among them is on the type of the referred compensation term. They both gave rise to suitable results with very small differences among the corresponding closed-loop responses. Thus, the implementation simplifications earned through the desired compensation are concluded to have a negligible cost on the system performance, passing the bill rather to the closed-loop analysis.

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References

Bhat, S.P., & Bernstein, D.S. (2005). Geometric homogeneity with applications to finite-time stability. *Mathematics of Control, Signals and Systems, 17*(2), 101–127.

Brogliato, B., Lozano, R., Maschke, B., & Egeland, O. (2007). *Dissipative systems analysis and control* (2nd ed.). London: Springer.

Colbaugh, R., Barany, E., & Glass, K. (1997). Global regulation of uncertain manipulators using bounded control. In *Proceedings of the 1997 International Conference on Robotics & Automation* (pp. 1148–1884). Albuquerque, NM: IEEE.

Cruz-Zavala, E., Nuño, E., & Moreno, J.A. (2017). Finite-time regulation of fully-actuated Euler-Lagrange systems without velocity measurements. In *Proceedings of the 56th Conference on Decision and Control* (pp. 6750–6755). Melbourne, Australia: IEEE.

Hong, Y., Xu, Y., & Huang, J. (2002). Finite-time control for robot manipulators. *Systems & Control Letters, 46*(4), 243–253.

Kawski, M. (1990). Homogeneous stabilizing feedback laws. *Control Theory and Advanced Technology, 6*, 497–516.

Kelly, R. (1997). PD control with desired gravity compensation of robotic manipulators: a review. *The International Journal of Robotics Research, 16*(5), 660–672.

Kelly, R., Santibáñez, V., & Loría, A. (2005). *Control of robot manipulators in joint space.* London: Springer.

Khalil, H.K. (2002). *Nonlinear Systems* (3rd ed.). Upper Saddle River: Prentice Hall.

Loría, A., Kelly, R., Ortega, R., & Santibáñez, V. (1997). On global output feedback regulation of Euler-Lagrange systems with bounded inputs. *IEEE Transactions on Automatic Control, 42*(8), 1138–1143.

M'Closkey, R.T., & Murray, R.M. (1997). Exponential stabilization of driftless nonlinear control systems using homogeneous feedback. *IEEE Transactions on Automatic Control, 42*(5), 614–628.

Mendoza, M., Zavala-Río, A., Santibáñez, V., & Reyes, F. (2015). A generalized PID-type control scheme with simple tuning for the global regulation of robot manipulators with constrained inputs. *International Journal of Control, 88*(10), 1995–2012.

Mendoza, M., Zavala-Río, A., Santibáñez, V., & Reyes, F. (2015a). Output-feedback proportional-integral-derivative-type control with simple tuning for the global regulation of robot manipulators with input constraints. *IET Control Theory & Applications, 9*(14), 2097–2106.

Michel, A.N., Hou, L., & Liu, D. (2008). *Stability of dynamical systems.* Boston: Birkhäuser.

Nuño, E., & Ortega, R. (2018). Achieving consensus of Euler-Lagrange agents with interconnecting delays and without velocity measurements via passivity-based control. *IEEE Transactions on Control Systems Technology, 26*(1), 222–232.

Ortega, R., Loría, A., Kelly, R., & Praly, L. (1994). On passivity-based output feedback global stabilization of Euler-Lagrange systems. In *Proceedings of the 33th Conference on Decision and Control* (pp. 381–386). Lake Buena Vista, FL: IEEE.

Sanyal, A.K. & Bohn, J. (2015). Finite-time stabilisation of simple mechanical systems using continuous feedback. *International Journal of Control, 88*(4), 783–791.

Takegaki, M. & Arimoto, S. (1981). A new feedback method for dynamic control of manipulators. *Journal of Dynamic Systems, Measurement and Control, 103*(2), 119–125.

Tomei, P. (1991). Adaptive PD controller for robot manipulators. *IEEE Transactions on Robotics and Automation, 7*(4), 565–570.

Zamora-Gómez, G.I., Zavala-Río, A., & López-Araujo, D.J. (2017). Observer-less output-feedback global continuous control for the finite-time and exponential stabilization of mechanical systems with constrained inputs. *European Journal of Control, 36*, 30–42.

Zavala-Río, A., & Fantoni, I. (2014). Global finite-time stability characterized through a local notion of homogeneity. *IEEE Transactions on Automatic Control, 59*(2), 471–477.

Zavala-Río, A., & Santibáñez, V. (2007). A natural saturating extension of the PD-with-desired-gravity-compensation control law for robot manipulators with bounded inputs. *IEEE Transactions on Robotics, 23*(2), 386–391.

Zavala-Río, A., & Zamora-Gómez, G.I. (2017). Local-homogeneity-based global continuous control for mechanical systems with constrained inputs: finite-time and exponential stabilization. *International Journal of Control, 90*(5), 1037–1051.

Zhao, D., Li, S., Zhu, Q., & Gao, S. (2010). Robust finite-time control approach for robotic manipulators. *IET Control Theory and Applications, 4*(1), 1–15.

Appendix A. Proof of Corollary 3.2

The global asymptotic stability follows from Proposition 3.1. Thus, all that remains to be proven is the (local) exponential stability property. In this direction, let us consider the scalar function $V_2(x_1, x_2, x_3) = V_1(x_1, x_2, x_3) + \epsilon x_1^T H(x_1 + q_d)x_2 - \epsilon\epsilon_0 x_2^T B^{-1} x_3$, with $V_1(x_1, x_2, x_3)$ as defined through Eq. (19) (with $\ell = 1$), *i.e.*

$$V_2(x_1, x_2, x_3) = \frac{1}{2} x_2^T H(x_1 + q_d) x_2 + \int_{0_n}^{x_1} s_1^T(K_1 z) dz + \mathcal{U}_{\rm ol}(x_1 + q_d) - \mathcal{U}_{\rm ol}(q_d) - g^T(q_d) x_1 + \int_{0_n}^{x_3} s_2^T(K_2 z) B^{-1} dz + \epsilon x_1^T H(x_1 + q_d) x_2 - \epsilon\epsilon_0 x_2^T B^{-1} x_3$$

where ϵ and ϵ_0 are positive constants such that

$$\epsilon < \min\{\epsilon_1, \epsilon_2\} \tag{A1}$$

$$\epsilon_0 > k_C \varrho_1 + \mu_M \tag{A2}$$

with

$$\epsilon_1 = \left[\frac{\bar{k}_{1m}\bar{k}_{2m}\mu_m}{\bar{k}_{2m}\mu_M^2 + \bar{k}_{1m}(\epsilon_0/b_m)^2}\right]^{1/2} \quad , \quad \epsilon_2 = \frac{\bar{k}_{1m}\gamma_{22}\tilde{k}_{2m}}{\bar{k}_{1m}\gamma_{22}\gamma_{33} + \bar{k}_{1m}(\gamma_{23}/2)^2 + \gamma_{22}(\gamma_{13}/2)^2}$$

$$\bar{k}_{1m} = \min_j\{\bar{k}_{1j}\} \quad , \quad \bar{k}_{2m} = \min_j\{k_{2j}/b_j\} \quad , \quad b_m = \min_j\{b_j\} \tag{A3}$$

$$\gamma_{22} = \epsilon_0 - k_C\varrho_1 - \mu_M \quad , \quad \gamma_{13} = k_{2M} + (k_{1M} + k_g)\epsilon_0/(b_m\mu_m) \tag{A4a}$$

$$\gamma_{23} = \epsilon_0[\bar{a}_M + k_C\varrho_2/(b_m\mu_m)] \quad , \quad \gamma_{33} = \epsilon_0 k_{2M}/(b_m\mu_m) \tag{A4b}$$

$$\tilde{k}_{2m} = \min_j\{k_{2j}a_j/b_j\} \quad , \quad k_{1M} = \max_j\{k_{1j}\} \tag{A5a}$$

$$k_{2M} = \max_j\{k_{2j}\} \quad , \quad \bar{a}_M = \max_j\{a_j/bj\} \tag{A5b}$$

μ_m, μ_M, k_C and k_g as defined through (2) and Assumptions 2.1–2.3; ϱ_2 is any positive constant;

$$\varrho_1 = \max_{x_1\in\mathcal{Q}_1}\|x_1\| = \left[\sum_{j=1}^{n}\left[\min\{\bar{b}_{1j}/\bar{k}_{1j}\, ,\, L_{1j}/k_{1j}\}\right]^2\right]^{1/2} \tag{A5c}$$

and

$$\mathcal{Q}_1 = \mathcal{Q}_{11} \cap \mathcal{Q}_{12} = \{x_1 \in \mathbb{R}^n : |x_{1j}| \le \min\{\bar{b}_{1j}/\bar{k}_{1j}\, ,\, L_{1j}/k_{1j}\},\ j = 1, \ldots, n\} \tag{A6a}$$

$$\mathcal{Q}_{11} = \{x_1 \in \mathbb{R}^n : |x_{1j}| \le \bar{b}_{1j}/\bar{k}_{1j},\ j = 1, \ldots, n\} \tag{A6b}$$

$$\mathcal{Q}_{12} = \{x_1 \in \mathbb{R}^n : |x_{1j}| \leq L_{1j}/k_{1j},\, j = 1, \ldots, n\} \tag{A6c}$$

From the proof of Theorem 3.1 (particularly, from inequality (27)), we have that $V_2(x_1, x_2, x_3) \geq \frac{\mu_m}{2}\|x_2\|^2 + S_1(x_1) + \int_{0_n}^{x_3} s_2^T(K_2 z)B^{-1}dz - \epsilon|x_1^T H(x_1 + q_d)x_2| - \epsilon\epsilon_0|x_2^T B^{-1}x_3|$, with $S_1(x_1)$ as defined through Eqs. (24) (with $\ell = 1$). More precisely, by observing that $S_1(x_1) = \sum_{j=1}^n \bar{k}_{1j}x_{1j}^2/2$ on $\mathcal{Q}_{11}$ (recall (A6b)) and $s_2(K_2x_3) = K_2x_3$ on $\mathcal{Q}_{31} = \{x_3 \in \mathbb{R}^n : |x_{3j}| \leq L_{2j}/k_{2j},\, j = 1, \ldots, n\}$, we have that, on $\mathcal{Q}_{11} \times \mathbb{R}^n \times \mathcal{Q}_{31}$:

$$\begin{aligned}
&V_2(x_1, x_2, x_3) \\
&\quad \geq \frac{\mu_m}{2}\|x_2\|^2 + \sum_{j=1}^n \frac{\bar{k}_{1j}}{2}x_{1j}^2 + \sum_{j=1}^n \frac{k_{2j}}{2b_j}x_{3j}^2 - \epsilon|x_1^T H(x_1 + q_d)x_2| - \epsilon\epsilon_0|x_2^T B^{-1}x_3| \\
&\quad \geq \frac{\mu_m}{2}\|x_2\|^2 + \frac{\bar{k}_{1m}}{2}\|x_1\|^2 + \frac{\bar{k}_{2m}}{2}\|x_3\|^2 - \epsilon\mu_M\|x_1\|\|x_2\| - \frac{\epsilon\epsilon_0}{b_m}\|x_2\|\|x_3\| \\
&\quad = \frac{1}{2}\begin{pmatrix}\|x_1\| \\ \|x_2\| \\ \|x_3\|\end{pmatrix}^T Q_1 \begin{pmatrix}\|x_1\| \\ \|x_2\| \\ \|x_3\|\end{pmatrix}
\end{aligned}$$

with

$$Q_1 = \begin{pmatrix} \bar{k}_{1m} & -\epsilon\mu_M & 0 \\ -\epsilon\mu_M & \mu_m & -\epsilon\epsilon_0/b_m \\ 0 & -\epsilon\epsilon_0/b_m & \bar{k}_{2m} \end{pmatrix}$$

($\bar{k}_{1m}$, $\bar{k}_{2m}$ and b_m as defined through expressions (A3)) where Assumption 2.1 has been considered, and since (A1) $\implies \epsilon < \epsilon_1 \implies Q_1 > 0$, we get

$$V_2(x) \geq c_1\|x\|^2 \tag{A7}$$

$\forall x \in \mathcal{Q}_{11} \times \mathbb{R}^n \times \mathcal{Q}_{31}$, with $c_1 = \lambda_m(Q_1)/2 > 0$. On the other hand, by analogously observing that in view of (34) we have $s_1(K_1x_1) = K_1x_1$ on $\mathcal{Q}_{12}$ (recall (A6c)), we get, under the consideration of (23) and Assumption 2.1, that, on $\mathcal{Q}_{12} \times \mathbb{R}^n \times \mathcal{Q}_{31}$:

$$\begin{aligned}
&V_2(x_1, x_2, x_3) \\
&\quad = \frac{1}{2}x_2^T H(x_1 + q_d)x_2 + \frac{1}{2}x_1^T K_1 x_1 + \mathcal{U}_{\mathrm{ol}}(x_1 + q_d) - \mathcal{U}_{\mathrm{ol}}(q_d) - g^T(q_d)x_1 \\
&\qquad\qquad + \sum_{j=1}^n \frac{k_{2j}}{2b_j}x_{3j}^2 + \epsilon x_1^T H(x_1 + q_d)x_2 - \epsilon\epsilon_0 x_2^T B^{-1}x_3 \\
&\quad \leq \frac{\mu_M}{2}\|x_2\|^2 + \frac{k_{1M}}{2}\|x_1\|^2 + \frac{k_g}{2}\|x_1\|^2 + \frac{\bar{k}_{2M}}{2}\|x_3\|^2 + \epsilon\mu_M\|x_1\|\|x_2\| + \frac{\epsilon\epsilon_0}{b_m}\|x_2\|\|x_3\| \\
&\quad = \frac{1}{2}\begin{pmatrix}\|x_1\| \\ \|x_2\| \\ \|x_3\|\end{pmatrix}^T Q_2 \begin{pmatrix}\|x_1\| \\ \|x_2\| \\ \|x_3\|\end{pmatrix}
\end{aligned}$$

where

$$Q_2 = \begin{pmatrix} k_{1M} + k_g & \epsilon\mu_M & 0 \\ \epsilon\mu_M & \mu_M & \epsilon\epsilon_0/b_m \\ 0 & \epsilon\epsilon_0/b_m & \bar{k}_{2M} \end{pmatrix}$$

with $\bar{k}_{2M} = \max_j\{k_{2j}/b_j\}$ (and k_{1M} as defined through expressions (A5)). From simple developments, one can further verify that (A1) $\implies \epsilon < \epsilon_1 \implies Q_2 > 0$, whence we get

$$V_2(x) \leq c_2\|x\|^2 \tag{A8}$$

$\forall x \in \mathcal{Q}_{12} \times \mathbb{R}^n \times \mathcal{Q}_{31}$, with $c_2 = \lambda_M(Q_2)/2 > 0$. Furthermore, the derivative of V_2 along the closed-loop system trajectories is given by

$$\begin{aligned}
&\dot{V}_2(x_1, x_2, x_3) \\
&\quad = x_2^T H(x_1 + q_d)\dot{x}_2 + \frac{1}{2}x_2^T \dot{H}(x_1 + q_d, x_2)x_2 + [s_1(K_1x_1) + g(x_1 + q_d) - g(q_d)]^T \dot{x}_1 \\
&\qquad + s_2^T(K_2x_3)B^{-1}\dot{x}_3 + \epsilon x_1^T H(x_1 + q_d)\dot{x}_2 + \epsilon x_1^T \dot{H}(x_1 + q_d, x_2)x_2 + \epsilon\dot{x}_1^T H(x_1 + q_d)x_2 \\
&\qquad - \epsilon\epsilon_0 x_2^T B^{-1}\dot{x}_3 - \epsilon\epsilon_0\dot{x}_2^T B^{-1}x_3 \\
&\quad = x_2^T[-C(x_1 + q_d, x_2)x_2 - g(x_1 + q_d) + g(q_d) - s_1(K_1x_1) - s_2(K_2x_3)] \\
&\qquad + \frac{1}{2}x_2^T \dot{H}(x_1 + q_d, x_2)x_2 + [s_1(K_1x_1) + g(x_1 + q_d) - g(q_d)]^T x_2 \\
&\qquad + s_2^T(K_2x_3)B^{-1}[-As_3(x_3) + Bx_2] \\
&\qquad + \epsilon x_1^T[-C(x_1 + q_d, x_2)x_2 - g(x_1 + q_d) + g(q_d) - s_1(K_1x_1) - s_2(K_2x_3)] \\
&\qquad + \epsilon x_1^T[C(x_1 + q_d, x_2) + C^T(x_1 + q_d, x_2)]x_2 + \epsilon x_2^T H(x_1 + q_d)x_2 \\
&\qquad - \epsilon\epsilon_0 x_2^T B^{-1}[-As_3(x_3) + Bx_2] \\
&\qquad - \epsilon\epsilon_0 x_3^T B^{-1}H^{-1}(x_1 + q_d)[-C(x_1 + q_d, x_2)x_2 - g(x_1 + q_d) + g(q_d) - s_1(K_1x_1) - s_2(K_2x_3)] \\
&\quad = -s_2^T(K_2x_3)B^{-1}As_3(x_3) - \epsilon x_1^T[s_1(K_1x_1) + g(x_1 + q_d) - g(q_d)] - \epsilon x_1^T s_2(K_2x_3) \\
&\qquad + \epsilon x_2^T C(x_1 + q_d, x_2)x_1 + \epsilon x_2^T H(x_1 + q_d)x_2 + \epsilon\epsilon_0 x_2^T B^{-1}As_3(x_3) - \epsilon\epsilon_0 x_2^T x_2 \\
&\qquad - \epsilon\epsilon_0 x_3^T B^{-1}H^{-1}(x_1 + q_d)[-C(x_1 + q_d, x_2)x_2 - g(x_1 + q_d) + g(q_d) - s_1(K_1x_1) - s_2(K_2x_3)]
\end{aligned}$$

where Eqs. (3) have been applied. By (25), (2), Assumptions 2.1, 2.2 and 2.3.2, Remark 2.1, observing that —in view of (34)— $s_3(x_3) = x_3$ on $\mathcal{Q}_{32} = \{x_3 \in \mathbb{R}^n : |x_{3j}| \leq L_{3j},\ j = 1, \ldots, n\}$, and defining $\mathcal{Q}_3 = \mathcal{Q}_{31} \cap \mathcal{Q}_{32}$, $\mathcal{B}_2 = \{x \in \mathbb{R}^n : \|x_2\| \leq \varrho_2\}$ for any

$\varrho_2 > 0$, and $\mathcal{Q}_1$ as in (A6a), we get that, on $\mathcal{Q}_1 \times \mathcal{B}_2 \times \mathcal{Q}_3$:

$$
\begin{aligned}
&\dot{V}_2(x_1, x_2, x_3) \\
&\quad \leq \; -x_3^T K_2 B^{-1} A x_3 - \epsilon \sum_{j=1}^{n} \bar{k}_{1j} x_{1j}^2 + \epsilon |x_1^T K_2 x_3| + \epsilon |x_2^T C(x_1 + q_d, x_2) x_1| + \epsilon |x_2^T H(x_1 + q_d) x_2| \\
&\qquad + \epsilon\epsilon_0 |x_2^T B^{-1} A x_3| - \epsilon\epsilon_0 x_2^T x_2 + \epsilon\epsilon_0 |x_3^T B^{-1} H^{-1}(x_1 + q_d) C(x_1 + q_d, x_2) x_2| \\
&\qquad + \epsilon\epsilon_0 |x_3^T B^{-1} H^{-1}(x_1 + q_d)[g(x_1 + q_d) - g(q_d)]| + \epsilon\epsilon_0 |x_3^T B^{-1} H^{-1}(x_1 + q_d) K_1 x_1| \\
&\qquad + \epsilon\epsilon_0 |x_3^T B^{-1} H^{-1}(x_1 + q_d) K_2 x_3| \\
&\quad \leq \; -\tilde{k}_{2m} \|x_3\|^2 - \epsilon \bar{k}_{1m} \|x_1\|^2 + \epsilon k_{2M} \|x_1\| \|x_3\| + \epsilon k_C \varrho_1 \|x_2\|^2 + \epsilon \mu_M \|x_2\|^2 + \epsilon\epsilon_0 \bar{a}_M \|x_2\| \|x_3\| \\
&\qquad - \epsilon\epsilon_0 \|x_2\|^2 + \frac{\epsilon\epsilon_0 k_C \varrho_2}{b_m \mu_m} \|x_2\| \|x_3\| + \frac{\epsilon\epsilon_0 k_g}{b_m \mu_m} \|x_1\| \|x_3\| + \frac{\epsilon\epsilon_0 k_{1M}}{b_m \mu_m} \|x_1\| \|x_3\| + \frac{\epsilon\epsilon_0 k_{2M}}{b_m \mu_m} \|x_3\|^2 \\
&\qquad = -\begin{pmatrix} \|x_1\| \\ \|x_2\| \\ \|x_3\| \end{pmatrix}^T Q_3 \begin{pmatrix} \|x_1\| \\ \|x_2\| \\ \|x_3\| \end{pmatrix}
\end{aligned}
$$

($\tilde{k}_{2m}$, k_{2M}, $\bar{a}_M$ and ϱ_1 as defined through expressions (A5)) with

$$
Q_3 = \begin{pmatrix} \epsilon \bar{k}_{1m} & 0 & -\epsilon\gamma_{13}/2 \\ 0 & \epsilon\gamma_{22} & -\epsilon\gamma_{23}/2 \\ -\epsilon\gamma_{13}/2 & -\epsilon\gamma_{23}/2 & \tilde{k}_{2m} - \epsilon\gamma_{33} \end{pmatrix}
$$

[γ_{22} (being positive in view of (A2)), γ_{13}, γ_{23} and γ_{33} as defined through expressions (A4)] and since (A1) $\implies \epsilon < \epsilon_2 \implies Q_3 > 0$, we get

$$
\dot{V}_2(x) \leq -c_3 \|x\|^2 \tag{A9}
$$

$\forall x \in \mathcal{Q}_1 \times \mathcal{B}_2 \times \mathcal{Q}_3$, with $c_3 = \lambda_m(Q_3) > 0$. Thus, from the simultaneous satisfaction of inequalities (A7)–(A9) on $\mathcal{Q}_1 \times \mathcal{B}_2 \times \mathcal{Q}_3$, we conclude —by (Khalil, 2002, Theorem 4.10)— that the origin $(x_1, x_2, x_3) = (0_n, 0_n, 0_n)$ is a (locally) exponentially stable equilibrium of the closed-loop system, whence the proof is completed.

Appendix B. On inequalities (37)

Observe that on $\{\varsigma \in \mathbb{R} : 0 < |\varsigma| \leq 2B_{gj}/k_g\}$ we have that (recall from Corollaries 3.1 and 3.2 that $0 < \beta_1 \leq 1$):

$$
|\varsigma| \leq \frac{2B_{gj}}{k_g} \iff |\varsigma|^{1-\beta_1} \leq \left(\frac{2B_{gj}}{k_g}\right)^{1-\beta_1} \iff k_{1j}^{\beta_1} \left(\frac{2B_{gj}}{k_g}\right)^{\beta_1 - 1} |\varsigma| \leq |k_{1j}\varsigma|^{\beta_1}
$$

while from (37a) we have, for all $\varsigma \neq 0$, that:

$$
\begin{aligned}
(37a) \iff k_g (2B_{gj})^{(1-\beta_1)/\beta_1} |\varsigma|^{1/\beta_1} < k_{1j} |\varsigma|^{1/\beta_1} &\iff k_g^{\beta_1} (2B_{gj})^{1-\beta_1} |\varsigma| < k_{1j}^{\beta_1} |\varsigma| \\
&\iff k_g |\varsigma| < k_{1j}^{\beta_1} \left(\frac{2B_{gj}}{k_g}\right)^{\beta_1 - 1} |\varsigma|
\end{aligned}
$$

From these developments we thus get, on $\{\varsigma \in \mathbb{R} : 0 < |\varsigma| \leq 2B_{gj}/k_g\}$, that: (37a) $\Longrightarrow$ $k_g|\varsigma| < |k_{1j}\varsigma|^{\beta_1}$, and consequently, for all $\varsigma \neq 0$, that: (37a) $\Longrightarrow$ $\min\{k_g|\varsigma|, 2B_{gj}\} < |k_{1j}\varsigma|^{\beta_1}$, whence, under the additional consideration of (37b), we get that: (37) $\Longrightarrow$ $\min\{k_g|\varsigma|, 2B_{gj}\} < \min\{|k_{1j}\varsigma|^{\beta_1}, M_{1j}\} = |\sigma_{1j}(k_{1j}\varsigma)|$, $\forall\varsigma \neq 0$.