A Note on Optimizationof Chaos Suppression via Robust Asymptotic feedback: Accounting control cost.

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This paper deals with the chaos suppression for oscillators in canonical form. The underlying idea is to optimize the robust chaos supression by accounting the control cost. The robust chaos supression is attained by the robust asymptotic feedback while the optimization is solved via Riccatti ecuations. A finite horizon is arbitrarily settled and the suppression is achieved at this time by means of optimal control problem. This scheme allows to take into account the energy that is wasted by the controller and the closed-loop performance of states. Some experimental results show the features of the approach when a High-Gain observer is added in order to have available the complete state vector.

1. Introduction

Nowadays the studies of chaotic systems are still having interest from many fields, like physics, chemistry, electronics, biology, control theory even economy. The chaos suppression problem exhibits an important advance of chaos theory. From the point of view of control theory, the chaos suppression is treated as stabilization of a nonlinear system and a lot of works dealing with stabilization, robust stabilization, secure communication, etc. had been published in the recent decade. Some of these works exploit the geometric control theory to obtain a (suitable) linear equivalent system and then apply the standard methods of linear control. As the chaotic system is not fully linearizable an useful way to tackle the problem is the definition of the canonical form obtained by Lie-based derivative operator [5], [8], [10]. Most of chaotic systems can be transform to this canonical form aided by a nonlinear diffeomorphic map, see for instance [5] and [10].

Currently two questions arises from the chaos suppression problem. The first one question is about how to set an arbitrary time in which the stabilization of system be successfully done. The latter is how to estimate (and possibly to reduce) the control effort wasted in perform the stabilization. Bowong and Kakmeni [3] have reported a scheme for the compute the "time duration" on robust asymptotic feedback for suppressing chaotic behavior. Their work opens this research direction, but their method is overcomplex, implementation is difficult, and implies to solve by simulation a given differential equation at each step of control. Moreover, the control cost has not been reported and an experimental verification shows that high control cost is involved in their approach. The present contribution tackles this two questions and the results shows that is feasible under some conditions set the time of convergence and take into account the effort wasted.

The rest of the paper is organized as follows. The problem formulation is presented in next section. In section 3 the ideal robust controller is constructed. Section 4 is devoted to tackle the problem under estimation of states. Section 5 introduces, by simulations, the features of controller. Section 6 shows the behavior of a real system under this class of controllers. Finally Section 7 present the conclusions of the present document.

2. Problem Formulation

Consider the chaotic system in the canonical form:

$$\begin{cases} \dot{x}_i = x_{i+1}, \; i = 1, 2, ..., p-1, \\ \dot{x}_p = \varsigma(x, \upsilon) + \gamma(x, \upsilon)u, \\ \dot{\upsilon} = \xi(x, \upsilon), \\ y = x_1 \end{cases} \tag{1}$$

where $x \in \mathbb{R}^p$, $\upsilon \in \mathbb{R}^{n-p}$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the state variables, unobservable variables, system input and output, respectively. $\varsigma(x, \upsilon)$ and $\gamma(x, \upsilon)$ are unknown nonlinear functions smooth functions.

The objective is to design a robust feedback controller for system (1) achieving some optimal objective and achieving the trajectory $y = 0$ in a (may be given) finite time.

For the uncertain chaotic system (1) we take the following assumptions

Assumption 1. Only the system output $y = x_1$ is available for feedback.

Assumption 2. $\gamma(x, \upsilon)$ is bounded away from zero.

Assumption 3. The sign of nonlinear function $\gamma(x, \upsilon)$ is known and and estimate $\hat{\gamma}(x)$ of $\gamma(x, \upsilon)$ is available for feedback.

Assumption 4. System (1) is minimum phase; i.e., the subsystem $\dot{\upsilon} = \xi(x, \upsilon)$ is stable.

By defining $\delta(x, \upsilon) = \gamma(x, \upsilon) - \hat{\gamma}(x)$, $\Theta(x, \upsilon, u) = \varsigma(x, \upsilon) + \delta(x, \upsilon)u$, and $\eta = \Theta(x, \upsilon, u)$ [6]. Thus the system (1) takes the form

$$\begin{cases} \dot{x}_i = x_{i+1}, \; i = 1, 2, ..., p-1, \\ \dot{x}_p = \eta + \hat{\gamma}(x)u, \\ \dot{\eta} = \Xi(x, \eta, \upsilon, u, \dot{u}) \\ \dot{\upsilon} = \xi(x, \upsilon), \\ y = x_1 \end{cases} \tag{2}$$

First step in our approach is to consider the transitive of states only. To this end, a quadratic performance criterion, qualifying the transient trajectory toward the desired end point ($y = 0$), is defined as [4]

$$J(x, u) = x(t)^{\intercal} Q_f x(t) + \int_{t_0}^{T} x(t)^{\intercal} Q x(t) dt \tag{3}$$

where $t_0 \geq 0$ is the time which control starts and $T > t_0$ is the time when the system (2) achieves the desired trajectory ($y = 0$); $Q > 0$, $Q_f \geq 0$ are symmetric matrices.

It can be proof that system (2) is dynamically externally equivalent to system (1).

3. Ideal robust control

Theorem 1 *The control law*

$$u(x) = \frac{1}{\hat{\gamma}(x)}\left[-\eta - \tfrac{1}{2}B^{\intercal}P(t)x\right],\ t_0 \le t \le T \tag{4}$$

where T is given, $P(t)$ is a symmetric positive matrix solving the Riccati equation

$$\begin{array}{l} -\dot{P} = A^{\intercal}P + PA - PBB^{\intercal}P + Q \\ P(T) = Q_f \end{array} \tag{5}$$

with $A \in \mathbb{R}^{p\times p}$ and $B \in \mathbb{R}^{p\times p}$ defined as

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

stabilizes the system (2) in the following sense $x \to 0$ as $t \to T < \infty$, with a suitable matrices Q and Q_f, moreover the closed loop performs a value of the functional (3) $J(x,u) = x_0^{\intercal}P(0)x_0$, where x_0 is the initial condition of system.

Proof. Substituting control law (4) in (2) the closed-loop can be written as

$$\left\{ \begin{array}{l} \dot{x} = \left(A - \tfrac{1}{2}BB^{\intercal}P(t)\right)x \\ \dot{\eta} = \Xi(x,\eta,\upsilon,u,\dot{u}) \end{array} \right. \tag{6}$$

Defining the Lyapunov function

$$V(t) = x(t)^{\intercal}P(t)x(t) \tag{7}$$

and evaluate his time derivative

$$\dot{V}(t) = \dot{x}(t)^{\intercal}P(t)x(t) + x(t)^{\intercal}\dot{P}(t)x(t) + x(t)^{\intercal}P(t)\dot{x}(t)$$

substituting (5) and (6) and skipping the time dependence

$$\begin{array}{rcl} \dot{V} & = & x^{\intercal}\left(A - \tfrac{1}{2}BB^{\intercal}P\right)^{\intercal}Px + x^{\intercal}P\left(A - \tfrac{1}{2}BB^{\intercal}P\right)x \\ & & +x(t)^{\intercal}\left(-A^{\intercal}P - PA + PBB^{\intercal}P - Q\right)x(t) \end{array}$$

finally we have

$$\dot{V}(t) = -x(t)^{\intercal}Qx(t) \tag{8}$$

which is negative definite, thus $x(t)$ converges to zero, and Q determines how much fast converges. Now, Integrating (8) from t_0 to T with respect t and using (7) we have

$$x(T)^{\intercal}P(T)x(T) - x(0)^{\intercal}P(0)x(0) = -\int_{t_0}^{T} x(t)^{\intercal}Qx(t)dt$$

then $J(x,u) = x_0^{\intercal}P(0)x_0$ ∎

Remark 1 *The control law (4) is defined only for the interval* $t_0 \leq t \leq T$*, and the stabilization is achieved for some matrices* Q *and* Q_f *if solution of Riccati equation (5) exist in such interval. The proposed feedback requires availability of the complete state.*

A more complete approach can be settled introducing the following definitions

$$\begin{array}{l} \tilde{B} = \left[\begin{array}{cccc} 0 & \cdots & 0 & \hat{\gamma}(x) \end{array}\right]^{\intercal} \\ u_u = -\frac{\eta}{\hat{\gamma}(x)} \end{array} \tag{9}$$

we can call u_u the part "unavoidable" of control, that is the control force necessary to compensate the nonlinear function η in the system. In this way, as a second step of design, the control effort can be added to the cost function. That is, we can redefine the performance criterion to include the "avoidable" control effort in a quadratic criterion (3) to have

$$\tilde{J}(x, \tilde{u}) = x(t)^{\intercal} Q_f x(t) + \int_{t_0}^{T} \left[x(t)^{\intercal} Q x(t) + \tilde{u}^{\intercal} R \tilde{u}\right] dt \tag{10}$$

with a given symmetric matrix $R > 0$, $\tilde{u} := u - u_u$. Hence the control law becomes

$$u = \left[-\frac{\eta}{\hat{\gamma}(x)} + \tilde{u}\right]$$

Therefore the closed loop takes the form

$$\left\{ \begin{array}{l} \dot{x} = Ax + \tilde{B}\tilde{u} \\ \dot{\eta} = \Xi(x, \eta, \upsilon, u, \dot{u}) \end{array} \right. \tag{11}$$

that allows to set the standard LQ problem as [4]

$$\begin{array}{l} \min\limits_{u} \tilde{J}(x, \tilde{u}) \\ s.t. \;\; \dot{x} = Ax + \tilde{B}\tilde{u} \end{array}$$

whose solution for controller is given by

$$\tilde{u} = -R^{-1} B^{\intercal} P(t) x, \;\; t_0 \leq t \leq T$$

where $P(t)$ is now the solution of the following Riccati equation

$$\begin{array}{l} -\dot{P} = A^{\intercal} P + PA - P\tilde{B}R^{-1}\tilde{B}^{\intercal} + Q \\ P(T) = Q_f \end{array} \tag{12}$$

and a value of functional (10) given by

$$\tilde{J}(x, \tilde{u}) = x_0^{\intercal} P(0) x_0 \tag{13}$$

Notice that, at this point, the approach requires full knowledge of states. Nevertheless, according with Assumption 1, only x_1 in available for feedback. In the next section, the full knowledge situation is relaxed by paying higher control cost due to state estimation.

4. Suboptimal robust control

Because Assumption 1, only x_1 is available for feedback. Then, the construction of an observer becomes suitable to get a suboptimal robust controller. We take the observer from [1], [7] and [9].

$$\begin{cases} \dot{\hat{x}}_i = \hat{x}_{i+1} + C^i_{p+1}\rho^i(x_1 - \hat{x}_1), \\ i = 1, 2, ..., p-1, \\ \dot{\hat{x}}_p = \hat{\eta} + \hat{\gamma}(\hat{x})u + C^p_{p+1}\rho^p(x_1 - \hat{x}_1), \\ \dot{\hat{\eta}} = \rho^{p+1}(x_1 - \hat{x}_1) \end{cases}$$

where $(\hat{x}$,$\hat{\eta})$ are estimated values of $(x$,$\eta)$ respectively

$$C^i_{p+1} = \frac{(p+1)!}{i!(p+1-i)!}$$

and ρ is the so-called high-gain parameter. For a sufficiently large value of the high-gain parameter ρ, the dynamics of estimation error converge exponentially to zero, see [7].

By using the defined estimates, the feedback control law (4) can be written as

$$u(\hat{x}) = \frac{1}{\hat{\gamma}(\hat{x})}\left[-\hat{\eta} - \tfrac{1}{2}B^{\intercal}P(t)\hat{x}\right], \ t_0 \le t \le T \tag{14}$$

and we complete the control after transient with the control

$$u(\hat{x}) = \frac{1}{\hat{\gamma}(\hat{x})}\left[-\hat{\eta} - \tfrac{1}{2}B^{\intercal}\bar{P}\hat{x}\right], \ t > T \tag{15}$$

where $\bar{P}$ is a constant positive definite matrix such that the system (2) still in $y = 0$ for $t > T$, thus $\bar{P}$ is also a tuning parameter.

Notice that the control cost of the suboptimal robust controller is higher than the ideal one. The waste of energy increases due to the estimation, and have the form:

$$\tilde{J}(x, \tilde{u}) = x_0^{\intercal}P(0)x_0 + \int_{t_0}^{T} (x - \hat{x})^{\intercal}\, Q\, (x - \hat{x})$$

in consequence, the choice of parameters becomes important and present a trade-off between optimization and estimation.

5. Illustrative example

We take as example to illustrate the behavior of the proposed controller, the Chua's dynamical system, a system easy to implement physically [2]. Chua's circuit is widely know as a system that exhibits a strange attractor and having several configurations of it. The dynamics of Chua's circuits are given by the differential equations [2]:

$$\begin{cases} \dot{x}_1 = \gamma_1\,(x_2 - x_1 - f(x_1)) + u \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -\gamma_2 x_2 \end{cases} \tag{16}$$

where u is the control input and

$$f(x_1)\begin{cases} bx_1 + a - b & x_1 > 1 \\ ax_1 & |x_1| \leq 1 \\ bx_1 - a + b & x_1 < -1 \end{cases}$$

Typical value of the system parameters $\gamma_1 = 9$, $\gamma_2 = 100/7$, $a = -8/7$ and $b = -5/7$ creates chaotic behavior in the dynamical system (16). The implementation of the controller (14) on Chua's system allows to show that it is capable to suppress chaos in dynamical systems whose Lie derivative could not be defined at some points belonging to its domain. In Chua's system case, the Lie derivative is not defined at $(\pm 1, 0, 0)$ [12]. The observer can be written as

$$\begin{cases} \dot{\hat{z}}_1 = \hat{\eta} + u + 2\rho(z_1 - \hat{z}_1) \\ \dot{\hat{\eta}} = \rho^2(z_1 - \hat{z}_1) \end{cases} \tag{17}$$

for this case we can compare the performance with the work of Bowong and Kakmeni [3], They use the same observer but a different controller based on a Lyapunov methodology. The advantage that they present is that the control time is explicitly computed. In our case the control time is arbitrarily settled and the suitable parameters founded.

The initial conditions of system (16) in both cases were $(x_1(0), x_2(0), x_3(0)) = (0.1, 0, 0)$ and for the observer (17) $(\hat{z}_1(0), \hat{\eta}(0)) = (0.1, 0.128)$. The high gain parameter value was chosen as $\rho = 20$. The controller parameter were

$$Q = 1 \times 10^6, Q_f = 1000$$
$$\bar{P} = 1000$$

and the finite horizon is settled in $T = 0.3$. Figure (1) present the behavior of first state of Chua's system for the proposed controller (x_1 Solid line) and the Bowong's one (x_1^c Dotted line). Both systems achieves the zero at $t = 0.3$, that is both converges to the objective set, but the behavior of the proposed control looks better. A zoom in the time line shows details and differences of the evolution for both systems. Figure (2) shows the second and third states for both systems. The system controlled by the proposed controller converges faster to the origin.

6. Experimental implementation

Now, in order to illustrate the real behavior of the approach, the Malasoma's oscillator is physically implemented. The model is based on sprott's work [13], and have three states and have a nonlinear part due to a multiplicative signal. The analysis of chaos can be found in [11]. The dynamical system is given by:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -\alpha x_3 - x_1 + x_1 x_2 + u \end{cases} \tag{18}$$

where α is a parameter that belonging to the set $2.08 < \alpha < 2.51$ allows the chaotic behavior of system. In order to have a physical realization of Malasoma's oscillator We

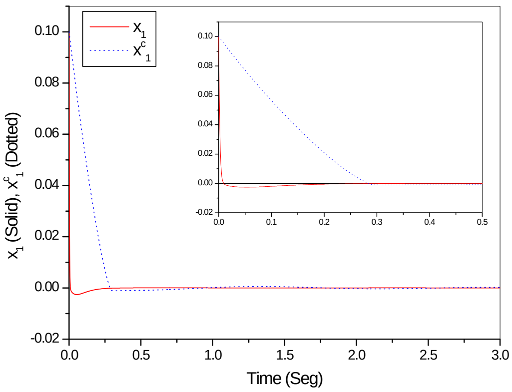

Figure 1. x_1, x_1^c States from two equals Chua's systems for two different controllers

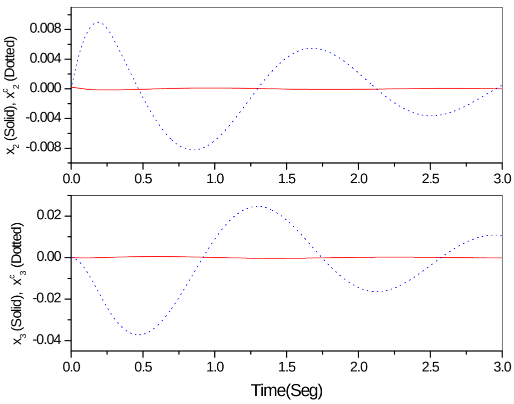

Figure 2. States for the two systems

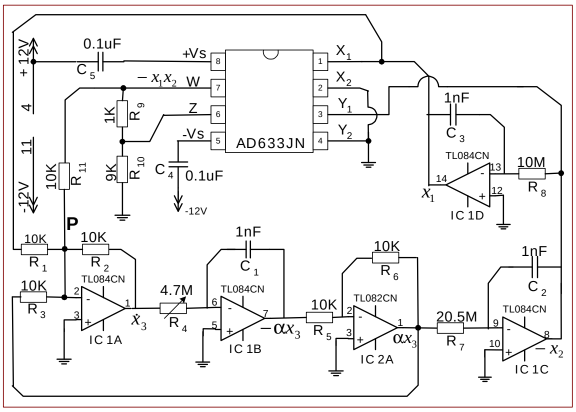

Figure 3. Electronic realization of the Malasoma's oscillator

used the IC AD633JN, which is a four-quadrant multiplier. It includes high impedance X and Y inputs and a high impedance input Z (called summing node). The summing node allows the addition of two or more multiplier outputs. The differential inputs X and Y are converted into differential currents by means of internal voltage-to-currents converters. The product of these currents is performed by the multiplying core and a buried Zener reference provides an overall scale factor of $10V$. Then the sum of $(XY)/10 + Z$, where $X = X_1 - X_2$ and $Y = Y_1 - Y_2$ is applied to the amplifier output. When the Z is not used, it is grounded. Thus, the output of the multiplier is given by

$$W = \frac{(X_1 - X_2)(Y_1 - Y_2)}{10} \frac{(R_9 + R_{10})}{R_9} + Z$$

where Z is the optional summing input of the multiplier, X_1, and Y_1 are the inputs entering to the multiplier and corresponds to the variables to be multiplied, the inputs X_2, and Y_2 are grounded, R_9 and R_{10} are resistors whose value are chosen to compensate the effect of the scaling factor, given by the division of 10. The Figure (3) shows the diagram of the circuit. Note that the chaotic circuit also includes five amplifiers for integration and inversion operations The TL082CN and TL084CN are junction field effect transistors, JFET's, input opamp's. Each opamp incorporates well-matched high-voltage JFET and bipolar transistors in the same integrated circuit. The device features are high slew rates, low input bias and offset currents, and low offset voltage temperature coefficient. This is significant to reduce sensitivity to circuit parameter values. In this manner, the bifurcation of the circuit can be handled.For a better control the observer is turned on before the control acts thus the control law takes the form

$$u(\hat{x}) = \begin{cases} 0 & t < t_0 \\ \frac{1}{\hat{\gamma}(\hat{x})}\left[-\hat{\eta} - \frac{1}{2}B^\intercal P(t)\hat{x}\right] & t_0 \leq t \leq T \\ \frac{1}{\hat{\gamma}(\hat{x})}\left[-\hat{\eta} - \frac{1}{2}B^\intercal \bar{P}\hat{x}\right] & t > T \end{cases} \tag{19}$$

Also We used the dSpace Real Time Interface CP1104, reading only state x_1 of oscillator and sending the control signal to the corresponding node. In order to the control signal must be saturated at $\pm 10V$. In all experiments the observer is turned on at $t = 0$ and the controller at $t_0 = 10$ with a finite horizon of 0.1, that is $T = 10.1$; using a integration step of $Ts = 0.001$, The weighting matrices are given as

$$Q = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, Q_f = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

and the observer gain as $\rho = 135$, finally the corresponding tuning parameter for $t > T$ is

$$\bar{P} = \begin{bmatrix} 1 \times 10^{-5} & 1 \times 10^{-5} \\ 1 \times 10^{-5} & 1 \times 10^{-5} \end{bmatrix}$$

Figure (4) shows the behavior when the observer signals are saturated as

$$\begin{aligned} \hat{x} &\to \pm 10 \\ \hat{\eta} &\to \pm 0.5 \end{aligned}$$

notice that stabilization around the origin is achieved in the given horizon. In the Figure (4) is also plot a zoom of the controller interval in order to view the acceptable behavior taking into account that the system was observed from state x_1. Finally the Figure (4) shows the control action, the dotted lines pointed the control horizon. After $t = 10.1$ the control signal is used only for confine near to zero the state. Figure (5) shows the phase portrait of oscillator when the control action is applied, notice that only x_1 versus x_2 is plotted, but from the equation (18) we know that x_3 is also near to zero.Figure (6) shows the behavior when the observer signals are not saturated but the controller is saturated at $\pm 4V$. Notice that stabilization around the origin is also achieved in the given horizon. In the Figure (6) is also plot a zoom of the controller interval in order to view the acceptable behavior. Finally the Figure (6) shows the control action, the dotted lines pointed the control horizon. Figure (7) shows the phase portrait of oscillator when the control action is applied, notice that only x_1 versus x_2 is plotted and there are near to zero.

7. Conclusions

In this contribution, an optimal robust feedback control is designed to suppress chaotic behavior on dynamical system in strict-feedback form. The problem is solved for the full knowledge of states and for the estimation of states as well, both consider the partial linearization of the dynamical system. The controller takes into account the behavior of transient response and the controller effort. The proposed strategy leads to construct a controller that optimizes the control cost and at the same time allows to set the time for suppress the chaotic behavior. Two examples, shows the features of the proposed controller by simulation and by physical realization of system. The construction of the controller requires minimal information of system and have several tuning parameters that shows the compromises between time of suppress an cost, and between estimation of states and cost.

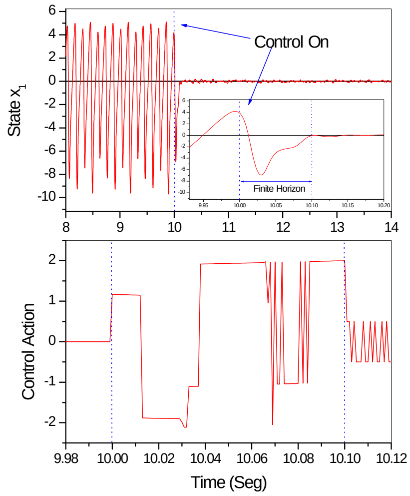

Figure 4. Chaos Suppression in the first state of Malasoma's Oscillator, zoom at finite horizon and controller behavior.

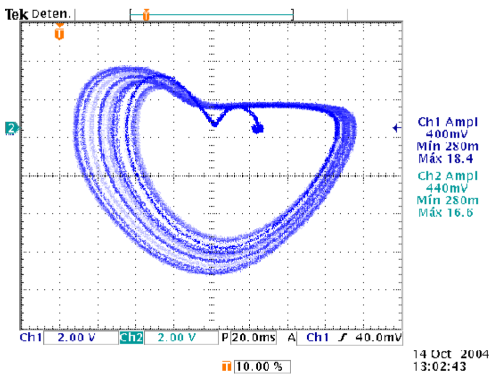

Figure 5. Phase portriat of Malasoma's Oscillator when controller is applied.

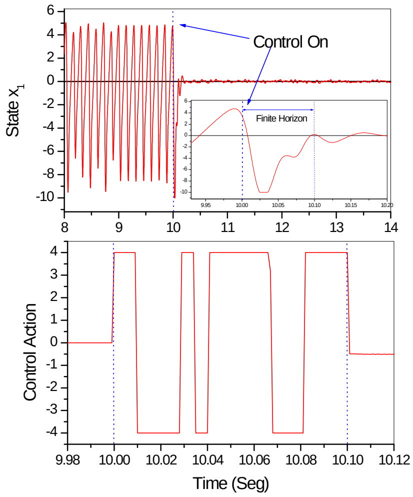

Figure 6. Behavior of Malasoma's under the Control action plotted

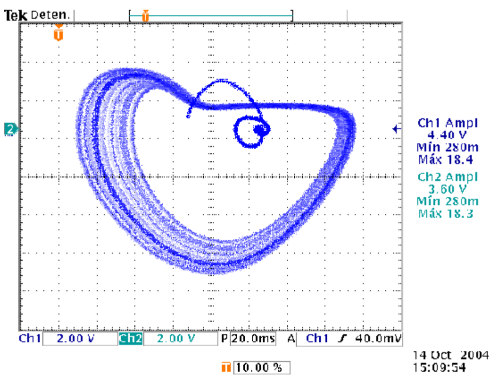

Figure 7. Malasoma's phase portrait under control

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