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## Multiscroll attractors by switching systems

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In this paper, we present a class of three-dimensional dynamical systems having multiscrolls which we call *unstable dissipative systems* (UDSs). The UDSs are dissipative in one of its components but unstable in the other two. This class of systems is constructed with a switching law to display various multiscroll strange attractors. The multiscroll strange attractors result from the combination of several unstable “one-spiral” trajectories by means of switching. Each of these trajectories lies around a saddle hyperbolic stationary point. Thus, we describe how a piecewise-linear switching system yields multiscroll attractors, symmetric or asymmetric, with chaotic behavior.

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**The construction of dynamical systems showing chaotic behavior is relevant in diverse scientific disciplines, for instance, biology and ecology, complex chemical networks, circuit and communication systems, etc. An important issue is to generate chaos via reliable methods using simple dynamical systems. This issue allows one to match the mathematical complexity of chaos with requirements in other disciplines and motivates scientific research in recent years. An interdisciplinary impact is regarding the motion of trajectories in a set of subdomains each one having its own equilibrium. For example, let us consider the motion of flocks looking for distinct feeding sites. Flock trajectories can be described by the motion from one site to another and each site can be interpreted as an equilibrium contained in a subdomain within the migratory region. Chaos can appear by conceiving the flock motion dynamics as a piecewise-linear (PWL) system whose trajectories are in subdomains belonging to the migratory region. Moreover, the environmental conditions can be seen as switching laws defining the border between subdomains. Here, we show how PWL systems with switching laws can generate multiscroll chaotic attractors. The simplicity of our method allows to generate symmetric or asymmetric multiscroll attractors.**

### I. INTRODUCTION

Over the past two decades, self-sustained chaotic systems generating multiscroll attractors have attracted the attention of many scientists. The term *multiscroll attractors* is used to refer to three or more scrolls in an attractor. A prevalent approach has been to modify a system that originally produces double-scroll attractors such that multiscrolls arise,

like Chua and Lorenz systems, among others.<sup>1-3</sup> Usually, a characteristic of these modified multiscroll systems is that they have more equilibria than scrolls. Although the reader can find a summary of different approaches to yield multiscroll chaotic attractors in Refs. 4 and 5, we briefly discuss the relevant contributions on our concern.

Multiscroll attractors can be induced by adding breaking points. As a matter of fact, Chua's double scroll is one of the most extensively studied examples of chaotic behavior. This system was modified at the nonlinear resistor with additional breaking points by Suykens and Vandewalle.<sup>6,7</sup> The modified system can be seen as a generalization of the Chua's system. In the same spirit, the Brockett's system was modified by Aziz-Alaoui.<sup>8</sup> Both Suykens–Vandewalle's and Aziz-Alaoui's proposals yield multiscroll attractors. More recently, multiscroll chaotic attractors with five equilibria exhibited a pair of double-wing chaotic attractors.<sup>9</sup>

Moreover, multiscroll attractors might be induced by switching in piecewise systems. In fact, Lü<sup>10-12</sup> presented chaos generators using a switching PWL controller. Multiple merged basins of attraction are generated in which an orbit holds on a single scroll until the control forces it to enter into a different one. This approach considers just one equilibrium point at each attractor. In addition, switching in piecewise systems has shown to yield very interesting chaotic behaviors<sup>13</sup> and multiscroll attractors are also generated by thresholding.<sup>14</sup>

The effect of PWL functions on dynamics can yield multiscroll attractors. Yalçın and co-workers<sup>15</sup> propounded a linear system with a PWL function in order to generate a *family of scroll grid* attractors. This approach allows to yield scrolls in arbitrary location in  $\mathbb{R}^3$  by having only one equilibrium point for each scroll.

In this work, we propound a class of three-dimensional dynamical systems. This class is called *unstable dissipative systems* (UDSs) because it is dissipative in one of its com-

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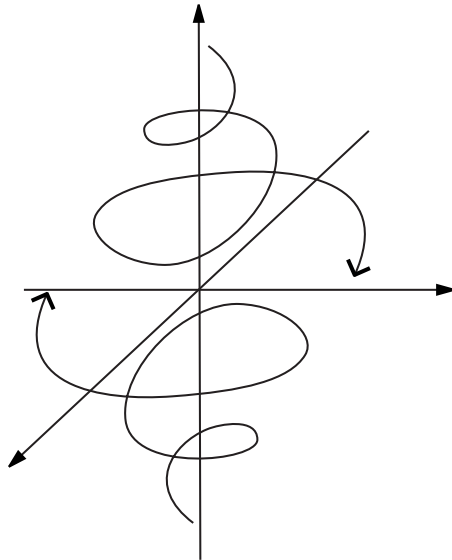


FIG. 1. Saddle hyperbolic stationary point.

ponents while unstable in the other two. The UDSs are constructed with a switching law to obtain various multiscroll strange attractors. The strange multiscroll attractors appear as a result of the combination of several unstable “one-spiral” trajectories. Each of these trajectories lies around a saddle hyperbolic stationary point. This is shown by numerical results which describe how the dynamical changes by switching around the hyperbolic points arise to multiscroll attractors. Our results contribute to extend (i) how a switching system shows multiscroll strange attractors and (ii) some of the routes to multiscroll chaos and bifurcation phenomena. Thus, the evolution of the dynamics and mechanism for the development of multiscroll strange attractors are discussed.

The rest of this work is organized as follows. In Sec. II, the UDS and the switching law are presented to create multiscroll chaotic attractors. Some numerical examples of multiscroll attractors using the proposed approach are given in Sec. III. Conclusions are given in Sec. IV.

## II. SYSTEM DESCRIPTION AND PROBLEM STATEMENT

In the same spirit that in Refs. 10, 15, and 16 we consider the class of linear system given by

$$\dot{\chi} = A\chi + B, \tag{1}$$

$\chi = [x_1, x_2, x_3]^T \in \mathbb{R}^3$  is the state variable,  $B = [b_1, b_2, b_3]^T \in \mathbb{R}^3$  stands for a real vector, and  $A \in \mathbb{R}^{3 \times 3}$  denotes a linear operator given as follows:

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}. \tag{2}$$

We assume the following to generate the multiscroll attractors. The system (1) is dissipative and unstable. Henceforth, the matrix  $A$  is such that the sum of its eigenvalues is negative. The equilibrium of system (1),  $\chi^* = -A^{-1}B \in \Omega \subseteq \mathbb{R}^3$ , is a saddle hyperbolic point. Then, the linear system is character-

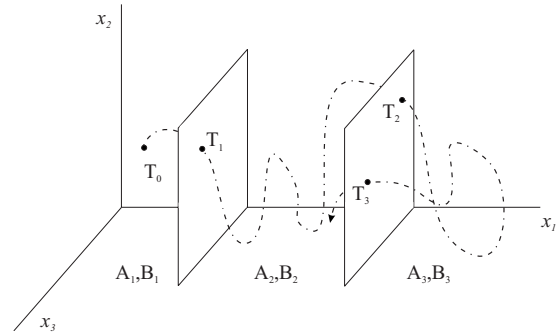


FIG. 2. Generation of multiscroll chaotic attractors under domains  $\mathcal{D}_i$ . Note that flow is governed by the corresponding pair  $A_i, B_i$ .

ized by the set of eigenvalues  $\Lambda: \text{spec}(A)$  and our assumptions are satisfied as *one eigenvalue is a negative real number and two eigenvalues are complex numbers with positive real part but the overall sum is real negative scalar*. The characteristic polynomial of Eq. (2) is given by

$$g(\lambda) = \lambda^3 - \tau\lambda^2 + \gamma\lambda - \delta, \tag{3}$$

where  $\tau = \text{Tr}(A)$ ,  $\gamma = \alpha_{11}\alpha_{22} + \alpha_{11}\alpha_{33} + \alpha_{22}\alpha_{33} - \alpha_{13}\alpha_{31} - \alpha_{23}\alpha_{32} - \alpha_{12}\alpha_{21}$ , and  $\delta = \det(A)$ . The classical Descartes’ rule of signs is a useful tool to state how many positive or negative roots you can expect of polynomial  $g(\lambda)$ . Thus, since system (1) is assumed dissipative, a direct implication is that  $\tau = \text{Tr}(A) = \sum_{i=1}^3 \lambda_i < 0$ . Additionally, the system (1) has a saddle point at the equilibria as  $\delta = \det(A) < 0$ . Moreover,  $\gamma > 0$  in Eq. (3) is required to ensure that the real root of  $g(\lambda)$  is negative. Hence, under the above rationale, there are no positive real roots because all coefficients in Eq. (3) are positive. Such implications give us the possibility that one of the three eigenvalues is a negative real value and the other two are complex with a real positive part. As a summary,  $\tau < 0$ ,  $\delta < 0$ , and  $\gamma > 0$  imply two possibilities: (a) all roots of  $g(\lambda)$  are real negative or (b)  $g(\lambda)$  has one real negative root and two complex roots with real positive part. Since the first choice results on a stable equilibrium point, we are interested in case (b) because the component related with the negative real eigenvalue is attracting and the two complex eigenvalues are responsible for the steady outward slide, as is shown in Fig. 1. Henceforth, we concentrate our discussion on system (1) on case (b). Next, we describe a switching law to generate a UDS which displays a very interesting behavior as multiscroll attractors.

We are interested in a switched system (SW), constituted by a set of systems in the form of Eq. (1) governed by a switching law  $S_i$ , with  $i = 1, \dots, n$  and  $n \geq 2$ . Each system  $S_i$  has a domain  $\mathcal{D}_i \subseteq \mathbb{R}^3$ , containing the equilibrium  $\chi_i^* = -A_i^{-1}B_i$ , such that  $\bigcap_{i=1}^n \mathcal{D}_i = \emptyset$  and  $\bigcup_{i=1}^n \mathcal{D}_i = \Omega \subseteq \mathbb{R}^3$ . Then, the switching law governs the SW dynamics by changing the equilibria from  $\chi_i^*$  to  $\chi_j^*$ ,  $i \neq j$ , when the flow  $\Phi^t: \mathcal{D}_i \rightarrow \mathbb{R}^3$  crosses from the  $i$ th to the  $j$ th domain. The simplest way is that one of the components  $x_j$  defines parallel planes through the one coordinate of the triple  $(x_1, x_2, x_3) \in \mathbb{R}^3$  and the vector  $B_i$  changes as the flow  $\Phi^t$  reaches the border (see Fig. 2). As we shall see below, this very simple configuration allows for the generation of multiscroll attractors.

### III. MULTISCROLL CHAOTIC ATTRACTORS

Once we have stated the problem, seeking clarity, next our results are shown via study cases. Study case (i) shows how the multiscroll attractors are derived as the matrix  $A$  is the same at all domains  $\mathcal{D}_i$ ; that is only the vector  $B_i$  changes. Study case (ii) considers that both  $A_i$  and  $B_i$  change as the flow  $\Phi^t$  goes into the corresponding domain  $\mathcal{D}_i$ . Finally, study case (iii) shows how asymmetric multiscroll chaotic attractors are yielded.

*Study case (i):* In order to have a SW, we should define at least two systems in the form (1). In what follows, without losing generality, we start by illustrating multiscroll attractor generation using the simplest case, i.e., two systems. After that, we add one more system to show how the number of scrolls increases proportionally to the number of systems in SW. For two systems, a switching law is given in terms of only one state, which defines the hyperplanes as in Fig. 2. A convenient approach to build the matrices  $A$  and  $B$  is based on the linear ordinary differential equation (ODE) written in the jerky form  $\ddot{x} + \alpha_{33}\dot{x} + \alpha_{32}x + \alpha_{31}x + \beta = 0$ , where  $\alpha_{33}, \alpha_{32}, \alpha_{31}, \beta \in \mathbb{R}$ .<sup>11,12,16</sup> Note that coefficients of jerky equation can be arbitrary real scalars. Thus, the dynamics of a system in the SW can be represented in state space as Eq. (1), where the matrix  $A$  is found to be

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_{31} & -\alpha_{32} & -\alpha_{33} \end{pmatrix} \quad (4)$$

and  $B = (b_1, b_2, b_3)^T = (0, 0, -\beta)^T$ . Details on the analysis of this jerky equation have been reported in Ref. 11, whereas numerical studies have been provided in Refs. 12 and 16. Now, we are interested in coefficients for the jerky equation satisfying the conditions  $\tau = \text{Tr}(A) = \sum_{i=1}^3 \lambda_i < 0$ ,  $\gamma = \alpha_{23}\alpha_{32} > 0$ , and  $\delta = \det(A) < 0$ , such that  $g(\lambda)$  has one real negative root and two complex roots with real positive part. Notice that  $\chi^*$  belongs to a one-dimensional (1D) submanifold implying that only 1D n-scrolls can be generated. Therefore, we can choose the entries of matrix  $A$  to be  $\alpha_{31} = 1.5$ ,  $\alpha_{32} = \alpha_{33} = 1.0$ , and  $\beta = -1.0$  to generate a double scroll with the switching law defined as follows:

$$SW = \begin{cases} B = (0, 0, -\beta)^T, & \text{if } x_1 \geq 0.35 \\ B = (0, 0, 0)^T, & \text{otherwise.} \end{cases} \quad (5)$$

Also note that the matrix (2) is not restricted to the form derived from jerky equation. This provides richer possibilities on the scroll generation. Next, in order to show that entries of matrix  $A$  can be chosen arbitrarily, but satisfying conditions discussed in Sec. II, we consider the matrix  $A$  as

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & \alpha_{22} & 1 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}, \quad (6)$$

where we might choose  $\alpha_{22} = -0.2465$ ,  $\alpha_{31} = -6.8438$ ,  $\alpha_{32} = -2.006$ ,  $\alpha_{33} = -1.1102$ , and  $b_1 = b_2 = 0$ . Then we have

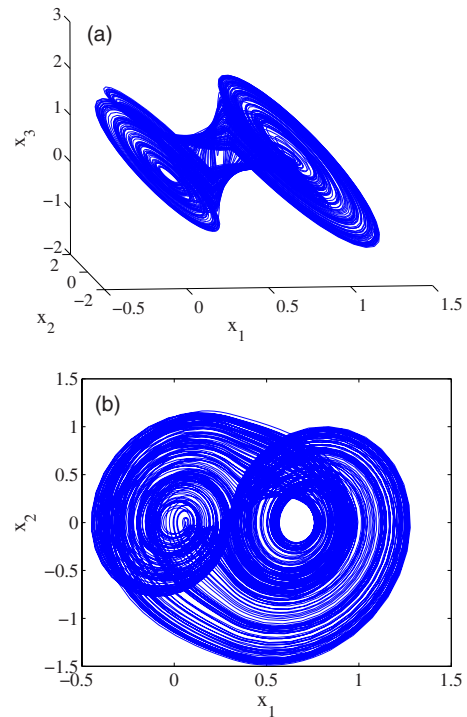


FIG. 3. (Color online) (a) The double-scroll chaotic attractor generated by Eq. (7) is shown. (b) The projection on the plane  $(x_1, x_2)$ .

$\delta_1 = -7.9540$ ,  $\tau_1 = -1.3567$ , and  $\gamma_1 = 3.2797$ . Moreover, let us assume that for the first system  $S_1$ , the vector  $B_1 = (0, 0, b_3)^T$  with  $b_3 = 5.5355$ . Thus, the equilibrium of the system  $S_1$  is  $\chi_1^* = (0.6042, 0, 0.6042)$ . For the second system  $S_2$ ,  $B_2 = (0, 0, 0)^T$ , which means that the system  $S_2$  has equilibrium  $\chi_2^*$  at the origin  $(0, 0, 0)$ . The switching law is defined as follows:

$$SW_2 = \begin{cases} B_1, & \text{if } x_1 \geq 0.3021 \\ B_2, & \text{otherwise.} \end{cases} \quad (7)$$

Figure 3(a) shows the numerical results of the chaotic double-scroll attractor generated by the system SW under Eqs. (6) and (7). Figure 3(b) depicts the projection of the attractor on the plane  $(x_1, x_2)$ . A mathematical analysis for the SW system can be done as in Ref. 11. Now, if we add a third system  $S_3$  into SW, it is possible to generate an attractor with triple scroll. For the third system, we have chosen the follow vector  $B_3 = (0, 0, -5.5355)^T$ . Notice that  $B_3 = -B_1$ , which implies  $\chi_1^* = -\chi_3^*$ . This issue is intentionally defined to illustrate the symmetry in scrolls. Now the switching law is as follows:

$$SW_3 = \begin{cases} B_1, & \text{if } x_1 \geq 0.3021 \\ B_2, & \text{if } -0.3021 < x_1 < 0.3021 \\ B_3, & \text{if } x_1 \leq -0.3021. \end{cases} \quad (8)$$

Figure 4(a) shows the chaotic multiscroll attractor generated by the system SW under Eqs. (6)–(8). Figure 4(b) shows the

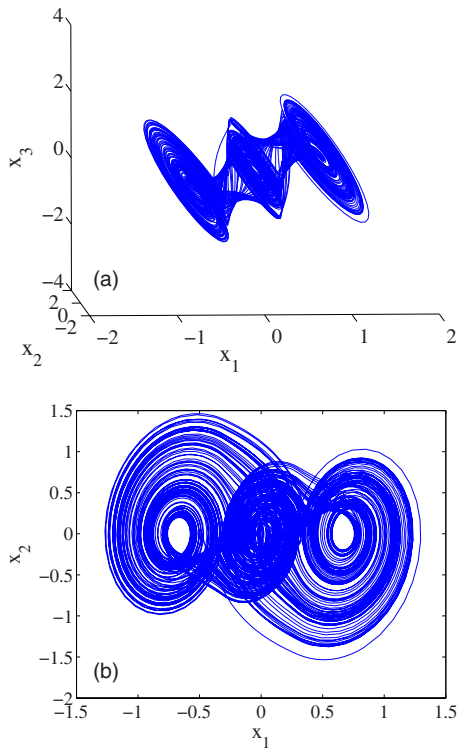


FIG. 4. (Color online) (a) The multiscroll chaotic attractor generated by Eq. (8) is shown. (b) The projection on the plane  $(x_1, x_2)$ .

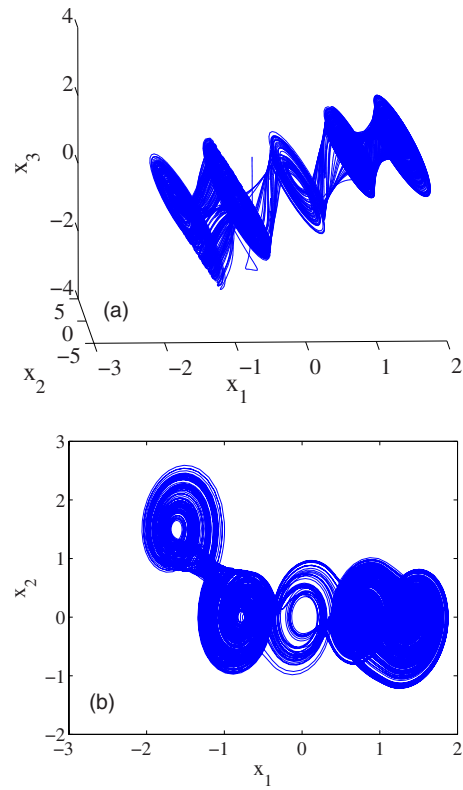


FIG. 5. (Color online) (a) The chaotic attractor generated by Eq. (9) is shown. (b) The projection on the plane  $(x_1, x_2)$

projection of the attractor on the plane  $(x_1, x_2)$ . So, it is possible to yield several as attractors as required by adding systems in form (1) into SW. For example, the asymmetric quintuple-scroll chaotic attractor might be given by the next vectors: for the systems  $S_1$ ,  $B_1=(0011.1699)^T$  with equilibrium located at  $\chi_1^*=(1.4043, 0, 1.4043)$ ; for  $S_2$ ,  $B_2=(006.1455)^T$  with  $\chi_2^*=(0.7726, 0, 0.7726)$ ; for  $S_3$ ,  $B_3=(000)^T$  with  $\chi_3^*=(0, 0, 0)$ ; for  $S_4$ ,  $B_4=(00-6.1455)^T$  with  $\chi_4^*=(-0.7726, 0, -0.7726)$ ; finally, for  $S_5$ ,  $B_5=(-1.5 -0.3698 -9.1545)^T$  with  $\chi_5^*=(-1.6325, 1.5, -0.8929)$ . The switching rule for these five systems is

$$SW_5 = \begin{cases} B_1, & \text{if } x_1 \geq \kappa_2 \\ B_2, & \text{if } \kappa_1 < x_1 < \kappa_2 \\ B_3, & \text{if } -\kappa_1 \leq x_1 \leq \kappa_1 \\ B_4, & \text{if } x_1 < -\kappa_1 \text{ and } x_2 < \kappa_2 \\ B_5, & \text{if } x_1 < -\kappa_1 \text{ and } x_2 \geq \kappa_2, \end{cases} \quad (9)$$

where  $\kappa_1=0.3863$  and  $\kappa_2=1.0884$ . Figure 5(a) shows the asymmetric quintuple-scroll chaotic attractor given by Eq. (9). The projection of the multiscroll chaotic attractor on the plane  $(x_1, x_2)$  is shown in Fig. 5(b).

*Study case (ii):* The following paragraphs are devoted to show the effect of changes on the entries of matrix  $A$  into the dynamics of SW system in form (1). Although the case considering a switching law as Eq. (7) is interesting to chaos generation, the effect of differences in matrix  $A$  with only two domains is not relevant due to its similarity discussed

above. Then, we focus on the case with at least three domains. We proceed maintaining the same values of entries for  $B$  [as in the case (i)] and changing the entries for  $A$ . For example, let us consider the system SW as follows. The first system  $S_1$  has matrices defined by

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -0.2465 & 1 \\ -8.0521 & -2.006 & -1.1102 \end{pmatrix}, \quad (10)$$

$$B_1 = \begin{pmatrix} 0 \\ 0 \\ 5.5355 \end{pmatrix},$$

from where  $\delta_1=-9.1623$ ,  $\tau_1=-1.3567$ , and  $\gamma_1=3.2797$  with its equilibrium at  $\chi_1^*=(0.6042, 0, 0.6042)$ ; the second system  $S_2$  becomes

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -0.2465 & 1 \\ -6.8438 & -2.006 & -1.1102 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (11)$$

resulting in  $\delta_2=-7.9540$ ,  $\tau_2=-1.3567$ ,  $\gamma_2=3.2797$ , and the equilibrium  $\chi_2^*=(0, 0, 0)$ ; finally, a third system  $S_3$  is considered to generate symmetric or asymmetric attractors depending on the position of the equilibrium point. The third system  $S_3$  for the symmetric case is the following:

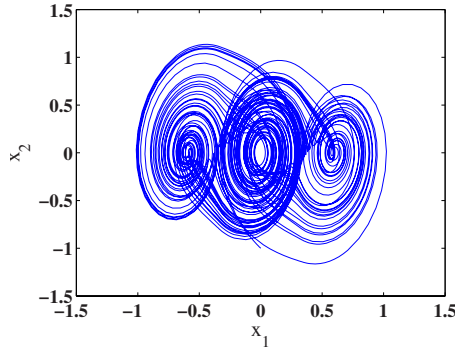


FIG. 6. (Color online) Symmetric case: this figure shows the projection on the plane  $(x_1, x_2)$  of the chaotic attractor generated by Eq. (13), i.e.,  $\chi_1^* = -\chi_3^*$ .

$$A_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -0.2465 & 1 \\ -8.0521 & -2.006 & -1.1102 \end{pmatrix}, \tag{12}$$

$$B_3 = \begin{pmatrix} 0 \\ 0 \\ -5.5355 \end{pmatrix}.$$

Notice that  $\chi_1^* = -\chi_3^*$ . Now, the switching law for the symmetric multiscroll attractor is defined as follows:

$$SW_3 = \begin{cases} S_1, & \text{if } x_1 \geq 1/3 \\ S_2, & \text{if } -1/3 < x_1 < 1/3 \\ S_3, & \text{if } x_1 \leq -1/3. \end{cases} \tag{13}$$

In Fig. 6 is shown the projection on the plane  $(x_1, x_2)$  of the chaotic attractor generated by Eqs. (10)–(13). Next, for the asymmetric case, we locate  $\chi_1^* \neq -\chi_3^*$  under the same switching law (13). The projection on the plane  $(x_1, x_2)$  of the multiscroll chaotic attractor is shown in Fig. 7.

*Study case (iii):* Actually, it is possible to yield as many multiscroll chaotic attractors as we wish to, for instance, the asymmetric quintuple-scroll chaotic attractor given by the next systems. For first system  $S_1$ , its matrix  $A$  is redefined to be

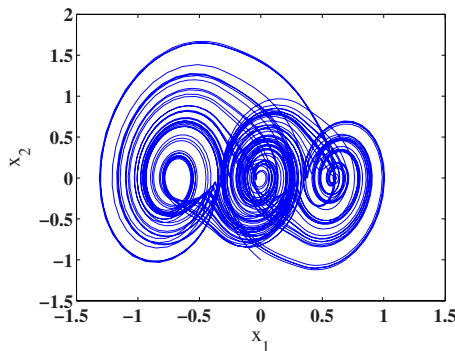


FIG. 7. (Color online) Asymmetric case: this figure shows the projection on the plane  $(x_1, x_2)$  of the chaotic attractor generated by Eq. (13), i.e.,  $\chi_1^* \neq -\chi_3^*$ .

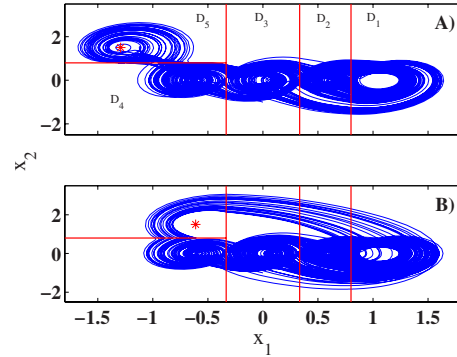


FIG. 8. (Color online) Asymmetric case: this figure shows the projection on the plane  $(x_1, x_2)$  of the chaotic attractor generated in study case (iii). Note that the hyperplanes are projected as straight lines for each switch  $S_i, i=1 \dots 5$  in Eq. (16). Also note that equilibrium point is denoted with \* in order to show the effect of position of equilibrium at domain  $\mathcal{D}_5$ .

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -0.2465 & 1 \\ -8.9394 & -2.006 & -1.1102 \end{pmatrix}, \tag{14}$$

$$B_1 = \begin{pmatrix} 0 \\ 0 \\ 11.1699 \end{pmatrix},$$

where  $\delta_1 = -10.0496$ ,  $\tau_1 = -1.3567$ ,  $\gamma_1 = 3.2797$ , and its equilibrium  $\chi_1^* = (1.1115, 0, 1.1115)$ ; for the second system  $S_2$  we consider  $A_1 = A_2$  and  $B_2 = (0.061455)^T$ , which means  $\chi_2^* = (0.6115, 0, 0.6115)$ . The third system  $S_3$  is taken as

$$A_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -0.2465 & 1 \\ -8.7429 & -2.006 & -1.1102 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{15}$$

where  $\delta_3 = -9.8531$ ,  $\tau_3 = -1.3567$ ,  $\gamma_3 = 3.2797$ , and its equilibrium is at  $\chi_3^* = (0, 0, 0)$ . The fourth system  $S_4$  becomes such that  $A_4 = A_1$  and  $B_4 = (0, 0, -6.1455)^T$ , which means that equilibrium is at  $\chi_4^* = (-0.6115, 0, -0.6115)$ . The matrices for the last system  $S_5$  are given by  $A_5 = A_1$  and  $B_5 = (-1.5, -0.3698, -9.1545)^T$ , which locate the equilibrium at  $\chi_5^* = (-1.2921, 1.5, -0.5525)$ . The switching law for this quintuple-scroll chaotic system is chosen as

$$SW_5 = \begin{cases} S_1, & \text{if } x_1 \geq \kappa_2 \\ S_2, & \text{if } \kappa_1 < x_1 < \kappa_2 \\ S_3, & \text{if } -\kappa_1 \leq x_1 \leq \kappa_1 \\ S_4, & \text{if } x_1 < -\kappa_1 \text{ and } x_2 < \kappa_2 \\ S_5, & \text{if } x_1 < -\kappa_1 \text{ and } x_2 \geq \kappa_2, \end{cases} \tag{16}$$

where  $\kappa_1 = 1/3$  and  $\kappa_2 = 4/5$ . The projection of the quintuple-scroll chaotic attractor on the plane  $(x_1, x_2)$  for the asymmetric case is shown in Fig. 8(a).

Note that if the position of  $\chi_5^*$  is moved on its first component  $\chi_5^*(1)$  near the first component of fourth equilibrium  $\chi_4^*$  [i.e.,  $\chi_5^*(1) \rightarrow \chi_4^*(1)$ ], the quintuple scroll disappears [see Fig. 8(b)]. This fact can be explained in terms of the closeness of equilibrium  $\chi_5^*$  to the border of  $\mathcal{D}_4$  and its relation with

oscillation rate induced by matrix  $A_5$ . That is, the position of the equilibrium point  $\chi_5^*$  of  $S_5$  is close to the border defining switching  $\chi_5^* = (-0.6116, 1.5, -0.6115)$ . In order to depict this, Fig. 8(b) includes the projections of the hyperplanes delimiting the border of each domain  $\mathcal{D}_i$ . This last discussion shows that a necessary condition to generate each scroll from SW in form (2) is that the equilibrium point presents a one negative real eigenvalue and two complex eigenvalues with positive real part. However, intuitively, such a condition is necessary but not sufficient, because it is required to have a sufficiently large region where the scroll can occur.

#### IV. CONCLUSIONS

This work presents a study of very simple switched systems yielding multiscroll chaotic attractors. The degrees of freedom to yield the multiscrolls are two: (i) entries of affine matrices  $A$  and  $B$  and (ii) the borders of each domain defining hyperplanes, where each switching law is established. In other words, by defining UDS within different domains and each domain containing one equilibrium point. The paper contributes at the issue of constructing switching systems SW with chaotic behavior. Our contribution has a potential application at interdisciplinary science as, for example, the trajectories of flocks looking for feeding or resting places are interpreted as PWL systems and the hyperplanes are given by borders of distinct environmental conditions at the migration region.

A natural progression is to address the following issue: (a) the equilibrium  $\chi_i^* \in \mathbb{R}^n$  defined by a pair  $(A_i, B_i)$  lies outside the borders of domain  $\mathcal{D}_i$  or (b) a pair  $(A_i, B_i)$  induces multiple equilibria at domain  $\mathcal{D}_i$ . This makes sense, for instance, if matrix  $A \in \mathbb{R}^{m \times n}$ ,  $n \neq m$ . Additionally, (c) the coupling between switched systems opens the possibility to

study what kind of synchronization occurs, e.g., multimodal synchronization,<sup>17</sup> asymmetric synchronization, forced synchronization,<sup>18</sup> or any other.<sup>19</sup>

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