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INFINITE FAMILIES OF HYPERBOLIC PRIME KNOTS WITH ALTERNATION NUMBER 1 AND DEALTERNATING NUMBER n

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Abstract

For each positive integer n we will construct a family of infinitely many hyperbolic prime knots with alternation number 1, dealternating number equal to n, braid index equal to $n+3$ and Turaev genus equal to n.

1 Introduction

After the proof of the Tait flype conjecture on alternating links given by Menasco and Thistlethwaite in [25], it became an important question to ask how a non-alternating link is "close to" alternating links [15]. Moreover, recently Greene [11] and Howie [13], independently, gave a characterization of alternating links. Such a characterization shows that being alternating is a topological property of the knot exterior and not just a property of the diagrams, answering an old question of Ralph Fox "What is an alternating knot? ".

Kawauchi in [15] introduced the concept of alternation number. The *alternation number of a link diagram D* is the minimum number of crossing changes necessary to transform D into some (possibly non-alternating) diagram of an alternating link. The *alternation number of a link L*, denoted $alt(L)$, is the minimum alternation number of any diagram of L. He constructed infinitely many hyperbolic links with Gordian distance far from the set of (possibly, splittable) alternating links in the concordance class of every link.

Another related invariant, the *dealternating number*, was introduced by Adams et al. [4]. The *dealternating number of a link diagram D* is the minimum number of crossing changes necessary to transform D into an alternating diagram. The *dealternating number of a link L*, denoted $dalt(L)$, is the minimum dealternating number of any diagram of L. A link L with dealternating number k is also called *k-almost alternating* and we say that a link is *almost alternating* if it is 1-almost alternating. It is immediate from their definitions that $alt(L) \leq dalt(L)$ for any link L.

In 1978, W. Thurston proved that every knot is either a torus knot, a satellite knot, or a hyperbolic knot and these categories are mutually exclusive. Adams et al. proved that a prime almost alternating knot is either torus knot or hyperbolic knot [4], this generalizes Menasco's proof of the same fact in the case of alternating links [24]. Moreover they also demonstrated that the result does not extend to almost alternating links or to 2-almost alternating knots or links.

Another invariant which is used as an obstruction to the knot being alternating is the *Turaev genus of a knot*: Given a knot diagram D of a knot K, Turaev [30] associated a closed orientable surface embedded in S^3, called the *Turaev surface* (see also [21], [9]), From it the *Turaev genus*, denoted by $g_T(K)$, was defined as the minimal number of the genera of the Turaev surfaces of all diagrams of K [9].

Several authors that have worked with these invariants, for instance Abe and Kishimoto gave examples where the alternation number equals the dealternating number [3]. In particular, they determined dealternating numbers, alternation numbers and Turaev genus for a family of closed positive 3-braids. They also showed that there exist infinitely many positive knots with any dealternating number (or any alternation number) and any braid index.

On the other hand, recently Lowrance demonstrated that there exist families of links for which the difference between certain alternating distances is arbitrarily large [20]. In order to obtain this result he gave three families of knots; the first one denoted $F(W_n)$ consists of iterated Whitehead doubles of the figure-eight knot, the second one $F(\hat{T}(p,q))$ consists of links obtained by changing certain crossings of torus links. The last family $F(T(3,q))$ consists of the $(3,q)$-torus knots. In particular, $F(W_n)$ are satellite knots with alternation number one and dealternating number arbitrarily large, where for each positive integer n there exists a knot K such that $alt(K) = 1$ and $n \leq dalt(K)$.

In addition to the results given in [20] and [3], for each n, we give a infinite family of hyperbolic prime knots such that $alt(K) = 1$ and $dalt(K) = n$, instead of just one knot. Moreover in each family $dalt(K) = g_T(K) = n$.

The content of this paper is organized of the following form: In section 2 we recall definitions and results needed later, in particular the Khovanov width and the relation between the Khovanov width, the Turaev genus and dealternating number. In section 3 we will introduce the family of knots $\mathcal{D}$ and we will prove they have dealternating number n and alternation number 1. Finally, in section 4 we will prove that the elements of $\mathcal{D}$ are prime hyperbolic knots and we will estimate their braid index.

2 Preliminary

A link L is a disjoint union of circles embedded in S^3, a knot K is a link with one component. Let $T(p,q)$ denote the (p,q)-torus link and U the unknot. Throughout this paper, all links are oriented and we will follow the notation used by Lowrance [19].

Khovanov [17] introduced an invariant of links, called the Khovanov homology, which is a bigraded $\mathbb{Z}$-module with homological grading i and polynomial (or Jones) grading j so that $Kh(L) = \bigoplus_{i,j} Kh^{i,j}(L)$ and whose graded Euler characteristic is the Jones polynomial.

The support of $Kh(L)$ lies on a finite number of slope 2 lines with respect to the bigrading. Therefore, it is convenient to define the δ-grading by $\delta = j - 2i$ so that $Kh(L) = \bigoplus_{\delta} Kh^{\delta}(L)$. Also, all the δ-gradings of $Kh(L)$ are either odd or even. Let δ_{min} and δ_{max} be the minimum and the maximum δ-grading, respectively, where $Kh(L)$ is nontrivial. Then $Kh(L)$ is said to be $[\delta_{min}, \delta_{max}]$-thick, and the Khovanov width of L is defined as

$$w_{Kh}(L) = \frac{1}{2}(\delta_{max} - \delta_{min}) + 1.$$

If D is a diagram for L, then denote the Khovanov homology of L by either $Kh(L)$ or $Kh(D)$. Similarly, let $w_{Kh}(L)$ and $w_{Kh}(D)$ equivalently denote the Khovanov width of L. If $\mathbb{F}$ is a field, then let $Kh(L;\mathbb{F})$ denote $Kh(L) \otimes \mathbb{F}$ and $w_{Kh}(L;\mathbb{F})$ denote the width of $Kh(L;\mathbb{F})$.

Let L be an oriented link, and let C be a component of L, denote by l the linking number of C with its complement $L - C$. Let L' be the link L with the orientation of C reversed, and let D be a diagram for L and D' be the diagram D with the component C reversed. Denote the number of negative and positive crossings in D by $neg(D)$ and $pos(D)$, respectively, where the sign of a crossing is as in Figure 1.

Each $Kh^{i,j}(D)$ can be obtained by suitable normalization from a homology group of the following form:

$$Kh^{i,j}(D) := H^{i+neg(D),j-pos(D)+2neg(D)}(D).$$

Since D' is the diagram D with the component C reversed, it follows that

$$pos(D') = pos(D) - 2l \text{ and } neg(D') = neg(D) + 2l.$$

Therefore, we have that for $i, j \in \mathbb{Z}$ there are isomorphisms of groups

$$Kh^{i,j}(D') = Kh^{i+2l,j+6l}(D). \tag{1}$$

Considering the δ-grading and setting $s = neg(D') - neg(D)$ it follows that:

$$Kh^{\delta}(D') = Kh^{\delta+s}(D). \tag{2}$$

Let D_+, D_-, D_v and D_h be diagrams of links that agree outside a neighborhood of a distinguished crossing as in Figure 1 and define $e = neg(D_h) - neg(D_+)$. There are long exact sequences relating the Khovanov homology of each of these links, as indicated in Theorem 2.1. Khovanov [17] implicitly describes these sequences. The graded versions are taken from Rasmussen [28] and Manolescu-Ozsvath [22].

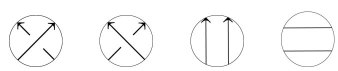


Figure 1: The crossings D_+, D_-, D_v, D_h respectively.

Theorem 2.1. *[17] There are long exact sequences relating the Khovanov homology of D_+, D_-, D_v and D_h as follows:*

$$\cdots Kh^{i-e-1,j-3e-2}(D_h) \to Kh^{i,j}(D_+) \to Kh^{i,j-1}(D_v) \to Kh^{i-e,j-3e-2}(D_h) \to \cdots$$

and

$$\cdots Kh^{i,j+1}(D_v) \to Kh^{i,j}(D_-) \to Kh^{i-e+1,j-3e+2}(D_h) \to Kh^{i+1,j+1}(D_v) \to \cdots .$$

When only the $\delta = j - 2i$ grading is considered, the long exact sequence become

$$\cdots Kh^{\delta-e}(D_h) \xrightarrow{f_+^{\delta-e}} Kh^{\delta}(D_+) \xrightarrow{g_+^{\delta}} Kh^{\delta-1}(D_v) \xrightarrow{h_+^{\delta-1}} Kh^{\delta-e-2}(D_h) \to \cdots$$

and

$$\cdots Kh^{\delta+1}(D_v) \xrightarrow{f_-^{\delta+1}} Kh^{\delta}(D_-) \xrightarrow{g_-^{\delta}} Kh^{\delta-e}(D_h) \xrightarrow{h_-^{\delta-e}} Kh^{\delta-1}(D_v) \to \cdots .$$

Lowrance pointed out that Theorem 2.1 directly implies the following Corollary:

Corollary 2.1. *[19] Let D_+, D_-, D_v and D_h be as in Figure 1. Suppose $Kh(D_v)$ is $[v_{min}, v_{max}]$-thick and $Kh(D_h)$ is $[h_{min}, h_{max}]$-thick. Then $Kh(D_\pm)$ is $[\delta^{\pm}_{min}, \delta^{\pm}_{max}]$-thick, where*

$$\delta^{\pm}_{min} = \begin{cases} min\{v_{min} \pm 1, h_{min} + e\} & \text{if } v_{min} \neq h_{min} + e \pm 1 \\ v_{min} + 1 & \text{if } v_{min} = h_{min} + e \pm 1 \text{ and } h_{\pm}^{v_{min}} \text{ is surjective} \\ v_{min} - 1 & \text{if } v_{min} = h_{min} + e \pm 1 \text{ and } h_{\pm}^{v_{min}} \text{ is not surjective,} \end{cases}$$

and

$$\delta^{\pm}_{max} = \begin{cases} max\{v_{max} \pm 1, h_{max} + e\} & \text{if } v_{max} \neq h_{max} + e \pm 1 \\ v_{max} - 1 & \text{if } v_{max} = h_{max} + e \pm 1 \text{ and } h_{\pm}^{v_{max}} \text{ is injective} \\ v_{max} + 1 & \text{if } v_{max} = h_{max} + e \pm 1 \text{ and } h_{\pm}^{v_{max}} \text{ is not injective.} \end{cases}$$

The following results show the relation between the Khovanov width, the Turaev genus and the dealternating number. Lemma 2.1 was proved by Manturov [23] and Champanerkar, Kofman and Stoltzfus [8], and Corollary 2.2 was proved by Abe and Kishimoto.

Lemma 2.1. *[23][8] Let K be a knot then we have*

$$w_{Kh}(K) - 2 \leq g_T(K).$$

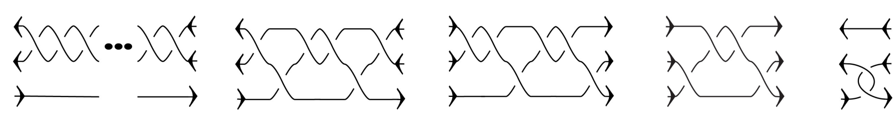
Corollary 2.2. *[3] Let L be a non-split link then we have*

$$g_T(L) \leq dalt(L).$$

In the following section we will describe a family of knots, $\mathcal{D}$, with dealternating number arbitrarily large while the alternation number is one. In order to do this, we will use the Khovanov width of some knots to obtain a lower bound for the delternating number (also for the Turaev genus) of knots in $\mathcal{D}$.

3 Families of Knots with $alt(K) = 1$ and $dalt(K) = n$

Previously, in [12] the author introduced the family of knots $\mathcal{D}$ as an example of knots with alternation number equal to one, also the Alexander polynomial of these knots was obtained. Let us now recall some basic definition and notation taken from [6] to describe the family $\mathcal{D}$. A 3-tangle is a 1-manifold properly embedded in a 3-ball and the set of 3-braids is a subfamily of 3-tangles. Given a 3-braid B, there exists a finite sequence of integers $a_1, \ldots, a_n$, such that B admits a diagram of the form $\mathcal{T}(a_1, \ldots, a_n)$, where $\mathcal{T}(a_1, \ldots, a_n)$ indicates $|a_1|$ crossings of the two uppermost strands, followed by $|a_2|$ crossings of the two lowermost strands, and then $|a_3|$ crossings of the two uppermost strands, and so on, with the following sign convention. For odd i, positive values of the a_i indicate that the uppermost strand passes over the middle strand, whereas for even i, a positive value of a_i indicates that the lowermost strand passes over the middle strand. This notation is illustrated in Figure 2, where examples of 3-braid diagrams which are endowed with an orientation are given.



$\mathcal{T}(2l+1)$ $\quad$ $\mathcal{E}^2$ $\quad$ $\mathcal{T}(2,-1,2,-1)$ $\quad$ $\mathcal{T}(0,-1,2,-1)$ $\quad$ c

Figure 2: Oriented 3-tangles diagrams; $\mathcal{T}(2l+1)$, $\mathcal{E}^2$, $\mathcal{T}(2,-1,2,-1)$, and $\mathcal{T}(0,-1,2,-1)$ are 3-braids while the 3-tangle c is not a 3-braid. The 3-braid $\mathcal{T}(2l+1)$ has $2l+1$ crossings, $\mathcal{E}^2$ is a full twist and $\mathcal{T}(2,-1,2,-1)$ is another diagram for $\mathcal{E}^2$. The orientation of $\mathcal{T}(2,-1,2,-1)$ and $\mathcal{T}(0,-1,2,-1)$ is the usual for 3-braids

Given two 3-tangles diagrams A and B we denote the concatenation of them by $A \cdot B$. It is clear that a diagram $\mathcal{T}(a_1, \ldots, a_n)$ equals the concatenation of diagrams $\mathcal{T}(a_1) \cdot \mathcal{T}(0, a_2) \cdot \ldots \cdot \mathcal{T}(0, a_n)$, if n is even, or $\mathcal{T}(a_1) \cdot \mathcal{T}(0, a_2) \cdot \ldots \cdot \mathcal{T}(a_n)$, if n is odd.

An 3-braid is said to be alternating if, and only if, it admits an alternating diagram, that is, a diagram $\mathcal{T}(a_1, \ldots, a_n)$ such that $a_i \geq 0$ for all $i = 1, 2, \ldots, n$ or $a_i \leq 0$ for all $i = 1, 2, \ldots, n$. As an example, the 3-braid diagrams in Figure 2, except $\mathcal{T}(2l+1)$, are not alternating.

The family $\mathcal{D}$ is defined as follows:

$$\mathcal{D} = \{N(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})\,|\, l+1, n \in \mathbb{N}\},$$

where $\mathcal{T}(2l+1)$, c and $\mathcal{E}^2$ are defined as above, and N is the usual closure of 3-tangles (see Figures 2 and 3).

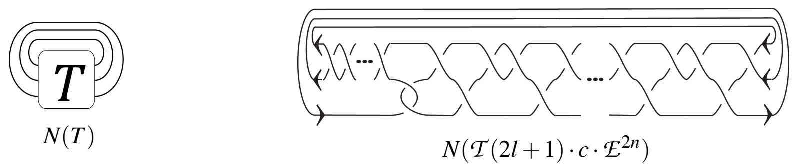


Figure 3: The N closure of a 3-tangle T denoted by $N(T)$ and $N(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})$.

Now by using the Khovanov width of some 3-closed braids it will be proved that if $K = N(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})$ then $dalt(K) = n$. In [19] Lowrance determined the Khovanov width of closed 3-braids based upon Murasugi's classification of closed 3-braids up to conjugation. In particular, by using our notation but with the usual orientation of 3-braids (from left to right) we rewrite Proposition 4.8 part (1) of Lowrance.

Proposition 3.1. *[19] If $n > 0$ and $m \geq 0$, then $Kh(N(\mathcal{T}(m)\cdot \mathcal{E}^{2n}))$ is $[4n+m-3, 6n+m-1]$-thick and $w_{Kh}(N(\mathcal{T}(m)\cdot \mathcal{E}^{2n})) = n+2$.*

Let $\sigma(L)$ be the signature of a link L [27], where the right-hand trefoil knot has signature -2. In [19] Lowrance showed that the results in [22] implies that if L is alternating then $Kh(L)$ is $[-\sigma(L)-1, -\sigma(L)+1]$-thick and $w_{Kh}(L) = 2$. He considered it in a more general situation and we will use it to obtain the Khovanov width of $K \in \mathcal{D}$.

Lemma 3.1. *If $K = N(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})$ then $w_{Kh}(K) = n+2$.*

Proof. Let $K = N(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})$ and D_+ the diagram $N(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})$. Resolve the crossing in the neighborhood marked with a circle to obtain D_v and D_h (see Figure 4) and take $m = 2l+1$. So D_v is a diagram for $N(\mathcal{T}(m)\cdot \mathcal{E}^{2n})$, which is a closed 3-braid without the usual orientation and D_h is a diagram for $N(\mathcal{T}(m)\cdot \mathcal{T}(0,-1)\cdot \mathcal{E}^{2n})$, which has the usual orientation for 3-braids. In order to obtain δ_{min} and δ_{max} of $Kh(D_+)$ we will calculate δ_{min} and δ_{max} of $Kh(D_v)$ and $Kh(D_h)$, respectively.

Let D_v^* be the diagram $N(\mathcal{T}(m)\cdot \mathcal{E}^{2n})$ with the usual orientation for 3-braids, and, by Proposition 3.1, $Kh(D_v^*)$ is $[4n+m-3, 6n+m-1]$-thick. Note that D_v has a component with reverse orientation to D_v^*, also note that $neg(D_v^*) = 0$ and $neg(D_v) = 4n$, then (2) implies that $Kh^{\delta}(D_v) \cong Kh^{\delta+s}(D_v^*)$ therefore $Kh(D_v)$ is $[m-3, 2n+m-1]$-thick.

Resolve D_h at the crossing of $\mathcal{T}(0,-1)$ to obtain D_{h_v} and D_{h_h}. Note that $D_{h_v} = D_v'$ then $Kh(D_{h_v})$ is $[4n+m-3, 6n+m-1]$-thick. Also note that D_{h_h} is a diagram for $T(2,m)$, which is alternating, then $Kh(D_{h_h})$ is $[-\sigma(T(2,m))-1, -\sigma(T(2,m))+1]$-thick where $\sigma(T(2,m)) = -m+1$ and therefore $Kh(D_{h_h})$ is $[m-2, m]$-thick. As $neg(D_{h_h}) - neg(D_h) = 4n$ by Corollary 2.1 the group $Kh(D_h)$ is $[4n+m-2, 6n+m]$-thick.

Now, $e = neg(D_h) - neg(D_+) = -4n$, since $(m-3) \neq (4n+m-2)+e+1$ and $(2n+m-1) \neq (6n+m)+e+1$ Corollary 2.1 implies that $Kh(D_+)$ is $[m-2, 2n+m]$-thick. Hence, $w_{Kh}(N(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})) = n+2$. □

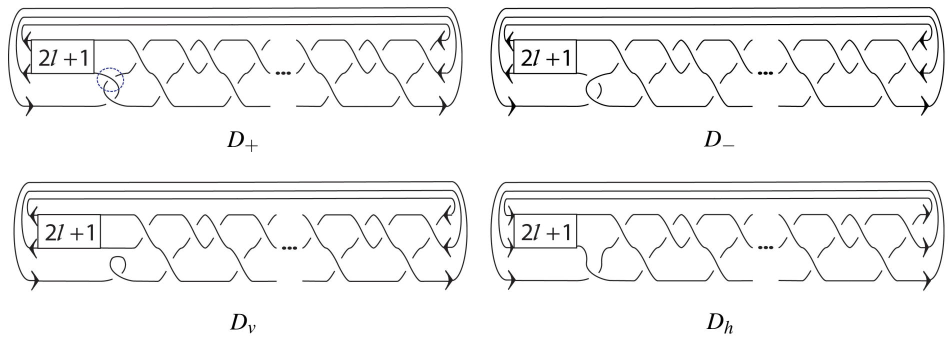


Figure 4: Diagrams D_+, D_-, D_v, and D_h. In D_+ it is marked the neighborhood that differs from D_-,D_v and D_h, moreover D_+ is a diagram for $N(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})$.

Using the provided information, one can prove the following result.

Theorem 3.1. *For all $n \in \mathbb{N}$ there exists an infinite knot family in $\mathcal{D}$, namely $N(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})$ with $l \in \mathbb{N}\cup\{0\}$, such that if $K \in \mathcal{D}$ then $alt(K) = 1$ and $dalt(K) = g_T(K) = n$.*

Proof. Let $n \in \mathbb{N}$ and $K = N(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})$ with $l \in \mathbb{N}\cup\{0\}$. Since $\mathcal{E}^2$ is a full twist then $N(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n}) = N(\mathcal{T}(2l+1)\cdot \mathcal{E}^{2n}\cdot c)$.

Note that:

$$\begin{aligned} \mathcal{E}^{2n} &= (\mathcal{T}(1,-1,2,-1,1))^n \\ &= (\mathcal{T}(2,-1,2,-1))^n \\ &= (\mathcal{T}(2)\cdot\mathcal{T}(0,-1,2,-1))^n \\ &= (\mathcal{T}(2))^n\cdot(\mathcal{T}(0,-1,2,-1))^n \\ &= \mathcal{T}(2n)\cdot(\mathcal{T}(0,-1,2,-1))^n. \end{aligned}$$

Therefore,

$$\begin{aligned} N(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2n}) &= N(\mathcal{T}(2l+1)\cdot\mathcal{T}(2n)\cdot(\mathcal{T}(0,-1,2,-1))^n\cdot c) \\ &= N(\mathcal{T}(2l+1+2n)\cdot(\mathcal{T}(0,-1,2,-1))^n\cdot c), \end{aligned}$$

where $\mathcal{T}(2l+1+2n)$ is alternating and $(\mathcal{T}(0,-1,2,-1))^n$ is non-alternating.

The diagram $N(\mathcal{T}(2l+1+2n)\cdot\mathcal{T}(0,-1,2,-1)^n\cdot c)$ can be rewritten as diagram D, see Figure 5. The n crossings that we will change in order to obtain an alternating diagram are marked in D. Since D is another diagram for K, it follows that

$$dalt(K) \leq n. \tag{3}$$

On the other hand, Lemma 3.1 states that $w_{Kh}(K) = n+2$ and, by Lemma 2.1 and Corollary 2.2, we have the inequalities:

$$n = w_{Kh}(K) - 2 \leq g_T(K) \leq dalt(K). \tag{4}$$

From inequalities (3) and (4) we may conclude that $g_T(K) = dalt(K) = n$.

Now, since that $dalt(K) = n$ it follows that K is not alternating and by definition $alt(K) \geq 1$. Furthermore, the knot K has the diagram D_+, which by one crossing change is transformed into D_-, see diagrams in Figure 4. Since D_- is a diagram of an alternating knot, it follows that $alt(K) = 1$. □

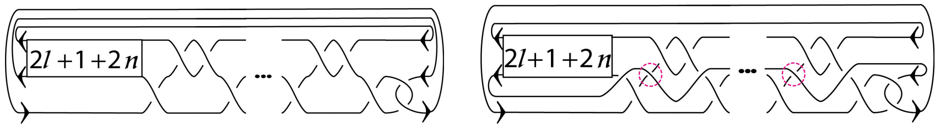


Figure 5: Two equivalent diagrams: The first one is $N(\mathcal{T}(2l+1+2n)\cdot\mathcal{T}(0,-1,2,-1)^n\cdot c)$ and the second one is the diagram D.

Now for each integer n we have a knot family with dealternating number and Turaev genus equal to n and alternation number equal to 1. In the following section we will prove that these knots are hyperbolic prime knots and also we will obtain their braid index.

4 Hyperbolic Prime Knots

We will show that the knots in $\mathcal{D}$ are hyperbolic prime knots. Let $br(K)$ denote the bridge number of the knot K.

Lemma 4.1. *If K is a knot in $\mathcal{D}$ then $br(K) = 3$.*

Proof. The knot K has a diagram with three bridges, and so $br(K) \leq 3$. Suppose $br(K) \leq 2$, it follows that K is alternating. However, since $K \in \mathcal{D}$ it implies that K is non-alternating then $br(K) = 3$. □

Lemma 4.2. *If K is a knot in $\mathcal{D}$ then it is prime.*

Proof. Suppose that K is non-prime then K is the connected sum of non-trivial knots K_1 and K_2. In [29] it is proved that $br(K) = br(K_1) + br(K_2) - 1$. By Lemma 4.1 we have $br(K) = 3$, thus as K_1 and K_2 are non-trivial then $br(K_1) = br(K_1) = 2$ and therefore they are alternating knots. Further, since the connected sum of alternating knots is an alternating knot it implies that $K \notin \mathcal{D}$. Hence K is a prime knot. □

The Γ-polynomial, which was defined in [16], is the common zero-th coefficient polynomial of both; the HOMFLYPT polynomial, $P(L;y,z) \in \mathbb{Z}[y^{\pm 1}, z^{\pm 1}]$, and the Kauffman polynomial, $F(L;a,b) \in Z[a^{\pm 1}, b^{\pm 1}]$. The Γ-polynomial $\Gamma(K) \in \mathbb{Z}[x^{\pm 1}]$ of a knot K is calculated by the following formulas:

- $\Gamma(U) = 1$ where U is the unknot,
- $-x\Gamma(K_+) + \Gamma(K_-) = (1-x)x^{-lk(K' \cup K'')}\Gamma(K')\Gamma(K'')$,

where $(K_+; K_-; K' \cup K'')$ is a skein triple such that K_+, K_-, K', K'' are knots and $lk(K' \cup K'')$ is the linking number of K' and K'' (see Figure 1 where $D_v = K' \cup K''$).

The y-*span* $P(L;y,z)$ is the difference between the maximum and the minimum degrees of the $P(L;y,z)$ polynomial in the variable y , and *span* $\Gamma(L)$ is the difference between the maximum and the minimum degrees of the $\Gamma(L)$ polynomial; and the *braid index*, $b(L)$, of a link L is the minimal number of strands of any braid whose closure is equivalent to L. The Morton-Franks-Williams inequality, which was proved in [26] and [10], relates the y-span $P(L;y,z)$ and the braid index as follows:

$$\frac{1}{2} y\text{-}span\, P(L;y,z) + 1 \leq b(L) \tag{5}$$

In particular for the Γ-polynomial:

$$span\,\Gamma(L)+1 \leq b(L). \tag{6}$$

In order to obtain the braid index for the knots in $\mathcal{D}$ we calculate the following.

Proposition 4.1. *We have* $\Gamma(T(2,2r+1)) = (r+1)x^{-r} - rx^{-(r+1)}$, *where* $r \in \mathbb{N}$.

Proof. We will prove by using induction on l, for $r = 1$

$$\begin{aligned}\Gamma(T(2,3)) \quad &= x^{-1}\Gamma(U) - x^{-1}(1-x)x^{-lk(U\cup U)}\Gamma(U)\Gamma(U)\\ &= x^{-1} - x^{-1}(1-x)x^{-1}\\ &= 2x^{-1} - x^{-2}.\end{aligned}$$

Let us see the case $r = n+1$; by induction hypothesis we have

$$\begin{aligned}\Gamma(T(2,2(n+1)+1)) \quad &= x^{-1}\Gamma(T(2,2l+1)) - x^{-1}(1-x)x^{-lk(U\cup U)}\Gamma(U)\Gamma(U)\\ &= x^{-1}((n+1)x^{-n} - nx^{-(n+1)}) - x^{-1}(1-x)x^{-(n+1)}\\ &= (n+2)x^{-(n+1)} - (n+1)x^{-(n+2)}.\end{aligned}$$

□

Yamada [31] gave an upper bound for the braid index.

Lemma 4.3. *[31] Let L be a link and D a diagram of L and let* $o(D)$ *the number of Seifert circles of D. Then we have* $b(L) \leq o(D)$.

According to the results on section:

Lemma 4.4. *If* $K = N(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})$ *then* $b(K) = n+3$ *for all* $l,n \in \mathbb{N}$.

Proof. Let $K = N(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})$, by using the formulas to obtain the Γ-polynomial and Proposition 4.1 we have the following:

$$\begin{aligned}&\Gamma(N(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n}))\\ &= x^{-1}\Gamma(T(2,2l+1)) - x^{-1}(1-x)x^{-lk(T(2,2l+1+2n)\cup U)}\Gamma(T(2,2l+1+2n))\Gamma(U)\\ &= x^{-1}((l+1)x^{-l} - lx^{-(l+1)}) - x^{-1}(1-x)x^{-(-2n)}((l+n+1)x^{-(l+n)} - (l+n)x^{-(l+n+1)})\\ &= -lx^{-(l+2)} + (l+1)x^{-(l+1)} + (l+n)x^{-l+n-2} - (2l+2n)+1)x^{-l+n-1} + (l+n+1)x^{-l+n}.\end{aligned}$$

Then, if $l \neq 0$ the $span\,\Gamma(K) = n+2$, and by inequality (6) we have that $b(K) \geq n+3$. By Lemma 4.3 and diagrams in Figure 6 we have $b(K) \leq n+3$, therefore $b(K) = n+3$. □

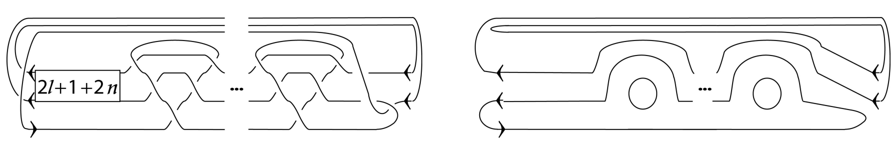


Figure 6: Another diagram of $N(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})$ and their Seifert circles.

To summarize, we have the following result.

Theorem 4.1. *If K is a knot in* $\mathcal{D}$ *then it is hyperbolic.*

Proof. Let K be a knot in $\mathcal{D}$. Lemmas 4.2 and 4.1 imply that K is prime and has bridge index 3. It is known that the prime knots with $br(K) \leq 3$ that are not torus knots, are hyperbolic [14], then K is torus knot or hyperbolic. Furthermore, by Theorem 3.1 we have that K has alternation number one and the only torus knots with alternation number one are 8_{19} and 10_{124} [1]. However, Lemma 4.4 implies that if $l \neq 0$ then $4 \leq b(K)$ and it is known (see [7]) that $b(8_{19}) = b(10_{124}) = 3$ this implies that K is different to both 8_{19} and 10_{124}. Therefore K is hyperbolic. In the case $l = 0$ and $n \geq 2$, by Theorem 3.1 the dealternating number is greater than 1 and as $dalt(8_{19}) = dalt(10_{124}) = 1$ (see [4]) then K is hyperbolic. Finally, in the case when $l = 0$ and $n = 1$ it is easy to see that K is different to both 8_{19} and 10_{124} and therefore K is hyperbolic. □

5 Conclusions

In this paper we introduced the family $\mathcal{D} = \{N(\mathcal{T}(2l+1) \cdot c \cdot \mathcal{E}^{2n}) \,|\, l+1, n \in \mathbb{N}\}$ which is an infinite family of prime hyperbolic knots with alternation number one, dealternating number n and Turaev genus n. Furthermore we obtained the Γ-polynomial for $K \in \mathcal{D}$ and we used it to calculate the braid index of K. We obtained that for any knot $K \in \mathcal{D}$ with $l \in \mathbb{N}$ the Morton-Franks-Williams inequality is sharp, in particular the Morton-Franks-Williams inequality applied to Γ-polynomial is sharp too.

The family $\mathcal{D}$ was constructed by using a family of 3-braids and the specific 3-tangle c. In this context, what will it happen with the alternation and the dealternating numbers if the family of 3-braids is changed by another one with suitable orientation? Or what will it happen if the 3-tangle c is changed for another one?

On the other hand, it is known that the Turaev genus is closely related to algebraic invariants. For a knot K, Bae and Morton [5] and separately Dasbach et al. [9] showed that $g_T(K) \leq c(K) - spanV_K(t)$, where $c(K)$ is the crossing number and $V_K(t)$ the Jones polynomial of the knot K. Lickorish and Thistlethwaite [18] introduced the concept of an adequate link, which is a generalization of an alternating link and Abe in [2] showed that if $g_T(K) < c(K) - spanV_K(t)$ then L is not adequate. For some knots in $\mathcal{D}$ the inequality holds but, does the inequality hold for any knot $K \in D$? Will it be true that for each integer n the knots $N(\mathcal{T}(2l+1) \cdot c \cdot \mathcal{E}^{2n})$ satisfy that the difference between $c(K) - spanV_K(t)$ and $g_T(K)$ is greater than n?

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