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POSGRADO EN CONTROL Y SISTEMAS DINÁMICOS

# Invariant densities and phase transition phenomenon in random maps in the interval

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# Contenido

# Notation

| | |
|---|---|
| $\mathbb{C}$ | Complex numbers set |
| $\mathbb{R}$ | Real numbers set |
| $\mathbb{N}$ | Natural numbers set |
| $\mathbb{Z}$ | Integer numbers set |
| $\varnothing$ | Empty set |
| $A^c$ | Complement set of $A$ |
| $T\|_A$ | Restrictions of T to the set $A$ |
| $T^{-1}(A)$ | Pre-image of the set $A$ under the transformation $T$ |
| $T'(x)$ | Derivative of $T$ with respect to $x$ |
| $\mathcal{C}^n$ | Set of $n-$times differentiable functions |
| $T^n$ | $n$-th iteration of the transformation $T$ |
| $\mu \ll \nu$ | Measure $\mu$ being absolutely continuous with respect to the measure $\nu$ |
| $\|\cdot\|$ | Absolute value |
| $\|\|\cdot\|\|$ | Norm $L^1$ |
| $\|\|\cdot\|\|_p$ | Norm $L^p$ |

## Disclaimer about gender-neutral language

Whenever we cite the work of an author throughout this thesis, we will deliberately refer to them by utilizing the gender-neutral pronoun "they" in order to avoid using a pronoun they do not identify with.

# Resumen


Invariant densities, intermittency, and phase-transition phenomenon in random maps in the interval.

**Palabras clave:** transición de fase, mapeos del intervalo, medida invariante absolutamente continua (m.i.a.c.), operadores de Perron-Frobenius, decaimiento de correlaciones, exponente de Lyapunov, mapeos conectados.

En esta tesis proponemos una definición del fenómeno de transición de fase con respecto a un parámetro, en el sentido de la existencia o no existencia de una medida invariante absolutamente continua respecto a la medida de Lebesgue, en mapeos en el intervalo, y mostramos algunos ejemplos deterministas muy conocidos en los que sucede. Más adelante, definimos una clase de mapeos aleatorios en el intervalo que están constituidos, cada uno, por una colección de cardinalidad no numerable de mapeos no expansivos, y de otros estrictamente expansivos, cuya probabilidad de incidencia posee dependencia continua en un parámetro $\gamma$. A partir de esta interacción entre mapeos expansivos y contractivos surgen condiciones para las cuales se presentan regímenes en promedio expansivos o contractivos. Para esta clase de sistemas mostramos evidencia numérica donde se observa a través de densidades empíricas, exponentes de Lyapunov, coeficientes de autocorrelación, entre otras, cómo cambia la dinámica resultante de la interacción entre mapeos no expansivos y expansivos, y da lugar al fenómeno de transición de fase, en el sentido de la no existencia a la existencia de una m.i.a.c. en función del parámetro característico del sistema. Después presentamos un resultado que establece las condiciones necesarias y suficientes para la existencia de una m.i.a.c. en esta clase de mapeos aleatorios, así como un procedimiento para encontrar el valor crítico del parámetro que caracteriza la incidencia este fenómeno.

# Abstract


Invariant densities, intermittency, and phase-transition phenomenon in random maps in the interval.



**Keywords:** phase transition, maps in the interval, absolutely continuous invariant measure (a.c.i.m.), Perron-Frobenius operators, decay of correlations, Lyapunov exponent, connected maps.



In this thesis, we propose a definition of the phase transition phenomenon with respect to a parameter, in the sense of non-existence or existence of an absolutely continuous invariant measure with respect to the Lebesgue measure, in maps in the interval, and we show some well-known deterministic examples of its occurrence. Later on, we define a class of random maps in the interval, which are individually conformed by a collection of a continuum non-expansive of maps, as well as strictly expansive maps, whose weighting probabilities has a continuous dependence on a parameter $\gamma$. From this interplay between expanding and contracting dynamics, the conditions for the arising of expanding in mean or contracting in mean regimes take place. For this class of systems we present numerical evidence, where the reader can observe by means of empirical densities, Lyapunov exponents, autocorrelation coefficient, among others; how the resulting dynamics of the interplay between non-expansive and strictly expansive maps changes, and gives rise to the phenomenon of phase transition, in the sense of the non-existence to the existence of an a.c.i.m. as a function of the characteristic parameter of the system. Next, we present a result that establishes the sufficient and necessary conditions for the existence of an a.c.i.m. in this class of random maps, as well as a method for computing the critical value of the parameter that characterizes the incidence of this phenomenon.

# Contents

# Chapter 1

# Introduction

A dynamical system is a mathematical representation of the movement, or change present in a kinetic physical entity during a window of time. For its analysis, it is necessary to study how will it behave under a certain rule of evolution, and broadly speaking, we can identify two main outcomes for its asymptotic state: it will stop or "stabilize" in a determined regular behavior (a fixed point, periodic orbits or a limit cycle), or it will evolve in such a way that for increasingly longer time frames it is erratic or virtually unpredictable (due to lack of computational power or time). For the case where it does not stabilizes, and when it is required to know its typical behavior (or if there is one), we can study the evolution of a typical initial condition for very large times; or we can study the evolution of a "large" set of initial conditions, that is, the study of the statistical behavior of its orbits. In the case of ergodic systems, these two approaches are equivalent. This is a fundamental result in measure theory, referred to as the Birkhoff's pointwise ergodic theorem (we go into further details ahead in Chapter 2.6).

In this thesis, our main focus are the maps in the interval, specifically, discrete time transformations of the unit interval $[0, 1]$, which is our state space. The main topic that is discussed here is that of under what conditions in general, the orbits of a map will typically and asymptotically (or for almost every initial condition) distribute according to a certain distribution (or probability density) function. The study of these dynamical systems implies the study of the evolution of an initial density, which is attained by defining the action of an operator in the set or space of density functions, and this resulting progression or sequence of functions represents the dynamics in the state space. Therefore, to establish the existence of an invariant measure with respect to the dynamics is equivalent to find a fixed point for this action [6]. This action is defined as an operator on a space of functions, known as the Perron-Frobenius operator. The existence of this invariant measure sets the starting point for studying the statistical properties of the dynamical systems.

One of the first works regarding the existence of a stationary density associated to the dynamics of a map in the interval, is the result published in [2], in which the existence of an absolutely continuous (with respect to the Lebesgue measure) invariant measure for a class of piecewise expansive maps in the interval is proven. The central idea around this is to prove that the Perron-Frobenius operator associated to the dynamical system has a fixed point in a determined subspace of the continuous functions, which is the bounded variation functions space (see Chapter 2.4 for further details), given that in this subspace the operator is contractive, and therefore it is possible to prove this utilizing nowadays-classical techniques related to fixed point theorems.

The expansive maps in the interval have posed a big topic of interest regarding the study of dynamical systems, particularly about their statistical properties, due to the fact that these systems exhibit a wide variety of behaviors, which arises from dynamics represented through relatively simple equations. This is specially useful for the modeling and analysis of complex physical phenomena, for instance, the modeling the phenomenon of turbulence in fluids dynamics [1]. A nowadays classical example of almost-everywhere expansive dynamics within the interval $[0,1]$ (except for one point) are the Manneville-Pomeau maps. This family of maps was defined by Manneville and Pomeau as a relatively simple model for the study of the phenomenon of intermittency [1], and it has a continuous dependency on a real-valued parameter, i.e., it is defined for the values of its characteristic parameter $\alpha \geq 0$, and it is defined by the following expression:

$$T_\alpha(x) = \begin{cases} x(1+2^\alpha x^\alpha), & 0 \leq x \leq 0.5 \\ 2x-1, & 0.5 \leq x \leq 1. \end{cases}$$

This system has an indifferent fixed point at $x = 0$, and the trajectories get "stuck" near this point, meaning they tend to perform long excursions due to the weak expansiveness around this region of the state space.

The change of the statistical behavior of this system as a function of $\alpha$ undergoes different dynamical regimes. For example, it is known that it has an absolutely continuous invariant measure for $0 < \alpha < 1$, whose density is a power-law function [3]; a constant function corresponding to an uniform distribution for $\alpha = 0$, and it has a $\delta$-distribution (i.e., the orbit accumulates in one point) for some initial conditions like $x = 1$ and $x = 0$, and it is also the limit distribution as $\alpha \to \infty$. In this sense, we say that a phase transition phenomenon is present here (see Chapter 3 for the definition and further discussion), since the system experiences a major change in the asymptotic distribution of its orbits.

As we will see in Chapter 2.7, for systems that exhibit a non-stable behavior in which we cannot specify the exact point of landing of an initial condition for sufficiently large times, we work with measures that quantify the incidence of trajectories along every region in the state space, and allows us to establish a certain probability of re-entry in a region of the state space. Thus, it is clear that we are dealing with uncertainty. And although the orbits of these systems can be understood and mathematically treated as independent random variables (mostly when speaking of mixing dynamics (see Chapter 2.8)), they are not, since every state of the trajectory is directly dependent on the previous one. However, the statistical behaviors that we mentioned before (specifically the existence of an a.c.i.m.), that are main subjects for research in dynamical systems theory, can be retrieved in a way that entails the action of well-defined independent random variables (see Chapter 4.3) involving relatively simple expressions, through the study of random dynamical systems. This constitutes an advantage in the sense that brings the possibility to transfer a major portion of the difficulty of analysis to the probabilistic realm, which is by no means simpler, but offers a whole other set of tools [37, 38].

The research around the existence of an a.c.i.m. in random dynamical systems is relatively new, but rather extensive. Concerning the random maps in the interval, it includes semi-Markov piecewise linear maps with probability dependent probabilities [4, 5], generalizations to skew and pseudo-skew products [9, 14, 15], random $\beta-$transformations [29], systems equipped with a continuum of transformation [16, 20] and even goes further around the question of the existence of S.R.B. measures [19] (see Chapetr 4.3.5) and the development of methods for computing the set of attainable densities according to the weighting probabilities of the dynamics ([5]). In this thesis, our work is focused in exploring under what conditions a class of random maps in the interval can exhibit the phenomenon of phase transition, in the sense of non-existence to existence of an a.c.i.m. (or viceversa). Here, we recall two families of random maps we proposed in a previous work [17], with the goal of obtaining relatively simple models characterized by a single real-valued parameter that affects the probability component of the system, that assimilate some of the previously mentioned statistical properties, like the existence of an a.c.i.m, a phase transition or intermittency. The main contribution in this proposal lies in defining a clsss of systems that allow non-strictly expansive components, and that can exhibit a phase transition phenomenon in the sense we brought up earlier on (i.e., having an absolutely continuous invariant measure). We further study this phenomenon in a class of dynamical systems knows as connected maps, for which we provide the formal definition later on. The remainder of this thesis is organized as follows.

In Chapter 2 we introduce the fundamental concepts related to the basic tools of analysis involved in our work. We retrieve some theory about the function spaces $L^p$ and of bounded variation, as well as the definition of the Perron-Frobenius operator and concepts related to the statistical properties of dynamical systems, like ergodicity, mixing, decay of correlations and Lyapunov exponents. Furthermore, and we briefly discuss the nowadays classic Lasota-Yorke theorem.

Next, in Chapter 3 we give the formal definition of the phase transition phenomenon that we are proposing, as well as a definition for the critical value involved in the occurrence of this phenomenon in maps in the interval. Then, we cite a few examples of deterministic maps in the interval that present the phase transition phenomenon and provide brief discussions about the nature of the behavior observed in their dynamics.

In Chapter 4 we establish the context about random dynamics theory in which our investigation lies. We provide a definition of random maps in the interval along with the general expression for the random Perron-Frobenius operator associated to it. We also present the zero-one probability laws of Borel-Cantelli and Hewitt-Savage, and gather some of the most important results concerning the existence of an a.c.i.m. in random dynamics in one-dimensional systems. At the end of this chapter we discuss the numerical data we obtained from simulations of the systems we proposed that helped us establish the questions we give an answer to here.

In Chapter 5 we define the class of random maps in the interval for which later, we provide sufficient conditions for the existence of a.c.i.m., with an approach based on the Lasota-Yorke's theorem. Later, from a probabilistic approach we establish a set of sufficient conditions for this class of random maps to have an a.c.i.m., as well as another sufficient and necessary conditions to not have it. These two joint results conform our main theorem, in which we prove that this class of random maps in the interval experiences a phase transition phenomenon in the sense of non-existence to existence of an a.c.i.m. (or viceversa).

Later on, we describe and briefly discuss an original work in progress in Chapter 6, which is the setting for a new type of deterministic dynamical system, in which we have found numerical evidence of the occurrence of the phenomenon of phase transition with respect to a parameter. This setting aims to explore the resulting behavior derived form the interaction of two dynamical systems with mutually contrasting behavior.

Finally, we address in Chapter 7 some possible perspectives for the future of our work, considering the limitations that our theoretical results have. We also discuss briefly an original work in progress on a deterministic system equipped with a parameter $\alpha$. We found that for an infinite but countable values of $\alpha$, all the points in its state space are part of an $n-$cycle dynamics.

After the concluding remarks, there is a list of our publications, participation in congresses, preprints and attendance to math schools.

# Chapter 2

# Preliminaries

A dynamical system is a pair $(X,T)$, where $X$ is the state space, equipped with a transformation $T$ of the space $X$ on itself, that acts as a temporal evolution. From now on, it will be denoted by the couple $(X,T)$. Here, $X$ will be a compact metric space.

In this thesis we will restrict ourselves to discrete dynamics, that is, we will consider a map with evolution in discrete time. Let $T : X \to X$ be a function that determines the new state of the system in one time-step. If $x_0 \in X$ is the state of the system at time zero, then the state at the first time-step is $x_1 = T(x_0)$, and more generally, the state at the time-step $n$ is recursively given by iteration $x_n = T(x_{n-1})$. This often is written as $x_n = T^n(x_0)$ with $T^n = T \circ T \circ ... \circ T$ ($n$ times), where $\circ$ is the symbol for function composition: $(T \circ g)(x) \equiv T(g(x))$. The sequence $\{T^n(x_0)\}_{n \in \mathbb{Z}^+}$ is known as the *forward orbit of the trajectory of the initial condition* $x_0$; and if the inverse function of $T$ exists, then the orbit is as well defined for negative $n$. For further details, on dynamical systems, see [7].

## 2.1 Maps in the interval

The focus of our work is centered in dynamical systems defined on the interval $X = [0,1]$, whose image is on itself. Despite their relative simplicity, the maps in the interval do exhibit a wide variety of behaviors. Some of these behaviors of interest are intermittency, period-doubling regimes and chaos, and have been widely studied through, now iconic examples, as the logistic map, the Manneville-Pomeau maps, or the tent maps. A basic, general class of dynamics in the interval, which is very relevant in research, are the *expansive maps in the interval.*

**Definition 2.1** (expansive maps in the interval)**.** *Let* $X = [0,1]$*, and* $0 = a_0 < a_1 < \cdots < a_K = 1$ *be a sequence of numbers in the interval* $a_i \in [0,1]$ *For every* $i = 0, \ldots, K$*,* *such that they conform a partition of* $[0,1]$. *The transformation* $T:[0,1]\to[0,1]$ *is said to be a expansive map in the interval if it satisfies:*

1. *The map* $T$ *restricted to each element of the partition,* $T_i := T|_{[a_i,a_{i+1}]}$ *is such that* $T_i \in \mathcal{C}^2$.

2. *There exists a constant* $\Lambda > 1$ *such that* $\inf_{x\in[0,1]}|T'(x)| \geq \Lambda > 1$.

One of the classic and simplest examples of deterministic maps in the interval is the doubling map, or , (Figure 2.1):

**Example 2.1.** *The map in the interval* $x_{n+1} = 2x_n \pmod 1$ *is defined as*

$$T(x) = 2x \pmod 1 := \begin{cases} 2x, & 0 \leq x \leq 0.5 \\ 2x-1, & 0.5 < x \leq 1. \end{cases} \tag{2.1}$$

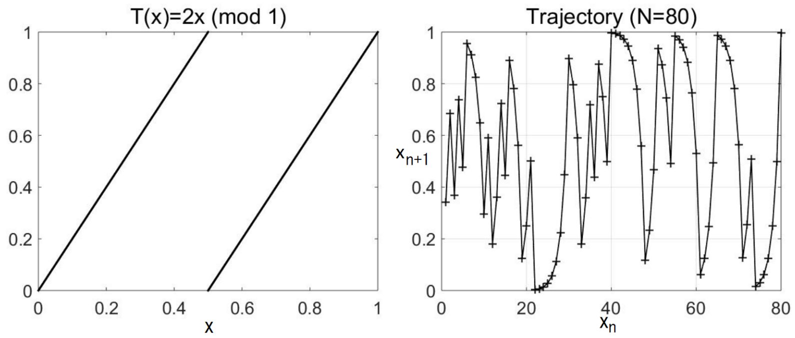



Figure 2.1: Transformation in the interval $T(x) = 2x \pmod 1$ (left) and a sample of eighty iterations of its typical trajectory (right).

*As we can see, the map in this example is piecewise* $\mathcal{C}^2$ *in the partition given by* $\{a_0 = 0, a_1 = 0.5, a_2 = 1\}$ *which defines the intervals* $\{[0,0.5],(0.5,1]\}$. *And since it is a linear map, the constant* $\Lambda = 2$.

**Example 2.2.** *Given the rational map* (Figure 2.2)

$$T(x) = \begin{cases} 5\left(1-\frac{2}{x+2}\right), & 0 \leq x \leq 0.5 \\ 3\left(1-\frac{1}{x+0.5}\right), & 0.5 < x \leq 1, \end{cases} \tag{2.2}$$

*as the previous example, it is piecewise* $\mathcal{C}^2$ *in the partition given by* $\{a_0 = 0, a_1 = 0.5, a_2 = 1\}$, *and from its derivative, we can determine that the constant* $\Lambda$ *equals* $\frac{4}{3}$.

The maps in these examples show an "erratic" or "unpredictable" behavior of the trajectories for almost any initial condition. Nonetheless, for Example 2.1, it

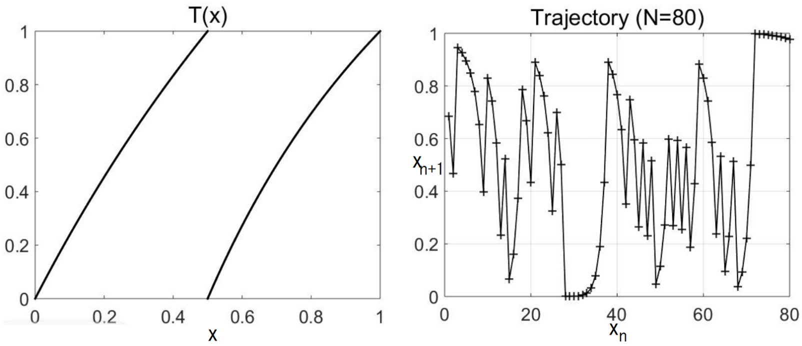



Figure 2.2: Transformation in the interval from the example 2.2 (left) and a sample of eighty iteration of its typical trajectory (right).

is known that almost every orbit (with respect to the uniform distribution) under the action of this system becomes uniformly distributed all over the interval $[0,1]$ (see [6], Chapter 3), which is a characteristic of the uniformly distributed stochastic processes. As for Example 2.2, it is a completely different distribution. Notice that for these previous examples, every point has two pre-images, therefore the maps are not invertible.

These two systems in Examples 2.1 and 2.2 exhibit the type of unstable behavior we mentioned earlier, and these dynamics are representative of the fact that studying the behavior of a particular orbit can be a hefty task if one is trying to predict the exact position of the trajectory at a given time (mostly large times). Yet, it is only natural to ask if there exists a big-scale behavior, intrinsic to the dynamics. In other words, if an inherent distribution of the points in the orbit arises when $n \to \infty$. We will give further details below, but we need first to provide some preliminary definitions.

One way to explore empirically the statistical or probabilistic properties of these dynamical systems is through the construction of histograms, made by the computational calculation of a large number of iterations of the system. These histograms can help to visualize the asymptotic frequency of the trajectories landing in different regions of the state space [8]. This is implemented in the following way. The state space $[0,1]$ is divided in $j$ disjoint discrete intervals such that describe a partition and the $i$-th interval is (the extreme point $x = 1$ is omitted):

$$\left[\frac{i-1}{j}, \frac{i}{j}\right) \qquad i = 1, ..., j$$

Then, given a initial state of the system $x_0$ the orbit is computed for large $N$

$$x_0, T(x_0), T^2(x_0), ..., T^N(x_0),$$

where $N$ represents the size of the sample utilized for plotting it, with $N \gg 2j$ [8]. A histogram computed for a partition of the space state sufficiently fine and a sufficiently large number of iterations, can reveal the approximate shape that the invariant density function associated to a dynamical system asymptotically takes, whenever it possesses one. Considering the doubling map in Example (2.2), we can observe in Figure 2.3 how the asymptotic frequency eventually shapes itself into the invariant density function associated with the map in the interval, which, as demonstrated later on, it actually is.

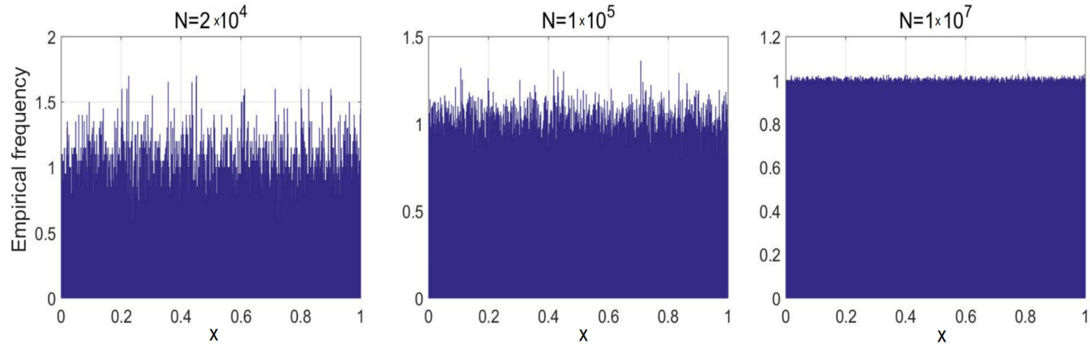



Figure 2.3: Histograms for the transformation $T(x) = 2x \pmod 1$, for $2 \times 10^4$ (left), $1 \times 10^5$ (center) and $1 \times 10^7$ iterations, where the interval $[0,1]$ being divided into $j$ =1000 bins. These, gradually approximate a uniform density function which is known to be invariant under this system (see [6], Chapter 3).

Due to the fact that in a histogram the state space is divided into $j$ equal intervals, and the height of each bar quantifies the relative incidence of the orbit in that interval; the empirical measure $\hat{m}(A)$ of a subset $A$ of $[0,1]$ is given by:

$$\hat{m}\left(A(x)\right) = \frac{1}{j}\sum_{i=1}^{j}\hat{h}(x)\chi_{\Delta_i}(x),$$

where $\hat{h}(x)$ is the height of each bar, $\chi_{\Delta_i}(x)$ is the indicator function corresponding to the subdivision $\Delta_i$, and by dividing by $j$ the histogram is normalized for the whole state space to have a total sum of 1. The indicator function takes value of 1 for all $x \in \Delta_i$ and 0 otherwise. All the histograms shown in this thesis are plotted considering a sample size of $N = 1 \times 10^7$ and a value of $j = 1000$ for uniformly sized bins.

As we can see, if we had an infinite number of segments $\Delta_i$, the empirical measure given by a histogram would be a probability measure. We provide further details about measures and space measures in the following section.

## 2.2 Measure and measure spaces

By convenience, in the following, we will include some general concepts about measure, measure spaces and measurable dynamical systems. This content is a compilation about the most relevant topics necessary to make this thesis self-contained. This preliminary theory is standard and one can look deeper on any particular topic, in [6], [7], [8], [37] or [28] for instance.

**Definition 2.2** ($\sigma$-algebra)**.** *Let $X$ be an arbitrary non-empty set. One class of subsets $\Sigma$ of $X$ is called* $\sigma$-algebra *on $X$ if it satisfies the following conditions:*

1. *If $A \in \Sigma$ implies $A^c \in \Sigma$, where $A^c \equiv \{x \in X : x \notin A\}$ is the complement of $A$ in $X$.*
2. *If $A_n \in \Sigma, n = 1, 2, \ldots$ implies that the union $\bigcup_{n=1}^{\infty} A_n \in \Sigma$.*
3. *$X \in \Sigma$.*

**Definition 2.3** (Measure)**.** *A non-negative function of sets $\mu$ to real values (including possibly $\infty$) defined on a $\sigma$-algebra $\Sigma$ is called* measure *if*

- $\mu(\varnothing) = 0$.
- $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$,

*for any finite or infinite sequence $\{A_n\}$ of pair-wise disjoint sets of $\Sigma$, i.e., $A_i \cap A_j = \varnothing$, $i \neq j$.*

**Definition 2.4** (Measure space)**.** *The ordered triplet $(X, \Sigma, \mu)$ is called a measure space if a $\sigma$-algebra is given $\Sigma$ over a set $X$ and a measure $\mu$ defined on $\Sigma$. If the measure is not explicitly expressed, the ordered pair $(X, \Sigma)$ is called a measurable space and any $A \in \Sigma$ is called $\Sigma$-measurable set, or simply, measurable set.*

**Definition 2.5** ($\sigma$-finite space)**.** *A measure space $(X, \Sigma, \mu)$ is said to be* $\sigma$-finite *if $X$ is a measurable union of of its subsets with finite measure, namely:*

$$X = \bigcup_{n=1}^{\infty} A_n, \quad A_n \in \Sigma, \quad \mu(A_n) < \infty, \quad n = 1, 2, \ldots$$

**Definition 2.6** (Finite measure space)**.** *A measure space $(X, \Sigma, \mu)$ is said to be* finite *if $\mu(X) < \infty$. Particularly, if $\mu(X) = 1$, then the measure space is known as a* probability space *or* normalized space measure.

**Definition 2.7** (Measurable function)**.** *Let $(X, \Sigma, \mu)$ be a measure space. A real-valued (or complex-valued) function $f : X \to \mathbb{R}$ ($\mathbb{C}$ resp.) is said to be* measurable *if $f^{-1}(G) \in \Sigma$ for every open set $G \subset \mathbb{R}$ (or $\mathbb{C}$), where $f^{-1}(G) \equiv \{x \in X : f(x) \in G\}$ is the inverse image of $G$ by $f$.*

**Remark 2.1.** *More generally, a transformation $X \to Y$ from a measurable space $(X, \Sigma)$ to a measurable space $(Y, \mathcal{A})$, is said to be* measurable *if $T^{-1}(A) \in \Sigma$ for each $A \in \mathcal{A}$. Thus, a measurable function $f$ is a measurable transformation from $(X, \Sigma)$ to $(\mathbb{R}, \mathcal{B})$, where $\mathbb{R}$ is the set of the real numbers and $\mathcal{B}$ a Borel-$\sigma$-algebra. If $X_1$ and $X_2$ are topological spaces with their respective Borel $\sigma-$algebras $\mathcal{B}_1$ and $\mathcal{B}_2$, then a continuous transformation $T : X_1 \to X_2$ is a* Borel-measurable *transformation. In particular, a continuous function on a topological space is a measurable function.*

One of the most relevant mathematical objects in this thesis is the density function associated to the distribution of the orbits, and before defining it, we need to define the concept of random variable. For a transformation $T$, a $T-$invariant probability density function is commonly associated to an absolutely continuous (with respect to the Lebesgue measure) invariant measure in the sense that their existence provide a statistical analysis tool for the study of dynamical systems.

**Definition 2.8** (Random variable)**.** *A random variable is a transformation $\xi$ from the sample space $X$ to the real numbers set, namely*

$$\xi : X \to \mathbb{R},$$

*such that for any real number $x \in \mathbb{R}$, and any $\sigma$-algebra $\Sigma$ of $X$*

$$\{\omega \in X : \xi(\omega) \leq x\} \in \Sigma,$$

*i.e., $\xi^{-1}$ is measurable whenever it is defined.*

**Definition 2.9** (Probability density function)**.** *A random variable $\xi$ is said to have a density function $F(x)$, where $F$ is a Lebesgue-integrable function, if the density of $\xi$ with respect to the reference measure $\mu$ is given by*

$$\nu(\xi \in A) = \int \chi_{\xi^{-1}(A)}(x) d\nu(x) = \int_A F(x) d\mu(x),$$

*where $\nu$ is the measure of the probability density function of $\xi$ on a set $A$, for any measurable set $A \in \Sigma$.*

**Definition 2.10** (Measure-preserving transformation)**.** *Let $(X, \Sigma, \mu)$ measure space. A measurable transformation $T : X \to X$ is said to be* measure-preserving *$\mu$, or alternatively, the measure $\mu$ is said to be $T$-invariant, if*

$$\mu(T^{-1}(A)) = \mu(A), \quad \forall A \in \Sigma.$$

**Remark 2.2.** *In practice, it is hard to verify that a transformation is measure-preserving for every measurable set on the previous definition, in application, if $\mu(T^{-1}(A)) = \mu(A)$, is verified for every measurable set on a subclass $\pi$ which is closed under the intersection operation of its members, and is $\Sigma$-generating, that is, $\Sigma$ is the smallest $\sigma-$algebra that contains $\pi$, then $\mu$ is invariant under $T$. The subclass $\pi$ with this property is known as a $\pi-$system [6].*

With the purpose of spatial average of certain functions, or their expected value with respect to the invariant measure of the system, it is important to establish the following definition.

**Definition 2.11** (Simple function). *A measurable function $f : X \to [0, \infty)$ on a measurable space is called* simple function *if its range is conformed by a finite amount of points. In other words, $f$ is a simple function if*

$$f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}, \quad A_i \in \Sigma,$$

*where $\alpha_i$ are the distinct values the function can take and $\chi_{A_i}$ is the indicator function of $A_i$.*

For physical measurements, quite often a probability distribution of a physical quantity is considered. Let $(T, X)$ be a dynamical system, and let $X$ be a phase space with finite measure $\mu(X) < \infty$ and let $A$ be a subset of $X$. Instead of distinguishing the deterministic properties of the individual orbits, the probabilistic properties are considered by observing the frequencies of the first $n$ terms of the orbit $\{T^n(x)\}$ of an initial point $x$ which enters $A$ for every natural numbers $n$. In order to calculate the frequencies. Let $\chi_A$ be the *indicator function* of $A \subset X$:

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$$

Then, the frequency for a given $n$ is exactly $\frac{1}{n}\sum_{i=0}^{n-1} \chi_A(T^i(x))$.

**Definition 2.12** (Asymptotic frequency of $A \in \Sigma$). *Consider a subset $A \in \Sigma$, the* asymptotic frequency *or* time average *of $A$ is given by the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(T^i(x)),$$

*which, in case it exists, measures how frequently the orbit visits $A$, or the* time average *for which the dynamics circulates by $A$.*

The limit mentioned in this definition exists under specific circumstances, given by the Birkhoff's pointwise ergodic theorem; which states as well, that for a $T$-invariant ergodic measure $\mu$, this limit is equivalent to the spatial mean given by that measure $\mu$.

One of the most important concepts in this thesis are the *absolutely continuous measures*, which we define in the following.

**Definition 2.13** (Absolutely continuous measure)**.** *Let $\mu$ be a measure on a $\sigma$-algebra $\Sigma$ and let $\nu$ be an arbitrary measure on $\Sigma$; $\nu$ can be positive, real or complex. The measure $\nu$ is said to be absolutely continuous with respect to $\mu$ and we write*

$$\nu \ll \mu$$

*if $\nu(A) = 0$ for each $A \in \Sigma$ such that $\mu(A) = 0$. If $\nu \ll \mu$ and $\mu \ll \nu$ both hold, then the measures $\nu$ and $\mu$ are said to be equivalent, written as $\mu \cong \nu$.*

**Theorem 2.1** (Radon-Nikodym theorem, [6])**.** *Let $(X, \Sigma, \mu)$ be a measure space $\sigma$-finite and let $\nu$ be a measure real-valued (or complex-valued) such that it is absolutely continuous with respect to $\mu$. Then, there exists a unique function $f : X \to \mathbb{R}$ (or $\mathbb{C}$), $\mu$-integrable such that*

$$\nu(A) = \int_A f(x)d\mu(x), \quad \forall A \in \Sigma. \tag{2.3}$$

*$f$ is referred to as the Radon-Nikodym derivative and is denoted by $\frac{d\nu}{d\mu}$.*

**Remark 2.3.** *If a determined property that involves the points in the measure space is preserved except for a set of zero measure, then this property is said to be true* almost everywhere (a.e.). *The notation $\mu$-*a.e. *(or simply a.e.) when $\mu$ is implicit, is sometimes utilized if the property holds almost everywhere with respect to the measure $\mu$ [6].*

## 2.3 L($\mu$) spaces

Since we will study the behavior of densities with respect to the evolution of dynamics, it is important to establish the space where this evolution occurs, that is, spaces of functions. To get started, the concept of variation is fundamental for the definition of compactness on $L^1$ spaces, which in turn takes relevance since the Perron-Frobenius operators are defined on them.

**Definition 2.14** ($L^p$ and $L^\infty$ spaces)**.** *Let $(X, \Sigma, \mu)$ be a finite-measure space. Let $p$ be a real number such that $1 \le p < \infty$. The family of all functions $\mu$-measurable of real-valued (or complex-valued) $f : X \to \mathbb{R}$ (o $\mathbb{C}$) satisfying*

$$\int_X |f|^p d\mu < \infty,$$

*is denoted by $L^p$. The space $L^\infty(X, \Sigma, \mu)$ is defined as the family of all measurable $\mu$-a.e. functions.*

**Definition 2.15** ($L^p$ norm)**.** *The number $\|f\|_p = \left(\int_X |f|^p d\mu\right)^{1/p}$ is called $L^p$* norm of *$f \in L^p$ for $p < \infty$; and the number $\|g\|_\infty = \text{ess sup}_{x\in X}|g(x)|$ is referred to as* the $L^\infty$ norm of *$g \in L^\infty$.*

**Definition 2.16** (Essential supremum)**.** *The* ess $\sup_{x\in X}$ *of an arbitrary function* $f \in L^p$ *is the supremum of a function* $\mu$*-a.e.:*

$$\operatorname*{ess\,sup}_{x\in X} f(x) = \inf\{M : \mu(\{x : f(x) > M\}) = 0\}$$

Under the characteristics of Definition 2.15, $L^p(X, \Sigma, \mu)$ is a *Banach space*, namely, it is a complete, normed space [6].

The *dual space*, or simply, *dual*, by definition is the space of all bounded linear functionals [6]. The next theorem characterizes the dual space of $L^p$.

**Theorem 2.2** ([6])**.** *Let* $1 \le p < \infty$*. The dual of* $L^p(X, \Sigma, \mu)$ *is isomorphic to* $L^q(X, \Sigma, \mu)$*, where* $1/p + 1/q = 1$*, for* $p > 1$ *and* $q = \infty$ *if* $p = 1$*.*

*The dual relationship between* $f \in L^p$ *and* $g \in L^q$ *is given by*

$$\langle f, g\rangle \equiv \int_X fg d\mu,$$

*which satisfies the* Cauchy-Hölder inequality

$$|\langle f, g\rangle| \le ||f||_p ||g||_q, \quad \forall f \in L^p,\ g \in L^q.$$

**Remark 2.4.** *Sometimes one can write* $L^p$ *instead of* $L^p(X, \Sigma, \mu)$ *when the measure space stays implicit,* $L^p(X)$*,* $L^p(\mu)$ *or* $L^p(\Sigma)$ *when the respective elements of the measure space are inferred.*

## 2.4 Bounded variation

Another important concept for the class of dynamics considered in this thesis is the concept of bounded variation. And the subclass of functions defined on a closed interval $X = [a, b]$, of a normed space satisfying the bounded variation property.

**Definition 2.17** (Variation)**.** *Let* $f$ *be a real-valued or complex-valued function defined on an interval* $[a, b]$*. The variation of* $f$ *on* $[a, b]$ *is the non-negative number (it can be infinite)*

$$\bigvee_{[a,b]} f = \sup\left\{\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| : a = x_0 < x_1 < \ldots < x_n = b\right\},$$

*where the supremum is taken for all the possible partitions of* $[a, b]$*. If* $\bigvee_{[a,b]} f < \infty$*,* $f$ *is said to be of bounded variation on* $[a, b]$*.*

**Definition 2.18** (Variation on $L^1$)**.** *Let* $f \in L^1([a, b])$*. Then, its variation on* $[a, b]$ *is defined as*

$$\bigvee_{[a,b]} f = \inf \left\{ \bigvee_{[a,b]} g : g(x) = f(x), \forall x \in [a,b], \mu - a.e. \right\},$$

*If* $\bigvee_{[a,b]} f < \infty$*, then* $f$ *is said to be of bounded variation on* $[a,b]$*.*

**Example 2.3.** *The variation in* $[0,1]$ *for the density function*

$$h(x) = \chi_{[0,\frac{1}{4})}(x) + \frac{1}{3}\chi_{[\frac{1}{4},\frac{1}{3})}(x) + 3\chi_{[\frac{1}{3},\frac{1}{2}]}(x) + \frac{4}{9}\chi_{[\frac{1}{2},1]}(x),$$

*is computed choosing* $x_0 = x \in \left[0, \frac{1}{4}\right)$, $x_1 = x \in \left[\frac{1}{4}, \frac{1}{3}\right)$, $x_3 = x \in \left[\frac{1}{3}, \frac{1}{2}\right)$, *and* $x_4 = x \in \left[\frac{1}{2}, 1\right]$*. Given that it is a simple function, we can see that the supremum is attained by the choice of this set of* $x_i$*, otherwise some of the differences will be zero. Therefore:*

$$\bigvee_{[0,1]} f = \left|1 - \frac{1}{3}\right| + \left|\frac{1}{3} - 3\right| + \left|3 - \frac{4}{9}\right| = \frac{44}{9}.$$

## 2.5 Compactness and quasi-compactness

An important result for the class of expansive maps on the interval utilizes a theorem of fixed point of an operator applied on a dense subset of space of functions $L^1$, and a required condition for the existence of the fixed point in this aforementioned theorem is compactness. It is defined in the following.

**Definition 2.19** (Strong precompactness and compactness)**.** *Let* $(X, \Sigma, \sigma)$ *be a measure space and let* $\mathcal{F}$ *be a subset of* $L^1(X)$*. The set* $\mathcal{F}$ *is said to be* strongly precompact *if for every sequence* $\{f_n\} \subset \mathcal{F}$ *there exists a subsequence* $\{f_{n_i}\}$ *of* $\{f_n\}$ *such that*

$$\lim_{i\to\infty} \|f_{n_i} - \bar{f}\| = 0$$

*for some* $\bar{f} \in L^1$*. Moreover, if* $\mathcal{F}$ *is closed in* $L^1$*, then* $\mathcal{F}$ *is said to be* compact*.*

**Definition 2.20** (Weak topology)**.** *The weak topology of measures is a topology of weak convergence of a sequence of measures* $\{\mu_n\}$ *to a measure* $\mu$ *if and only if*

$$\int_X g d\mu_n \to \int_x g d\mu,$$

*where* $g : X \to \mathbb{R}$ *with the norm*

$$\|g\|_C := \sup_{x\in X} |f(x)|.$$

*This is also sometimes referred to as vague convergence.*

**Definition 2.21** (Precompactness and weak compactness). *The set $\mathcal{F}$ is said to be* weakly precompact *if every sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence $\{f_{n_i}\}$ that weakly converges to some $\bar{f} \in L^1(X)$, i.e.,*

$$\lim_{i\to\infty}\langle f_{n_i}, g\rangle = \langle \bar{f}, g\rangle, \forall g \in L^\infty(X).$$

*Moreover, if $\mathcal{F}$ is closed in the weak topology of $L^1(X)$, then $\mathcal{F}$ is said to be* weakly compact*.*

Let $P : V \to V$ be a linear bounded operator on a Banach space $(V, \|\cdot\|_V)$.

**Definition 2.22** (Compactness). *A linear bounded operator $P$ is said to be* compact *if it maps bounded sets to precompact sets.*

**Definition 2.23** (Quasi-compactness). *$P$ is said to be quasi-compact if there exists a positive integer $r$ and a compact operator $K$ such that*

$$\|P^r - K\|_V < 1.$$

In this thesis, one of the main tools we utilize for the study of the evolution of density functions under the action of a transformation, is the Perron-Frobenius operator, which, in general is not compact in $L^1$ [6]. The following theorem is a classic and fundamental result, which states sufficient conditions for guaranteeing the quasi-compactness of bounded linear operators in $L^1$. As we will see, this property is fundamental for proving the existence of a fixed point of the Perron-Frobenius operator, which in turn translates to the existence of an invariant density for the respective system associated to it.

**Theorem 2.3** ([2], Ionescu-Tulcea and Marinescu). *Let $\Omega$ be a bounded region of $\mathbb{R}^N$, and let $(V, \|\cdot\|_V)$ be a Banach space such that $V$ is a dense vector subspace of $L^1(\Omega)$. Let $T : V \to V$ be a bounded linear operator with respect both to the norm $\|\cdot\|_V$ and the norm $\|\cdot\|$. Assume that*

(i) *if $f_n \in V$ for $n = 1, 2, ...,$ $f \in L^1(\Omega)$, $\lim_{n\to\infty}\|f_n - f\| = 0$, and $\|f_n\|_V \leq M$ for $n$, then $f \in V$ and $\|f\|_V \leq M$, where $M$ is a constant;*

(ii) $\sup_{n\geq 0}\{\|T^n f\|/\|f\| : f \in V, f \neq 0\} < \infty$;

(iii) *there exist $k \geq 1$, $0 < \alpha < 1$, and $\beta < \infty$ such that*

$$\|T^k f\|_V \leq \alpha\|f\|_V + \beta\|f\|, \quad \forall f \in V;$$

(iv) *if $V_0$ is a bounded subset of $(V, \|\cdot\|_V)$, then $T^k V_0$ is precompact in $L^1(\Omega)$.*

*Then, $\Lambda$ (the set of the eigenvalues of $T$ with modulus 1) has only a finite number of elements, $D(\lambda) = \{f \in V : Tf = \lambda f\}$ (the eigenspace of $T$ associated with eigenvalue $\lambda$) is finite dimensional for each $\lambda \in \Lambda$, and $T : (V, \|\cdot\|_V) \to (V, \|\cdot\|_V)$ is quasi-compact.*

**Definition 2.24** (Markov operator)**.** *A linear operator $P : L^1(\mu) \to L^1(\mu)$ if it is closed under the set of density functions $\mathcal{D} := \{f \in L^1(X, \Sigma, \mu) : f \geq 0, \|f\| = 1\}$ in $L^1(\mu)$. Namely, if $P\mathcal{D} \subset \mathcal{D}$.*

The following result is a widely known and powerful tool utilized for determining the convergence of a sequence of the iterates of Markov operators $P : L^1 \to L^1$.

**Theorem 2.4** ([2],Kakutani-Yosida abstract ergodic theorem)**.** *Let $(X, \Sigma, \mu)$ be a measure space and let $P : L^1 \to L^1$ be a Markov operator. If for a given $f \in L^1$, the sequence of $\{A_n f\}$ of the Cesáro averages $\left(A_n f = \frac{1}{n} \sum_{i=0}^{n-1} P^i f\right)$ is weakly precompact in $L^1$, then it converges strongly to some $f^* \in L^1$, which is a fixed point of $P$, that is,*

$$\lim_{n \to \infty} \|A_n f - f^*\| = 0$$

*and $Pf^* = f^*$. Furthermore, if $f$ is a density function, then so it is $f^*$, so that $f^*$ is a stationary density of $P$.*

Therefore, if a Markov operator (in particular, the Perron-Frobenius operator) is quasi-compact, it maps bounded sets to precompact sets, and the implication from this last theorem is that the sequence $\{A_n f\}$ converges strongly to a fixed point. These last notions about compactness and precompactness are necessary to provide a comprehensive extent on the proof of a classic result about the existence of a fixed point for the Perron-Frobenius operator on the Lasota-Yorke maps in the interval.

## 2.6 Ergodicity and mixing

The chaotic discrete dynamic systems do satisfy, by the definition of Devaney, the property of transitivity. This concept, in turn, is related to the idecomposibility of the transformation [6]; i.e., the restriction of the dynamics to a subset of the state space will eventually this subset.

If for a measure-preserving transformation $T : (X, \Sigma, \mu) \to (X, \Sigma, \mu)$ there exists a non-trivial set $A \in \Sigma$ different from $X$ such that $T^{-1}(A) = A$ (i.e., it is $T$−invariant), then $T^{-1}(A^c) = A^c$. Therefore, we have that the dynamics of $T$ can be decomposed in two: $T|_A : A \to A$ and $T|_{A^c} : A^c \to A^c$. If this cannot happen, then for $T$ the property of ergodicity holds [6].

**Definition 2.25** (Ergodicity)**.** *Let $(X, \Sigma, \mu)$ be a measure space. A measurable transformation $T : X \to X$ is said to be* ergodic *if every invariant set $A \in \Sigma$ of $T$ is such that $\mu(A) = 0$ or that $\mu(A^c) = 0$. In other words, $T$ is ergodic if and only if its invariant sets or their complements are equivalent to the empty set a.e. Such sets are known as the trivial subsets of $X$.*

**Theorem 2.5.** [6] *Let $(X, \Sigma, \mu)$ be a measure space and let $T : X \to X$ be a non-singular transformation. Then $T$ is ergodic if and only if for every bounded measurable function $f : X \to \mathbb{R}$,*

$$f(T(x)) = f(x), \quad \forall x \in X \quad \mu - a.e$$

*implies $f(x)$ is a constant function (a.e.).*

An alternate equivalent condition for egodicity is given by the next theorem:

**Theorem 2.6.** [6] *Let $(X, \Sigma, \mu)$ be a probability measure space and let $T : X \to X$ be measure preserving. Then, $S$ is ergodic if and and only if for all $A, B \in \Sigma$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu\left(T^{-i}(A) \cap B\right) = \mu(A)\mu(B).$$

The definitions of mixing and weak mixing provide the most robust notions about the unpredictability for large times for deterministic dynamics, regarding how much the system can "spread" all over the state space any measurable set under its action. Or in other words, what is the likeliness of the points in any given set to visit any other given set, with respect to the invariant probability measure of the system.

**Definition 2.26** (Mixing)**.** *Let $(X, \Sigma, \mu)$ be a probability measure space and let $T : X \to X$ be a measure preserving transformation. $T$ is said to be* mixing *if for every $A, B \in \Sigma$,*

$$\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B).$$

**Definition 2.27** (Weakly mixing)**.** *Let $(X, \Sigma, \mu)$ be a probability measure space and let $T : X \to X$ be a measure preserving transformation. $T$ is said to be* weakly mixing *if for every $A, B \in \Sigma$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \left|\mu(T^{-i}(A) \cap B) - \mu(A)\mu(B)\right| = 0.$$

**Theorem 2.7** (Birkhoff's Point-wise ergodic theorem)**.** *Let $\mu$ be a probability measure on $X$ which is invariant under a transformation $T : X \to X$. Then, for any integrable function $f$ defined on $X$ and almost every $x \in X$, the time average*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T_i(x)),$$

*exists and is denoted by $\hat{f}(x)$. Also*

$$\hat{f}(T(x)) = \hat{f}(x), \forall x \in X \quad \mu - a.e.$$

*Moreover, if $T$ is ergodic, then $\hat{f}$ is the constant function $\int_X f d\mu$.*

Next, we show an example of a system with an invariant measure that is ergodic but not mixing.

For further details about this result, see Chapter 3 in [6].

**Example 2.4.** *Consider the map* $T(x) = x + \frac{1}{3}$ (mod 1)*:*

$$T(x) = \begin{cases} x + 1/3, & 0 \le x \le \frac{2}{3} \\ x - 2/3, & \frac{2}{3} < x \le 1. \end{cases} \tag{2.4}$$

*It is known to have a 3-cycle for every initial condition (see Example 3.1.2 in [13]), and one can also check that the third iterate* $T^3(x)$ *is equal to the identity function. Then, the invariant measure* $\mu_1(A)$*, induced by the density function* $f_1(x) = \frac{1}{3}\chi_{\{x_0\}}(x) + \frac{1}{3}\chi_{\{T(x_0)\}}(x) + \frac{1}{3}\chi_{\{T^2(x_0)\}}(x)$ *can be expressed as*

$$\mu_1(A) = \int_A f_1(x) d\mathrm{m},$$

*where* m *is the Lebesgue measure. The measure* $\mu_1$ *of the only invariant set* $E = \{(x_0), T(x_0), T^2(x_0)\}$ *is exactly 1, and we can check that* $\mu_1(E^c) = 0$*. Therefore* $\mu_1$ *is* $T$*-ergodic.*

*We can check that* $\mu_1$ *is not mixing by considering* Definition 2.27*: the limit on the left hand side of the expression:*

$$\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B)$$

*does not exists, given that for every* $A \in \Sigma$ *and every* $n \in \mathbb{N}$, $T^{-3n}(A) = A$*. Thus, the equality does not hold.*

**Remark 2.5.** *In general, a system is not inherently ergodic, but with respect to its invariant measures. That is, if a system has more than one invariant measure, not all of them could be ergodic (or mixing).*

The next ergodic theorem expresses the concepts of ergodicity, weak mixing and mixing in a functional form. This is useful in order to confirm if the mixing properties hold for a specific transformation [6].

**Theorem 2.8.** *[6] Let* $(X, \Sigma, \mu)$ *be a probability measure space and let* $T : (X, \Sigma, \mu) \to (X, \Sigma, \mu)$ *be a measure-preserving transformation*

1. *The following expressions are equivalent:*

   (a) $T$ *is ergodic*

   (b) *For every* $f$, $g \in L^2(\mu)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \int_X (f \circ T^i) g d\mu = \int_X f d\mu \int_X g d\mu.$$

(c) *For every* $f \in L^2(\mu)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \int_X (f \circ T^i) f d\mu = \left( \int_X f d\mu \right)^2.$$

2. *The following expressions are equivalent:*

(a) $T$ *is weakly mixing*

(b) *For every* $f$, $g \in L^2(\mu)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \left| \int_X (f \circ T^i) g d\mu - \int_X f d\mu \int_X g d\mu \right| = 0.$$

(c) *For every* $f \in L^2(\mu)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \left| \int_X (f \circ T^i) f d\mu - \left( \int_X f d\mu \right)^2 \right| = 0.$$

(d) *For every* $f \in L^2(\mu)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \left[ \int_X (f \circ T^i) f d\mu - \left( \int_X f d\mu \right)^2 \right]^2 = 0.$$

3. *The following expressions are equivalent:*

(a) $T$ *is mixing*

(b) *For every* $f$, $g \in L^2(\mu)$,

$$\lim_{n\to\infty} \int_X (f \circ T^n) g d\mu = \int_X f d\mu \int_X g d\mu.$$

(c) *For every* $f \in L^2(\mu)$,

$$\lim_{n\to\infty} \int_X (f \circ T^n) f d\mu = \left( \int_X f d\mu \right)^2.$$

## 2.7 Perron-Frobenius operator

Let $(X, \Sigma, \mu)$ be a measure space and let $T : X \to X$ be a non-singular transformation. Given a function $f \in L^1$, we can define a real-valued measure

$$\mu_f(A) = \int_{T^{-1}(A)} f d\mu, \forall A \in \Sigma.$$

Since $T$ is non-singular, $\mu(A) = 0$ implies $\mu_f(A) = 0$. Thus, the Radon-Nikodym theorem implies that there exists a unique function $\hat{f} \in L^1$, such that

$$\mu_f(A) = \int_A \hat{f} d\mu, \forall A \in \Sigma.$$

**Definition 2.28** (Perron-Frobenius operator). *The operator $P : L^1 \to L^1$ defined by*

$$\int_A Pf d\mu = \int_{T^{-1}(A)} f d\mu, \quad \forall A \in \Sigma, \quad \forall f \in L^1,$$

*is known as the* Perron-Frobenius operator associated to $T$, *also denoted by* $P_T$.

An explicit expression for the Perron-Frobenius operator associated to maps on the interval, whose computation and proof can be read in [6], is the following:

$$P_T f(x) = \frac{d}{dx} \int_{T^{-1}([a,x])} f d\mathrm{m}, \quad x \in [a,b], \quad a.e.,$$

where m denotes the Lebesgue measure. Likewise, the Perron-Frobenius operator can be seen as a combination of an integral operator with a differential operator. Additionally, if the transformation $T : [a,b] \to [a,b]$ is differentiable and monotonic, from the previous equation, we obtain:

$$P_T f(x) = f(T^{-1}(x)) \left| \frac{d}{dx} T^{-1}(x) \right|,$$

or [7]:

$$P_T f(x) = \sum_{y,\, T(y)=x} \frac{f(y)}{|T'(y)|}$$

If a particular system is applied on a density as initial condition, instead of an individual point, then the successive densities are given by a linear integral operator, the Perron-Frobenius operator [8].

**Example 2.5.** *Recalling* Example 2.1 *(equation* (2.2)*), one can observe the evolution of the densities according to the successive iteration of the Perron-Frobenius operator asocciated to the transformation (piecewise monotonic and $\mathcal{C}^2$)* $T(x) = 2x$ (mod 1)*.*
*We have*

$$T^{-1}(x) = \frac{1}{2}x, \quad \forall x \in [0, 0.5], \quad T^{-1}(x) = \frac{x+1}{2}, \quad \forall x \in [0.5, 1],$$

*and with $P_T^0 f(x)$ as $\hat{f}(x) = 1 - x$, the first iteration of the Perron-Frobenius operator results as*

$$P_T f(x) = \frac{1}{2}\left(1 - \frac{1}{2}x\right) + \frac{1}{2}\left(1 - \frac{x+1}{2}\right) = \frac{3}{4} - \frac{1}{2}x = \hat{f}_2(x).$$

*By repeating the procedure with $\hat{f}(x) = \hat{f}_2(x)$ in order to compute the second iteration $P_S^2 f(x)$, we obtain $\hat{f}_2(x) = \frac{5}{8} - \frac{1}{4}x = \hat{f}_3(x)$. Then, by mathematical induction, we can compute an explicit expression for $P_S^n f(x)$:*

*First, lets assume*

$$P_T^n f(x) = \frac{2^n + 1}{2^{n+1}} - \frac{x}{2^n}.$$

*We can verify it for the first natural numbers:*

$$P_S^0 T(x) = 1 - x, \quad P_T f(x) = \frac{3}{4} - \frac{1}{2}x, \quad P_T^2 f(x) = \frac{5}{8} - \frac{1}{4}x.$$

*Now we assume $P_T^n f(x)$ and calculate $P_T^{n+1} f(x)$:*

$$\begin{aligned} P_T^{n+1} f(x) &= \frac{1}{2}\left(\frac{2^n + 1}{2^{n+1}} - \frac{1}{2^n}\left(\frac{1}{2}x\right)\right) + \frac{1}{2}\left(\frac{2^n + 1}{2^{n+1}} - \frac{1}{2^n}\left(\frac{x+1}{2}\right)\right) \\ &= \frac{2(2^n + 1) - 1}{2^{n+2}} - \frac{2x}{2^n + 2}, \end{aligned}$$

*which can be rewritten as*

$$P_T^{n+1} f(x) = \frac{2^{(n)+1} + 1}{2^{(n+1)+1}} - \frac{x}{2^{(n)+1}},$$

*which proves, in turn, that $P_T^n f(x)$ is correct. Then, by computing the limit when n goes to infinity we get:*

$$\lim_{n\to\infty} P_T^n f(x) = \frac{1}{2} \cdot \chi_{[0,1]}(x),$$

*which corresponds to a uniform density distribution.*

**Remark 2.6.** *This computation illustrates what can be seen empirically through histograms estimation we mentioned in Chapeter 2.1, and in* Figure 2.3*, the evolution of an initial density, coincides in the limit with the invariant density function associated to $T(x) = 2x \pmod 1$, which is a constant function.*

It is worth mentioning that in the previous example, any constant density function is an invariant density associated to the map $T(x) = 2x \pmod 1$, being a fixed point to the Perron-Frobenius operator $P_T$, that is, a function which satisfies the equation $P_T f^* = f^*$.

In order to illustrate the convergence of the Perron-Frobenius operator, according to Theorem 2.9, we show in Figure 2.4 the first ten iterations of the Perron-Frobenius operator applied to the initial function $P_T^0 f(x) = 1 - x$.

For the general case of the expansive maps on the interval, we will discuss a classic result on the existence of a invariant density under the action of the Perron-Frobenius operator. The proof if this result requires an argument of quasi-compactness of the operator from and a the fixed point theorem, when the operator acts on a particular subspace of $L^1(\mu)$.

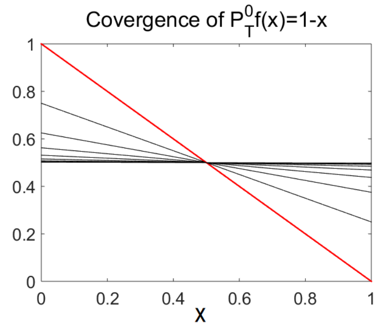



Figure 2.4: Sequence of functions obtained through successive iterations of the PF operator on the function $P_T^0 f(x)$ (in red) associated to the map in Example 2.1.

**Theorem 2.9** (Lasota-Yorke theorem, 1973 [2]). *Let* $T : [0,1] \to [0,1]$ *be a* $\mathcal{C}^2$*piece-wise function such that* $\inf |T'| > 1$. *Then, for any* $f \in L_1$, *the sequence*

$$\frac{1}{n} \sum_{k=0}^{n-1} P_T^k f$$

*converges in norm to a function* $f^* \in L^1$. *The limit function has the following properties:*

(1) $f \geq 0$, *then* $f^* \geq 0$.

(2) $\int_0^1 f^* dm = \int_0^1 f dm$.

(3) $P_T f^* = f^*$ *and consequently, the measure* $d\mu^* = f^* dm$ *is* $T$*-invariant.*

(4) *The function* $f^*$ *is of bounded variation; also, there exists a constant* $c$, *independent of the choice initial* $f$ *such that the variation of* $f^*$ *satisfies the inequality*

$$\bigvee_{[0,1]} f^* \leq c\|f\|. \tag{2.5}$$

The outline of the proof follows first from Theorem 2.3 (Ionescu-Tulcea and Marinescu), from which the quasi-compactness of the operator $P_T$ is given. Then, by applying Theorem 2.4 (Kakutani-Yosida) the existence of a fixed point $f^* \in L^1$ such that $P_T f^* = f^*$ is guaranteed.

The type bound we see in (2.5) is widely known, and it also has been utilized in the proof of further generalizations of this classic result, as we will recall later on in Chapter 4.3.

## 2.8 Decay of correlations

Let $(X, \Sigma, \mu)$ be a probability measure space and let $T : X \to X$ a non-singular transformation. Given two adequate functions $f$ and $g$ from a space of functions, which are frequently, in this context referred as *observables*. The differences

$$\int_X (g \circ T^n) f d\mu - \int_X f d\mu \int_X g d\mu$$

are called *correlation functions* of the observables $f$ and $g$. If $T$ preserves $\mu$ and it is mixing, then the correlation function decays up to zero when $n$ goes to infinity. The rate of decay of correlations measures the speed at which the dynamics of the system, determined by $T$ and $\mu$, becomes independent if the initial conditions [6].

**Definition 2.29** (Correlation coefficient). *Let $\mu$ be an invariant probability measure for a non-singular transformation $T : X \to X$ and let $n$ be a positive integer. For any $f \in L^1(\mu)$ and $g \in L^\infty(\mu)$ the quantity*

$$Cor(f,g,n) = \left| \int_X (g \circ T^n) f d\mu - \int_X f d\mu \int_X f d\mu \right| \tag{2.6}$$

*is referred to as the $n$-th correlation coefficient.*

In practice, the measure $\mu$ is generally unknown, and therefore, it is not possible to determine the correlation coefficient analytically. In [7], an expression for the correlation coefficient is provided, which is known as correlogram:

$$\widehat{Cor}(f,g,n,k) := \frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x))g(T^{i+k}(x)) - \frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x)) \cdot \frac{1}{n}\sum_{i=0}^{n-1} g(T^i(x)) \tag{2.7}$$

For an ergodic system, (2.7) is a consistent estimator for (2.6) [6, 7]. That is, $\widehat{Cor}(f,g,n,k) \to Cor(f,g,n)$ $\mu$-a.s.

**Example 2.6.** *Consider the map of the interval $T(x) = 3x \pmod 1$. Setting $f = g = \chi_{[0,1/3]}$, one can compute the decay of correlations with* (2.7) *(see* Figure 2.5*).*

## 2.9 Lyapunov exponents

Another important property to know about the chaotic systems is whether or not their trajectories depend in a very sensitive way on the initial conditions. Namely, if we start with two typical initial conditions (with respect to the typical invariant measure, if it exists), the distance between its orbits grows exponentially over time. The phenomenon about the growth of small errors is known as *sensitive dependence on the initial conditions* [7]. It is important to note that throughout this thesis we consider the space state $X$ as a subset of the real numbers $\mathbb{R}$.

Closely related with the sensitive dependence on initial conditions, is the concept of hyperbolicity, which characterizes the notion of unstable and stable fixed points.

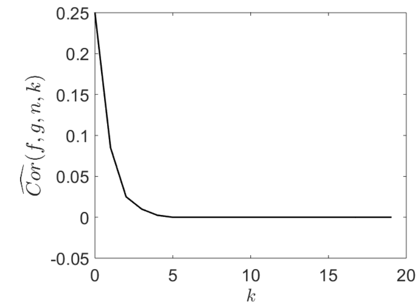



Figure 2.5: Coefficient of correlation for the map $T(x) = 3x \pmod 1$, with $f = g = \chi_{[0,1/3]}(x)$. One can observe how it resembles a exponentially decaying curve, which is expected, since this transformation is known to be exponentially mixing with respect to the Lebesgue measure.

Given that the dynamics are locally directed by the first derivative of a system near a fixed point (see [7] for further details), in a hyperbolic system, some sets of points are exponentially expanded (or contracted) fast by successive iterates of the map. The following definitions state the corresponding expressions for this concept.

**Definition 2.30** (Hyperbolic fixed point)**.** *The point $x_0$ is a hyperbolic fixed point of the map $T$ if $T(x_0) = x_0$ and $|T'(x_0)| \neq 1$.*

**Definition 2.31** (Attracting or repelling hyperbolic fixed point)**.** *There are two types of hyperbolic fixed points. A fixed point is said to be* hyperbolic attracting *if*

$$|T'(x_0)| < 1.$$

*A fixed point is said to be* hyperbolic repelling *if*

$$|T'(x_0)| > 1.$$

**Remark 2.7.** *A hyperbolic system, that is, with hyperbolic fixed points, is characterized by the predictability of its orbits (say if they converge to a fixed point, periodic cycles or to infinity), regardless of initial conditions. In a repelling hyperbolic system, two orbits cannot stay arbitrarily close to each other [7]. And in a attracting hyperbolic system, the orbits converge to the fixed point, for sufficiently large times.*

The following lemma, whose proof can be found in [7], ensures that in a hyperbolic dynamical system, the distance between two non-identical orbits at some iteration grows exponentially fast.

**Lemma 2.10.** *Consider a hyperbolic dynamical system. There is an $\epsilon > 0$ such that the following is true. Let $\{x_n\}$ and $\{y_n\}$ be any two orbits. Then either $x_k = y_k$ for all $n$, or there is at least one $k$ for which $|x_k - y_k| \geq \epsilon$.*

**Remark 2.8.** *In a compact system, the orbits can only separate some finite amount. That is why it is also important consider a rate of divergence between arbitrarily close points.*

**Definition 2.32** (expansive transformation)**.** *A transformation $T : X \to X$ is said to be expansive, if for every element in its space state $x \in X$, the following holds:*

$$\inf_{x \in X} |T'(x)| > 1.$$

The quantity $\lambda$ defined by:

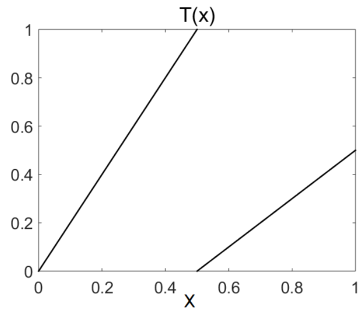
$$\lambda := \lim_{n \to \infty} \frac{1}{n} \log \left| \frac{d}{dx} (T^n(x)) \right| = \int \log |T'(\cdot)| d\mu, \tag{2.8}$$

is known as the Lyapunov exponent of the transformation $T$ for the measure $\mu$. Frequently, the Lyapunov exponent is interpreted as a quantitative indicator of the sensitive dependence on the initial conditions on specific chaotic systems [7]. From Birkhoff's theorem and the expression for the Lyapunov exponent [7], we obtain the following expression, which will be referred later on Chapter 4:

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \log \left| \frac{d}{dx} (T^n(x)) \right| = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |T'(T^j(x))|.$$

**Example 2.7.** *Consider the system*

$$T(x) = \begin{cases} 2x, & 0 \leq x \leq 0.5 \\ x - 0.5, & 0.5 < x \leq 1, \end{cases}$$




*which has invariant probability density function $f^*(x) = \frac{4}{3}\chi_{[0,1/2]}(x) + \frac{2}{3}\chi_{(1/2,1]}(x)$, which induces the measure $\mu_f(A) = m(A) \cdot f^*(x)$, for every subset $A$ of $[0,1]$, where $m$ is the Lebesgue measure. Then, we can compute analytically its Lyapunov exponent with (2.8), which yields:*

$$\lambda = \int_0^{1/2} \frac{4}{3} \log(2) \, dm + \int_{1/2}^{1} \frac{2}{3} \log(1) \, dm = \frac{2}{3} \log(2) \approx 0.46209...$$

*Therefore, having a strictly positive Lyapunov exponent, one can expect to observe the phenomenon of sensitive dependence on initial conditions, as we can see on* Figure 2.6.

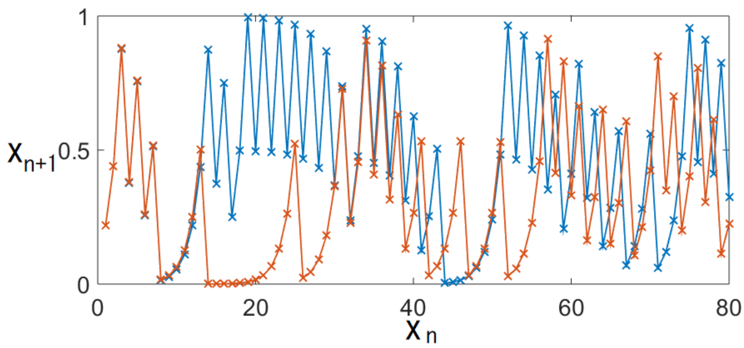



Figure 2.6: The first iterations of two arbitrary orbits for $T(x)$, $\{x_n\}$ and $\{y_n\}$, with $|x_0 - y_0| \leq 0.001$. We can see how these orbits stay relatively near to each other for a few iterations, but eventually they separate and display an asynchronous behavior, as expected in a system with a positive Lyapunov exponent.

In this previous example, if any two trajectories start arbitrarily close to the fixed point $x = 0$, we can compute that they tend to separate exponentially, at a rate of $2^n$. However, since the branch in $(1/2, 1]$ maps every incidence back to $[0, 1/2]$ without any further expanding (since it has derivative equal to 1), their eventual rate of separation is lower and is given by the Lyapunov exponent. It is important to note here that although the orbits considerably deviate from each other after a few iterations, they can eventually get relatively close, given that this dynamical system occurs in a compact set. The Lyapunov exponent is not a measure of separation between close initial conditions; it actually gives us the rate of separation of infinitesimally close arbitrary points.

Having defined the concepts and definitions we require in this thesis about deterministic dynamics (and maps on the interval in general), we will approach in the following chapter, definitions and results on random dynamics, concerning the exposition and discussion of the Chapter 4.

# Chapter 3

# Phase transition phenomenon

Within a physical or experimental context, the phenomena that can be modeled by means of a map in the interval, can undergo an abrupt change of physical or chemical properties, after which, its nature may change completely from the one that was present in its former state (see Chapter 1.2 in [6]). That is why, in the context of the dynamical systems, the expressions that model physical phenomena replicate this event in such a way that it is carried over as some type of singularity, usually understood as a sudden change of dynamical or statistical properties, which consequently reflect a relevant part of the nature of the physical phenomenon that a system under study system models.

In this chapter, we provide some context about the phase transition phenomenon in maps in the interval. The type of phase transition we are interested in, is that in which a map in the interval undergoes a major change in the statistical behavior of its orbits as a function of a characteristic parameter defined in its equations. We define this change as the transition from the existence to the non-existence (or viceversa) of an invariant density function with support on a set of positive Lebesgue measure, and therefore, of an a.c.i.m. In this scenario, the dynamics of the systems transits from a "stable" behavior, i.e., one that distributes the orbit over a finite set of points in the interval $[0,1]$, into a type of behavior that distributes the trajectories on relatively large portions of the interval, or usually referred to as "chaotic", being the case that they converge to a chaotic attractor. This phenomenon can also take place the other way around, having the transition from existence of the a.c.i.m. to the non-existence situation. We propose the following definitions to describe the phenomenon we study in this thesis:

**Definition 3.1** (Phase transition)**.** *Consider a dynamical system $T : (X, \Sigma) \to (X, \Sigma)$, with a continuous dependence on a real-valued parameter $\gamma \in F$ and two disjoints sets $G, H \subset F$ such that $\mathrm{m}(G \cup H) = \mathrm{m}(F)$. If $T$ has an absolutely invariant measure (with respect to the Lebesgue measure) for the values of $\gamma$ in either $G$ or $H$, but it does have an acim for the values in the other set, then the dynamical system*

*$T$ is said to experience a phase transition in the sense of non-existence to existence of an a.c.i.m.*

From this definition, it is sufficient to identify two regions of values of the parameter where the acim exists, and where it does not exist to point out the occurrence of the phase transition phenomenon. Furthermore, given that the dynamic behavior of $T$ is continuously dependent on $\gamma$, is implied that there exists a critical region where this transition occurs. We define it as follows:

**Definition 3.2** (Critical region)**.** *If a system $T$ as described in* Definition 3.1 *experiences a phase transition, and there exists a region of values of the parameter $\gamma$, such that it conforms the boundary set of both $G$ and $H$,i.e. $\gamma_c := \{\gamma \in F : \gamma \in (\partial G \cup \partial H)\}$, then the set $\gamma_c$ is said to be a critical region of the parameter associated to the phase transition in the dynamics of $T$.*

Our main interest lies in the study of the set of conditions that lead to the manifestation of the phase transition phenomenon in random maps in the interval. Nonetheless, we need to point out that up to our knowledge, and concerning random maps in the interval, there is no literature registering the incidence of this phenomenon, nor have been provided some conditions that yield to its occurrence in some class of random dynamics. On the other hand, this phenomenon, in the way we defined it, has been observed and reported in deterministic dynamics (not only maps in the interval). Therefore, we will cover in the following pages some of the most widely-known deterministic systems in one dimension which are known to exhibit this phenomenon.

## 3.1 Logistic map

A classic and well-known example that can illustrate the richness of behaviors that a map in the interval can exhibit, is the logistic map. This map, defined as $T_\mu(x) = \mu x(1-x)$ shows a behavior as a function of its characteristic parameter $\mu$, where $T_\mu(x) : [0,1] \to [0, \frac{\mu}{4}]$. The dynamics of $T_\mu(x)$ changes as the parameter $\mu$ passes through each of the values $1, 2, 3, 1+\sqrt{6}, \ldots$, called the *bifurcation points* [6], and the quantity, location and nature of the fixed or the periodic points changes when $\mu$ passes through each of them.

When $\mu > 1+\sqrt{6}$, the 2-cycle present in the dynamics for lower values, becomes repellent and a attracting 4-cycle is generated in its place. Actually, there exists a sequence $\{\mu_n\}$ of the period-doubling bifurcation values for the parameter $\mu$, with $\mu_0 = 3$, and $\mu_1 = 1+\sqrt{6}$, such that if $\mu_{n-1} < \mu \leq \mu_n$, then $T_\mu(x)$, has two repelling fixed points, one repelling $2^k$-cycle for $k = 1, 2, \ldots, n-1$, and one attracting $2^k$–cycle [6], see Figure 3.1.

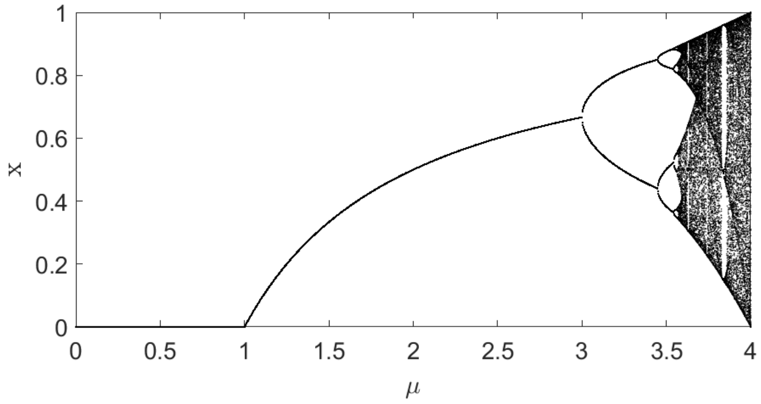



Figure 3.1: Bifurcation diagram for $T_\mu(x) = \mu x(1-x)$.

A fact that is very significant in the analysis of this system too, is the following limit, from the work of Feigenbaum ([31]):

$$\lim_{n\to\infty} \mu_n = 3.561547\ldots,$$

which is known as the *Feigenbaum number* for the logistic family of maps in the interval. Moreover, the sequence of ratios

$$\lim_{n\to\infty} r_n = \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} \approx 4.6692016091...,$$

denoted here as $\{r_n\}$ converges, as $n \to \infty$, to a quantity known as the *Feigenbaum constant* [6]. In this case, when the emergence of an infinite sequence of period doubling cascades as a function of a parameter occurs, it is generally associated with chaotic behavior, given that through the observation of bifurcation diagrams, an intermingling of period-doubling cascades is detected [31], for higher values of $\mu$ which yields into chaotic behavior at a specific value of this parameter.

This dynamical system belongs to a class of unimodal maps defined by G. Keller in [34], where they proof that this class of systems has an a.c.i.m. of positive entropy if and only if its Lyapunov exponent is strictly positive. Thus, one can propose a large number of transformations that undergo this phase transition and compute its Feigenbaum number. Given that continuous unimodal maps have a critical point where its derivative equals zero, they also are characterized for a contracting region near this point, and for values of its parameter for which this region contains a fixed point, they will be trivially stable.

Here, we consider that the phase transition phenomenon is occurring at each value of $\mu$ for which the change in the typical distribution of the trajectories imply that the dynamics underwent a transition from the non-existence to the existence of an a.c.i.m (or vice versa). For example, when $\mu \approx 3.56995$ the oscillations perceived in the trajectories are no longer of finite period, and it is considered to be the end of a period-doubling cascade [30], and the system experiences a phase transition in the direction of the non-existence to the existence of an a.c.i.m., according to Definition 3.1.

Thus, we will discuss next two families of transformations that have no contracting regions but also they do not satisfy the conditions of the Lasota-Yorke theorem and nonetheless, experience the phenomenon of phase transition (in the way we propose it) in an asymptotic manner.

## 3.2 W-maps

In [22] a family of 'W'- shaped maps whose behavior changes as according to the parameter $a$ is considered. As the family approaches a limit W map, the dynamics are either be described by a probability density function or by a singular point measure.

This family of W-maps they have an a.c.i.m. supported on the whole interval, with the particularity that its limiting dynamical behavior is captured by a singular measure as the parameter $a$ gets arbitrarily close to the zero value. Consider the family $\{W_a : 0 \leq a\}$ of maps $[0,1] \to [0,1]$ defined by

$$W_a(x) = \begin{cases} 1 - 4x, & x \in [0, 1/4) \\ (2+a)(x - 1/4), & x \in [1/4, 1/2) \\ 1/2 + a/4 - (2+a)(x - 1/2), & x \in [1/2, 3/4) \\ 4x - 3, & x \in [3/4, 1]. \end{cases} \tag{3.1}$$

In their article, only relatively small values of $a > 0$ are considered, as their interest lies in the limiting behavior of $W_a$ as $a \to 0$. For every value of $a > 0$, the map is piecewise linear, piecewise expansive and with minimal modulus of the slope equal to $2 + a$ (Figure 3.2).

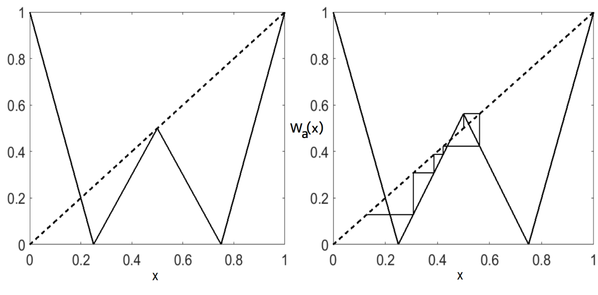



Figure 3.2: Map $W_0(x)$ (left), and a map $W_a(x)$ ($a = 0.1$), with a short cobweb diagram for the trajectory of $x = 1/2$ (right).

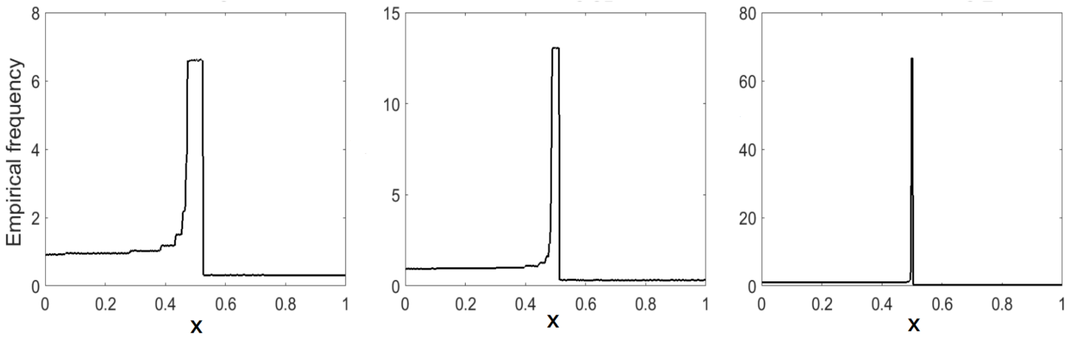



Figure 3.3: Normalized invariant densities for $W_a(x)$, with $a = 0.1$, $a = 0.05$ and $a = 0.01$ (from left to right).

Their main result is the following theorem, which establishes the weak convergence of the invariant measure $\mu_a$ to the measure $\mu_0$ as $a \to 0$. The measure $\mu_0$ has the density function

$$h_0 = \begin{cases} \frac{3}{2}, & x \in [0, 1/2) \\ \frac{1}{2}, & x \in [1/2, 1]. \end{cases}$$

**Theorem 3.1.** [22] *As $a \to 0$, the measures $\mu_a$ converge weakly to the measure*

$$\frac{2}{3}\mu_0 + \frac{1}{3}\delta_{\frac{1}{2}},$$

*where $\delta_{\frac{1}{2}}$ is the Dirac measure at the point $x = 1/2$.*

This theorem implies that the invariant probability density functions for the $W_a(x)$ are a combination of an absolutely continuous and a singular measure, and in turn as $a \to 0$, it is noticeably the growing resemblance of this invariant density to a Dirac measure giving in some sense, the spirit of a phase transition in the sense of existence to non-existence of an a.c.i.m., but only as an asymptotic phenomenon, as we can see in Figure 3.3.

In this case, there is no doubling-period cascade transition, given that for all values of $a$ here considered, $W_a(x)$ is expansive, and therefore, there is no attracting set of points to which the trajectories can converge. Rather, this occurred because of the existence of diminishing invariant neighborhoods of the critical point. And the standard bounded variation methods for proving the quasi-compactness of the Perron-Frobenius operator cannot be applied in this family of maps, due to the slopes are not uniformly bounded away from 2, and since this is a system with stochastic perturbations, the theory behind the Lasota-Yorke theorem is not applicable. For further details, see [23].

## 3.3 Manneville-Pomeau maps

There exists another very well-known example of maps in the interval with an interesting dynamical transition as a function of one parameter; it has been widely studied as a model that presents the phenomenon of intermittency as defined in [1]. This phenomenon exhibits a transition from a "well-behaved" or regular periodic behavior into a chaotic or "turbulent" one in dissipative dynamical systems for larger values of $\alpha$.

The most iconic model of maps in the interval that display such behavior, is the family of Maneville-Pomeau. These maps are characterized by a parameter $\alpha > 0$, and one of the key particularities is that this map is always expansive, except at a neutral fixed point, where hyperbolicity is lost (because $T'_\alpha(0) = 1$). The equations defining this maps are:

$$T_\alpha(x) = \begin{cases} x\,(1 + (2x)^\alpha)\,, & x \in [0, 1/2) \\ 2x - 1, & x \in [1/2, 1], \end{cases}$$

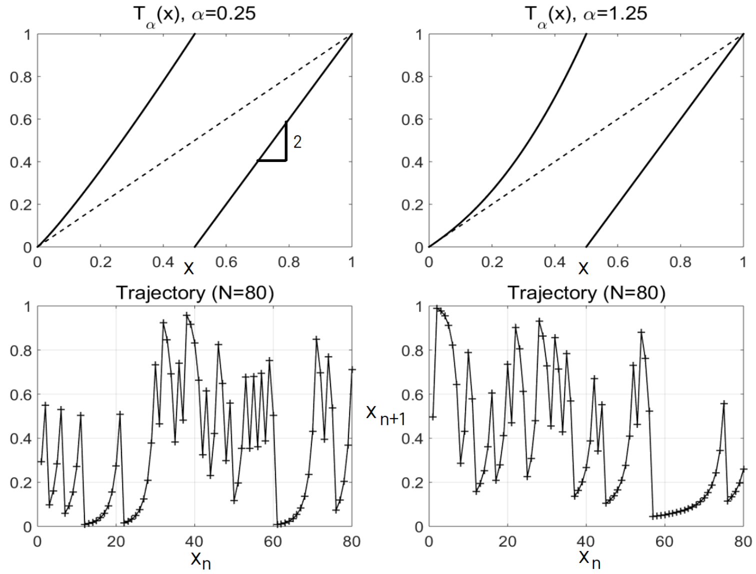



Figure 3.4: Manneville Pomeau map, with two different values of $\alpha$ (top) and a sample of its typical trajectory (bottom).

where $\alpha$ is the characteristic parameter, for which different ranges of values, imply a distinct asymptotic behavior of the dynamics in $T_\alpha(x)$. For $\alpha = 0$, this map is equivalent to the doubling map, for which it is known to preserve the Lebesgue measure, and the uniform distribution as its invariant density. When $\alpha \in (0, 1)$, $T_\alpha(x)$ has an finite invariant measure, which is commonly associated to the ergodicity and positivity of its typical Lyapunov exponent [32]; however, the Dirac measure at $x = 0$

is invariant as well [3]. And for $\alpha \geq 1$, there are no absolutely continuous invariant probability measures, whereas one still has a $\sigma-$finite, absolutely invariant measure.

As the value of $\alpha$ is increased, for the branch in $[0, 1/2)$, the limit map as $\alpha \to \infty$ is the identity map. For this reason, the trajectories in this range of values of $\alpha$ tend to perform long excursions in $[0, 1/2)$ (see Figure 3.4), associated to a laminar regime near $x = 0$ which coexists with a expansive or turbulent behavior near sufficiently away from $x = 0$ [32]. In this scenario, $T_\alpha(x)$ has a diverging invariant measure $\mu(x)$ near their indifferent fixed points, its invariant density $h(x)$ behaves as $h(x) \sim bx^{-\alpha}$ near $x = 0$, with $b$ is a positive real number and it is nonergodic [10]. We can see in Figure 3.5, how the limit density of $T_\alpha(x)$ as $\alpha \to \infty$ is the delta density in one point in $x \in [0, 1/2)$. Again, there is no doubling-period cascade transition here, and the phase transition in the sense of existence to non-existence of an a.c.i.m., only occurs in asymptotic manner. However, it is known that the decay of correlations is exponential for $\alpha = 0$ and polynomial for $\alpha \in (0, 1)$ (when the a.c.i.m. exists) and the convergence in Central Limit Theorems (CLTs) has been established [33], and the study of thermodynamic properties like Rényi entropy and topological pressure in [10] has shed light about the criticality of the values of these quantities at $\alpha = 1$, which is consistent at every instance with the change of the nature of the invariant density as the Manneville-Pomeau maps transit from and ergodic regime to a non-ergodic one. For further details, see the articles here cited and the references therein.

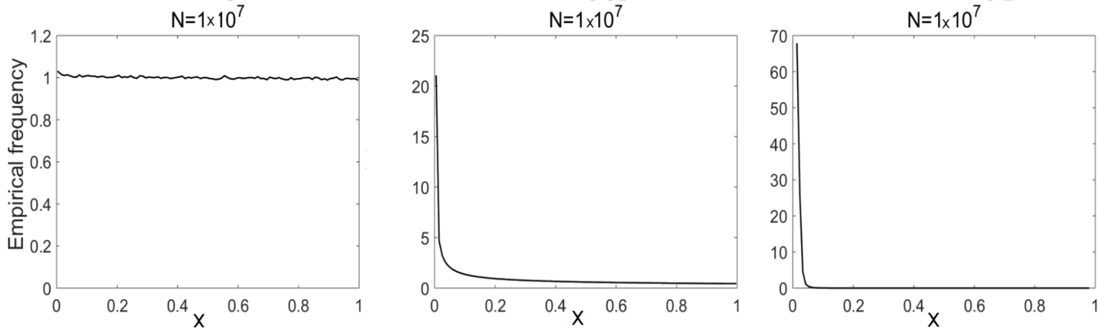



Figure 3.5: Empirical invariant densities $h_\alpha(x)$ for $T_\alpha(x)$, with $\alpha = 0.01$, $\alpha = 0.75$ and $\alpha = 5$ (from left to right). Due to the finite amount of iterations, the density for when $\alpha = 5$ appears to be a delta distribution. However, it is actually a power-law-like density near the origin; the delta distribution is the limit measure as $\alpha \to \infty$.

In the next chapter we will introduce the necessary concepts and notions about random dynamics in order to proceed towards the study of the phase transition phenomenon in random maps in the interval.

# Chapter 4

# Random dynamics

In order to model physical processes from "simple" models, a class of dynamical systems subject to a perturbations has been defined, due to the fact that, generally, the real physical phenomena are subject to noise, which in practice, or during implementation is often hard to quantify with certainty.

One way to introduce uncertainty and model this type of phenomena in pursuit of relatively simpler equations, is by means of random maps, which are discrete-time dynamical systems that are composed by a number of transformations, each equipped with a well-defined chance of occurring after every single iteration, given by a probability distribution. For more general cases and further details, we refer the reader to [12, 8].

First, we introduce in the following some basic notions regarding stochastic processes and probability theory. The measure space in this section is referred to as a probability space and the notation used in this section to describe it will be $(\Omega, \mathcal{F}, \mathbb{P})$, for the result space that takes vales in $\mathbb{R}$, the $\sigma$-algebra and the probability measure, respectively.

**Definition 4.1** (Stochastic process). *For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a stochastic process is a collection of random variables $\{X_t : t \in \Theta\}$ parameterized by a set $\Theta$, known as parametrical space, where the variables take values in $\Omega$, known as the space state.*

**Definition 4.2** (Independence). *Given a set of data $\{\omega_i\} \subset \mathcal{F}$, with $\mathcal{F} = \bigcup_{i=1}^{n} F_i$, and $i = 1, 2, ..., n$ for each $\omega_i \in \Omega$, where $F_i$ are the elements of the $\sigma-$algebra $\mathcal{F}$, and $i = 1, 2, ..., n$ on a probability measure space $(\Omega, \mathcal{F}, \mathbb{P})$, and if*

$$\mathbb{P}(F_n \cap F_m) = \mathbb{P}(F_n)\mathbb{P}(F_m), \quad n \neq m$$

*then the set of data is said to be independent.*

That is, the probability of observing two values $\omega_1$ and $\omega_2$ is simply the probability of observing $\omega_1$ times the probability of observing $\omega_2$. The random component when we define the random dynamical systems will only be considering independent identically distributed processes, since it is a natural way to model perturbations, and for the sake of simplicity.

Due to the fact that in the context of random variables we are dealing with a lot more of uncertainty than in deterministic sets of data, it is very important to distinguish on what terms we say that a random variable is "well behaved" in the sense of its convergence or divergence. We provide here some fundamental definitions on this regard.

**Definition 4.3** (Almost sure convergence). *We say that a sequence of random variables* $\{X_n\}_{n\geq 1}$ converges almost surely *to a random variable* $X$ *if*

$$N = \left\{\omega : \lim_{n\to\infty} X_n(\omega) \neq X(\omega)\right\} \textit{ and } \quad \mathbb{P}(N) = 0.$$

*Where* $N$ *is referred to as a* null set*, or a* negligible set*. Usually, almost sure convergence is abbreviated by writing*

$$\lim_{n\to\infty} X_n = X \quad \mathbb{P} - a.s.$$

**Definition 4.4** (Convergence in probability). *We say that a sequence of random variables* $\{X_n\}_{n\geq 1}$ converges in probability *to* $X$ *if for any* $\epsilon > 0$ *we have*

$$\lim_{n\to\infty} \mathbb{P}\left(\{\omega : |X_n(\omega) - X(\omega)| > \epsilon\}\right) = 0.$$

*This is also written*

$$\lim_{n\to\infty} \mathbb{P}\left(|X_n - X| > \epsilon\right) = 0.$$

The weakest type of convergence of random variables, but also the most commonly found in practice, is the convergence in distribution (also known as convergence in law); but it is also fundamentally relevant for the discussion of the Central Limit Theorem (CLT).

**Definition 4.5** (Cumulative distribution function). *If a random variable has a probability density function* $f_X(\omega)$*, then its corresponding cumulative distribution function (c.d.f.)* $F_X(\omega)$ *is the area under the probability density function (p.d.f.) denoted by*

$$F_X(\omega) = \mathbb{P}(X \leq x) = \int_{\{\omega\in\Omega:\omega\leq x\}} f_X(\omega) d\mathbb{P}$$

**Definition 4.6** (Convergence in distribution). *We say that a sequence of random variables* $\{X_n\}_{n\geq 1}$ converges in distribution *to* $X$ *if*

$$\lim_{n\to\infty} F_{X_n(\omega)} = F_X(\omega),$$

*for all* $\omega \in \Omega$ *at which* $F$ *is continuous.*

A fundamental result of probability theory that, roughly speaking, helps to justify our intuitive notions of probability is the large numbers law (LNL), which states that in the limit of infinite data $n \to \infty$ the mean of the sample converges to the expected value $\mathbb{E}(X_j)$, which is the same of each $j$, given that it is identically distributed. Namely, with a large enough quantity of data, the approximations increase their precision [38].

**Theorem 4.1** (Strong Law of Large Numbers)**.** *Let $\{X_n\}_{n\geq 1}$ be an independent, identically distributed (i.i.d.) process with law $\mathbb{P}$, defined on the same space with , with finite expected value $\tilde{\mu} = \mathbb{E}(X_j)$ and variance $\sigma^2 = \sigma^2_{X_j}\infty$. Let $S_n = \sum_{j=1}^{n} X_j$. Then*

$$\lim_{n\to\infty} \frac{S_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} X_j = \tilde{\mu}, \quad \mathbb{P}\text{-}a.s.$$

When dealing with dynamical systems, the strong large numbers law closely resembles the Birkhoff ergodic theorem, given that in this case, the time mean (equation 2.7) is analogous to the expected value in a stochastic process. When the convergence occurs in probability, we are dealing with the Weak Law of Large Numbers [37].

Another key result in probability theory is the central limit theorem (CLT), which helps to understand the rare of convergence of a random variable; from which a key observation needs to be noted: the only assumption about the distribution of the random variable $\{X_n\}_{n\geq 1}$ in question is that is has a finite variance. Therefore, if one is allowed to understand or redefine this random variable as the sum of many i.i.d. random variables with finite variances, it is possible to deduce that its distribution is approximately Gaussian, which opens the door to a more extensive estimations of its most important quantities.

**Theorem 4.2** (Central limit theorem)**.** *Let $\{X_n\}_{n\geq 1}$ be i.i.d. with $\mathbb{E}\left(X_j\right) = \tilde{\mu}$ and $\mathrm{Var}((X_j) = \sigma^2$ for all $j$, with $0 < \sigma^2 < \infty$. Let $S_n = \sum_{j=1}^{n} X_j$, and let $Y_n = \frac{S_n - n\tilde{\mu}}{\sigma\sqrt{n}}$. Then, $Y_n$ converges in distribution to $Y$, where $\mathcal{L}(Y) = N(0,1)$ is the Gaussian distribution with $\tilde{\mu} = 0$ and $\sigma^2 = 1$ :*

$$N(\tilde{\mu}, \sigma^2) \coloneqq f(\omega) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(\omega-\tilde{\mu})^2/2\sigma^2}.$$

It is proven in a slightly weaker version of the CLT that in the limit, for sufficiently-large sample sizes, there is convergence in distribution of $S_n/n$ towards $\tilde{\mu}$ at a rate of $\sqrt{n}$ (see [**?**, 37]), meaning that roughly speaking, the rate of convergence of the LNL is $\sqrt{n}$.

## 4.1 Zero-One probability laws

We need to provide some probabilistic results that are going to be necessary in this thesis for proving some of the results ahead. These zero-one laws provide the characteristics that a infinite sequence of random variables need to have in order to determine if a certain event (or subset of this sequence) has probability 0 or 1 of happening.

**Lemma 4.3** (Borel-Cantelli I). *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $A_n \in \mathcal{F}$, $n = 1, 2, ...$ If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ then $\mathbb{P}(A_n \text{ i.o.}) = 0$. "i.o." stands for* infinitely often.

Let $\mathcal{B}^{\infty}$ denote the Borel $\sigma-$field of subset of $\mathbb{R}^{\infty} = \{(x_1, x_2, ...) : x_i \in \mathbb{R}^1\}$ generated by events depending on finitely many coordinates. For example: for a random variable defined by the sum of two random variables $X_{1,2} \in [0,1]$, $\mathcal{B}^{\infty} = [0,1] \times [0,1]$.

**Lemma 4.4** (Hewitt-Savage zero-one law). *Let $X_1, X_2, ...$ be an i.i.d. sequence of random variables. If an event $A = \{(X_1, X_2, ...) \in B\}$, where $B \in \mathcal{B}^{\infty}$, is invariant under finite permutations $(X_{i_1}, X_{i_2}, ...)$ of terms of the sequence $(X_1, X_2, ...)$, that is, $A = \{(X_{i_1}, X_{i_2}, ...) \in B\}$ for any finite permutation $(i_1, i_2, ...)$ of $(1, 2, ...)$, then $\mathbb{P}(A) = 1$ or 0.*

In general, it is relatively easy to determine if an event in these type of infinite sequence of random variables satisfies the conditions for this results. What turns out to be more complicated is to point out which, 0 or 1, is the value that it will actually take [28].

## 4.2 Random maps

In [8] there are results about a type of dynamics where a non-singular deterministic transformation $T$ is considered, and at each instant of (discrete) time the state $x$ moves with probability $(1-\epsilon)$ to the next location $T(x)$, and with probability $\epsilon$ that the new state is given by the action of a random variable. They also define its Perron-Frobenius operator and proved its asymptotic stability as well as its convergence and uniqueness of its fixed point. They also treated the case of stochastic perturbations being constantly applied to a system; their result (see [8, 11]) implies that for a very broad class of transformations (including non-singular ones), the addition of a stochastic perturbation will cause the limiting sequence of densities to become asymptotically periodic, as a type of noise-induced order. This type of dynamics with a stochastic element taking part in its evolution provided a strong foundation for the proposition, formalization and study of the concept of random maps. In this thesis we are concerned with the study of one-dimensional random maps in the interval $[0,1]$.

For our setting, we consider $S$ to be an index set, and consider the family $\Gamma := \{\tau_k : \text{for } k \in S\}$ of measurable transformations of the interval into itself, that is, $\tau_k : I \to I$ for every $k \in S$. Consider also a sequence of $S$-valued random variables $\{\xi_n\}_{n=1}^{\infty}$ which are independent and identically distributed (i.i.d.) with probability law $\mathbb{P}$ on the space $(\Omega, \mathcal{F}, \mathbb{P})$.

**Definition 4.7** (Random maps in the interval)**.** *A random map in the interval which we denote by $T_{\xi,\Gamma}$, is a time discrete dynamical system in which, the state of the system at step $n+1$ is given by one of the transformations $\tau_k$ selected with respect to the random variable $\xi_n$. That is, given $\omega$ a realization of the random variable $\xi$, the trajectory corresponding to the initial condition $x \in I$ is given by,*

$$T_{\xi}^{n}(x) = \tau_{\omega_n} \circ \cdots \circ \tau_{\omega_1}(x). \tag{4.1}$$

Theorem 2.9 is a result of existence of the absolutely continuous measure for expansive maps in the interval, nonetheless, it does hold for completely deterministic dynamics. A natural question that arises is that if is it valid a random version of this theorem. This question is, in general, not completely answered. Nonetheless, there are several valid results for distinct forms of perturbations.

### 4.2.1 Random Perron-Frobenius operator

One can study this class of random maps in a more general framework of the skew-product (see for instance [26, 12]), but here for simplicity restrict ourselves to this particular situation. Next, if we consider $(\Omega, \mathcal{F}, \mathbb{P})$ to be a discrete finite probability space, with $|\Omega| = K$. The probabilities are given by $\mathbb{P}(\xi = i) = p_i$. With $\sum_{i=1}^{K} p_i = 1$. Then we say that the measure $\mu$ defined on $I$, is invariant under the random map $T_\xi$ if for every measurable set $A$, one has that,

$$\mu(A) = \sum_{i=1}^{K} p_i \mu(\tau_i^{-1} A). \tag{4.2}$$

In the case that $(\Omega, \mathcal{F}, \mathbb{P})$ is set to be a continuous probability space (i.e., when the cardinality of $\Omega$ is non-numerable), then the measure $\mu$ defined on $I$ is said to be invariant under the random map $T_\xi$ if for every measurable set $A$ one has that

$$\mu(A) = \int_{\Omega} \mu(T_{\xi}^{-1}(A)) \mathrm{d}\mathbb{P}.$$

Next, as we have done in the deterministic case, one can define the Perron-Frobenius operator associated to the random map $T_\xi$. Which, for the discrete case one has that for every $f \in L^1(\mu)$, the operator is given by

$$P_{T_\xi} f(x) = \sum_{i=1}^{K} \frac{p_i(\tau_i^{-1}(x)) f(\tau_i^{-1}(x))}{|\tau_i'(\tau_i^{-1}(x))|},$$

and an analogous expression can be defined for the case of $\Omega$ to be a continuous probability space, which is the following

$$P_{T_{\xi,\Gamma}} f(x) = \frac{1}{\theta - \gamma} \int_{\gamma}^{\theta} \frac{f(\tau_{\xi}^{-1}(x))}{|\tau_{\xi}'(\tau_{\xi}^{-1}(x))|} \chi_{\tau_{\xi}(I)}(x) d\Omega,$$

where $\gamma$, $\theta \in \{\xi_n\}_{n=1}^{\infty}$ are parameters which will set the boundaries in which the realization of $\xi$ associated to the random map $T_{\xi,\Gamma}$, occurs.

## 4.3 Existence of invariant densities associated to a random map

In this section we recall some of the most relevant results concerning the existence of an a.c.i.m in the setting of dynamical systems equipped with a random component. These results include settings which range from random transformations with position-dependent probabilities, skew-product transformations and a condition of mean expansiveness, expansive $\beta$-transformations and homeomorphisms in the circle. In these references there is a diverse range of techniques and considerations for the analysis of these type of transformations. However, in none of these settings is considered a case for the occurrence of a explicit change in the nature of the dynamics, namely, the conditions for a phase transition in the sense of non-existence to the existence of an a.c.i.m.

### 4.3.1 Position-dependent random maps

We present first a theorem from [4] for determining the existence of an absolutely continuous invariant measure associated to random maps in the interval $[0,1]$ on itself, that are piecewise one to one, nonsingular transformations on a partition $\mathcal{P}$ of the state space $X$, equipped with a function of weighting probabilities $p_k(x)$ which makes them position dependent. These random maps are also conformed by a finite number of transformations $\{\tau_k\}_{k=1}^{K}$. Moreover, Góra and Boyarsky propose a method, briefly described here, from [4] for computing the invariant densities of piece-wise linear semi-Markov (this concept is properly defined later on) and position-dependent random maps. A position-dependent random map $T{=}T_{(\Gamma,P)}$, where $\Gamma = (\tau_1, \tau_2, ..., \tau_K)$ is a collection of $K$ maps in the interval and $P = (p_1, p_2, ..., p_K)$ is a collection of the corresponding probabilities that are position-dependent, one has that $p_k(x) \geq 0$ for $k = 1, 2, ..., K$ and $\sum_{k=1}^{K} p_k(x) = 1$, for all $x \in I$. Thus, after each iteration, the random map $T$ comes from point $x$ to $\tau_k(x)$ with probability $p_k(x)$. For a fixed collection of maps $\Gamma$, $T$ can have distinct invariant probability density functions, depending on the choice of the weighting probabilities $P$ [5].

**Theorem 4.5** ([4]). *Let $T_{(\Gamma,P)}$ be the random map for which the following assumptions hold:*

- *There exist partitions $\mathcal{P} = \{I_1^{(k)}, ..., I_{qk}^{(k)}\}$, $k = 1, 2, ..., K$, such that each $\tau_{k_i} = \tau_k|_{I_i^{(k)}}$, with $i = 1, ..., q_k$, $k = 1, 2, ..., K$*
  1. *is monotonic,*
  2. *$\mathcal{C}^2$,*
  3. *$|\tau'_{k_i}| \geq \sigma > 1$, for some universal constant $\sigma$, for all $i$, and*
  4. *the functions assigning probabilities to each map $\tau_k$, $p_k(x)$, $k = 1, 2, ..., K$ are piece-wise $C^1$ functions.*

*Let $\delta = \min\{\lambda(I_i^{(k)}) : i = 1, ..., q_k, k = 1, 2, ..., K\}$ and $\beta_k = \sup_{x\in I} p_k(x)$, $k = 1, 2, ...K$, then for each $f \in BV(I)$,*

$$\bigvee_I (P_T f) \leq A \bigvee_I f + B \int_I |f| d\lambda,$$

*where*

$$A = \frac{2(\beta_1 + \beta_2 + ... + \beta_K)}{\sigma}, \; B = \frac{2(\beta_1 + \beta_2 + ... + \beta_K)}{\sigma\delta} + \max_{k=1,2,...,K} \sup_I \left| \left( \frac{p_k}{\tau'_k} \right)' \right|.$$

*If $A < 1$, then the random map $T$ has an absolutely continuous invariant density (a.c.i.m.) $\mu$. Moreover, the operator $P_T$ is quasi-compact.*

The proof of this theorem can be found in [4].
Now, the method for computing the invariant densities of random maps is specific for the piece-wise linear semi-Markov maps, and it is a generalization of the matrix solution to the inverse Perron-Frobenius problem for the deterministic version of these maps proposed by the same authors in [35]. We recall here the definition they provide:

**Definition 4.8** (Piece-wise linear semi-Markov map). *A map $\tau$ is a piece-wise linear semi-Markov map on a partition $\mathcal{P} = \{I_1, I_2, ..., I_q\}$ if any interval $I_i$ can be further partitioned into subintervals $\{J_1^{(i)}, ...J_{r_i}^{(i)}\}$ such that $\tau|_{J_r^{(i)}}$ is linear and its image is a union of a number of intervals of $\{I_1, I_2, ..., I_q\}$.*

### 4.3.2 Obtaining the invariant densities associated to the random map

For the piece-wise linear semi-Markov map $\tau$, the Perron-Frobenius operator consists on a matrix $\mathbb{M} = (m_{i,j})_{1\leq i,j\leq q}$, with

$$m_{i,j} = \frac{1}{\left|\tau'|_{J_r^{(i)}}\right|} \cdot \delta(i, r, j),$$

where $\delta(i,r,j) = 1$ if $\tau(J_r^{(i)}) \supset I_j$ and 0 otherwise. The invariant density of a piece-wise linear semi-Markov map is represented by a vector $f = [f_1, f_2, ..., f_q]$, $f_i = f|_{I_i}$, which is normalized by the requirement $\sum_{i=1}^{q} f_i = q$. Similarly, each weight assigned to the probabilities are assumed to be constant on the elements of the partition $\mathcal{P}$ and are represented by $p = [p(1), p(2), ..., p(q)]$, $p_i = p|_{I_i}$, $0 \le p(i) \le 1$. Under this notation, the $T$−invariant density is

$$f = \sum_{k=1}^{K} \mathbb{M}_k^{\dagger} diag(p_k) f, \tag{4.3}$$

where "†" denotes the transposed operator and diag$(p_k)$ is a diagonal matrix with elements $p_k(1), p_k(2), ..., p_k(q)$ in its diagonal. Given a fixed collection of piece-wise linear semi-Markov maps $\Gamma = (\tau_1, ..., \tau_K)$ and considering the set of attainable densities $\mathcal{A}_\Gamma^{pc}$ as the set of functions $f$ such that there exists a vector $P = (p_1(x), p_2(x), ..., p_K(x))$ that assigns a weigth to the probabilities, so $f$ is an invariant density of the random map $T_{(\Gamma,P)}$. In [5] it is proven that $\mathcal{A}_\Gamma^{pc}$ is convex and in the same paper, a method for computing its extreme points is given after assigning a vector of probabilities $P$ constituted by zeros and ones. The theorem is presented as follows.

**Theorem 4.6** (Extreme points in the set of attainable densities [5])**.** *Let $\Gamma = (\tau_1, ..., \tau_K)$ be a fixed collection of piece-wise linear semi-Markov maps defined on a common partition $\mathcal{P}$. Let $\mathcal{A}_\Gamma^{pc}$ the set of attainable densities that is being considered assigning constant weighting-probabilities on elements of $\mathcal{P}$. If $f$ is an extreme point of $\mathcal{A}_\Gamma^{pc}$, then $f$ corresponds to a weighting-probability $p_k = [p_k(1), ..., p_k(q)], k = 1, ..., K$, where the components of each $p_k$ are 0 or 1.*

For the proof, we refer to the reader to [5]. The next case, taken from the same paper, exemplifies Theorem 4.5:

**Example 4.1.** *Considering the two piece-wise linear semi-Markov maps $\tau_1, \tau_2$ that preserve the Lebesgue measure, such that the position-dependent random map constructed with these has more than one invariant density. Let $\tau_1, \tau_2$ defined on $[0,1]$ with a Markov partition $\mathcal{P} = \{[0,1/2],[1/2,1]\}$ and their corresponding Perron-Frobenius matrices*

$$\mathbb{M}_1 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \quad \mathbb{M}_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

*For the probabilities $p_1 = (p_1(1), p_1(2)) = (s,t)$, $p_2 = (1-p_1(1), 1-p_1(2))$ the invariant density $f = (f_1, 2-f_1)$ is obtained, normalized by the condition $f_1 + f_2 = 2$ :*

$$f_1 = \frac{8-2t}{8-t-s}, \quad f_2 = 2 - f_1.$$

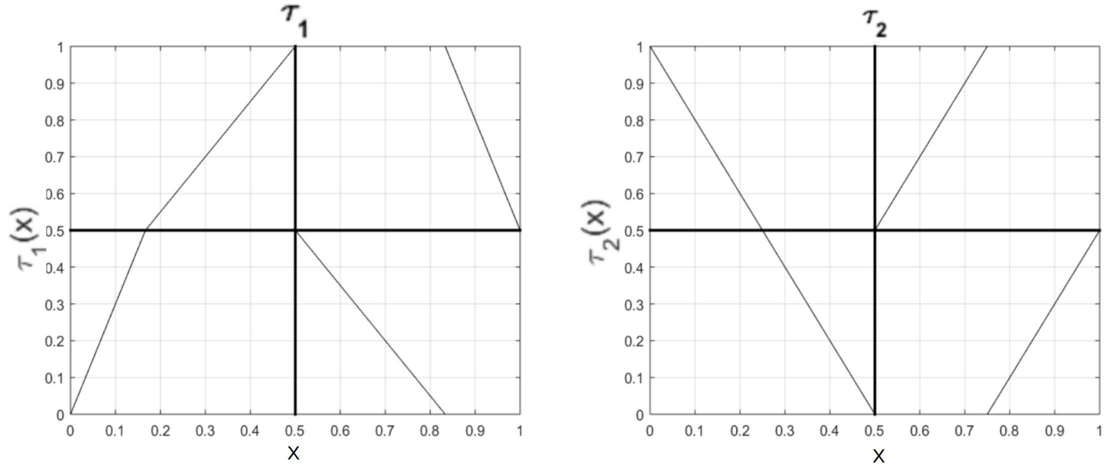



Figure 4.1: Possible representation of the configuration of the maps $\tau_1$ and $\tau_2$, based on their Perron-Frobenius matrices. The thick lines delimit the elements of $\mathcal{P}$.

*Note that* $f_1$ *can assume any value in the interval* $[6/7, 8/7]$*, with* $f_1((1,0)) = 6/7$ *and* $f_1((0,1)) = 8/7$.

**Remark 4.1.** *The results of the Theorems* 4.5 and 4.6*, and of the proposed method in* [5] *for obtaining the invariant densities hold for maps that are piecewise linear semi-Markov, piecewise* $\mathcal{C}^2$ *and piecewise monotonic, and that are equipped with weighting-probabilities functions assigned by their position in the state space.*

We need to precise that this result for the existence of acims in random maps, as well as some of the ones here presented, only consider random maps conformed by a finite set of transformation to choose from.

### 4.3.3 Random maps as a projection of skew-products

Given a measure preserving transformation $T$ and an $\omega$-valued random variable, $\xi$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and considering a model of a random dynamical system whose time evolution is given by

$$x_{n+1} = \tau_{\xi_{n+1}(\omega)}(x_n) \quad \text{for} \quad n \geq 1,$$

where $\xi_n = \xi \circ \sigma^{n-1}$. The systems that of T. Morita works with in [14] are known as a *skew-product transformation*, $T$ on $I \times \Omega$, defined by

$$T(x, \omega) = \tau_{\xi_1(\omega)}(x_n) \quad \text{for} \quad (x, \omega) \in (I \times \Omega)$$

Given that $\text{proj}_I \circ T^n(x, \omega) = \tau_{\xi_n(\omega)} \circ \tau_{\xi_{n-1}(\omega)} \circ \ldots \circ \tau_{\xi_1(\omega)}(x)$, they investigate the asymptotic behavior of the dynamical system $(T, m \times \mathbb{P})$ instead of the one-dimensional random dynamical system. They show that $\{\xi_n\}_{n=1}^{\infty}$ is a stationary set of dependent random variables, the skew product transformation has a number of $(m \times \mathbb{P})$-acims, according to the spectral decomposition of the set of random variables $\{\xi_n\}_{n=1}^{\infty}$.

Their main theorem in [9], states a result about the existence of a number of $(T, m\times\mathbb{P})$-acims, and about the ergodicity of these measures. The major assumption from which it follows is, one of expansiveness in mean, taking into account the respective probabilities of the constituting maps, from which is evident that even if a very contractive map conforms the random transformation, having a small enough weight probability may not make it contractive in mean.

This result, just like Theorem 4.7 next, does not impose restrictions on the derivative of the maps, other than being greater than zero, in absolute value. Nonetheless, they provide only sufficient conditions for the existence of the acim; besides, it only allows the Lebesgue measure of the intervals of monoticity being greater than zero.

### 4.3.4 Random maps with constant probabilities

S. Pelikan in [15] explores the behavior in dynamical systems on the unit square $[0,1]\times[0,1]$, by representing them through random maps of $[0,1]$. This construction, is understood as a pseudo-skew product. They provide sufficient conditions for a random map of one dimension to have an acim. The random maps they work with are Lasota-Yorke type, defined as in the classic Theorem 2.9 of 1973.

The random maps they study are similarly defined to those in [5], with the difference that the maps conforming it have position-dependent probabilities, and the ones in the work of S. Pelikan allow the absolute value of the derivative of the maps to be less than one. Their result is stated as follows:

**Theorem 4.7** (Existence of an acim [15]). *Let $T(x) = T_i(x)$ (with probability $p_i$), $i = 1, \dots, M$ be a random map of $[0,1]$, where each $T_i$ is a Lasota-Yorke map, and $\sigma \in \mathbb{R}_+$. If for all $x \in [0,1]$,*

$$\sum_{i=1}^{M} \frac{p_i}{|T_i'(x)|} \le \sigma < 1, \tag{4.4}$$

*then for all $f \in L^1([0,1], \mathrm{m})$:*

1. *The limit*
$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} P_T^j(f) = f^* \quad \text{exists in } L^1,$$
2. $P_T(f^*) = f^*$,
3. $\bigvee_0^1 f^* \le C\|f\|_1$ *for some constant $C > 0$ which is independent of $f$.*

This result, along with those published by P. Góra [5] are for random maps equipped with a finite set of transformations to choose from, in contrast with the ones included in the next reults of existence, and the class of maps we are interested in.

Is it worth noting that based on this theorem, we proved in [17] that a non-uniformly expansive random map with a parameter of probability has an acim but for a single value of its parameter $\alpha \in [0,1]$. We will provide more details ahead.

### 4.3.5 Existence of S.R.B. measures in random maps

The concept of a S.R.B. (Sinai-Ruelle-Bowen) measure has been proposed, formalized and studied for the analysis of local instability of attractors in dissipative systems [36]. Given the fact that a large number of dynamical systems admit more than one ergodic measure [7], it is logical to investigate which one is more important and whether it is "typical" under the consideration that most observable events are positive Lebesgue measure sets; or if the dynamics in a given system is better understood by events of zero Lebesgue measure sets (see [36] for extensive details). For random dynamical systems, a S.R.B. measure is defined as follows.

**Definition 4.9** (S.R.B. measure)**.** *A probability measure $\mu$ is* S.R.B. *on $\mathcal{X}$ for a random dynamical system, if for* $\mathbb{P}$-almost all $\omega \in \Omega$*, the set $B_\omega(\mu)$ of points $x \in \mathcal{X}$ such that*

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{f_{T^{k-1}\omega} \circ \cdots \circ f_\omega(x)} \to \mu \quad \text{vaguely, as} \quad n \to \infty$$

*has positive Lebesgue measure, being $B_\omega(\mu)$ known as the* random basin of $\mu$ [7].

A sequence of measures $\{\mu_n\}_{n=1}^{\infty}$ on a measure space $(\Omega, \mathcal{F})$ is said to converge vaguely to a measure $\mu$ if $\int_\Omega g d\mu_n \to \int_\Omega g d\mu$ for all continuous functions vanishing at infinity, i.e., $g(\omega) \to 0$ as $\omega \to \infty$.

In [19], they provide the proof to a theorem on the existence of S.R.B. measures for random Lasota-Yorke transformations $f : [0,1] \to [0,1]$, specifically, random $\beta-$transformations $f(x) = \beta x \pmod 1$, where $\beta$ is distributed accordingly to any stationary stochastic process in $(1, \infty)$. They define the stationary process as $f_{n+1} = f \circ T^n$, where $T$ is an automorphism of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\delta_x$ the Dirac measure at a point $x$.

**Theorem 4.8.** *Let $\mu_1, \ldots, \mu_r$ be the finite collection of all ergodic a.c.i.m.'s for the skew-product $F$ on $\Omega \times [0,1]$. Define $\nu_i$ as the projection on $[0,1]$ of the acim $\mu_i$. Then each $\nu_i$ is an absolutely continuous S.R.B. measure. Moreover,* $\mathbb{P}$-almost surely*, the union of their basins has total Lebesgue measure.*

In other words, this result states that the union of a finite number of S.R.B. measures on $[0,1]$ contains $\mathbb{P}$-almost every orbit of $F$, conformed by the union of their respective basins. While this theorem provides the conditions for a class of random Lasota-Yorke interval maps to admit finitely many ergodic absolutely continuous invariant probability measures, they also posed a question: "Under what assumptions one can obtain quasi-compactness of the Perron-Frobenius (PF) operator of the random $\beta$-transformation?" And this question was answered by Góra and Bahsoun in [29], for $\beta \in (1, \infty)$ giving the following lemma.

**Lemma 4.9** (Quasi-compactness of PF operator for random $\beta$-transformations)**.** *Let $I = [0,1]$ and $\tau_\beta : I \to I$ be a $\beta-$transformation with $\beta \in (1, \infty)$, we have*

$$\bigvee_I P_{\tau_\beta} \le A \bigvee_I f + \delta ||f||_1,$$

*where $A = 2/\beta$ and $\delta = 1$.*

With this lemma, they obtained a bound for the Perron-Frobenius operator which depends uniquely on $\beta$. Thus, the Perron-Frobenius operator is quasi-compacteness and it admits a fixed point in the space of bounded variation functions $BV(I)$.

### 4.3.6 Random maps equipped with a continuum of transformations

In [16], they prescind of the skew-product setting, and consider a class of random maps in the interval $\tau_t : \mathcal{X} \to \mathcal{X}$, equipped with a random parameter $t \in W$, whose space is allowed to have cardinality of continuum. In each iteration, a transformation is selected from a set $\{\tau_t : t \in W\}$, determined by this random parameter and by the probability density function $p(t,x) : W \times \mathcal{X} \to [0, \infty)$, which makes them position-dependent. The maps $\tau_t$ considered in this paper are assumed to be non-singular piece-wise monotone transformations $t$-measurable for every $x \in \mathcal{X}$.

They allow as well, the interval $[0,1]$ to be partitioned into subintervals by using a set of subindexes $\Lambda$.

Let $\Lambda$ be a countable or finite set and let $\Lambda_t \subseteq \Lambda$ for each $t \in W$, such that int$(I_{t,i})\cap$int$(I_{t,j})= \varnothing$, with $(i \neq j)$ and being $m$ the Lebesgue measure, we have $m\left([0,1] \setminus \bigcup_{i\in\Lambda_t} I_{t,i}\right) = 0$.

With this considerations, and being $(W, \mathcal{B}, \nu)$ a $\sigma-$finite measure space the parameter space, and $T = \{\tau_t, p(t,x), \{I_{t,i}\}_{i\in\Lambda} : t \in W\}$ the denomination for the class of random maps that T. Inoue works with, the function $g(t,x)$ is set to be:

$$g(t,x) = \begin{cases} p(t,x)/|\tau_t^{'}(x)|, & x \in \bigcup_i \operatorname{int}(I_{t,i}) \\ 0, & x \in [0,1] \setminus \bigcup_i \operatorname{int}(I_{t,i}) \end{cases}$$

Thus, for their main result they assume the following conditions:

(a) $\sup_{x\in[0,1]} \int_W g(t,x) \nu dt \le \alpha < 1$;

(b) There exists a constant $M$ such that $\bigvee_{[0,1]} g(t,\cdot)$ for $\nu$-a.s. $t \in W$, that is, there exists a $\nu-$measurable set $W_0 \subset W$ such that $\int_W p(t,x)\nu dt = 1$ and $\bigvee_{[0,1]} g(t,\cdot) < M$ for all $t \in W_0$.

**Theorem 4.10.** *Let $T = \{\tau_t, p(t,x), \{I_{t,i}\}_{i\in\Lambda} : t \in W\}$ be a random map. Assume that the random map $T$ satisfies conditions* (a) *and* (b) *above. Then $T$ has an invariant probability measure which is absolutely continuous with respect to Lebesgue measure.*

Similarly to the results in [5], [2], they prove the quasi-compactness of the Perron-Frobenius operator for this class of systems by classical techniques of bounded variation, and further, declaring Theorem 4.10 to be a generalization of these previous results. Moreover, analogously to the results in [9], given that the condition (a) is automatically satisfied if $\inf_{x\in[0,1]} |\tau_t'(x)| > 1$, it can be understood as a condition of expansiveness in mean, that can also be met for $\inf_{x\in[0,1]} |\tau_t'(x)| < 1$, if a suitable probability density function $p(t,x)$ is chosen.

### 4.3.7 Invariant measures on iterated function systems

Regarding a class of iterated function systems consisting of uncountably many homeomorphisms of the circle, and in a similar tenor that in [19] and [16], given that previously to this result it was assumed that the systems contained at most countably many transformations, G. Luczynska in [?] proved the existence of a unique invariant measure associated to the Markov operator $P$ corresponding to the aforementioned iterated function system.

Let $(S^1, d)$ be a metric space where $S^1$ denotes a unit circle with counterclockwise orientation. For $x, y \in S^1$, the distance between $x$ and $y$ is given by $d(x,y) := \min\{d[x,y], d[y-x]\}$.

Let $\Psi = \{S_\lambda\}_{\lambda\in[0,c]}$ be a family of orientation preserving homeomorphisms in the circle, such that $S_\lambda : S^1 \to S^1$ for every $\lambda \in [0,c]$, where $c \in (0,\infty)$ is fixed. The family $\Psi = \{S_\lambda\}_{\lambda\in[0,c]}$ defines an action of the semigroup of all compositions of the form $S_{\lambda_n,\dots,\lambda_1} := S_{\lambda_n} \circ S_{\lambda_{n-1}} \circ \cdots \circ S_{\lambda_1}$.

Let $p : [0,c] \to [0,1]$ be a probability density function; i.e., $\int_0^T p(\lambda)d\lambda = 1$ and

$$\mathbb{P}(\lambda \le t) = \int_0^t p(u)du, \quad \text{for all} \quad t \in [0,c]$$

for the random variable $\lambda$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $[0,c]$. They call the pair $(\Psi, p)$, an *iterated function system.*

The $e$-property for Markov operators is introduced by the following definition.

**Definition 4.10.** *The Markov operator $P$ satisfies the $e-$property if for every Lipschitz function $f : S \to \mathbb{R}$ and every $x \in S$ it holds*

$$\lim_{y\to x} \sup_{n\in\mathbb{N}} |U^n f(x) - U^n f(y)| = 0,$$

*where $U$ is such that $P$ is its dual.*

**Definition 4.11.** *Let the family $\Psi = \{S_\lambda\}_{\lambda\in[0,c]}$ be given. Let $p$ be a probability density function on $[0,c]$. We say that the family $\Psi$* acts minimally *if for every non-empty set $A \in \mathcal{B}(S^1)$, if it holds that $S_\lambda(A) \subset A$ for $\mathbb{P}-$a.e $\lambda$, then $\bar{A} = S^1$.*

The minimality of the family $\Psi$ is utilized to prove the $e$-property in [**?**] and the uniqueness of an invariant measure for a relevant Markov operator. They establish this dependence in the following theorem.

**Theorem 4.11.** *Let $\Psi^{-1} := \{S_\lambda^{-1}\}_{\lambda\in[0,c]}$ act minimally and let $p$ be a probability density function on $[0,c]$. Then the operator $P$ corresponding to the iterated function system $(\Psi,p)$ satisfies the e-property and admits a unique invariant measure.*

Their main result is the theorem which relates the minimality of the action and the uniqueness of the invariant measure for the corresponding iterated function system.

**Theorem 4.12.** *Let $\Psi = \{S_\lambda\}_{\lambda\in[0,c]}$ act minimally and let $p$ be a probability density function on $[0,c]$. Then the iterated function system $(\Psi,p)$ admits a unique invariant measure.*

In [**?**] the space $S^1$ is chosen for the sake of simplicity. They assure that instead of $S^1$, it could have been considered any other $1-$dimensional compact manifold with an order.

Once we outlined the state of the art about the existence of an invariant measure in a discrete-time system equipped with random dynamics, we recall next, the numerical evidence we obtained in two cases of study (see [17]) of random maps in the interval where a phase transition phenomenon takes place.

## 4.4 Random maps in the interval with no spontaneous phase transition

As we mentioned earlier, a setting regarding random transformations in which the phenomenon of phase transition in the sense of the non-existence to the existence of an a.c.i.m. takes place, has not been reported in the literature up to our knowledge. Our aim is to slightly widen the insight in which these systems are being studied. Analogously to the deterministic examples we recalled in the previous section, where the transition can be identified as a function of a single parameter, we propose here a setting where a single parameter $\gamma$ determines the nature of random component of the dynamics. In this case, this random component will be the range of the different maps conforming the transformation, which for simplicity, is considered to be uniform.

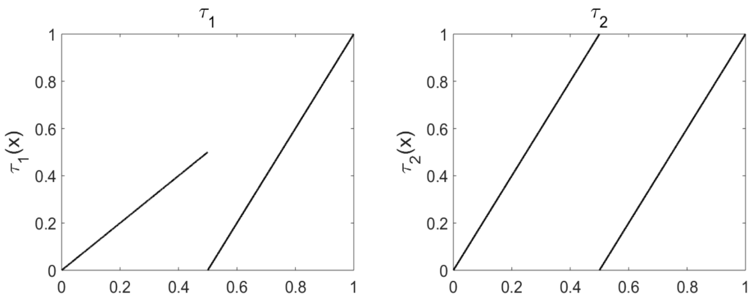



Figure 4.2: Graphs of $\tau_1$ and $\tau_2$.

In [17], we study a family of random maps in the interval exhibiting a phase transition phenomenon in the sense of non-existence to existence of an invariant density, and we presented two families of random maps in the interval that exhibits this phenomenon.

For our first case of study (a random map in the interval conformed by two transformations), let $(\Omega, \mathcal{F}, \mathbb{P})$ be the discrete probability space with $\Omega = \{1, 2\}$ and $\mathbb{P}$ be a Bernoulli distribution such that, for each realization of the random process, we fix an $\alpha \in [0, 1]$, and the probability of choosing the symbol 1 ($p_1$) is $\alpha$ and $1 - \alpha$ for the symbol 2 ($p_2$). We consider $\Gamma = \{\tau_1, \tau_2\}$ the set of maps in the interval which constitute $T_1$, our first case of random map in the interval. The maps $\tau_1$ and $\tau_2$ are defined as

$$\tau_1(x) = \begin{cases} x, & 0 \le x \le 0.5 \\ 2x - 1, & 0.5 < x \le 1 \end{cases}, \qquad \tau_2(x) = \begin{cases} 2x, & 0 \le x \le 0.5 \\ 2x - 1, & 0.5 < x \le 1. \end{cases} \tag{4.5}$$

For an arbitrary initial condition, each map is applied with probability $p_1$ and $p_2$, respectively. We determine $\alpha$ to be a parameter of probability, and we study how the dynamics of $T_1$ change along with the value of $\alpha$. Note that for the extreme values of $\alpha$ (0 and 1), the behavior is completely deterministic, being respectively $\tau_1$ the doubling map, which is known to preserve the Lebesgue measure; and $\tau_2$ being a transformation which contains the identity map on $\left[0, \frac{1}{2}\right]$, displays a $\delta$ density centered at one point on $\left[0, \frac{1}{2}\right]$, depending on the initial condition.

Therefore, given that this is a non-uniformly (due to the presence of the identity map in $\tau_1(x)$) expansive random transformation for $\alpha \in (0, 1]$, it is natural to ask if this random dynamics has an a.c.i.m., and in [17] we proved that it has no spontaneous phase transition, and thus we obtain the following theorem:

**Theorem 4.13.** *Let us consider the random map defined by the random composition of $\tau_1$ and $\tau_2$, as in (4.5), selected by a Bernoulli processes with parameter $\alpha \in [0, 1]$. This random map has an a.c.i.m. for all $\alpha \neq 1$, and thus, it has no phase transition.*

*Proof.* We use the theorem (4.7). Therefore, we need to compute $\sum_{i=1}^{2} \frac{p_i}{|\tau_i'(x)|}$. We have only two different maps and since the maps are piecewise linear, the slope is constant for both maps in two intervals. Then, for $x \in [0, 0.5]$, we have $\sum_{i=1}^{2} \frac{p_i}{|\tau_i'(x)|} \leq \frac{\alpha+1}{2}$, and for $x \in (0.5, 1]$, the sum $\sum_{i=1}^{2} \frac{p_i}{|\tau_i'(x)|} = 1/2$. In both cases it is strictly less than 1 whenever $\alpha \neq 1$, and thus the random map $T_1$ has an a.c.i.m. for all values of $\alpha$ except for $\alpha = 1$. Thus, the random map $T_1$ has an a.c.i.m. for all values of $\alpha$, except for $\alpha = 1$. □

Moreover, after proving the map $T_1$ has an acim for all values of $\alpha \in [0, 1)$, we used the method proposed in [5] for obtaining the invariant density explicitly, despite not explicitly being applicable, because of the non-uniformly expansive nature of $T_1$. Nevertheless, one can obtain the explicit expression for the density function associated to $T_1$, for $\alpha \in [0, 1)$. It is known that the invariant density of a piecewise linear Markov map is piecewise constant on the partition defined for the random map, and in the case of $T_1$, is clearly seen that this partition $\mathcal{P} = \{I_1 = [0, 0.5], I_2 = (0.5, 1]\}$, the way are defined in [5], is common for $\tau_1$ and $\tau_2$. We give the details about this computation as follows next.

The corresponding Perron-Frobenius operator for $T_1$ is a matrix given by

$$m_{ij} = \frac{1}{\left|\tau' \, I|_{I_i}\right|} \cdot \delta(i, j),$$

where the delta function is one if $I_j \subset \tau(I_i)$, and zero otherwise. The density vector will be given by $f = [f_1, f_2]^\dagger$, such that $f_1 + f_2 = 2$. Then, given that the matrices $\mathbb{M}_i^\dagger$ and $\text{diag}(p_k)$ are

$$\mathbb{M}_1^\dagger = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}, \quad \mathbb{M}_2^\dagger = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}, \quad \text{diag}(p_k) = \begin{bmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{bmatrix},$$

we are ready to compute the invariant densities for $T_1$, by the use of the equation (4.3), which yields

$$f_1 = \frac{2}{2-\alpha}, \quad f_2 = \frac{2-2\alpha}{2-\alpha}.$$

These expressions are completely consistent with the numerical simulation of the dynamics of $T_1$, due to the density represented by the histograms (Figure 4.3) converges to the expected invariant density prescribed by the method as a function of the number of iterations performed in each realization (see Figure 4.4).

Furthermore, we also verified that the associated measure induced by this density to be actually invariant under the random map $T_1$ as shown next, by using the equation (4.2).

$$\mu_f(A) = \frac{2}{2-\alpha} m(A_1) + \frac{2(1-\alpha)}{2-\alpha} m(A_2), \tag{4.6}$$

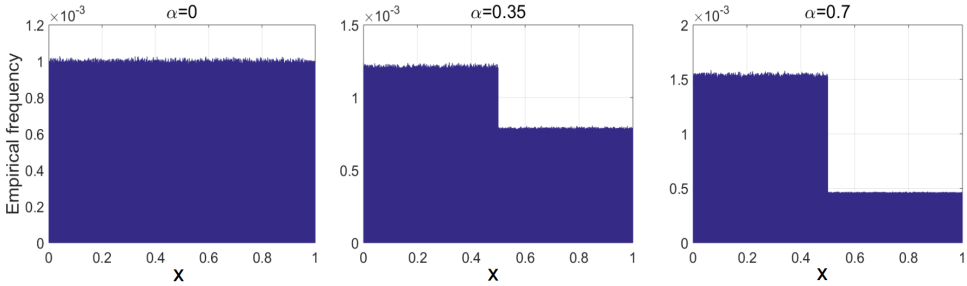



Figure 4.3: Empirical density function associated to the random map $T_1$ in the interval, for three different values of $\alpha$.

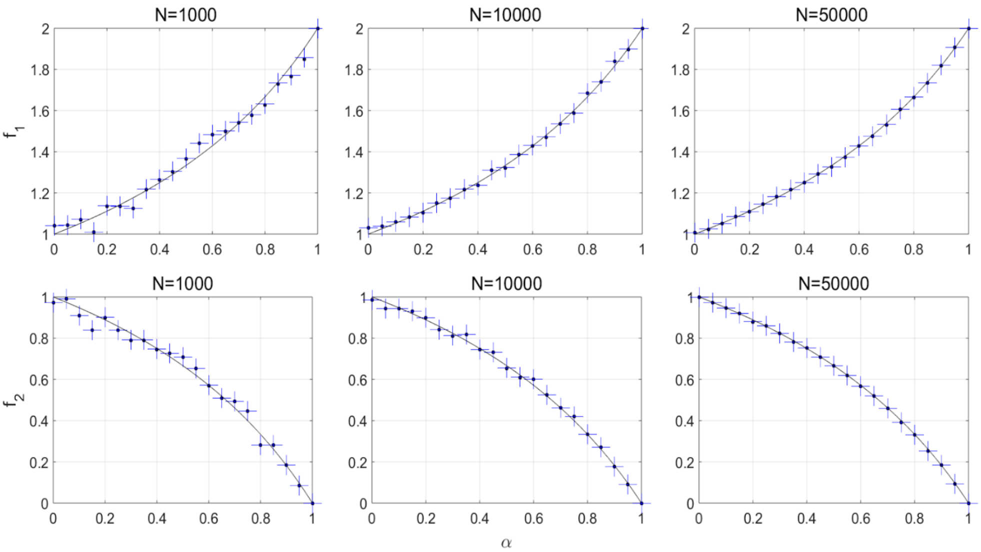



Figure 4.4: Convergence of the empirical density to the expected density function $f_i$ on $I_1$ (top) and $I_2$ (bottom) for 20 values of $\alpha$. The empirical density (crosses) converges to the density function obtained by the Góra-Boyarsky method (solid black line) as $N$, the number of iterations is increased.

where $m$ denotes the Lebesgue measure, and $A_i := A \cap I_i$ for $i = 1, 2$. Thus, we proceed to present the proof that (4.6) is preserved under the action of the random map of $T_1$, for $\alpha \in [0, 1)$.

**Proposition 4.1.** *Let the random map be defined as considered in Theorem 4.13. Then, the measure given by* (4.6) *is invariant under* $T_1$. *That is, the density* $f$ *obtained by the Góra-Boyarsky method is indeed, its invariant density.*

*Proof.* Let $A \subset [0, 1]$ be any Lebesgue measurable set. Let us define $B_{k,i} = \tau_k^{-1}(A) \cap I_i$, where $I_i$ are the elements of the partition of the interval, $I_1 = [0, 1/2]$ and $I_2 = (1/2, 1]$. Observe that, using the definition of $\mu_f$ given by (4.6), for each map $\tau_k$, one has that

$$\mu_f(\tau_k^{-1}(A)) = \frac{2}{2-\alpha} m(B_{k,1}) + \frac{2(1-\alpha)}{2-\alpha} m(B_{k,2}).$$

We want to check the invariant condition in the random case, which is given by (4.2). Since the weighting probabilities are $\alpha$ and $1-\alpha$ for $\tau_1$ and $\tau_2$, respectively, we have

$$\sum_{k=1}^{2} p_k \mu_f(\tau_k^{-1}(A)) = \alpha \mu_f(\tau_1^{-1}(A)) + (1-\alpha)\mu_f((\tau_2^{-1}(A))$$

$$= \alpha\Big[\frac{2}{2-\alpha} m(B_{1,1}) + \frac{2(1-\alpha)}{2-\alpha} m(B_{1,2})\Big] + (1-\alpha)\Big[\frac{2}{2-\alpha} m(B_{2,1}) + \frac{2(1-\alpha)}{2-\alpha} m(B_{2,2})\Big].$$

Observe that from the definition of the maps $\tau_k$'s and the definition of $A_i$, since they are a partition for the set $A$, one has that $m(B_{1,1}) = m(A) \cap I_1 = A_1$, $m(B_{1,2}) = \frac{1}{2}m(A) = \frac{1}{2}m(A_1) + \frac{1}{2}m(A_2)$, and $m(B_{2,1}) = m(B_{2,2}) = \frac{1}{2}m(A) = \frac{1}{2}m(A_1) + \frac{1}{2}m(A_2)$. And then, by plugging this expressions into the last equation one has that

$$\sum_{k=1}^{2} p_k \mu_f(\tau_k^{-1}(A)) = \frac{2}{2-\alpha} m(A_1) + \frac{2(1-\alpha)}{2-\alpha} m(A_2) = \mu_f(A).$$

Which finishes the proof. □

## 4.5 Random maps on the interval with spontaneous phase transition

Next, we present our second example, a family of random maps equipped with an uncountable quantity of maps to choose from at each iteration.

Let us consider a real number $\gamma \in [0, 2]$. Once $\gamma$ is fixed, it determines the random map by means of a set $\Gamma_1$, which contains the range of maps for the chosen value of the parameter $\gamma$. This set is defined by $\Gamma_1 := \{\tau_\beta : \beta \in [\gamma, 2]\}$, and each transformation $\tau_\beta : I \to I$ is given as follows,

$$\tau_\beta(x) = \begin{cases} \beta x, & 0 \le x \le 0.5 \\ 2x - 1, & 0.5 < x \le 1. \end{cases} \tag{4.7}$$

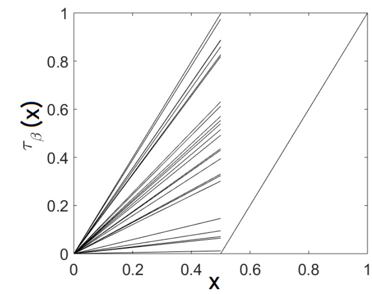



Figure 4.5: Representation of the random family with $\gamma = 0$. Here $\beta$ is allowed by the value of $\gamma$ to take values in the whole interval $[\gamma, 2] = [0, 2]$.

The value $\beta := \beta(\omega)$ is considered to be a realization of the random process $\{\xi_n\}$, as in Definition 4.1. Thus, our random maps are described by $T_\xi$ and its itinerary is given by the equation (4.1). We set the probability distribution $\mathbb{P}$ for $\{\xi_n\}$ to be the uniform distribution on $[\gamma, 2]$. For this family, the extreme cases are: on one side, when $\gamma = 2$, in this case the map is exactly the dyadic transformation $x_{n+1} = 2x_n \pmod 1$. Namely, it is the only map that can be chosen by the law $\mathbb{P}$, with $\beta \in \{[\gamma, 2]\} = 2$. On the other side, when $\gamma = 0$ then the probability distribution $\mathbb{P}$ is the uniform distribution on the whole interval $[0, 2]$ and the first branch of the maps are all the linear transformations with slope $\beta$, with values from 0 to 2.

For this family of random maps, we identify the phenomenon of phase transition as a consequence of an interplay of contracting and expansive transformations, being that the former induce the orbits to concentrate in a region or point in the interval, and the latter inducing the orbits to distribute all along the state space. In our setting for $T_2$, we determine the parameter $\gamma$ to control this interplay. In the following, we show some numerical simulations that exhibit the occurrence of this phase transition for the random map that took place from the random composition of the maps given by (4.7), as a mean to estimate its invariant density.

## 4.5.1 Numerical estimation of the invariant measure

Our first approach stood on the study about the existence of an invariant density for $T_2$. We can remit ourselves to existent results [19, 29], for values of $\gamma \ge 1$, given that, albeit the dynamic is guided by a random process, it is with probability 1. Therefore, we are interested in exploring the nature of the dynamics when there exists, almost surely, a permanent influence of the uncountable contracting maps against the expansive ones.

We calculated the typical empirical density for several vales of $\gamma$ by obtaining orbits of $N = 2.5 * 10^7$ iterations, in order to estimate the invariant density associated with the random map. We plotted this data in histograms of 1000 uniformly spaced bins. We could observe from these simulations that there exist a range of values of $\gamma \in [0, 0.27]$ for which the empirical density approaches typically to the $\delta$ function centered at 0. This suggests that the interaction between contracting and expansive maps is overweighted by the former, and the trajectories are almost certainly concentrated around the fixed point $x^* = 0$. For larger values of $\gamma$ a different empirical density shows up, one that distributes the orbit all along the unit interval. The important phenomenon that occurs here, is the apparent sudden change from non-existence to existence of the empirical density that becomes more uniformly distributed as the parameter $\gamma$ increases in value. This motivates us to estimate the critical value $\gamma_c$ for which the, 'so called', phase transition appears (see Figure 4.6).

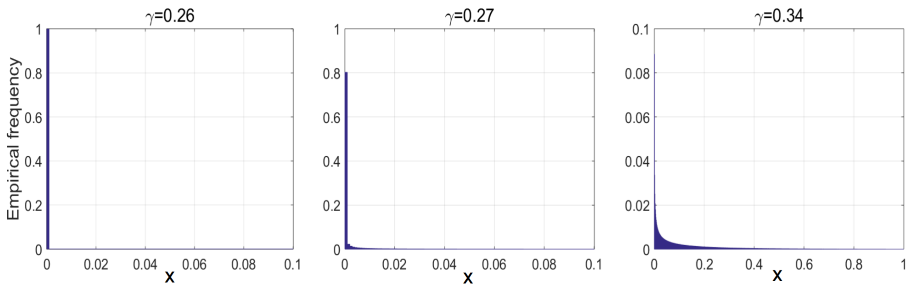



Figure 4.6: Typical empirical densities for different values of $\gamma$. Note that for $\gamma = 0.26$ and 0.27 we have zoomed in at $[0, 0.1]$ in order to make visible the density concentrating at 0. For $\gamma = 0.27$ the density starts to spread out all along the unit interval. This phenomenon is visible for $\gamma = 0.34$ for instance.

This behavior is replicated in the obtained numerical results of the empirical densities for several initial conditions and many realizations of the process $\{\xi_n\}$, and for many and each of values of $\gamma$ above the critical value. We give some examples in Figure 4.7.

In the next pages, we provide a summary of mostly numerical approaches by which, we explored how to estimate the critical value $\gamma_c$; which include the calculation of the Lyapunov exponent, an operator condition we took from [15] and accommodated to this case, the empirical measure computed from the histograms, a mean expansiveness condition we took [9] and also adapted to our case, and the decay of correlations. We performed this numerical exploration as a function of the change of the parameter $\gamma$.

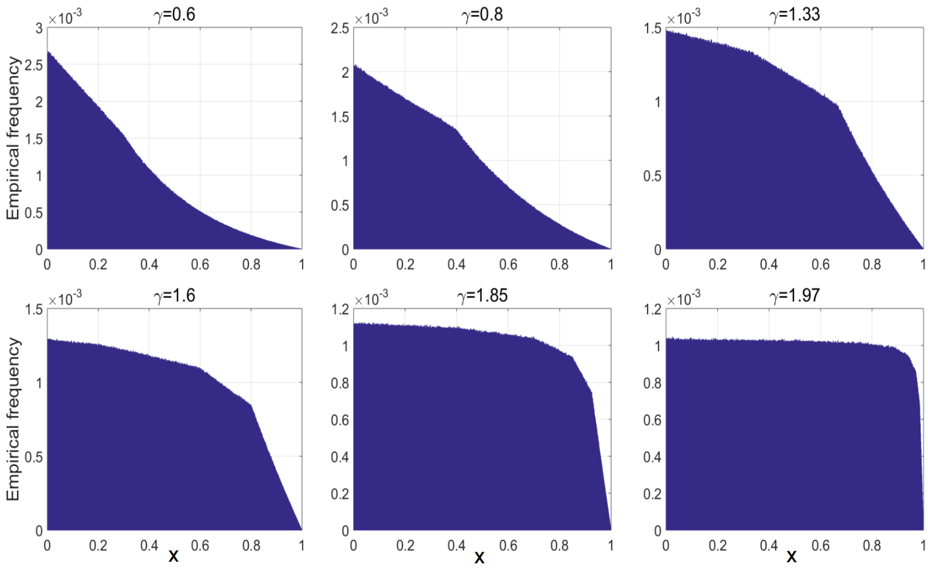



Figure 4.7: Typical empirical densities for different values of $\gamma$ above the critical value. The densities gradually tend to an uniform distribution as expected.

### 4.5.2 Lyapunov exponent

The Lyapunov exponent is often interpreted as a quantitative measure of the exponential separation of orbits, given a transformation $T$, the Lyapunov exponent associated to $T$ is given by

$$\lambda := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log|T'(T^j(x))|.$$

Here we use this definition to estimate the value of $\lambda$ associated to our random map $T_2$. We introduce the following estimator, given by

$$\widetilde{\lambda}_N(T_\xi) := \frac{1}{N} \sum_{j=0}^{N-1} \log|T'_\xi(T^j_\xi(x))|. \tag{4.8}$$

We computed an estimation of the Lyapunov exponent for the random map $T_2$ with an accuracy of $N = 2 \times 10^6$ iterations, using expression (4.8). We performed this for 200 values of $\gamma$ uniformly drawn from $[0,2]$. In Figure 4.8, for each $\gamma$ the plot shows the average of 100 realizations of 4.8. This quantity happens to display a change of sign; from negative to positive around $\gamma \simeq 0.26$, as its value is increased. It keeps growing for higher values of the parameter, up to $\widetilde{\lambda}_N(T_\xi) = \ln 2$, as expected for $\gamma = 2$.

In order to obtain a more thorough estimation of $\gamma_c$, we have estimated the Lyapunov exponent for 200 different values of $\gamma$ in the reduced interval $[0.26, 0.27]$

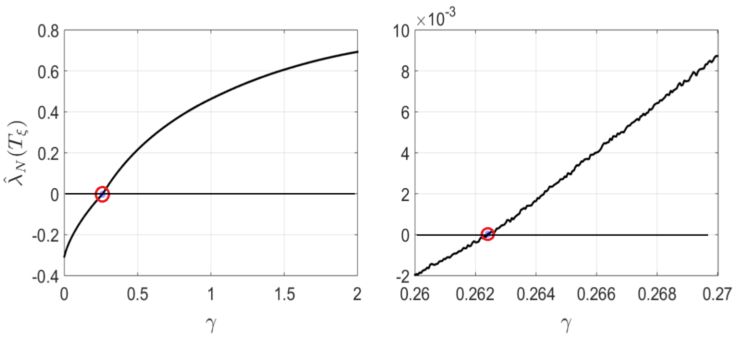



Figure 4.8: Average of (4.8) as a function of $\gamma$ for 100 realizations. The curve in the left hand side displays the average of the estimator of the Lyapunov exponent for $\gamma = 0$ up to $\gamma = 2$. On the right hand side, we have zoomed in around $\gamma_c \approx 0.263$. The data tip is pointing at the last negative value of the parameter for which the Lyapunov exponent is negative.

as well. The plot in the right hand side in Figure 4.8 shows the average of 100 realizations of $\widetilde{\lambda}_N(T_\xi)$, for each of the selected $\gamma$. Since a negative value of the Lyapunov exponent indicates that the trajectories of the systems do not tend to separate, or a sensitive dependence on initial conditions, its estimation gave us a bound for the value of $\gamma_c \approx 0.2625$, consistent to the numerical evidence of the invariant density in the previous section.

### 4.5.3 An operator condition

As shown previously, in [15], they give a sufficient condition for the existence of the a.c.i.m. in a random map equipped with finitely many Lasota maps. Similarly to the condition of expansiveness in [2], from the classic Lasota-Yorke theorem, the expression 4.4 ahead, is related to the definition of the Perron-Frobenius.

$$\sum_{i=1}^{m} \frac{p_i}{|T_i'(x)|} \leq \sigma < 1,$$

We adapted this condition to our case with $T_2$, by approximating it by an increasing quantity of transformations this map can be equipped with; and as seen early, $m$ is the number of maps conforming the random map, and $p_i$ the probability to chose $\tau_i$ with an uniform probability distribution. Here the slope of each of the $\tau_i$ is given by $\beta_i := \frac{i(2-\gamma)}{m} + \gamma$, for $i = 1, \ldots, m$. Consider $\Gamma_1$ of finite cardinality $m$, and thus $p_i = 1/m$ for the map $\tau_i$. We have done the estimation of (4.4) with $m = 1, 3, 5, 10, 20, 50, 100, 1000, 10^3, 10^4, 10^5, 10^6$, each considering 200 values of $\gamma$. For each of this estimations we plotted the results in Figure 4.9. From this condition, we obtained an upper bound of $\gamma_c \approx 0.42$, which goes in accordance with what we expected, but it is worth noting that it is a bound higher than the other estimations provided us.

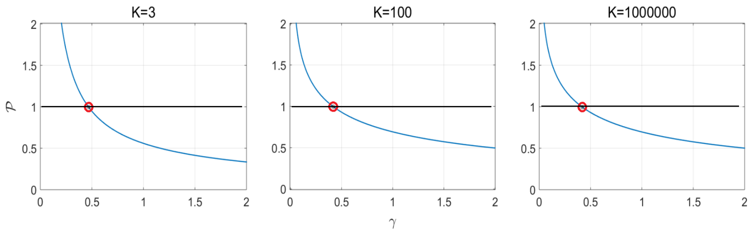



Figure 4.9: Estimation of condition (4.4) for $m = 3, 100$ and 1000000 of maps considered. The data tips point at the largest value of $\gamma$ for which the approximation of the sum in (4.4) is greater than 1. This gives us an upper bound of $\gamma_c \approx 0.42$.

### 4.5.4 Empirical measure of $I_2$

As a signature of the existence of an invariant density, one would have the appearance of mass at the state space, or inside intervals with positive Lebesgue measure, for typical realizations of the random map. For this reason, we did estimate the empirical probability for the trajectory of the random map to be in the interval $I_2 = [1/2, 1]$ (where the random interactions between contracting and expansive maps does not occur); taking into consideration 200 normalized histograms, which are a representation of the the empirical density function $\hat{h}_\gamma$. Using this empirical density, we calculated the following quantity

$$\int_{I_2} \hat{h}_\gamma dm = \widehat{\mathbb{P}}_\gamma([1/2, 1]). \tag{4.9}$$

In Figure 4.10, we show how the probability $\mathbb{P}_\gamma([1/2, 1])$ changes as $\gamma$ increases. For values of $\gamma \lessapprox 0.27$, the probability of incidence of the orbit in that region is practically equal to zero. And it increases up to $1/2$ when $\gamma = 1/2$, when the invariant density is expected to be uniform. As an estimation for the critical value $\gamma_c$, we searched for the smallest value of $\gamma$ for which the probability to be in $I_2$ is strictly positive, which is $\gamma \approx 0.28$. This remains consistent with all of our numeric results until now.

### 4.5.5 An expansiveness condition

The example 8.1 in [9] is a form of a random tent map, and since the main result in this paper provides a necessary and sufficient condition for the existence of the invariant density but in a skew-product setting, we tried to adapt their mean expansiveness condition to our case with $T_2$. We present here the results of this numeric

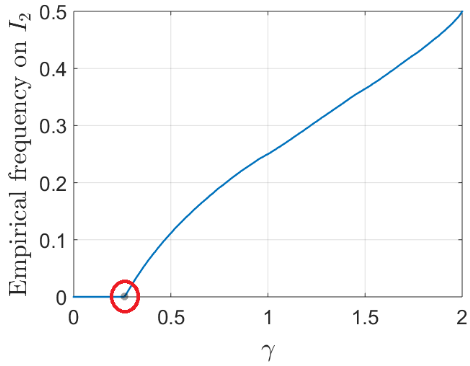



Figure 4.10: Estimation of condition (4.9) as function of $\gamma$. We look for the $\gamma$ for which for the first time the density in the second half is positive, which is $\gamma \approx 0.28$ (within the red circle).

test. As we have seen previously, this condition is true for a finite number of linear transformations in the interval. The condition, adjusted to $T_2$, is the following. If

$$\prod_{k=1}^{K} {\beta_k}^{p_k} > 1,$$

then the random map has an invariant density [9]. As before, $\beta_k$ denotes the slope of the linear transformation $\tau_k$ and $p_k$ its corresponding probability of being selected.

So, for our purposes we did not used directly this condition; we estimated $\gamma_c$ using a version of this condition when the transformations are all the linear maps with derivative $\beta \in [\gamma, 2]$. It considers the infimum of all the possible slopes, and since it can only be attained only in the interval $I_1 = [0, 1/2)$ of the maps, then we have the condition

$$\frac{1}{2-\gamma} \int_{\gamma}^{2} \log \beta d\beta > 0.$$

After solving this integral, we found an equation for the critical value $\gamma_c$, this yields:

$$\frac{1}{2-\gamma} [2(\log 2 - 1) + \gamma(\log \gamma - 1)] = 0,$$

whose solution, numerically obtained, is $\gamma^* \approx 0.2625828284...$ This gives us an upper bound for the critical value $\gamma_c$, which ultimately, is in conformity with the empirical estimations presented previously. Observe that this condition becomes singular when $\gamma = 2$, and in that case, the probability distribution can only select the map $\tau_\beta(x) = 2x$.

### 4.5.6 Decay of correlations

Another important statistical quantity we estimated for $T_2$, was the decay of correlations, using the equation (2.7). As we seen before in section 2.8, the expression for computing the correlation coefficient for a determined system and set of "test functions", or "observables", is closely related with the expression for quantifying the mixing property [6]. For $T_2$, we explored how the change in the decay of correlations varies with the value of $\gamma$, in order to determine if there exist a relation in this change around the critical value $\gamma_c$.

Let $\mu$ be an invariant probability measure for a non-singular transformation $T : X \to X$ and $n$ a positive integer. Let $\phi \in L^1$ and $\psi \in L^\infty$, then the $k$−th correlation coefficient of these test functions is given by,

$$\mathrm{Cor}(\phi, \psi, k) = \left| \int_X (g \circ T^k) f \mathrm{d}\mu - \int_X \phi \mathrm{d}\mu \int_X \psi \mathrm{d}\mu \right|,$$

it is clear that if $T$ preserves $\mu$ and is mixing, then the correlation function goes to zero when $k$ tends to infinite. Also, one can determine how fast the trajectories become independent of the initial conditions, by the rate of decay of the correlations. In other words, a fast decay in this rate is expected to be detected when we can see the "same kind" of typical trajectories, not being important what the initial conditions were. We remit the reader to [7, Chapter 5] for further details.

For practical issues, we estimated this rate of decay of correlations given finite samples of the dynamics. For this reason, we present here the definition of an estimator for the correlation coefficient, as a function of the number of iterations $k$. We consider actually, the auto-correlation function estimator, which for a given test observable $\phi$, and a sample of length $N$, it is defined by

$$\widehat{\mathrm{Cor}}(\phi, N, k) = \frac{1}{N} \sum_{i=0}^{N-1} \phi(T^i(x))\phi(T^{i+k}(x)) - \frac{1}{N} \sum_{i=0}^{N-1} \phi(T^i(x)) \cdot \frac{1}{N} \sum_{i=0}^{N-1} \phi(T^i(x)).$$

The convergence of $\widehat{\mathrm{Cor}}(\phi, N, k)$ to the auto-correlation function as $N$ goes to infinite, is assured by the Birkhoff ergodic theorem and should be because of the invariance of $\mu$. Here we computed the auto-correlation function estimator, as a function of $k$, and we computed it for several values of $\gamma$, wanting to detect notable changes in its behavior. We considered $N = 10^7 - k$ iterations for $k = 1, 2, ..., \Delta$, with $\Delta = 3000$. We made use of two test functions: $\phi_1(x) = T(x)$ the actual state of the system, and $\phi_2 = \chi_A$, the indicator function of a given interval $A$, which in this case we chose to be $A = [0, 0.05]$.

Through this numerical approach, we identified three different regimes for both of the auto-correlation estimators $\widehat{\mathrm{Cor}}(\phi_1, N, k)$ and $\widehat{\mathrm{Cor}}(\phi_2, N, k)$ as we gave higher values of the parameter. We provide the details of this scope, next.

First, for values of the parameter $\gamma \leq \gamma_c$ the auto-correlation function estimator, for each observable $\phi_1$ and $\phi_2$, goes rapidly to zero, at a rate non-identifiable as power-law-like nor exponential, but rather irregular. But this is expected, due to the random nature of the random map itself, and that its trajectories go to zero. For values of the parameter around the critical value, $\gamma \approx \gamma_c$, the empirical estimator of the auto-correlation function exhibits another rate of decay. In this situation, the rate of decay of the correlations resembles a power-law-like behavior (see Figure 4.11), for each observable $\phi_1$ and $\phi_2$. We plot the linear-linear and the log-log graphics. For the log-log graphic we adjust the curve to obtain the power law given by $\widehat{\mathrm{Cor}}(\phi_1, k) \sim \frac{c}{k^\eta}$ for a positive $c$ and $\eta \approx 0.4258$ and for $\phi_2$ the exponent is $\eta \approx 0.00247$.

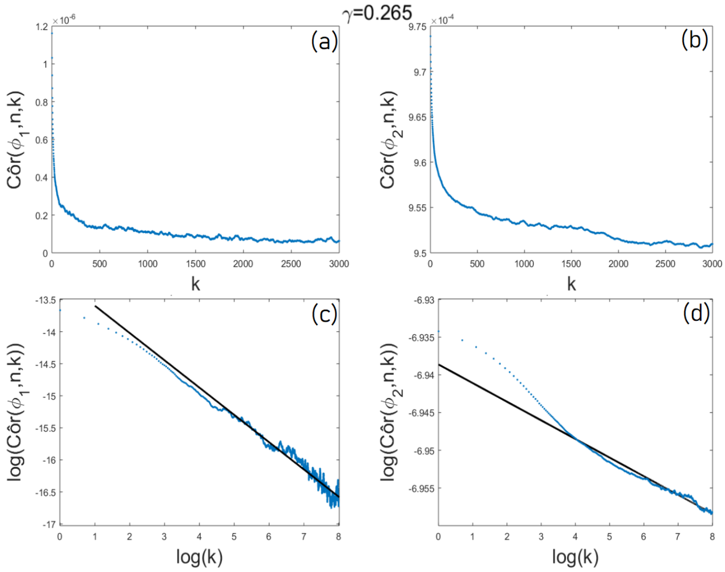



Figure 4.11: Linear-Linear plots for the auto-correlation estimator for observables $\phi_1$ (a) and $\phi_2$ (b) and its corresponding log-log plots in (c) and (d), respectively. The fitted line in the case of observable $\phi_1$ has slope approximately 0.4258 and for the observable $\phi_2$ is approximately 0.00247.

Lastly, for values of $\gamma > \gamma_c$, we expect an exponential rate of decay of correlations. First, since for $\gamma > 1$ all the transformations are expansive and thus the operator for each of them has the spectral gap property which implies the exponential decay of correlations. Here we shows the test for $\gamma = 0.3$, which we plot in Figure 4.12. Notice that the decay seems to be slightly faster than in the previous plot for $\gamma = 0.265$. This behavior sharpens for larger $\gamma$'s. With this evidence we show, at least numerically for two particular test functions, that this systems experiments a change in its rate of decay of correlations, around the critical value $\gamma_c$. Namely, the statistical properties of the system are altered at the phase transition point.

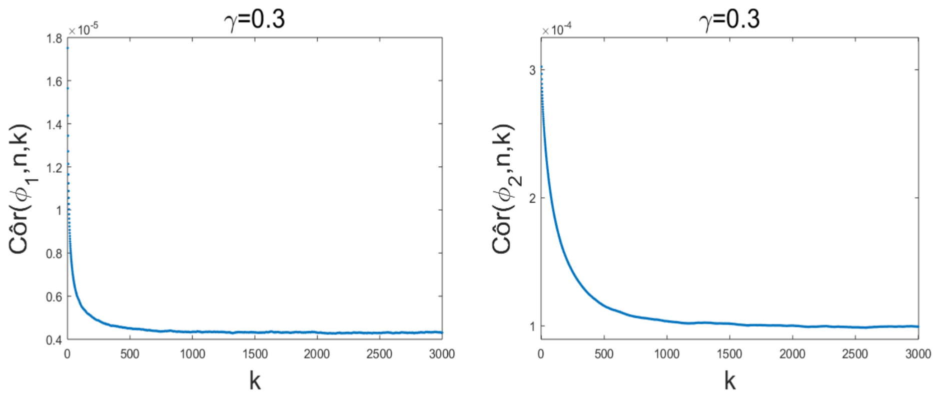



Figure 4.12: Exponential-like behavior for the auto-correlation estimator for each observable $\phi_1$ (left) and $\phi_2$ (right).

Thus, having established the concepts and existing results concerning to a.c.i.m.s in random maps, we will present in the following chapter our results regarding the existence and non-existing of a.c.i.m.s for a certain class of random maps in the interval, being our motivation, the two cases of study we explained here.

# Chapter 5

# Theoretical results

Next, we give two original results, concerning the first one, to the class of random maps defined in the previous section, states a set of sufficient conditions for this to have an a.c.i.m. as a function of its characteristic parameter $\gamma$. Under some additional conditions, the second theorem will prove how, as a function of $\gamma$, the map $T_{\xi,\Gamma}$ has an a.c.i.m., and therefore, a phase transition in the sense of non-existence to existence of an a.c.i.m..

## 5.1 Setting

Now we will define the class of random maps for which we will prove the existence an a.c.i.m., as well as the expression for its Perron-Frobenius operator, and the sufficient conditions for this to occur.

Consider a random map in the interval $T_{\xi,\Gamma}(x) : [0,1] \to [0,1]$, satisfying the following conditions:

$i)$ There exists a partition of $I = [0,1]$; $0 = a_0 < a_1 < \ldots < a_{r-1} < a_r = 1$ such that the restriction of $T_{\xi,\Gamma}(x)$ to the sub-interval $(a_{i-1}, a_i) = I_i$ is a $\mathcal{C}^2$ function, denoted by $T_\xi(x)\big|_{I_i} = T_{\xi_i}(x)$.

$ii)$ Each $T_{\xi_i}(x)$ is a family of functions with continuous dependence on a real-valued parameter $\gamma$, that is, $\{\tau_{i_\beta}\}_{\beta=\gamma}^{\theta}$, is a continuous sequence of functions; where $\tau_{i_\beta}$ corresponds to the $i$-th realization of the random variable $\xi$, defined on a continuous probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If for some $i$, $\tau_i$ is not dependent on $\beta$, the restriction of the map to this $I_i$ is deterministic.

$iii)$ For all $\beta$, there is a constant $s < \infty$ such that $\frac{\tau''_{i_\beta}(x)}{[\tau'_{i_\beta}(x)]^2} \leq s$

$iv)$$\mathbb{P}$ is considered to be an uniform distribution density; therefore $\mathbb{P}(\tau_{i_\beta}) = \frac{1}{\theta-\gamma}$.

$v)$ The functions $\frac{1}{|\tau'_\beta(x)|}$ are of bounded variation $\lambda(x)$-a.e, and we define $\phi_i(\beta) := \inf_{x \in I_i}\{|\tau'_\beta(x)|\}$

It is well known that the Perron-Frobenius operator for random maps in the interval with $\#(\Gamma) = K < \infty$ is expressed as

$$P_{T_{\xi,\Gamma}} f(x) = \sum_{k=1}^{K} \frac{f(\tau_k^{-1}(x)) p_k(\tau_k^{-1}(x))}{|\tau_k'(\tau_k^{-1}(x))|} \chi_{\tau_k(I)}(x),$$

and considering an uniform distribution for the maps $\tau_k \in \Gamma$, one obtains

$$P_{T_{\xi,\Gamma}} f(x) = \frac{1}{K} \sum_{k=1}^{K} \frac{f(\tau_k^{-1}(x))}{|\tau_k'(\tau_k^{-1}(x))|} \chi_{\tau_k(I)}(x).$$

In this case, since the $\Gamma$ is an uncountable set of transformations, we can prove that defining for the discrete case $\beta_k = \gamma + \frac{k}{K}(\theta - \gamma)$, as $K \to \infty$, we can remit ourselves to the classical definition of integrals, thus, we can express $P_{T_{\xi,\Gamma}} f(x)$ as

$$P_{T_{\xi,\Gamma}} f(x) = \frac{1}{\theta - \gamma} \int_{\gamma}^{\theta} \frac{f(\tau_\beta^{-1}(x))}{|\tau_\beta'(\tau_\beta^{-1}(x))|} \chi_{\tau_\beta(I)}(x) d\beta, \quad \beta \in (\gamma, \theta] \subset R.$$

**Remark 5.1.** *By the definition of the conditions that $T_{\xi,\Gamma}$ hold and the construction of its Perron-Frobenius operator, it is necessary to point out that we do not assume conditions on the derivative of each $\frac{1}{\tau_{i_\beta}}$, other than its variation. That is, for this setting it could be possible to include random transformations equipped with a continuum of maps with contracting direction.*

## 5.2 Lasota-Yorke's approach

In the following, we take the nowadays classic bounded variation approach from [2] to prove the quasi-compactness of the Perron-Frobenius operator corresponding to the class of random maps in the interval $T_{(\xi,\Gamma)}$, which in turn, implies the existence of the fixed point of the operator, and the existence of an a.c.i.m., and as we will show in the following, one of the sufficient conditions for this result (the expansiveness in mean), is stated as a function of the parameter $\gamma$.

### 5.2.1 Existence theorem

**Theorem 5.1.** *Let $\Gamma$ be an uncountable set of transformations in the interval, let $T_{(\xi,\Gamma)}(x) : [0,1] \to [0,1]$ be a random map satisfying the $i) - v)$ conditions, and let $P_T$ be the Perron-Frobenius operator associated with $T_{(\xi,\Gamma)}(x)$. Then, for all density functions $f \in L^1(I)$,*

$$\bigvee_I P_T^n f(x) \le Q \bigvee_I f(x) + \frac{R}{1 - Q}, \quad \forall n \in N, \tag{5.1}$$

*with*

$$R = 2\mathbb{P}\max_{1\le i\le r}\left\{\int_\gamma^\theta\left[s+\frac{1}{m(\hat{I}_i)}d\beta\right]\right\},\quad \text{if}\quad Q = 2\mathbb{P}\max_{1\le i\le r}\left\{\int_\gamma^\theta\frac{d\beta}{\phi_i(\beta)}\right\}<1.$$

*Proof.* Consider the variation of $P_T$:

$$\bigvee_I P_T f(x) = \bigvee_I \mathbb{P}\sum_{i=1}^r\int_\gamma^\theta\frac{f(\tau_{\beta_i}^{-1}(x))}{|\tau'_\beta(\tau_{\beta_i}^{-1}(x))|}\chi_{\tau_{\beta_i}(I)}(x)d\beta.$$

By properties of variation, the Yorke inequality, and setting $\hat{I}_i = \tau_{\beta_i}(I_i)$ we have

$$\begin{aligned}\bigvee_I P_T f(x) &\le \mathbb{P}\sum_{i=1}^r\int_\gamma^\theta\bigvee_{I_i}\frac{f(\tau_{\beta_i}^{-1}(x))}{|\tau'_\beta(\tau_{\beta_i}^{-1}(x))|}\chi_{\hat{I}_i}(x)d\beta\\
&\le 2\mathbb{P}\sum_{i=1}^r\int_\gamma^\theta\left(\bigvee_{\hat{I}_i}\frac{f(\tau_{\beta_i}^{-1}(x))}{|\tau'_\beta(\tau_{\beta_i}^{-1}(x))|}+\frac{1}{\lambda(\hat{I}_i)}\int_{\hat{I}_i}\frac{f(\tau_{\beta_i}^{-1}(x))}{|\tau'_\beta(\tau_{\beta_i}^{-1}(x))|}dx\right)d\beta\\
&\le 2\mathbb{P}\sum_{i=1}^r\int_\gamma^\theta\left(\sup_{x\in\hat{I}_i}\left\{\frac{1}{|\tau'_\beta(\tau_{\beta_i}^{-1}(x))|}\right\}\bigvee_{\hat{I}_i}f(\tau_{\beta_i}^{-1}(x))+\int_{\hat{I}_i}\frac{f(\tau_{\beta_i}^{-1}(x))}{\frac{d}{dx}|\tau'_\beta(\tau_{\beta_i}^{-1}(x))|}dx\right.\\
&\left.+\frac{1}{\lambda(\hat{I}_i)}\int_{\hat{I}_i}\frac{f(\tau_{\beta_i}^{-1}(x))}{|\tau'_\beta(\tau_{\beta_i}^{-1}(x))|}dx\right)d\beta.\end{aligned}$$

By condition $iii)$, setting $y=\tau_{\beta_i}^{-1}(x)$, and $\phi_i(\beta) := \inf_{x\in I_i}\{|\tau'_\beta(x)|\}$

$$\begin{aligned}\bigvee_I P_T f(x) &\le 2\mathbb{P}\sum_{i=1}^r\int_\gamma^\theta\frac{d\beta}{\phi_i(\beta)}\bigvee_{I_i}f(y)+2\mathbb{P}\sum_{i=1}^r\int_\gamma^\theta\left(\left[s+\frac{1}{m(\hat{I}_i)}\right]\int_{I_i}f(y)dy\right)d\beta\\
&\le 2\mathbb{P}\max_{1\le i\le r}\left\{\int_\gamma^\theta\frac{d\beta}{\phi_i(\beta)}\right\}\bigvee_I f(y)+2\mathbb{P}\max_{1\le i\le r}\left\{\int_\gamma^\theta\left[s+\frac{1}{m(\hat{I}_i)}\right]\right\}\int_I f(y)dy\\
&= Q\bigvee_I f(y)+R.\end{aligned}$$

□

Thus, under the assumptions stated in Theorem 2.1, (5.1) holds, and the random maps on the interval $T_{(\xi,\Gamma)}(x)$ have an a.c.i.m., as a standard consequence of an inequality of this type.

## 5.3 Probabilistic approach

Since we are dealing with random transformations, featuring a well-defined probability law in its dynamics, we proceed on its analysis about the existence and non-existence of an a.c.i.m. with probability theory. Next, we present our results on this regard.

### 5.3.1 Non-existence theorem

The following result is based on the probability that the trajectories of the random map accumulate on a delta distribution, centered at a fixed point. The next two lemmas will be useful when we prove this.

**Lemma 5.2.** *Let $T_{(\xi,\Gamma)}$ be a random map, which satisfies the $i) - v)$ conditions, having a fixed point $x^*$ such that $\tau_{\beta_i}(x^*) = x^*$, $\forall \beta_i \in (\gamma,\theta]$. Then,*

$$\left|\frac{d}{dx}\left(\tau_{\beta_k}(x)\circ\tau_{\beta_j}(x)\right)\Big|_{x=x^*}\right| = \left|\frac{d}{dx}\left(\tau_{\beta_k}(x)\right)\Big|_{x=x^*}\right|\cdot\left|\frac{d}{dx}\left(\tau_{\beta_j}(x)\right)\Big|_{x=x^*}\right|.$$

*Proof.*

$$\begin{aligned}\left|\frac{d}{dx}\left(\tau_{\beta_k}(x)\circ\tau_{\beta_j}(x)\right)\Big|_{x=x^*}\right| &= \left|\frac{d}{dx}\left(\tau_{\beta_k}(x)\right)\cdot\frac{d}{dx}\left(\tau_{\beta_j}(x)\right)\circ\left(\tau_{\beta_k}(x)\right)\Big|_{x=x^*}\right| \\ &= \left|\frac{d}{dx}\left(\tau_{\beta_k}(x)\right)\Big|_{x=x^*}\right|\cdot\left|\frac{d}{dx}\left(\tau_{\beta_j}(x)\right)\Big|_{x=x^*}\right|.\end{aligned}$$

□

Now, we want to determine the asymptotic hyperbolicity of the $n-$th composition of the random map $T_{(\xi,\Gamma)}$ evaluated a the fixed point that for all $\beta$, the $\tau_{\beta_k}(x)$ share. In other words, we want to compute what is probability that the fixed point $x^*$ is stable. We have

$$\mathbb{P}\left(\left|\frac{d}{dx}\left(\tau_{\beta_1}(x)\circ\ldots\circ\tau_{\beta_n}(x)\right)\Big|_{x=x^*}\right|<1\right), \tag{5.2}$$

which, by Lemma 5.2, (5.2) can be rewritten as

$$\mathbb{P}\left(\prod_{k=1}^{n}\left|\frac{d}{dx}\left(\tau_{\beta_k}(x)\right)\Big|_{x=x^*}\right|<1\right).$$

Then, taking the natural logarithm in both sides of the inequality yields:

$$\mathbb{P}\left(\sum_{k=1}^{n}\ln\left|\frac{d}{dx}\left(\tau_{\beta_k}(x)\right)\Big|_{x=x^*}\right|<0\right). \tag{5.3}$$

This means that if the probability in expression (5.3) is equal to 1, the fixed point $x^*$ is $\mathbb{P}$-almost certainly stable.

Next, we define a random variable $Y_k(\beta)$

$$Y_k(\beta) := \ln \left| \frac{d}{dx} \left(\tau_{\beta_k}(x)\right)\big|_{x=x^*} \right|.$$

From this, we have that (5.3) is the probability of the sum of the random variable $Y_k(\beta)$ being strictly less than zero. Given that the expected value of the sum of $n$ independent random variables is equal to the sum of the expected values of these random variables [39]. Considering $Y_k(\beta)$ to be identically distributed for every iteration $k$, we have:

$$\mathbb{E}\left(\sum_{k=1}^{n} Y_k(\beta)\right) = n\mathbb{E}\left(Y_k(\beta)\right). \tag{5.4}$$

Now, we proceed to state the following theorem, which considers a different class of random maps $T_{(\xi,\Gamma)}$ that Theorem 5.1 considers. In this case, the systems here considered will almost certainly exhibit a "stable" behavior, in the sense that all trajectories will converge to a fixed point.

**Theorem 5.3.** *Consider the random maps $T_{(\xi,\Gamma)}$, satisfying the $i) - iii)$ conditions. Assume the maps $\tau_{beta_k}$ are i.i.d. from a stochatic process with a law of finite variance. Furthermore:*

*$a)$ For only one $\tau_\beta(x)|_{I_i}$ there is only one $x^* \in I_i$ such that, $\tau_\beta(x^*) = x^*$, $\forall \beta \in [\gamma, \theta]$, with $x^* \neq a_{i-1}, a_i, \forall i \neq 0, r$*

*$b)$ Considering the random variable $Y_k(\beta)$, there exists one value $\gamma_1 \in [\gamma, \theta]$ such that*

$$\mathbb{E}_{\gamma_1}\left(Y_i(\beta)\right) < 0,$$

*where $\mathbb{E}_{\gamma_1}$ denotes the expected value of $Y_k(\beta)$ for $\gamma_1$.*

*Then, the trajectories of $T_{(\xi,\Gamma)}$ will almost certainly converge to $x^*$, thus, the invariant density is a delta distribution centered at this point for every value of the parameter $\gamma$ such that $\mathbb{E}_\gamma \leq \mathbb{E}_{\gamma_1}$.*

*Proof.* From the Central Limit Theorem, we have that the sum $S_n = \left(\sum_{k=1}^n Y_k(\beta)\right)$ converges in distribution to a random variable with Gaussian distribution and expected value $n\mathbb{E}_{\gamma_1}\left(Y_i(\beta)\right) < 0$.

Then, since $S_n$ is a sum of random variables, the event of $S_n$ being strictly less than zero an infinitely number of times is invariant under a finite number of permutations. Therefore, from the Hewitt-Savage zero-one law, its probability is either 0 or 1.

Finally, given that $\mathbb{E}_{\gamma_1}(S_n) < 0$, we have $\mathbb{P}(S_n < 0) > 0$, then it is almost surely 1, which implies that the $n$-th composition of the random map, has almost certainly has a derivative with absolute value strictly lower than 1 evaluated in its fixed point. So the trajectories will converge to $x^*$ with probability of one. And with this, the proof is complete. □

**Remark 5.2.** *In the condition a) we state that $x^*$ can be either equal to $a_0$ or $a_r$, since $\frac{d}{dx}\left(\tau_\beta(x)\right)\big|_{x=x^*=a_0,a_r}$ is always equal or greater than 0, which implies that there exists a $\delta > 0$, such that, $\forall \hat{\delta} \in (0, \delta]$, we have that $(x+\hat{\delta}) \in I_0$, $\mathrm{Im}\Big(\tau_\beta(x+\hat{\delta})\big|_{I_0}\Big) \subset I_0$, and given $(x-\hat{\delta}) \in I_r$, $\mathrm{Im}\Big(\tau_\beta(x-\hat{\delta})\big|_{I_r}\Big) \subset I_r$. For $i \neq 0, r$ this does not hold, and therefore the dynamics of the random map end up lying outside the element of the partition $I_i$, and the almost certain convergence of the trajectories to $x = x^*$ is not guaranteed.*

**Remark 5.3.** *Additionally, we need to investigate the nature of the dynamics of the random map if there were more than one interval $I_i$ with one fixed point holding b) and c) conditions, or a countable number $k$ of fixed points in $I_i$, which hold b) and c) conditions.*

### 5.3.2 Existence theorem

Analogously, we state a result for the existence of the a.c.i.m. based on Theorem 5.3, but with conditions for having, almost certainly an expansive map, i.e.,

$$\mathbb{P}\left(\left|\frac{d}{dx}\left(\tau_{\beta_1}(x) \circ \ldots \circ \tau_{\beta_n}(x)\right)\big|_{x=x^*}\right| > 1\right) = 1,$$

for some $\gamma_2$ such that $\mathbb{E}_{\gamma_2}\left(Y_i(\beta)\right) > 0$ holds. This means we are considering now a condition that sets the $n$-th composition of the random map in Theorem 5.3 as expansive, with derivative greater than 1, in absolute value; from which we can state the following corollary.

**Corollary 5.3.1.** *Consider the random maps $T_{(\xi,\Gamma)}$, satisfying the conditions in Theorem 5.3 but for $\mathbb{E}_{\gamma_2}\left(Y_i(\beta)\right)$ being strictly positive instead negative. Then, almost certainly, the map $T_{(\xi,\Gamma)}$ has an a.c.i.m. for all $\gamma$ such that $\mathbb{E}_{\gamma}\left(Y_i(\beta)\right) \geq \mathbb{E}_{\gamma_2}\left(Y_i(\beta)\right)$; by Theorem 2.9.*

It is clearly seen that for a random map satisfying Theorem 5.3 and Corollary 5.3.1, there exists a critical value (or range of values) $\gamma_c$ for which a phase transition, in the sense of non-existence to existence, takes place. Therefore, we give the following result.

**Corollary 5.3.2.** *If for a random map $T_{(\xi,\Gamma)}$ exist values $\gamma_1$ and $\gamma_2$ such that it satisfies Theorem 5.3 and Corollary 5.3.1, then there exist $\gamma_c$ such that $\mathbb{E}_{\gamma_1}(Y_i(\beta)) < \mathbb{E}_{\gamma_c}(Y_i(\beta)) < \mathbb{E}_{\gamma_2}(Y_i(\beta))$ and $\mathbb{E}_{\gamma_c}(Y_i(\beta)) = 0$; which is the critical value of $\gamma$ associated to the phase transition in the dynamics of the random map $T_{(\xi,\Gamma)}$.*

### 5.3.3 Examples

**Example 5.1.** *Let us consider the family of random maps considered in subsection 4.5. In the following, we show what is the range of values for which the random map in the interval (4.7) has and does not have an a.c.i.m.*

*Since the distribution of the maps in the system (4.7) is uniform, we have that the probability density function of the random variable $\tau_\beta(x)$ is $f_X(\beta) = \frac{1}{2-\gamma}\chi_{[\gamma,2]}(\beta)$, which has a cumulative distribution function $\mathbb{P}(\beta \le X) = \frac{x-\gamma}{2-\gamma}\chi_{[\gamma,2]}(x)$.*

*Then, we compute the transformation $Y := X \to \ln(X)$. Thus, we have*

$$\mathbb{P}(\beta \le X) = \mathbb{P}(\ln(\beta) \le \ln(X)) = \frac{e^y - \gamma}{2-\gamma}\chi_{[\ln(\gamma),\ln(2)]}(y) = \mathbb{P}(y \le Y).$$

*Which is the cumulative distribution function of the random variable $Y(\beta)$. From this we can obtain its probability density function $f_Y(\beta) = \frac{e^y}{2-\gamma}\chi_{[\ln(\gamma),\ln(2)]}(\beta)$ and compute its expected value*

$$\mathbb{E}(Y(\beta)) = \frac{1}{2-\gamma}\int_{\ln(\gamma)}^{\ln(2)} ye^y dy = \frac{2(\ln(2)-1) - \gamma(\ln(\gamma)-1)}{2-\gamma}.$$

*Now, setting $\mathbb{E}(Y(\beta)) = 0$ we have the conditions of Corollary 5.3.2, and we obtain the value of $\gamma_c \approx 0.2625828...$, which is the value associated to the phenomenon of phase transition as in Definition 3.1.*

*Therefore, for values of $\gamma < \gamma_c$ the trajectories converge to the almost-surely stable fixed point $x^* = 0$. And for values of $\gamma > \gamma_c$ the random map (4.7) has almost surely an a.c.i.m., which is consistent with the numerical evidence we presented earlier.*

In the setting of Theorem 5.3 we are allowed to consider any distribution of the random maps from an i.i.d. process with finite variance. In the next example we consider this situation.

**Example 5.2.** *Consider the family of random maps in the previous example, with a triangular distribution of the maps, given by*

$$f_X(\beta) = \frac{2(x-\gamma)}{(2-\gamma)(c-\gamma)}\chi_{[\gamma,c)}(\beta) + \frac{2(2-x)}{(2-\gamma)(2-c)}\chi_{[c,2]}(\beta),$$

*where c is the mode of this distribution; which has a c.d.f.*

$$\mathbb{P}(\beta \leq X) = \frac{2}{(2-\gamma)(c-\gamma)}\left(\frac{x^2+\gamma^2}{2} - \gamma x\right)\chi_{[\gamma,c)}(x)$$
$$+\frac{2}{(2-\gamma)(2-c)}\left(\frac{c^2-x^2}{2} + 2(x-c)\right)\chi_{[c,2]}(x).$$

*Then, we compute the transformation* $Y := X \to \ln(X)$. *Thus, we have*

$$\mathbb{P}(y \leq Y) = \frac{2}{(2-\gamma)(c-\gamma)}\left(\frac{e^{2y}+\gamma^2}{2} - \gamma e^y\right)\chi_{[\ln(\gamma),\ln(c))}(y)$$
$$+\left(\frac{2}{(2-\gamma)(2-c)}\left(\frac{c^2-e^{2y}}{2} + 2(e^y-c)\right) + \left(1 - \frac{(2-c)}{2-\gamma}\right)\right)\chi_{[\ln(c),\ln(2)]}(y).$$

*From this we can obtain its probability density function*

$$f_Y(\beta) = \frac{2}{(2-\gamma)(c-\gamma)}\left(e^{2y} - \gamma e^y\right)\chi_{[\ln(\gamma),\ln(c))}(y)$$
$$+\frac{2}{(2-\gamma)(2-c)}\left(2e^y - e^{2y}\right)\chi_{[\ln(c),\ln(2)]}(y),$$

*which has an expected value*

$$\mathbb{E}(Y(\beta)) = \frac{2}{(2-\gamma)(c-\gamma)}\left(0.25\left(c^2(2\ln(c)-1) - \gamma^2(2\ln(\gamma)-1)\right)\right)$$
$$-\frac{2}{(2-\gamma)(c-\gamma)}\left(\gamma\left(c(\ln(c)-1) - \gamma(\ln(\gamma)-1)\right)\right)$$
$$+\frac{2}{(2-\gamma)(2-c)}\left(2\left(2(\ln(2)-1) - c(\ln(c)-1)\right) - 0.25\left(4(\ln(4)-1) - c^2(2\ln(c)-1)\right)\right).$$

*Setting* $\mathbb{E}(Y(\beta)) = 0$ *will give us an explicit expression for* $\gamma_c$ *as a function of* $c$.

In the next example, we compute first $\phi_i(\beta)$ for the criterion for Theorem 5.1, and we compare it to the criterion for the setting with Theorem 5.3 and Corollary 5.3.1.

**Example 5.3.** *Consider the map, defined for* $\gamma \in (-0.5, 0.75]$, *and* $\beta \in [\gamma, 0.75]$, *which has a fixed point in* $x = 1/3$:

$$T_\xi(x) = \begin{cases} x + \frac{3}{4}, & 0 \leq x \leq 0.25 \quad (I_1) \\ 6\beta x(x - \frac{1}{3}) + x, & 0.25 < x \leq 0.5 \quad (I_2) \\ x - 0.5, & 0.5 < x \leq 1 \quad (I_3), \end{cases} \tag{5.5}$$

*which has a derivative given by*

$$T'_\xi(x) = \begin{cases} 1, & 0 \le x \le 0.25 \quad (I_1) \\ 12x\beta - 2\beta + 1, & 0.25 < x \le 0.5 \quad (I_2) \\ 1, & 0.5 < x \le 1 \quad (I_3). \end{cases}$$

*Therefore, for uniformly distributed* $\beta \in [\gamma, 0.75]$*, we have* $\phi_1(\beta) = |12x\beta - 2\beta + 1|_{x=\frac{1}{3}} = |2\beta + 1|$. *Then, we compute* $Q$*:*

$$Q = 2\mathbb{P} \max_{1 \le i \le r} \left\{ \int_\gamma^\theta \frac{d\beta}{\phi_i(\beta)} \right\}$$

$$= \frac{1}{0.75 - \gamma} \int_\gamma^{0.75} \frac{d\beta}{2\beta + 1} = \frac{1}{0.75 - \gamma} \ln\left(\frac{1.25}{\gamma + 1/2}\right),$$

$$Q < 1 \quad \textit{if} \quad \gamma > 0.28579...$$

*This criterion provides us only an upper bound for* $\gamma_c$*, which we can calculate exactly for the criterion established by Theorem 5.3 and Corollary 5.3.1. We know that:*

$$\phi_1(\beta) = |2\beta + 1| \in [2\gamma + 1, 2.5].$$

*Then, following the steps as the previous examples, we obtain an exact expression to compute* $\gamma_c$ *from the probability density function of the random variable* $Y := \ln(\phi_1(\beta))$ *:*

$$\mathbb{E}(Y) = 0 \Longleftrightarrow \frac{1}{2.5 - (2\gamma + 1)} \int_{\ln(2\gamma+1)}^{\ln(2.5)} y e^y dy = 0, \quad \textit{then:}$$

$$\frac{2.5\,(\ln(2.5) - 1) - (2\gamma + 1)(\ln(2\gamma + 1) - 1)}{2.5 - (2\gamma + 1)} = 0.$$

*Therefore,* $\gamma_c \approx -0.473399915...$

It is worth recalling that despite the noticeably relaxed bounds that the criterion from Theorem 5.1 yields, with respect to the criterion established by Theorem 5.3 and Corollary 5.3.1; these two results consider a different class of random maps. Theorem 5.1 does not need to assume there is only one fixed point, nor that the random action of the system occurs only in one of the elements of the partition $I_i$. On the other hand, the probabilistic approach of Theorem 5.3 allows us to consider the distributions of the maps to be any discrete or continuous distribution of an i.i.d. process with finite variance, not only uniform distributions necessary for the result of Theorem 5.1.

# Chapter 6

# Phase transition phenomenon in connected maps

In this chapter, we describe a type of dynamical systems we are proposing. This work in progress tackles on the topic of the phase transition phenomenon of an absolutely continuous measure of dynamical systems connected through a hole. The dynamics of systems with holes terminates when the orbit falls in the holes, therefore, its most known results are on the study of the invariant sets on this systems. A particularly transparent survey of results for dynamical systems with holes is [40]. For the type of dynamics we propose, the orbit does not terminate after landing in one of the holes, instead, it continues with a different transformation.

For the setting, let $I = [0,1]$ and consider two maps $T_1, T_2 : I \to I$. Associated to each map, we define two closed intervals $H_1 = H_{T_1} \subset I$ and $H_2 = H_{T_2} \subset I$ which we call *holes*. We are interested in the dynamics that arises in the following way: we iterate an initial condition taken from a uniform distribution on $I$ in one of the two maps, say $T_1$, until the trajectory visits its respective hole $H_1$. Then, the value is taken to the hole corresponding to the other transformation $H_2$ by means of the linear transformation $\tau_1 : H_1 \to H_2$ given by:

$$\tau_1(x) = \min(H_2) + \frac{(\max(H_2) - \min(H_2))(x - \min(H_1))}{\max(H_1) - \min(H_1)},$$

and we iterate now the map $T_2$ with initial condition $\tau_1(x)$ until the value of the iteration falls on $H_2$. When this happens, we take that value and apply the linear map $\tau_2 : H_2 \to H_1$ given by:

$$\tau_2(x) = \min(H_1) + \frac{(\max(H_1) - \min(H_1))(x - \min(H_2))}{\max(H_2) - \min(H_2)}.$$

We continue this way, changing the iterated map when the trajectory visits the hole that corresponds to each map.

Under this setting, we found cases where the associated *absolutely continuous measure* (depicted only numerically by the histograms we present here) may experience a phase transition phenomenon as in Definition 3.1 at a specific location and/or size of the holes; or at specific values of parameters of the transformations involved. We have explored this systems numerically, with the idea of analyzing the resulting behavior of two systems with contrasting dynamics.

For instance, one of the most notable configurations we explored is the one conformed by the family of Manneville-Pomeau maps $T_1(x)$ and a contracting map in the interval $T_2(x)$, defined as:

$$T_1(x) = \begin{cases} x(1+(2x)^\alpha), & x < 0.5 \\ 2x-1, & x \geq 0.5 \end{cases}$$

$$T_2(x) = \frac{x}{2} + \frac{1}{4}.$$

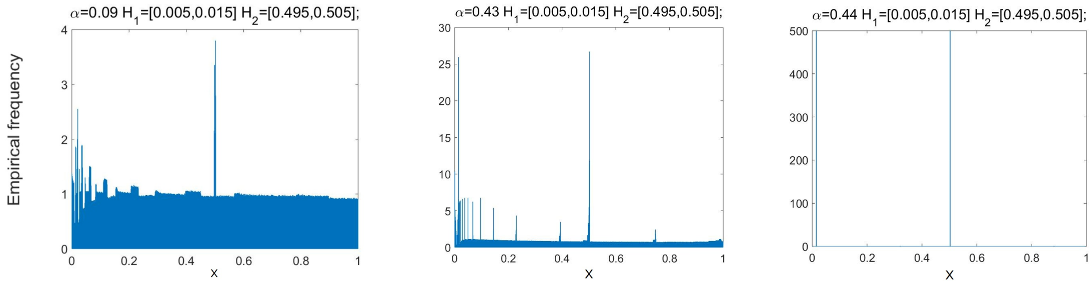
As we mentioned in Chapter 3, $T_1(x)$ has an ergodic regime for $0 < \alpha < 1$, and tends to accumulate almost every orbit near $x = 0$ for $\alpha > 1$. For $T_2(x)$ all orbits are trivially stable due to the contractive fixed point $x = 0.5$. We decided to connect their dynamics in order to explore what kind of behavior arises as an outcome of their interplay.

One notable behaviors here takes place when $H_1$ is placed in the interval $[0, 0.5)$. In this setting, due to the contracting nature of $T_2(x)$, $H_2$ can only be placed in such a way that it contains the fixed point $x^* = 0.5$, otherwise the trajectories will be trivially and asymptotically accumulated at this equilibrium point.

For every realization we fixed $H_1$ and $H_2$. Then, we explored the change in the resulting dynamics for different values of the parameter $\alpha$ for $T_1(x)$. Here, an interesting phenomenon arises, in which the histogram for the dynamics of this connected system shows that for a range of values of $\alpha$, the existence of a density function with support in all the interval $[0, 1]$. As we increased further the value of $\alpha$, this density experiences an abrupt change, and it becomes a couple of delta functions; one centered at a point in $H_1$, and the other centered at $x = \frac{1}{2}$, typical of a 2-cycle. In the following we show an example of this phenomenon.

We set $H_1 = [0.005, 0.015]$, and $H_2 = [0.495, 0.505]$. For values of $\alpha < 0.43$ the histograms show the existence of a density function with support in all the interval $[0, 1]$. The system experiences a sudden change of the density, for values of $\alpha \geq 0.44$ (see Figure 6.1c). This sudden change in the dynamics as a function of $\alpha$, lead us to think that there is a "critical value" for $\alpha \in [0.43, 0.44]$, which we will refer to as $\alpha_c$.

This same phenomenon is also observed for different placements and sizes of $H_1 \subseteq [0, 0.5)$; such that for lesser values of $\alpha$, the histograms exhibit a density function which implies the existence of a absolutely continuous measure (with respect




(a) $\alpha = 0.09$ (b) $\alpha = 0.43$ (c) $\alpha = 0.44$

Figure 6.1: Densities for the connected system conformed by the Manneville-Pomeau map for three values of $\alpha$ and a contracting system. We can note how the orbits accumulate in $[0, 0.5)$ for $\alpha < 0.44$ and at the contracting fixed point, but still preserving incidence on the rest of the state space. As $\alpha$ approaches 0.44 the incidence increases in a set of points in $[0, 0.5)$ ans in $x = 0.5$. For $\alpha > 0.44$ the density becomes one pair of delta functions centered at a point in $H_1$ and the fixed contracting point $x = 0.5$.

to the Lebesgue measure), and once $\alpha$ is increased up to a certain "critical value", this density function becomes a density conformed by two delta distributions in likely, a 2-cycle.

This aforementioned "criticality" of the parameter, according to our simulations, is dependent on the placement of $H_1$, given that for placements of $H_1$ closer to the point $x = 0$, the critical value for $\alpha$ is lower. This is expected, since for the deterministic setting of the Manneville-Pomeau maps, the trajectories tend to accumulate progressively more near $x = 0$, as $\alpha > 0$ keeps increasing. This is evidence that in this setting of connected systems, there exists a value of $\alpha$ for which almost surely, the trajectories being returned from $H_2$ to $T_1(x)$ will hit $H_1$ right after the first iteration, which in turn, makes them come back to $T_2(x)$, originating therefore this 2-cycle. We noticed as well, that for smaller sizes of $H_1$, the critical value of $\alpha$ is higher, which is expected by this reasoning.

From these observations, we propose the following conjecture.

**Conjecture 6.0.1.** *Considering the connected dynamical system conformed by the Manneville-Pomeau maps $T_1(x)$ with parameter $\alpha > 0$, and the contracting map $T_2(x) = \frac{x}{2} + \frac{1}{4}$, and the holes:*

$$H_1 \subseteq [0, 0.5)$$
$$H_2 = [a, b],\ a < 0.5 < b$$

*There exists a critical value of $\alpha$, "$\alpha_c$" such that for every $H_1 \subseteq [0, 0.5)$ and every $\alpha < \alpha_c$ the connected dynamical system has an absolutely continuous measure with* *respect to the Lebesgue measure, and for every* $\alpha > \alpha_c$ *it has not an a.c.m. Therefore, it experiences a phase transition phenomenon, in the sense of the existence to the non-existence of an a.c.m., as in Definition 3.1.*

Along with setting, we have observed a similar phase transition phenomenon by connecting two systems with an stable 3-cycle, where the abrupt change in the associated density function occurs as a function of the placement of one the holes. However, the details of this work in progress goes beyond the reach of this thesis.

# Chapter 7

# Open problems

In Chapter 5 we showed two original results. Theorem 5.1 considers a class of random dynamical systems equipped with an uncountable set $\Gamma$ of uniformly distributed transformations $\{\tau_{\beta_i}\}_{i=1}^{\infty}$. In Example 5.3 we compared the upper bound that yields from Theorem 5.1, to the exact expression that Theorem 5.3 and Corollary 5.3.1 provide for the value of $\gamma_c$. Thus, it seemed as if the results from Theorem 5.3 and Corollary 5.3.1 were more powerful; however, this is not the case, since the conditions of Theorem 5.3 require the random dynamics to occur in only one of the elements $I_i$ of the partition of the state space, as well as the existence of only one fixed point in $I_i$, shared by every $\tau_{\beta_i}$. Therefore, regarding random maps equipped with a continuum of transformations, the following questions naturally arise:

1. Is it possible to prove the existence of the phase transition phenomenon for a class of random maps where the random dynamics occur in more than one of the of the elements $I_i$ of the partition of the state space, and have a probability distribution other than the uniform?

2. If the random dynamics occur in more than one of the of the elements $I_i$ and there is more than one fixed point that the maps $\tau_{\beta_i}$ share, satisfying the conditions of Theorem 5.3, at which one of these fixed points the trajectories will converge and what is the probability for each?

3. Is there a set of conditions for random maps in the interval with no fixed points that guarantees its convergence to a stable periodic $n$−cycle? Or that guarantees the existence of an a.c.i.m. (other than for homeomorphisms of the unit circle)?

4. What is the asymptotic behavior of the dynamics of random maps with a fixed point whose position has a continuous dependence on the realization of the random variable that defines the distribution of all the $\tau_{\beta_i}$?

5. What is the asymptotic behavior of the dynamics of random maps that satisfy the conditions of Theorem 5.3 without condition $a$), i.e., if we allow $x^* = a_{i-1}, a_i, \forall i \neq 0, r$? What additional conditions will it require for the orbits to remain as almost surely convergent to $x^*$?

6. Can another versions of the Central Limit Theorem be applied to our setting in order to further generalize our results (e.g., considering a CLT for non i.i.d. processes)?

There is as well another open problem we are working on, outside the context of dynamical systems operating with more than one transformation. We describe it in the following.

For the deterministic map in the interval $I = [0,1]$, equipped with a parameter $\alpha \in [0,2]$:

$$T(x) = \begin{cases} \tau_1(x) = \alpha(x-0.5)+1, & 0 \leq x \leq 0.5 \\ \tau_2(x) = (2-\alpha)(x-0.5), & 0.5 < x < 1 \\ 0, & x = 1, \end{cases} \tag{7.1}$$

we have the following theorem:

**Theorem 7.1.** *Consider the system* (7.1)*. Then, the* $(n+1)$*-th composition* $T^n(x)$ *built as* $T^{(n+1)}(x) = \tau_1^n(x) \circ \tau_2(x)$ *is equal to the identity function, for the values of the parameter* $\alpha \notin \{0,2\}$ *which are positive real roots of the polynomial:*

$$x^n = \sum_{i=0}^{n-1} x^i \tag{7.2}$$

*Proof.*

$$\begin{aligned}
\tau_1(x) &= \alpha x - \frac{\alpha}{2} + 1 \\
\tau_1^2(x) &= \alpha^2 x - \frac{\alpha^2}{2} + \frac{\alpha}{2} + 1 \\
\tau_1^3(x) &= \alpha^3 x - \frac{\alpha^3}{2} + \frac{\alpha^2}{2} + \frac{\alpha}{2} + 1 \\
&\vdots \\
\tau_1^n(x) &= \alpha^n x - \frac{\alpha^n}{2} + \frac{1}{2}\sum_{i=0}^{n-1} \alpha^i + 1.
\end{aligned}$$

Then,

$$\begin{aligned}
\tau_1^n(x) \circ \tau_2(x) &= \alpha^n \left((2-\alpha)(x-0.5)\right) - \frac{\alpha^n}{2} + \frac{1}{2}\sum_{i=0}^{n-1} \alpha^i + 1 \\
&= x\left(2\alpha^n - \alpha^{n+1}\right) + \frac{\alpha^{n+1}}{2} - \frac{3}{2}\alpha^n + \frac{1}{2}\sum_{i=0}^{n-1} \alpha^i + 1.
\end{aligned}$$

Thus, we set $\tau_1^n(x) \circ \tau_2(x) = x + 0$, and obtain two equations

$$\frac{\alpha^{n+1}}{2} - \frac{3}{2}\alpha^n + \frac{1}{2}\sum_{i=0}^{n-1} \alpha^i + 1 = 0, \tag{7.3}$$

and

$$x\left(2\alpha^n - \alpha^{n+1}\right) = x. \tag{7.4}$$

The polynomial in terms of $\alpha$ in equation (7.3) can be rewritten as

$$\frac{1}{2}(\alpha - 2)\left(\alpha^n - \sum_{i=0}^{n-1} \alpha^i = 0\right),$$

and after dividing equation (7.4) by $x$, it can be rewritten as

$$2\alpha^n = \alpha^{n+1} + 1,$$

which is an alternative form of (7.2), that can be obtained by manipulating it as follows:

$$\begin{aligned} \left(x^{n+1}\right) &= x\left(\sum_{i=0}^{n-1} x^i\right) \\ x^{n+1} + 1 &= \sum_{i=1}^{n} x^i + 1 \\ &= x^n + \sum_{i=0}^{n-1} x^i \\ &= 2x^n. \end{aligned}$$

Therefore, $T^n(x) = x$ holds only if the value of the parameter $\alpha$ is the $(n-1)$–th multinacci number, for all $n \in \mathbb{N}$, and every point in $I$ is part of a $n$–cycle. □

For this system, we computed an invariant density, for m-almost every $\alpha$. This density function is given by the expression

$$f^*(x) = \frac{(\alpha - 1)}{\left(\ln\left(\frac{\alpha}{2-\alpha}\right)\right)\left(x\left(\alpha - 1\right) + 1 - 0.5\alpha\right)},$$

which is a fixed point for its Perron-Frobenius operator, i.e., $P_T f^*(x) = f^*(x)$. With this, we computed its Lyapunov exponent

$$\lambda = \int_I \log|T'(\cdot)| f^*(x) d\text{m} = 0.$$

Having a zero Lyapunov exponent is at least, uncommon for a system with an invariant density with support on a set with positive Lebesgue measure. Thus, we want to explore why is this the case. Moreover, we want to build a generalization of this system one-to-one, with no fixed points, and with one expanding branch and a contracting one, where the discontinuity is at any point in $I$, in order to determine under what conditions (if there are any) the behavior observed in system (7.1) can be retrieved and further studied.

# Chapter 8

# Concluding remarks

In this thesis, we discussed the topic of the existence of an a.c.i.m. for a class of random maps in the interval. This question has been addressed in the variety of settings we presented here. Nonetheless, the conditions for the incidence of the phenomenon of phase transition in the sense of Definition 3.1 have not been discussed in the literature. Our two original results here, consider a class of systems with a continuous dependence on parameter that determines the range of the possible transformations to choose from. This setting allows us to provide a method for computing, either a bound for the critical value of the parameter at which the place transition takes place, or its exact value. With this, we establish a first step into the research of the incidence a phase transition phenomenon with respect to a probabilistic parameter, in random maps in the interval.

# Contributions

## Publications

1. Cesar Maldonado and Ricardo A. Pérez Otero. Phase transition in piecewise linear random maps in the interval. *Chaos* 31, 093112, 2021

## Congresses

1. Participation in 54 Congreso Nacional de la Sociedad Matemática Mexicana with the conference: "Fenómeno de transición de fase en mapeos aleatorios en el intervalo lineales por partes", from October 18-October 22, at the Benemérita Universidad Autónoma de Puebla.

2. Participation in 54 Congreso Nacional de la Sociedad Matemática Mexicana with the conference: "Fenómeno de transición de fase en mapeos conectados en el intervalo", from October 23-October 28, at the Centro Universitario de Ciencias Exactas e Ingenierías (CUCEI) de la Universidad de Guadalajara.

## Attendance to Math Schools

1. Participation in the 2019 Houston Summer School in Dynamical Systems from May 30-June 6, 2019 at the University of Houston.

## Preprints

1. Cesar Maldonado, Hugo Alberto Nieto, and Ricardo A. Pérez Otero. From chaos to order on connected dynamical systems, 2023. Preprint.

2. Cesar Maldonado and Ricardo A. Pérez Otero. Phase transition phenomenon in a class of random maps in the interval, 2023. Preprint.

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