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Generalized multistable structure via chaotic synchronization and preservation of scrolls

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Abstract

Switched systems are capable of generating chaotic multi-scroll behavior in \mathbb{R}^3 by means of a control signal. This signal regulates an equilibrium position of the system and is defined according to the number of scrolls that is displayed by the attractor. Thus, if two systems are controlled by different signals, They exhibit a different number of scrolls. Multistability can be created by a pair of unidirectionally coupled unstable dissipative switched linear systems. A theoretical study of this phenomenon is performed with the jerky equations. Generalized synchronization is observed in numerical simulations of the master-slave system with different control signals. The

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proposed configuration preserves the number of scrolls and can possess an arbitrary large number of coexisting chaotic multi-scroll attractors.

Keywords: Multistability; chaotic multiscroll systems; generalized synchronization; piece-wise linear systems.

1. Introduction

Complex dissipative nonlinear systems often exhibit the coexistence of multiple stable equilibrium states for the same set of parameters. Some states may be chaotic, while others are periodic or fixed points. The coexistence of multiple attractors is one of the most exciting phenomena in nonlinear dynamics, which was observed in various systems, including electronic [1], lasers [2, 3], mechanical [4], chemical [5] and biological [6, 7] systems. Multistability, from the particular case of the coexistence of two attractors has been used for different applications, the most known is the generation of flip flops, widely used in memory storage devices. It has been recently shown that flip flops can be constructed using the base Chua's circuit by means of standard interconnection between NAND gates [8] or using a bistability mode [9]. Multistability may be of particular relevance to biological systems, which switch between discrete states, generate oscillatory responses, or "remember" transitory stimuli [10].

Multistability often appears in coupled systems due to increasing complexity, when two or more systems join together. The emergence of multiple stable regions depends strongly on one (or the combination) of two factors: the coupling interconnection (linear, nonlinear, unidirectional or mutually coupled); the coupling strength. Dissipative weakly coupled systems can

present a very large number of coexisting attractors[4]. In other systems, the emergence of multistability often results in a loss of synchronization, e.g., in the formation of clusters in oscillators ensemble [11, 12] and in coupled logistic [13] and Hénon [14, 15] maps. Multistability in unidirectionally coupled identical systems is accompanied by different kinds of intermittent synchronization dependent on the coupling parameter [16, 17, 18].

In this work, we study the emergence of coexisting attractors in two multi-scroll chaotic systems coupled in a master-slave configuration, when the master and the slave systems display different number of scrolls. There are different approaches to obtain multi-scroll chaotic attractors. One of them is by switching a piece-wise linear (PWL) system [19] and controlling the stability of its equilibrium points. Other approaches imply the modification of the Chua's system [20, 21] by replacing the nonlinear term with different nonlinear functions [22, 23]. Here, we consider unstable dissipative systems [24], a class of 3-D dynamical systems that present multiple scrolls. This class of systems is constructed with a switching law to display various multi-scroll strange attractors, which result from a combination of several unstable “one-spiral” trajectories by means of changing the equilibria. Switching systems have been commonly used to switch only one system or the control signal. However there have been some approaches in which, instead of stabilizing the system the idea is to preserve or generate a specific dynamic [25]. Here we will show how multistability can be obtained using coupled switched systems.

The article is arranged as follows: Section 2 presents the theory that involves the unstable dissipative systems and the generation of multiple scrolls; Sec-

tion 3 describes the type of coupling used; Section 4 introduces the multimodal structures generated; the numerical results to prove the theory are depicted in Section 5; and conclusions are drawn in Section 6.

2. Multiscroll attractors by unstable dissipative systems

We consider the class of autonomous systems of linear differential equations given as follows:

$$\dot{\chi} = \mathbf{A}\chi + \mathbf{B}, \quad (1)$$

where $\chi = [x_1, x_2, x_3]^T \in \mathbb{R}^3$ is the state variable, $\mathbf{B} = [\beta_1, \beta_2, \beta_3]^T \in \mathbb{R}^3$ stands for a real vector, $\mathbf{A} = [\alpha_{ij}] \in \mathbb{R}^{3 \times 3}$ denotes a linear operator and the equilibrium point is located at $\chi^* = -\mathbf{A}^{-1}\mathbf{B}$, which has a stable E^s and unstable E^u manifold. A Saddle equilibrium points, which connects stable and unstable manifolds play an important role in chaos creation, because they are responsible for successive stretching and folding processes. The stretching causes the system trajectories to exhibit sensitive dependence on initial conditions, whereas the folding creates a complicated microstructure [26]. The saddle points of a chaotic PWL system in \mathbb{R}^3 can be divided basic categories according to their eigenvalues $\Lambda = \{\lambda_i, \lambda_j, \lambda_k\} \in \mathcal{C}$: (i) The saddle points that are stable in one direction only and unstable or oscillatory in the other two [24], i.e., it has one negative real eigenvalue ($Re\{\lambda_i\} < 0, Im\{\lambda_i\} = 0$), and two complex conjugated eigenvalues ($Re\{\lambda_{j,k}\} > 0, Im\{\lambda_{j,k}\} \neq 0$). (ii) The saddle points that are stable in two directions and unstable in only one, i.e., the dissipative components are oscillatory ($Im\{\lambda_{j,k}\} \neq 0$ and $Re\{\lambda_{j,k}\} < 0$), while the unstable component corresponds to a real positive eigenvalue ($Re\{\lambda_i\} > 0, Im\{\lambda_i\} = 0$). If the system given

by Eq. (1) has a saddle equilibrium point responsible for unstable and stable manifolds and the sum of its eigenvalues is negative, then the system is called an *unstable dissipative system* (UDS). According to the above speculations, we define in the same way as [27] two different types of UDS, *Type I* and *Type II* as follows:

Definition 2.1. *A system given by Eq. (1) in \mathbb{R}^3 with eigenvalues λ_i , $i = 1, 2, 3$, is said to be an UDS Type I, if $\sum_{i=1}^3 \text{Re}\{\lambda_i\} < 0$ and one eigenvalue is positive real and the other two are complex conjugate with a negative real part.*

Definition 2.2. *A system given by Eq. (1) in \mathbb{R}^3 with eigenvalues λ_i , $i = 1, 2, 3$, is said to be an UDS Type II, if $\sum_{i=1}^3 \text{Re}\{\lambda_i\} < 0$ and one eigenvalue is positive real and the other two are complex conjugate with a negative real part.*

The above definitions imply that the UDS *Type I* is dissipative in one of its components, but unstable in the other two, which are oscillatory. The converse is the UDS *Type II*, which are dissipative and oscillatory in two of its components, but unstable in the other one. Since we are interested in the UDS *Type I*, the next corollary is important to note what kind of behaviors are possible in the system given by (1).

Corollary 2.3. *Let the system (1) be a UDS Type I. Then, the system has a stable manifold $E^s = \text{span}\{\lambda_1\} \subset \mathbb{R}^3$ and another, unstable manifold $E^u = \text{span}\{\lambda_2, \lambda_3\} \subset \mathbb{R}^3$ and the following statements are true:*

(a) *All initial condition $\chi_0 \in \mathbb{R}^3 - E^s$ leads to a unstable orbit that goes to infinity.*

(b) All initial condition $\chi_0 \in E^s$ leads to a stable orbit that settles down at χ^* and the system does not generate oscillations.

(c) The basin of attraction \mathcal{D} is $E^s \subset \mathbb{R}^3$.

Now, let us consider Lur -type systems of the form:

$$\dot{\chi}(t) = \mathbf{A}\chi(t) + \mathbf{B}u(t), \quad (2)$$

$$y(t) = \mathbf{G}\chi(t), \quad (3)$$

$$u(t) = F(y(t)), \quad (4)$$

where $\mathbf{G} \in \mathbb{R}^3$, $F : \mathbb{R} \rightarrow \mathbb{R}$. In the control system, Eq. (2) represents a plant with control input signal $\mathbf{B}u(t)$, Eq. (3) is the system output (observable states), and Eq. (4) is the control signal which is a function of the output. Since the system (2) is unstable (because it present a saddle equilibrium point), it is necessary to design a control signal u capable of stabilizing the system in a specific region. Different F functions have been reported [28, 29]. Our goal is to design a function F for Eq. (2) that generates a class of 3-D dynamical systems with chaotic oscillations, i.e., the flow $\Phi(\chi(t))$ of the system Eq. (2) is trapped in an attractor \mathbb{A} by means of modifying the function F . This class of systems can display various multi-scroll strange attractors as a result of the combination of several unstable ‘‘one-spiral’’ trajectories by means of u . In the other words, we are interesting in functions that can yield multi-scroll attractors constitute by a commuted signal, $u = S_i$, ($i = 1, \dots, n$ and $n \geq 2$). Each signal S_i has a domain $\mathcal{D}_i \subset \mathbb{R}^3$, containing the equilibrium point $\chi_i^* = -\mathbf{A}^{-1}\mathbf{B}S_i$. Then, the function F governs the dynamics of system (2) by changing the equilibria from χ_i^* to χ_j^* , $i \neq j$, when

the flow $\Phi^t : \mathcal{D}_i \rightarrow \mathbb{R}^3$ crosses from the i -th to the j -th domain. According to the above discussion it is possible to define a chaotic multi-scroll system based on UDS *Type I* as follows:

Definition 2.4. *A system given by (2) in \mathbb{R}^3 and with equilibrium points χ_i^* ($i = 1, \dots, n$ and $n \geq 2$) is said to be a chaotic multi-scroll system, if each χ_i^* contains oscillations around and the flow $\phi(\chi_0)$ generates an attractor $\mathbb{A} \subset \mathbb{R}^3$.*

The easiest chaotic system based on the UDS is defined with a step function F commutated in two values, S_1 and S_2 . Since the system Eq. (2) has two equilibria, a double scroll appears. The function F is defined as a PWL function. Adding more PWL segments S_i to the step function F , makes possible to obtain a number of scrolls proportional to the number of signals S_i . For simplicity, the function F is given in terms of one state only, which defines the borders of domains as parallel planes orthogonal to one axis, that is achieved by defining $\mathbf{G} = [1, 0, 0]^T$.

$$F(x_1(t)) = \begin{cases} S_1, & \text{if } x_1 \in \mathcal{D}_1, \\ S_2, & \text{if } x_1 \in \mathcal{D}_2, \\ \vdots, & \vdots \\ S_n & \text{if } x_1 \in \mathcal{D}_n, \end{cases} \quad (5)$$

where S_1, S_2, \dots, S_n are constants, preserving the following order $S_1 < S_2 < \dots < S_n$. In order to illustrate our approach, we consider the particular case of the linear ordinary differential equation (ODE) written in the equation of a jerky form as $\ddot{x} + \alpha_{33}\dot{x} + \alpha_{32}x + \alpha_{31}x + \beta_3 = 0$, representing the state space of Eq. (1), where the matrix \mathbf{A} and the vector \mathbf{B} are given as follows:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_{31} & -\alpha_{32} & -\alpha_{33} \end{pmatrix}; \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ \beta_3 \end{pmatrix}, \quad (6)$$

with the coefficients $\alpha_{31}, \alpha_{32}, \alpha_{33}, \beta_3 \in \mathbb{R}$. We should note that the \mathbf{A} matrix is not restricted to the form derived from a jerky equation. The jerky form is just a convenient approach to build the matrices \mathbf{A} and \mathbf{B} . The characteristic polynomial of matrix \mathbf{A} given by Eq. (6) takes the following form:

$$\lambda^3 + \alpha_{33}\lambda^2 + \alpha_{32}\lambda + \alpha_{31}. \quad (7)$$

The classical Descartes' rule of signs is a useful tool to state how many positive or negative roots can be expected from polynomial Eq. (7). Thus, since the system Eq. (6) is assumed dissipative, a direct implication yields $-\alpha_{33} = \text{Tr}(\mathbf{A}) = \sum_{i=1}^3 \text{Re}\{\lambda_i\} < 0$. Additionally, the system Eq. (6) has a saddle point at the equilibria as $-\alpha_{31} = \det(\mathbf{A}) < 0$. Moreover, $\alpha_{32} > 0$ in Eq. (7) is required to ensure that the real root of Eq. (7) is negative. Hence, under the above rationale, there are no positive real roots, because all coefficients in Eq. (7) are positive. Such implications give us the possibility that one of the three eigenvalues is a negative real value and the other two are complex with a real positive part.

3. Multimodal generalized synchronization

Consider a pair of unidirectionally coupled systems given as follows:

$$\dot{\chi}^m = \mathbf{A}\chi^m + \mathbf{B}u^m, \quad (8)$$

$$\dot{\chi}^s = \mathbf{A}\chi^s + \mathbf{B}u^s + \mathbf{C}(\chi^m - \chi^s), \quad (9)$$

where the vectors $\chi^m = (x_1^m, x_2^m, x_3^m)^T \in \mathbb{R}^3$ and $\chi^s = (x_1^s, x_2^s, x_3^s)^T \in \mathbb{R}^3$ are the state variables of the master and slave systems, respectively, $\mathbf{B} = (0, 0, 1)^T \in \mathbb{R}^3$, and $\mathbf{C} \in \mathbb{R}^{3 \times 3}$ is a constant matrix. In this work, we focus on the case when each system given by Eqs. (8) and (9) represent a different number of scrolls, i.e., $\#S_m < \#S_s$, when \mathbf{C} is the null matrix and $u^m \neq u^s$. $\#S_m$ and $\#S_s$ stand for the number of scrolls that display the master and the slave system, respectively. When the systems are coupled, \mathbf{C} is defined by just one real number as follows: $(c_{22}) \neq 0$, otherwise $(c_{ij}) = 0$, $i, j = 1, 2, 3$.

Using this kind of coupling, it is possible to explore generalized synchronization between a pair of different chaotic systems. The chaotic synchronization occurs whenever the master and the slave system flows are related by a function, that is:

$$\lim_{t \rightarrow \infty} |\chi^m(t) - h(\chi^s(t))| = 0 \quad (10)$$

for every different pair of initial conditions $\chi^m(0) = \chi_0^m$ and $\chi^s(0) = \chi_0^s$ inside a ball of dissipation \mathcal{D} (basin of attraction). For the case of generalized synchronization, the function h is difficult to find, but fortunately there are several approaches to determine generalized synchronization. One approach is by considering an auxiliary system χ^a proposed in [30], which is a replica of the slave system. Using the auxiliary system, it is easy to detect generalized synchronization, meaning that

$$\lim_{t \rightarrow \infty} |\chi^s(t) - \chi^a(t)| = 0. \quad (11)$$

This is true, if there is no multistability and the system Eqs. (8) and (9) exhibit generalized synchronization. For the case of multistability, the basin of attraction is comprised by several basins of attraction $\mathcal{D} = \bigcup_{k=1}^v \mathcal{D}_k$, where v represents a number of stable modes. Now, initial conditions for the slave and the auxiliary systems play an important role to detect general synchronization between the master and the slave systems.

$$\lim_{t \rightarrow \infty} |\chi^s(t) - \chi^a(t)| = \begin{cases} 0, & \text{for } \mathcal{D}_i(\chi_0^s) = \mathcal{D}_j(\chi_0^a), \\ d_{ij} > 0, & \text{for } \mathcal{D}_i(\chi_0^s) \neq \mathcal{D}_j(\chi_0^a), \end{cases} \quad (12)$$

where $\chi_0^s \in \mathcal{D}_i = \mathcal{D}_i(\chi_0^s)$ and $\chi_0^a \in \mathcal{D}_j = \mathcal{D}_j(\chi_0^a)$, $i, j = 1 \dots m$ and $d \in \mathbb{R}$ is an arbitrary distance. There exist a c_0 value that guarantees generalized synchronization between the coupled systems given by Eqs. (8) and (9), $c_0 \leq (c_{22})$.

4. Generalized multistable structure

A multistable structure is created by the coupled system given by Eqs. (8) and (9), each in the jerky form Eq. (6) with $\beta_3 = 1$ in both the master and the slave systems. The coupled system is explicitly written as follows:

$$\begin{aligned}
\dot{x}_1^m &= x_2^m, \\
\dot{x}_2^m &= x_3^m, \\
\dot{x}_3^m &= -\alpha_{31}x_1^m - \alpha_{32}x_2^m - \alpha_{33}x_3^m + u^m(x_1^m), \\
\dot{x}_1^s &= x_2^s, \\
\dot{x}_2^s &= x_3^s + c_{22}(x_2^m - x_2^s), \\
\dot{x}_3^s &= -\alpha_{31}x_1^s - \alpha_{32}x_2^s - \alpha_{33}x_3^s + u^s(x_1^s).
\end{aligned} \tag{13}$$

Theorem 4.1. *Let $\mathfrak{S}_m = \{S_1^m, S_2^m, \dots, S_{\#S_m}^m\}$ and $\mathfrak{S}_s = \{S_1^s, S_2^s, \dots, S_{\#S_s}^s\}$ be the sets conformed by the control signals u^m and u^s , respectively. A given system in the form of Eq. (13) exhibits multistability if $\mathfrak{S}_m^* \subset \mathfrak{S}_s$, and the number of multiple basins of attraction is $\#S_s - \#S_m + 1$, with $\#S_s > \#S_m$ and $\mathfrak{S}_m^* = \{S_1^m + \delta, S_2^m + \delta, \dots, S_{\#S_m}^m + \delta\}$, where $\delta \in \mathbb{R}$ is the displacement from one basin of attraction to another.*

Proof: The solution of the fifth linear differential equation of the system (13) can be rewritten as follows:

$$x_2^s(t) = e^{-c_{22}t} \int_0^t e^{c_{22}\tau} (c_{22}x_2^m(\tau) + x_3^s(\tau)) d\tau + e^{-c_{22}t} x_2^s(0). \tag{14}$$

Since we are interested in adiabatically stable solutions, $\lim_{t \rightarrow \infty} e^{-c_{22}t} x_2^s(0)$. Therefore, we neglect the last term in Eq. (14). The next step is to integrate by parts Eq. (14), resulting in

$$x_2^s(t) = x_2^m(t) + \frac{1}{c_{22}} x_3^s(t) - \frac{1}{c_{22}} e^{c_{22}\tau} \int_0^t e^{c_{22}\tau} (c_{22}x_2^m + x_3^s) d\tau. \tag{15}$$

Using the fact that $\dot{x}_2^m = x_3^m$ and by applying integration by parts, the term with the integral in Eq. (15) can be developed as

$$\frac{1}{c_{22}} e^{c_{22}\tau} \int_0^t e^{c_{22}\tau} (c_{22}\dot{x}_2^m + \dot{x}_2^s) d\tau = \frac{1}{c_{22}} x_3^m(t) + \frac{1}{c_{22}^2} e^{c_{22}\tau} \int_0^t e^{c_{22}\tau} (\dot{x}_2^m - \dot{x}_3^s) d\tau.$$

From Eq. (15), we obtain

$$x_2^s(t) = x_2^m(t) + \frac{1}{c_{22}} (x_3^m(t) - x_3^s(t)) + E(t), \quad (16)$$

where $E(t) = \frac{1}{c_{22}^2} e^{c_{22}\tau} \int_0^t e^{c_{22}\tau} (\dot{x}_2^m - \dot{x}_3^s) d\tau$. Assuming that *i*) $c_{22} \gg 1$, without loss of generality $E(t) \approx 0$, *ii*) $c_{22} \gg |x_3^m(t) - x_3^s(t)|$ gives as a consequence $(x_3^m(t) - x_3^s(t))/c_{22} \approx 0$. Under the above assumptions and after a transient time, Eq. (16) becomes $x_2^m(t) \approx x_2^s(t)$. Now, if $\rho_1(t) = x_1^m(t) - x_1^s(t)$, $\rho_2(t) = x_2^m(t) - x_2^s(t)$, $\rho_3(t) = x_3^m(t) - x_3^s(t)$, then we obtain

$$\begin{aligned} \dot{\rho}_1(t) &= \dot{x}_1^m(t) - \dot{x}_1^s(t) = x_2^m(t) - x_2^s(t) \approx 0, \\ \dot{\rho}_2(t) &= \dot{x}_2^m(t) - \dot{x}_2^s(t) = x_3^m(t) - x_3^s(t) \approx 0, \\ \dot{\rho}_3(t) &= \dot{x}_3^m(t) - \dot{x}_3^s(t), \end{aligned}$$

that results in the difference between the first states of the master and the slave systems as $x_1^m(t) - x_1^s(t) = K(x_1^m(0), x_2^m(0))$.

If we consider that $x_1^s \approx x_1^m + K(x_1^m(0), x_2^m(0))$, $x_2^s \approx x_2^m$, then for the sixth state of the system (13), we obtain

$$\dot{x}_3^s \approx -\alpha_{31}x_1^s - \alpha_{32}x_2^m - \alpha_{33}x_3^s + u^s(x_1^s). \quad (17)$$

By calculating the fixed points of the last equation, after transients we obtain $x_1^m = (1/\alpha_{31})u^m(x_1^m)$, $x_2^m = 0$, $x_3^s = 0$ resulting in

$$u^s(x_1^s) = u^m(x_1^m) + 1.5K(x_1^m(0), x_2^m(0)). \quad (18)$$

The last expression establishes that the u^s signal applied to the slave system of Eq. (13) is given as a function of the u^m signal applied to the master system of Eq. (13), but shifted by a constant, which depends on the initial conditions $x_1(0)$ and $x_2(0)$. Besides, if we substitute u^s from Eq. (18) into Eq. (17), the fixed points of the slave system are $(2/3u^m(x_1^m) + K(x_1(0), x_2(0), 0))$, i.e., the slave has the same fixed points as the master, but shifted by a constant K dependent on the initial conditions. The u^m signals are restricted to take consecutive values of u^s , i.e., if $\mathfrak{S}_m = \{S_1^m, S_2^m, S_3^m\}$, $\mathfrak{S}_s = \{S_1^s, S_2^s, S_3^s, S_4^s, S_5^s\}$, and $\mathfrak{S}_m \subset \mathfrak{S}_s$ then the possibilities are $\{S_1^m = S_1^s, S_2^m = S_2^s, S_3^m = S_3^s\}$, $\{S_1^m = S_2^s, S_2^m = S_3^s, S_3^m = S_4^s\}$, and $\{S_1^m = S_3^s, S_2^m = S_4^s, S_3^m = S_5^s\}$, but never $\{S_1^m = S_1^s, S_2^m = S_3^s, S_3^m = S_4^s\}$. Therefore, the multistable behavior is justified.

5. Numerical Results

In order to illustrate our approach, we consider the master system from Eq. (8) given by the \mathbf{A} matrix and the \mathbf{B} vector as follows

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.5 & -1 & -1 \end{pmatrix}; \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (19)$$

and the control signal given by a real function $u_i^m : \mathbb{R} \rightarrow \mathbb{R}$ (henceforth i denotes a number of scrolls that the orbit follows). The master system is

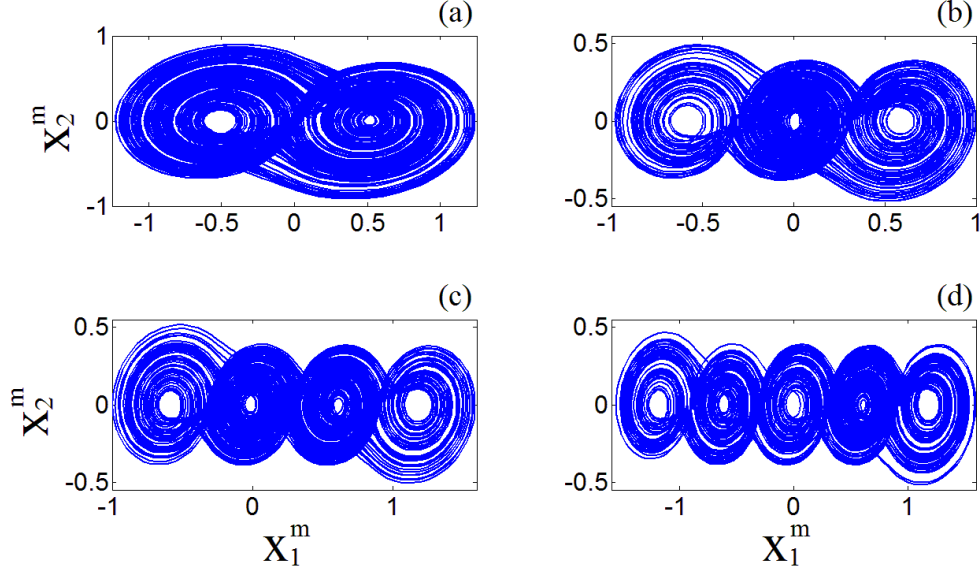


Figure 1: Projection of chaotic multi-scroll attractors onto (x_1^m, x_2^m) plane, generated by different control signal u_i : (a) Eq. (20), (b) Eq. (21), (c) Eq. (22), and (d) Eq. (23).

based on UDS *type I* because the \mathbf{A} matrix has three eigenvalues: one negative real eigenvalue $\lambda_1 = -1.2041$, and the others are complex eigenvalues with positive real part $\lambda_2 = 0.1020 + i1.1111$, $\lambda_3 = 0.1020 - i1.1111$. The system is dissipative due to $\sum_{i=1}^3 Re\{\lambda_i\} < 0$. Thus, a double-scroll system can be given by Eqs. (8) and (19), and u_2^m as follows

$$u_2^m = \begin{cases} 0.9, & \text{if } x_1 \geq 0.3; \\ 0, & \text{otherwise.} \end{cases} \quad (20)$$

The equilibrium points of the system given by Eqs. (8), (19), and (20) are $\chi_1^* = (0.6, 0, 0)$ and $\chi_2^* = (0, 0, 0)$. Figure 1 (a) depicts the projection of the double-scroll chaotic attractor onto the (x_1^m, x_2^m) plane. It is possible to generate a triple-scroll attractor by modifying the u_i function. Thus, u_3^m is

given as follows

$$u_3^m = \begin{cases} 0.9, & \text{if } x_1 \geq 0.3, \\ 0, & \text{if } -0.3 < x_1 < 0.3, \\ -0.9, & \text{if } x_1 \leq -0.3. \end{cases} \quad (21)$$

Notice that $\chi_3^* = -\chi_1^*$. This issue is intentionally defined to illustrate the symmetry in scrolls. Figure 1 (b) shows the projection of the triple-scroll chaotic attractor onto the (x_1^m, x_2^m) plane by using Eqs. (8), (19), and (21). This approach can be extended in order to generate systems with any number of scrolls, i.e., quadruple and quintuple scroll chaotic attractors are produced by defining u_4 and u_5 in accordance with the following features:

$$u_4^m = \begin{cases} 1.8, & \text{if } x_1 \geq 0.9, \\ 0.9, & \text{if } 0.3 \leq x_1 < 0.9, \\ 0, & \text{if } -0.3 < x_1 < 0.3, \\ -0.9, & \text{if } x_1 \leq -0.3. \end{cases} \quad (22)$$

$$u_5^m = \begin{cases} 1.8, & \text{if } x_1 \geq 0.9, \\ 0.9, & \text{if } 0.3 \leq x_1 < 0.9, \\ 0, & \text{if } -0.3 < x_1 < 0.3, \\ -0.9, & \text{if } -0.9 < x_1 \leq -0.3, \\ -1.8, & \text{if } x_1 \leq -0.9. \end{cases} \quad (23)$$

The functions given by Eqs. (22) and (23) introduce other equilibrium points located at $\chi_4^* = (1.2, 0, 0)$ and $\chi_5^* = (-1.2, 0, 0)$, respectively. Figures 1 (c) and (d) show the projection of the quadruple-scroll and quintuple-scroll chaotic attractors onto the (x_1^m, x_2^m) plane, respectively.

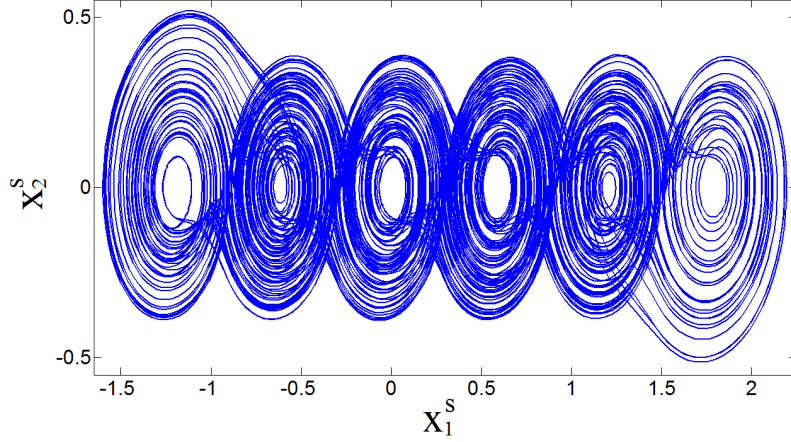


Figure 2: Projection of a chaotic multi-scroll attractor onto (x_1^s, x_2^s) plane, generated with the control signal given by Eq. (24).

Now, we consider the coupled system (13) given by the master system from Eqs. (8), (19), and (21), and the slave system from Eqs. (9) and (19) with

$$u_6^s = \begin{cases} 2.7, & \text{if } x_1 \geq 1.5, \\ 1.8, & \text{if } 0.9 \leq x_1 < 1.5, \\ 0.9, & \text{if } 0.3 < x_1 < 0.9, \\ 0, & \text{if } -0.3 < x_1 \leq 0.3, \\ -0.9, & \text{if } -0.9 < x_1 \leq -0.3, \\ -1.8, & \text{if } x_1 \leq -0.9. \end{cases} \quad (24)$$

Using the algorithm and based on the definition described by Wolf et.al [31], the maximum Lyapunov exponent for the system (19) is 0.10185, showing that the system is chaotic.

Figure 1 (b) shows the projection onto the (x_1^m, x_2^m) plane of the mas-

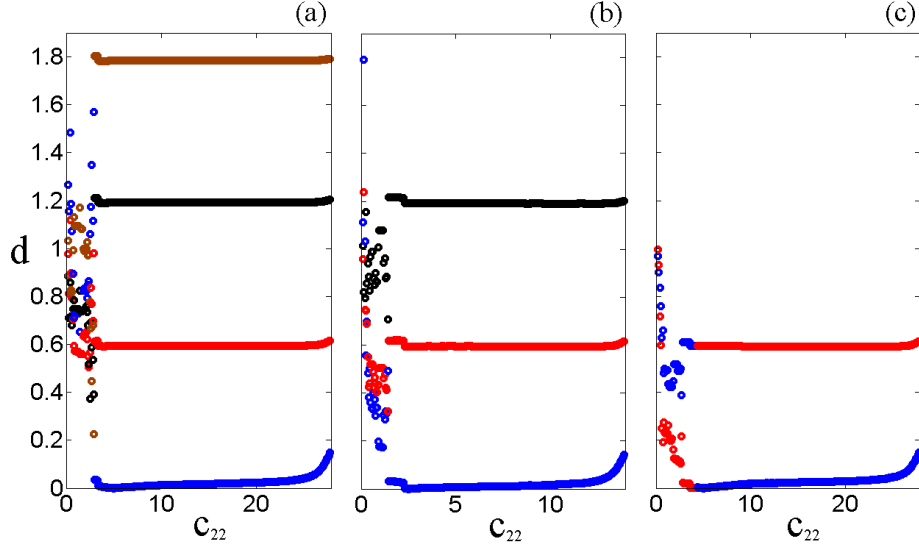


Figure 3: The d distance via c_{22} coupling parameter for master and slave systems given by (a) u_3^m and u_6^s , (b) u_4^m and u_6^s , and (c) u_5^m and u_6^s , respectively.

ter system, and Figure 2 shows the projection of the slave system onto the (x_1^s, x_2^s) plane when $c_{22} = 0$. The slave system displays six scrolls with equilibrium points χ_j^* for $j = 1, 2, 3, 4, 5$, given previously, and $\chi_6^* = (1.8, 0, 0)$. The sets $\mathfrak{S}_m = \{-0.9, 0, 0.9\}$ and $\mathfrak{S}_s = \{-1.8, -0.9, 0, 0.9, 1.8, 2.7\}$ have the cardinality $\sharp S_m = 3$ and $\sharp S_s = 6$, respectively. According to Theorem 4.1, this system presents multistability due to $\mathfrak{S}_m^* \subset \mathfrak{S}_s$, with $\delta = 0$ and there are $\sharp S_s - \sharp S_{m+1} = 4$ basins of attraction. By choosing arbitrary initial conditions for the slave system $\chi^s(0)$ and its auxiliary system $\chi^a(0)$ inside the ball of dissipation \mathcal{D} , we calculate the eucliden distance between their orbits in order to check the limit given by Eq. (11), where $d = \frac{1}{N} \sum_{j=1}^N \sqrt{(x_1^s(j) - x_1^a(j))^2 + (x_2^s(j) - x_2^a(j))^2 + (x_3^s(j) - x_3^a(j))^2}$. Figure 3 shows the d distance via the c_{22} coupling parameter when the master sys-

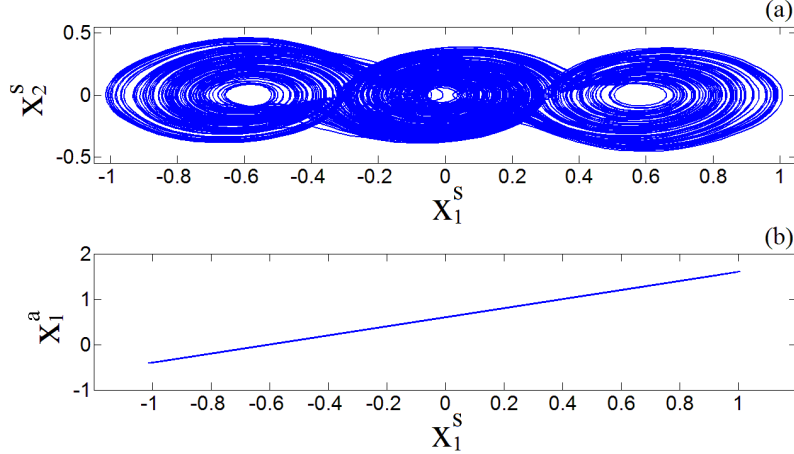


Figure 4: Generalized synchronization between master and slave systems. Projections of a chaotic multi-scroll attractor onto (a) (x_1^s, x_2^s) plane of slave system, and (b) (x_1^s, x_1^a) plane of slave and auxiliary systems.

tem and slave system are given by u_3^m and u_6^s , respectively. Figure 3 (a) shows $d = 0$ (marked in blue) for $c_{22} \in (4.5, 26)$ when the initial conditions belong to the same basin of attraction, and three distances $d \neq 0$ (marked in red, brown and black) when the initial conditions belong to a different basin of attraction, proving that there is multistability. Figure 3 (b) shows the d distance when the master and slave systems are given by u_4^m and u_6^s , respectively, thus exhibiting three basins of attraction. Finally, Fig. 3 (c) displays the d distance for u_5^m and u_6^s , thus giving two basins of attraction.

Figure 4 (a) shows the projection of the unidirectionally coupled slave system given by Eqs. (13) and (19) onto the (x_1^s, x_2^s) plane for $c_{22} = 5$. It can be seen that the number of scrolls of the master system is preserved in the slave system because u_3^m and u_6^s are given by Eqs. (21) and (24), respectively. Figure 4 (b) shows the projection of the slave and auxiliary

systems onto the (x_1^s, x_1^a) plane when their initial conditions belong to the same basin of attraction.

In the master-slave system, the number of scrolls of the master system is preserved in the slave system independently of the number of scrolls in the slave system. To illustrate this, the control signal u_6^s is changed as follows

$$u_{10}^s = \begin{cases} 8.1, & \text{if } x \geq 6.7, \\ 7.2, & \text{if } 5.1 \leq x_1 < 6.7, \\ 6.3, & \text{if } 4.5 \leq x_1 < 5.1, \\ 5.4, & \text{if } 3.9 \leq x_1 \leq 4.5, \\ 4.5, & \text{if } 3.3 \leq x_1 < 3.9, \\ 3.6, & \text{if } 2.1 \leq x_1 < 2.7, \\ 2.7, & \text{if } 1.5 \leq x_1 < 2.1, \\ 1.8, & \text{if } 0.9 \leq x_1 < 1.5, \\ 0.9, & \text{if } 0.3 \leq x_1 < 0.9, \\ 0, & \text{if } x_1 < 0.3. \end{cases} \quad (25)$$

Now, the slave attractor presents ten scrolls when it is uncoupled and its equilibrium points are given by $\chi_i = (6/10i, 0, 0)$, $i = 0, 1, \dots, 9$. The projection of the slave system onto the (x_1^s, x_2^s) plane is shown in Figure 5. The number of scroll is numerated from left to right. The sets $\mathfrak{S}_m = \{-0.9, 0, 0.9\}$ and $\mathfrak{S}_s = \{0, 0.9, 1.8, 2.7, 3.6, 4.5, 5.4, 6.3, 7.2, 8.1\}$ have the follow cardinality $\#S_m = 3$ and $\#S_s = 10$, respectively. According to Theorem 4.1, this system presents multistability due to $\mathfrak{S}_m^* \subset \mathfrak{S}_s$, with $\delta = 0.9$ and there are $\#S_s - \#S_{m+1} = 8$ basins of attraction.

Figure 6 shows the projections of eight attractors of the slave system onto the (x_1^s, x_2^s) plane obtained by changing the initial conditions only. The

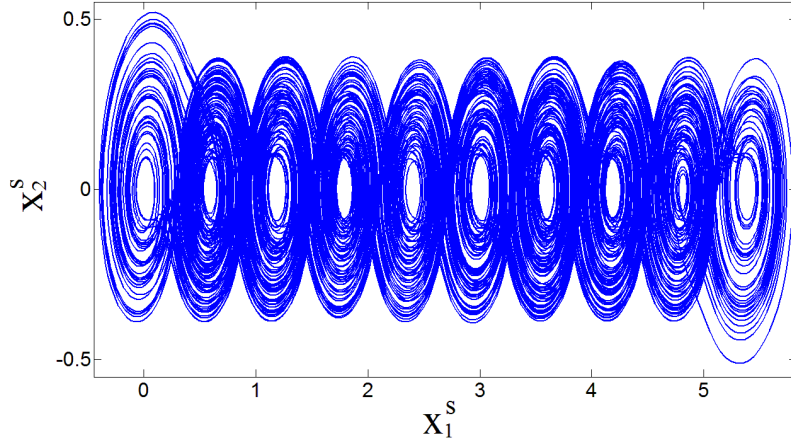


Figure 5: Projection of a chaotic multi-scroll attractor of the slave system onto (x_1^s, x_2^s) plane using the u_{10}^s control signal Eq. (25).

projection of the slave system trajectory for $c_{22} = 0$ is shown by very tenuous gray lines, resulting from the slave system before being forced. We can see that the slave system oscillates preserving the number of scrolls of the master system, giving rise to multistability phenomena.

Table 1 describes the attractors in Figure 6. The first column indicates the initial conditions of the slave system, the second column shows the number of scroll, and the third column indicates the attractor color.

Since we are interested in the study of multistability, we vary the initial conditions of the slave system on the (x_1^s, x_2^s) plane, while fixing the initial conditions of the master system at $\chi^m(0) = (1, 1, 1)$. Figure 7 shows eight basins of attraction onto a region of the plane $x_3^s = 0$ ($x_1^s \in [-1, 7]$ and $x_2^s \in [-2, 2]$) due to multistability of the coupled system (13) using u_3^m and u_{10}^s . The basin of attraction marked with blue dots corresponds to the attractor that covers the scrolls 1, 2 and 3 (see Figure 6). The basin of attraction

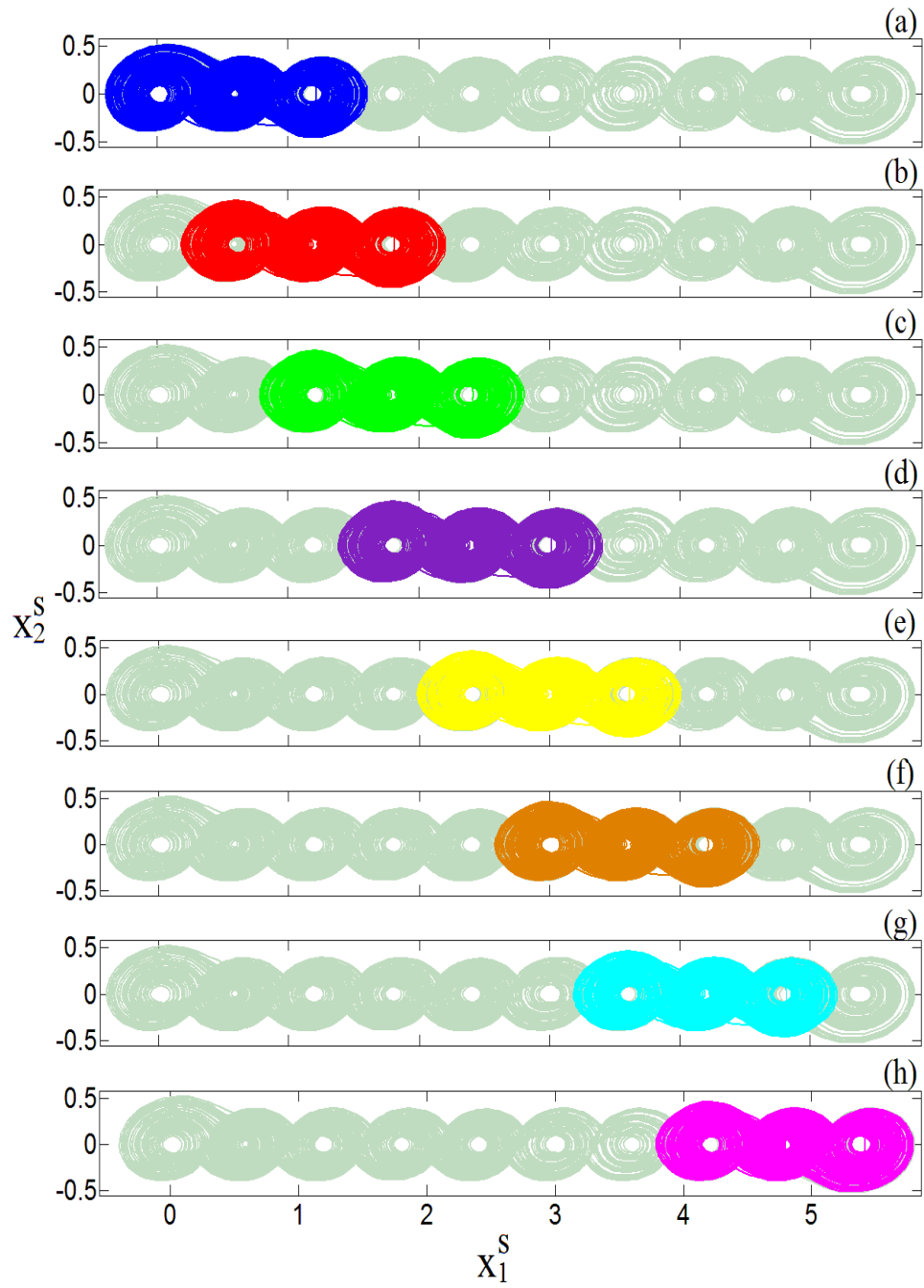


Figure 6: Projection of multiple coexisting chaotic multi-scroll attractors onto (x_1, x_2) plane.

Initial Conditions	Scrolls covered	Color
(0.1, 0.1, 0)	1, 2, 3	Blue
(0.6, 0.6, 0)	2, 3, 4	Red
(1.5, 1.5, 0)	3, 4, 5	Green
(2, 2, 0)	4, 5, 6	Purple
(2.5, 2.5, 0)	5, 6, 7	Yellow
(3, 3, 0)	6, 7, 8	Brown
(3.6, 3.6, 0)	7, 8, 9	Cyan
(5, 5, 0)	8, 9, 10	Pink

Table 1: List of initial conditions, number of scrolls and colors of the orbits in Figure 6 for the resulting slave system.

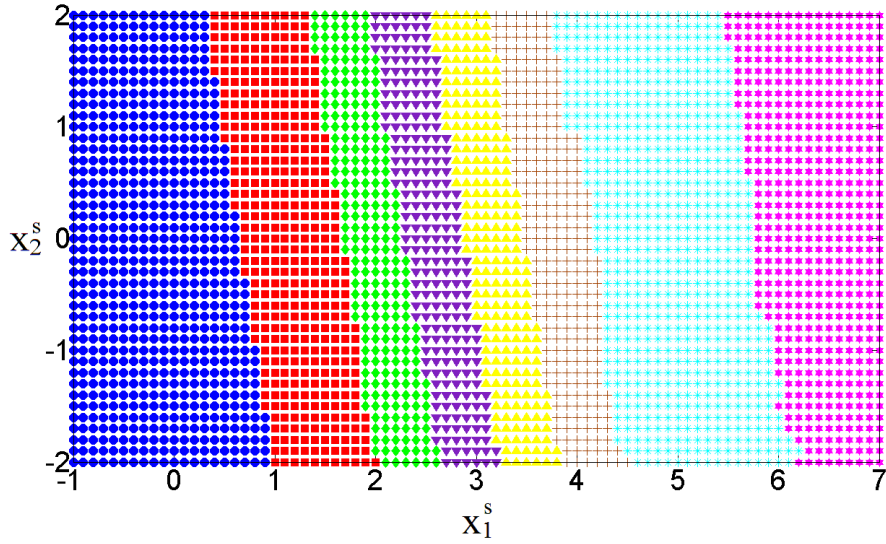


Figure 7: Eight basins of attraction onto plane $x_3^s = 0$ given by the couple system (13) using u_3^m and u_{10}^s .

marked with red dots corresponds to the attractor that covers the scrolls 2, 3 and 4. The colors of the basins of attraction in Figure 7 are related with the colors of the attractors shown in Figure 6.

6. Conclusions

In this work, we have evaluated a mechanism for constructing generalized multistable structures in a pair of unidirectionally coupled systems. Particularly, we have demonstrated the coexistence of multiple attractors in a chaotic multi-scroll system composed by two unidirectionally coupled unstable dissipative subsystems, when they present generalized synchronization. To illustrate this approach, the parameters of the jerky equation were tuned to satisfy the definition of UDS *type I*. The control signal used to generate a desired number of scrolls, was given by a step function. The chaotic behavior of the proposed system is justified by the Lyapunov exponent analysis. We have shown that the number of scroll of the master system is preserved in the slave attractor under our approach when generalized synchronization exits. That is, the master system determines the number of scroll in the slave system by means of synchronization, which induces preservation on the number of scrolls in the slave system. In a particular case, when the number of scrolls in the master system is less than the number of scroll in the slave system, the master-slave configuration results in multiple basins of attraction for the slave. If two identical systems (slave and auxiliary) are synchronized by a third different system (master), which is coupled unidirectionally with the first two, we say that there exists generalized synchronization between the master and the slave systems. But due to the coexistence of multiple attrac-

tors, the two identical systems (slave and auxiliary) can be led to the same or different basin of attraction depending on their initial conditions. This phenomenon is called multimodal generalized synchronization meaning that a functional relation between the master and the slave subsystems exists.

This approach may be further extended to generate: 1) coexistence of chaotic attractors by UDS *Type II* subsystems, and 2) coexistence of chaotic attractors by using other control signals u .

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References

- [1] S. Banerjee, *Coexisting attractors, chaotic saddles, and fractal basins in a power electronic circuit*, *IEEE Trans. Circuits Syst. I* **44**, 847–849, (1997).
- [2] V. N. Chizhevsky, *Coexisting attractors in a CO₂ laser with modulated losses*, *J. Opt. B: Quantum Semiclass. Opt.* **2**, 711–717, (2000).
- [3] A. N. Pisarchik, Yu. O. Barmenkov, and A. V. Kir'yanov, *Experimental demonstration of attractor annihilation in a multistable fiber laser*, *Phys. Rev. E* **68**, 066211-1/8, (2003).

- [4] U. Feudel, C. Grebogi, B. R. Hunt, and J. A. Yorke, *Map with more than 100 coexisting low-period periodic attractors*, *Phys. Rev. E* **54**, 71–81, (1996).
- [5] T. Wilhelm, *The smallest chemical reaction system with bistability*, *BMC Systems Biology* **3**, 90, (2009).
- [6] J. E. Ferrell Jr., *Feedback regulation of opposing enzymes generates robust, all-or-none bistable response*, *Current Biology* **18**, 244–245, (2008).
- [7] D. M. Eagleman, *Visual illusions and neurobiology*, *Nature* **2**, 920–926, (2001).
- [8] D. Cafagna and G. Grassi, *Chaos-based SR flip-flop via Chua’s circuit*, *Int. J. Bif. and Chaos*, **16**, 1521–1526, (2006).
- [9] E. Campos-Cantón, R. Femat, J.G. Barajas-Ramírez, and I. Campos-Cantón, *A multivibrator circuit based on chaos generation*, *Int. J. Bif. and Chaos*, **22**, 1250011, (2012).
- [10] D. Angeli, J. E. Ferrell, Jr., and E. D. Sontag, *Detection of multistability, bifurcations, and hysteresis in a large class of biological positive-feedback systems*, *PNAS*, **101**, 1822–1827, (2004).
- [11] A. Pikovsky, O. Popovych, and Y. Maistrenko, *Resolving clusters in chaotic ensembles of globally coupled identical oscillators*, *Phys. Rev. Lett.* **87**, 044102, (2001).
- [12] G. Osipov and M. Sushchik, *Synchronized clusters and multistability*

- in arrays of oscillators with different natural frequencies, Phys. Rev. E* **58**, 7198–7207, (1998).
- [13] R. Carvalho, B. Fernandez, and R. V. Mendes, *From synchronization to multistability in two coupled quadratic maps, Phys. Lett. A* **285**, 327–338 (2001).
- [14] V. Astakhov, A. Shabunin, W. Uhm, and S. Kim, *Multistability formation and synchronization loss in coupled Hénon maps: Two sides of the single bifurcational mechanism, Phys. Rev. E* **63**, 056212, (2001).
- [15] J. M. Saucedo-Solorio and A. N. Pisarchik, *Dynamics of unidirectionally coupled Hénon maps, Phys. Lett. A* (2011).
- [16] A. N. Pisarchik, R. Jaimes-Reátegui, J. R. Villalobos-Salazar, J. H. Garcia-Lopez, and S. Boccaletti. *Synchronization of chaotic systems with coexisting attractors. Phys. Rev. Lett.* **96**, 244102, (2006).
- [17] A. N. Pisarchik, R. Jaimes-Reátegui, J. H. García López, *Synchronization of multistable systems, Int. J. of Bif. and Chaos*, **18**, 1801-1819 (2008).
- [18] F. R. Ruiz-Oliveras and A. N. Pisarchik, *Synchronization of semiconductor lasers with coexisting attractors, Phys. Rev. E* **79**, 016202, (2009).
- [19] E. Campos-Cantón, I. Campos-Cantón, J. S. González Salas, and F. Cruz Ordaz, *A parameterized family of single-double-triple-scroll chaotic oscillations, Rev. Mex. Fis.* **54**, 411-415, (2008).

- [20] L.O. Chua, M. Komuro, T. Matsumoto, *The double scroll family*, *IEEE Transactions on Circuits and Systems* **33**, 1072–1118, (1986).
- [21] R. N. Madan, *Chua's Circuit: A Paradigm for Chaos*, World Scientific: Singapore, 1993.
- [22] J. A. K. Suykens and J. Vandewalle, *Generation of n-double scrolls (n=1;2;3;4; . . .)*, *IEEE Transactions on Circuits and Systems I* **40**, 861–867, (1993).
- [23] J. A. K. Suykens, A. Huang, and L. O. Chua, *A family of n-scroll attractors from a generalized Chua's circuit*, *International Journal of Electronics and Communications*, **51**, 131–138, (1997).
- [24] E. Campos-Cantón, J. G. Barajas-Ramírez, G. Solís-Perales, and R. Femat, *Multiscroll attractors by switching systems*, *Chaos* **20**, 013116, (2010).
- [25] L.J. Ontañón-García, E. Campos-Cantón, *Preservation of a two-wing Lorenz-like attractor with stable equilibria*, *Journal of the Franklin Institute*, in this issue, <http://dx.doi.org/10.1016/j.jfranklin.2013.04.018>.
- [26] R. L. Devaney, *A First Course in Chaotic Dynamical Systems: Theory and Experiment*, 1st edn. (Westview Press, 1992).
- [27] E. Campos-Cantón, R. Femat and G. Chen, *Attractors generated from switching unstable dissipative systems*, *Chaos*, **22**, 033121, (2012).

- [28] M. E. Yalcin, A.K. Suykens, J. Vandewalle and S. Ozoguz, *Families of scroll grid attractors*, *Int. J. of Bif. and Chaos*, **12**, 23-41 (2002).
- [29] S. Yu, J. Lu, H. Leung and G. Chen, *Design and implementation of n-scroll chaotic attractors from a general jerk circuit*, *IEEE Transactions on Circuits and Systems I* **52**, 1459–1478, (2005).
- [30] H. D. I. Abarbanel, N. F. Rulkov, and M. M. Sushchik, *Generalized synchronization of chaos: the auxiliary system approach*, *Phys. Rev. E* **53**, 4528–4535, (1996).
- [31] A. Wolf, J.B. Swift, H.L. Swinney and J. Vastano, *Determining Lyapunov exponents from a time series*, *Elsevier Science Publishers, Physica D*, 16, pp. 285-317, (1985).