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Attractors generated from switching unstable dissipative systems

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In this paper, we present a class of 3-D *unstable dissipative systems*, which are stable in two components but unstable in the other one. This class of systems is motivated by whirls, comprised of switching subsystems, which yield strange attractors from the combination of two unstable "one-spiral" trajectories by means of a switching rule. Each one of these trajectories moves around two hyperbolic saddle equilibrium points. Both theoretical and numerical results are provided for verification and demonstration. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4742338]

Biologic systems, weather systems, technology devices, and so on, generate oscillations and many of them have been mathematically modeled by a pair of first-order ordinary differential equations with suitable parameters to ensure oscillatory behaviors. Many of such nonlinear phenomena observed in nature or man-made devices have been described by piecewise-linear (PWL) systems due to the richness of their dynamical behaviors: limit cycles, homoclinic and heteroclinic orbits, strange attractors, etc. PWL systems are based on switching systems consisting of a set of subsystems and a switching signal which selects a subsystem to be active during an interval of time. Switching systems can properly characterize the structural variations of many practical systems through their operating processes. Also, the solutions of this class of systems can be integrated in closed forms when they are restricted to a region of the phase space. However, the analysis of the corresponding dynamics is never trivial. This paper is devoted to studying a mechanism of constructing chaos-generating systems based on PWL systems. This chaos generation mechanism may find useful applications in biologic systems and technological systems when chaos is desirable.

I. INTRODUCTION

Consider a system of nonlinear autonomous differential equations, capable of displaying chaotic behavior,

$$\dot{\mathbf{x}} = f(\mathbf{x}), \tag{1}$$

where $\mathbf{x} \in \mathbf{R}^n$ and $f : E \to \mathcal{A}$ with $E \supset \mathcal{A}$, in which E and $\mathcal{A}$ are open subsets of $\mathbf{R}^n$. Under well-posed conditions on f, system (1) has a unique solution starting from each point $\mathbf{x}_0 \in \mathbf{R}^n$ defined on a maximal interval of existence, $(a, b) \subset \mathbf{R}$. In general, it is not possible to solve the nonlinear system (1) analytically; however, a great deal of qualitative information about the local behaviors of its solutions can be determined near the equilibrium point $\mathbf{x}^*$ satisfying $f(\mathbf{x}^*) = 0$. The local behavior is determined by the Jacobian $Df(\mathbf{x}^*)$ of Eq. (1) and specifically its eigenvalues $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

An inverse problem is to generate chaos with strange attractors from linearly coupled systems. This contribution is devoted to such an inverse problem for a specific class of systems.

Definition 1. (Ref. 1) *An equilibrium point* $\mathbf{x}^*$ *of Eq. (1) is called a sink if all the eigenvalues of the matrix* $Df(\mathbf{x}^*)$ *have negative real parts; it is called a source if all the eigenvalues of* $Df(\mathbf{x}^*)$ *have positive real parts; it is called a saddle if it is a hyperbolic equilibrium point in the sense that* $Df(\mathbf{x}^*)$ *has at least one eigenvalue with a positive real part and at least one with a negative real part but no eigenvalues have zero real parts.*

Saddle equilibrium points, which connect to a stable manifold W^s and an unstable manifold W^u, are responsible of successive stretching and folding therefore play an important roll in generating chaos. The stretching causes the system trajectories to exhibit sensitive dependence on initial conditions whereas the folding creates the complicated microstructure.[2] The saddle points of a chaotic system in $\mathbf{R}^3$ can be characterized into two types according to its eigenvalues $\Lambda = \{\lambda_i, \lambda_j, \lambda_k\} \in \mathcal{C}$: (*i*) The saddle points that are stable in one of its components but unstable or oscillatory in the other two.[3] That is, the stable component is corresponding to a negative real eigenvalue; *i.e.*, $Re\{\lambda_i\} < 0, Im\{\lambda_i\} = 0$, whereas the unstable components are related with two complex conjugate eigenvalues; *i.e.*, $Re\{\lambda_k\} > 0$, $Im\{\lambda_k\} \neq 0$. (*ii*) The saddle points that are stable in two of its components but unstable in the another one. That is, the dissipative components are oscillatory: $Im\{\lambda_k\} \neq 0$ and $Re\{\lambda_k\} < 0$, while the unstable component corresponds to the real positive eigenvalue $Re\{\lambda_i\} > 0$, $Im\{\lambda_i\} = 0$. The Hartman-Grobman theorem and the stable manifold theorem show that the local behavior of nonlinear system (1), near an equilibrium point $\mathbf{x}^*$, is topologically determined by the behavior of a linear system,

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{2}$$

where $A = Df(\mathbf{x}^*) \in \mathbf{R}^{n\times n}$ and $\mathbf{x} \in U = N_\epsilon(x^*) \subset \mathbf{R}^n$. When there are multiple $x^* \in \mathbf{R}^n$, satisfying $f(x^*) = 0$, the same

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form of linear system given by Eq. (2) is related with each x^* in the corresponding neighborhood. If the system given by Eq. (2) has a saddle equilibrium point responsible for unstable and stable manifolds and the sum of its eigenvalues is negative, then the system is called an unstable dissipative system (UDS).

According to the above discussion, it is possible to define two types of UDS, and two types of corresponding equilibria.

Definition 2. A system given by Eq. (2) in $\mathbf{R}^3$ *with eigenvalues* λ_i, $i=1,2,3$, *is said to be an UDS Type I, if* $\sum_{i=1}^{3}\lambda_i<0$ *and one eigenvalue* λ_i *is negative real and the other two are complex conjugate with a positive real part.*

Definition 3. A system given by Eq. (2) in $\mathbf{R}^3$ *with eigenvalues* λ_i, $i=1,2,3$, *is said to be an UDS Type II, if* $\sum_{i=1}^{3}\lambda_i<0$ *and one eigenvalue is positive real and the other two are complex conjugate with a negative real part.*

For the corresponding equilibria, their two types are defined accordingly. The above definitions imply that the UDS *Type I* is dissipative in one of its components but unstable in the other two, which are oscillatory. The converse is the UDS *Type II*, which are dissipative and oscillatory in two of its components but unstable in the other one. Some hyperbolic chaotic dynamical systems in $\mathbf{R}^3$ may be related to these two types of UDS around equilibria; for instance, Chua's system[4] has two UDS *Type I* equilibria, symmetrically distributed, and another UDS *Type II* at the origin. Rössler's system[5] can also be characterized through UDS *Type I* and *Type II*, and similarly some other systems.[6–8] A characteristic of all these systems is that their scrolls are generated from UDS *Type I*. Recently, a class of 3-D dynamical systems was composed in Ref. 3 by constructing a system with a switching law to obtain various multiscroll attractors. Such systems are derived under the assumption of having an unique hyperbolic equilibrium point to each scroll, which belongs to UDS *Type I*. In general, chaotic dynamical systems have been constructed in two options: (i) considering both UDS *Type I* and *Type II*, for example Chua's systems; or (ii) only using UDS *Type I*, as those reported in Ref. 3.

In this work, we contribute to use only UDS *Type II* by constructing a class of 3-D dynamical systems. This class of dynamical systems generates strange attractors, which appear as a result of the combination of two unstable "one-spiral" trajectories. The trajectory of the attractor lies between two hyperbolic equilibrium points. This is shown by numerical simulations describing the dynamical changes via switching in between two UDS *Type II*.

II. SYSTEM DESCRIPTION AND PROBLEM STATEMENT

In general, in order to generate attractors via UDS *Type II* at equilibrium points distinct to origin, an affine linear system can be considered. Thus, from the same approach as that in Refs. 3, 9–11, we consider a class of affine linear systems given by

$$\dot{\chi} = A\chi + B, \tag{3}$$

where $\chi=[x_1,x_2,x_3]^T\in\mathbf{R}^3$ is the state vector, $B=[b_1,b_2,b_3]^T\in\mathbf{R}^3$ is a real vector, and $A\in\mathbf{R}^{3\times3}$ denotes a linear operator given as

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}. \tag{4}$$

Recall that if the matrix A has a negative sum of eigenvalues then it is dissipative. The equilibrium of system (3) is $\chi^* = -A^{-1}B\in\Omega\subset\mathbf{R}^3$ and it is a saddle hyperbolic point with a one-dimensional unstable manifold $W^u_{\chi^*}$ and a two-dimensional stable manifold $W^s_{\chi^*}$, see Figure 1. The dynamics of the affine linear system (3) is characterized by the set of eigenvalues $\Lambda : spec(A)$. Condition in Definition 3 is satisfied when one eigenvalue is a positive real number and the other two are complex conjugate numbers with a negative real part.

Theorem 1. *Let* δ,γ,τ *be real numbers defined by* $\delta\triangleq\det(A)$, $\gamma\triangleq-\alpha_{11}\alpha_{22}-\alpha_{11}\alpha_{33}-\alpha_{22}\alpha_{33}+\alpha_{13}\alpha_{31}+\alpha_{23}\alpha_{32}+\alpha_{12}\alpha_{21}$, $\tau\triangleq\mathrm{trace}(A)$, $p\triangleq-\gamma-\frac{\tau^2}{3}$ and $q\triangleq-\delta-\frac{\gamma\tau}{3}-\frac{2\tau^3}{27}$. *If* $\tau<0$ *and* $\delta>0$ *with* $q^2/4+p^3/27\geq0$ *then system (3) is UDS Type II with the unique equilibrium point at* χ^*.

Proof. Suppose $\tau<0$ and $\delta>0$. Since, by definition, $\tau=\mathrm{Trace}(A)=\sum_{i=1}^{3}\lambda_i<0$, system (3) is dissipative. Additionally, when $\delta=\det(A)>0$, system (3) has a saddle equilibrium, which is determined by the characteristic polynomial of Eq. (4), namely,

$$g(\lambda) = -\lambda^3+\tau\lambda^2+\gamma\lambda+\delta. \tag{5}$$

In fact, the classical Descartes' rule of signs implies that Eq. (5) has sings $(-,-,\mathrm{sign}(\gamma),+)$. That is, there is only one change of coefficient sign independently if $sign(\gamma)<0$ or $sign(\gamma)>0$. This implies that one of the three eigenvalues is a positive real value and the other two are (i) complex conjugate with a negative real part or (ii) negative real numbers. Thus, system (3) has a one-dimensional unstable manifold $W^u_{\chi^*}$ and a two-dimensional stable manifold $W^s_{\chi^*}$.

Define $y\triangleq\lambda-\tau/3$. By using the Cardano approach and the expansion of Taylor series, Eq. (5) takes the form,

$$g\left(y+\frac{\tau}{3}\right) = g\left(\frac{\tau}{3}\right)+g'\left(\frac{\tau}{3}\right)y+\frac{g''\left(\frac{\tau}{3}\right)}{2}y^2+\frac{g'''\left(\frac{\tau}{3}\right)}{6}y^3. \tag{6}$$

Evaluating g and its derivatives at $\frac{\tau}{3}$, the resulting equation is obtained as

$$y^3+py+q=0. \tag{7}$$

Then, Eq. (7) is solved by the change of variables, $y=u+v$ and $p=-3\,uv$, giving

$$u^3+v^3=-q; \quad u^3v^3=-\frac{p^3}{27}. \tag{8}$$

Combining both expressions in Eq. (8), one has $u^6+u^3q-\frac{p^3}{27}=0$. By defining $t=u^3$, one obtains

$$t^2+qt-\frac{p^3}{27}=0. \tag{9}$$

Its real solutions are

$$t_1 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}},$$
$$t_2 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \tag{10}$$

for $\frac{q^2}{4} + \frac{p^3}{27} \geq 0$. Due to the symmetry of the terms between u^3 and v^3 in system (8), $u^3 = t_1$ gives $u = \sqrt[3]{t_1}$. Then, there exist three roots: $u_1 = \sqrt[3]{t_1}$, $u_2 = w\sqrt[3]{t_1}$, and $u_3 = w^2\sqrt[3]{t_1}$, where $w = \frac{-1+i\sqrt{3}}{2}$ with $i = \sqrt{-1}$. Equivalently for v, one has three values: $v = \sqrt[3]{t_2}$, $v = w\sqrt[3]{t_2}$, and $v = w^2\sqrt[3]{t_2}$. Therefore, Eq. (5) has roots,

$$\lambda_1 = \sqrt[3]{t_1} + \sqrt[3]{t_2} + \tau/3,$$
$$\lambda_2 = w\sqrt[3]{t_1} + w^2\sqrt[3]{t_2} + \tau/3, \tag{11}$$
$$\lambda_3 = w^2\sqrt[3]{t_1} + w\sqrt[3]{t_2} + \tau/3.$$

Consequently, $\lambda_1 \in \mathbf{R}$ and $\lambda_{2,3} \in \mathbf{C}$. Finally, according to Descartes' Rule of Signs, one of the eigenvalues is positive real and the other two are complex conjugate with a negative real part, which completes the proof.

We are interested in a switching system (*SW*), which is constituted by two systems in the form of Eq. (3), S_i, $i = 1,2$, governed by a switching law. Each system S_i has a domain $\mathcal{D}_i \subset \mathbf{R}^3$, containing the equilibrium $\chi_i^* = -A_i^{-1}B_i$. Then, the switching law governs the *SW* dynamics which changes the equilibrium from χ_1^* to χ_2^*, or vice versa.

Definition 4. Let the two systems be given by Eq. (3) in $\mathbf{R}^3$ *with domains* $D_i \subset \mathbf{R}^3$ *and equilibria* $\chi_i^* \in D_i$, *for* $i = 1,2$. *Define a SW by two UDS Type II systems as follows:*

$$\dot{\mathbf{x}} = \begin{cases} A_1\mathbf{x} + B_1, & \text{if } \mathbf{x} \in D_1; \\ A_2\mathbf{x} + B_2, & \text{if } \mathbf{x} \in D_2. \end{cases} \tag{12}$$

Note that the system given by Eq. (12) induces a flow $\varphi^t, t \in \mathbf{R}$ in the phase space $\mathbf{R}^3$, such that the forward orbit of the initial point $\mathbf{x}_0$ is the set $\{\mathbf{x}(t) = \varphi^t(\mathbf{x}_0) : t \geq 0\}$. Assume that system (12) has a dissipation ball $\Omega \subset \mathbf{R}^3$ such that $\varphi^t(\Omega) \subset \Omega$ for all $t \geq 0$. This assumption will be guaranteed by suitably choosing the pair (A_i, B_i), $i = 1, 2$. The maximal attractor $\mathcal{A}$ of system (12) is the largest attracting invariant subset of Ω.

Proposition 1. Assume that system (12) oscillates between its two sub-systems. Then, for each $\mathbf{x}_0 \in \mathcal{A}$, there exists a flow $\varphi^t(\Omega) \subset D$ such that

(a) $\cup_{i=1}^2 D_i = D \subseteq \mathbf{R}^3$,

(b) $\cap_{i=1}^2 D_i = \emptyset, \cap_{i=1}^2 cl(D_i) \neq \emptyset$,

where $cl(\cdot)$ is the closure of a set.

The statement (a) is about the domain of system (12), which is constituted by two domains of UDS *Type II*. The statement (b) indicates that the flow φ^t is determined by only one vectorial field and moves from one domain to another one immediately. This is illustrated by Figure 2. As will be seen below, this very simple configuration allows the generation of strange attractors.

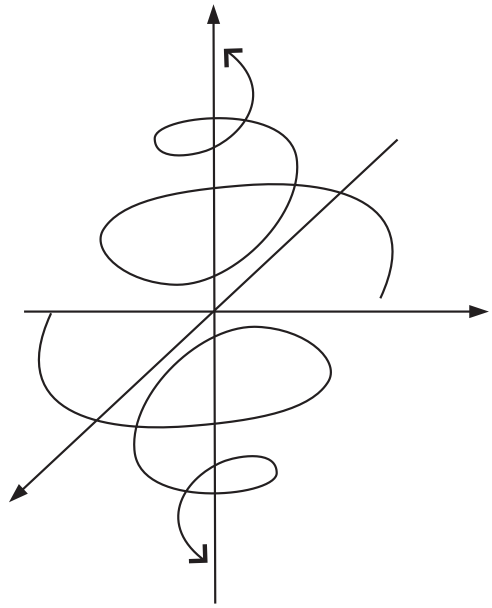

FIG. 1. Qualitative trajectories around a saddle hyperbolic equilibrium point of UDS *Type II*.

Proposition 2. The basin of attraction Ω of an attractor $\mathcal{A}$ given by system (12) is located between the two stable manifolds $W^s_{\chi_i^*}$, i = 1,2.

The above result is illustrated by Figure 3. Notice that in system (12), the unstable manifolds $W^u_{\chi_1^*}$ and $W^u_{\chi_2^*}$ guide the flow φ^t towards the equilibria χ_1^* and χ_2^*, respectively. Therefore, the orbit oscillates between the two equilibrium points. Thus, all initial points located outside the space between the manifolds $W^s_{\chi_i^*}$, $i = 1, 2$, become unstable. As a result, the basin of attraction Ω has the only possibility to be located between the manifolds $W^s_{\chi_i^*}$. Moreover, $W^s_{\chi_i^*} \cap \Omega = \emptyset$, $i = 1,2$, because starting from any initial point in $W^s_{\chi_i^*}$ the orbit goes to χ_i^*, $i = 1,2$.

III. ATTRACTORS GENERATED BY UDS TYPE II

The first case study is to show how attractors can be generated when the matrix A is identical in all domains $\mathcal{D}_i$; that is, only B_i are changed. The second case study is to consider both matrices A_i and vectors B_i be changed as the flow Φ^t goes into the corresponding domains $\mathcal{D}_i$, $i = 1,2$.

A convenient approach to building the matrices A and B is based on the linear ordinary differential equation (ODE) written in the jerky form: $\dddot{x} + \alpha_{33}\ddot{x} + \alpha_{32}\dot{x} + \alpha_{31}x + \beta = 0$

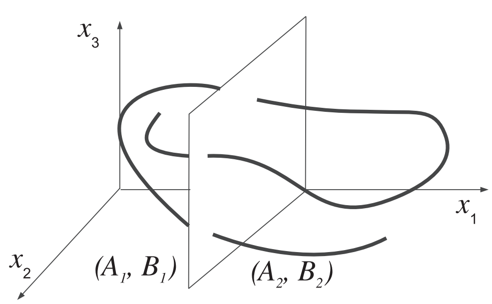

FIG. 2. Generation of attractors in domains D_i. The flow φ^t is governed by the corresponding pair (A_i, B_i), $i = 1,2$.

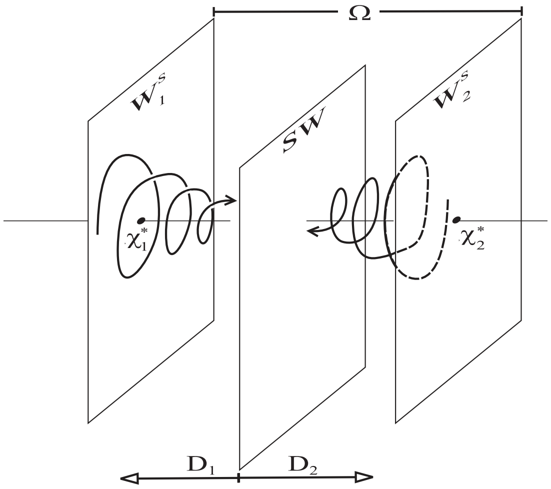

FIG. 3. The basin of attraction Ω of an attractor $\mathcal{A}$.

(see Refs. 10, 12, and 13), for which the *SW* can be represented in state space as Eq. (1), where the matrix A is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_{31} & -\alpha_{32} & -\alpha_{33} \end{pmatrix} \tag{13}$$

and $B = (b_1, b_2, b_3)^T = (0, 0, -\beta)^T$. Then, it follows from Eq. (13) directly that $\delta = -\alpha_{31}$, $\tau = -\alpha_{33}$, $\gamma = -\alpha_{32}$. By Theorem 1, the dynamical system based on matrix Eq. (13) is a UDS *Type II* if $\tau < 0$, $\delta > 0$ and $q^2/4 + p^3/27 \geq 0$. Therefore, one can choose the entries of matrix A to be $\alpha_{31} = -0.15$, $\alpha_{32} = 10$, $\alpha_{33} = 1.0$ and β to be zero or 10 so as to generate a system having two equilibria and one attractor, with the switching law defined as follows:

$$SW = \begin{cases} B_2 = (0, 0, -10)^T, & \text{if } x_1 \geq 1; \\ B_1 = (0, 0, 0)^T, & \text{otherwise.} \end{cases} \tag{14}$$

Also note that the matrix Eq. (4) is not restricted to the form derived from a jerky equation. This provides richer possibilities on chaos generation.

Figure 4 shows some numerical results of the attractor generated by the *SW* (Eqs. (13) and (14)). The attractor oscillates between two UDS *Type II*, near the switching surface $x_1 = 1$, and its equilibria are located at $\chi_1^* = (0, 0, 0)$ and $\chi_2^* = (66.6667, 0, 0)$. The largest Lyapunov exponent is 0.015. If the switching surface is located in the middle of the two equilibrium points, with $x_1 \simeq 33.333$, then the strange attractor collapses to a limit cycle. This approach does not yield oscillations around any equilibrium, different from the Rössler system whose oscillations are around an equilibrium *Type I*. Figure 5(a) shows a projection of the attractor onto the plane (x_1, x_2). In this figure, it is possible to see the effect of repulsion given by the subsystem (A_2, B_2). Figure 5(b) shows a projection of the attractor onto the plane (x_1, x_3). Figure 5(c) shows a projection of the attractor onto the plane (x_2, x_3).

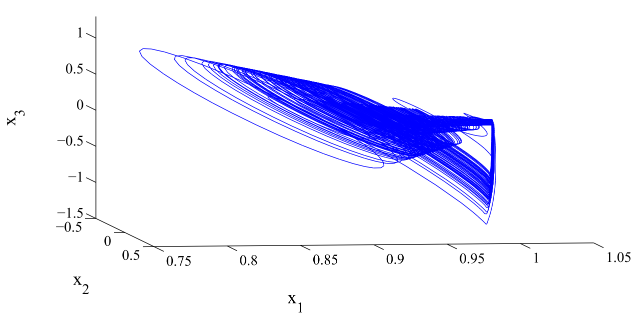

FIG. 4. The attractor generated from Eqs. (13) and (14).

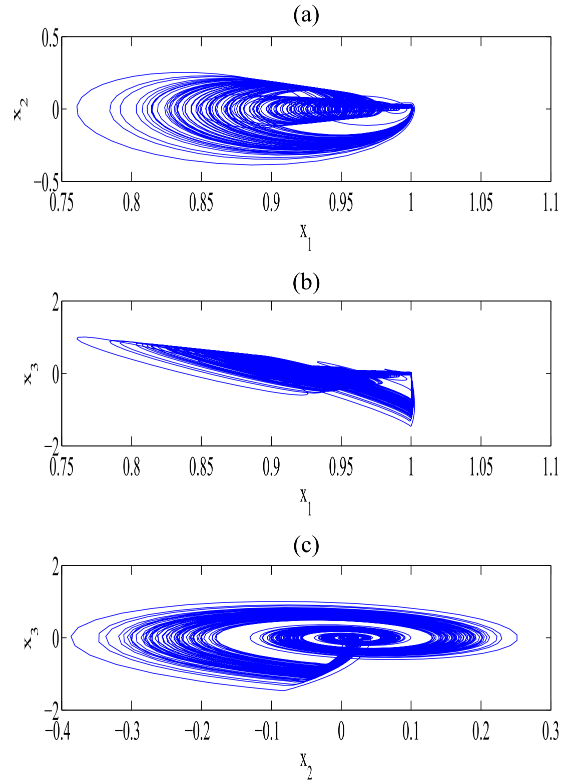

FIG. 5. Projections of the attractor onto planes: (a) (x_1, x_2), (b) (x_1, x_3), and (c) (x_2, x_3). The switching surface is defined at $x_1 = 1$. Because the system (A, B_2) is highly repulsive, the trajectories appear to bounce at the switching surface.

As seen above, the SW given by Eqs. (13) and (14) has two UDS *Type II*, with equilibrium points located at the origin and at (a, 0, 0) with $a > 1$. The parameter a is restricted to be $a > 1$ since the switching surface is located at (1,0,0), so a can be taken as a bifurcation parameter. Figure 6 shows

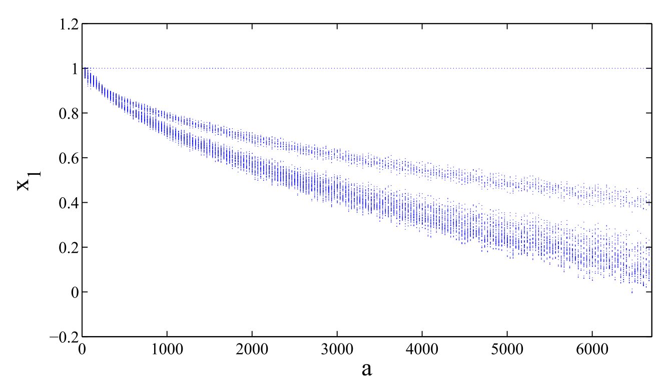

FIG. 6. Bifurcation diagram for $2 < a < 6700$.

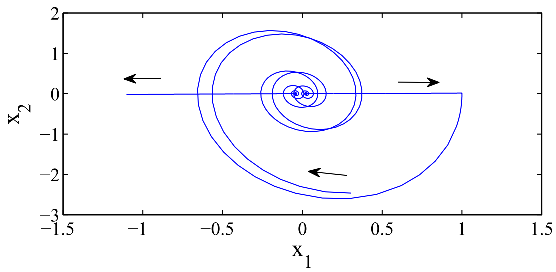

FIG. 7. The trajectory escapes along $W^u_{\chi_1}$ for $a = 6800$.

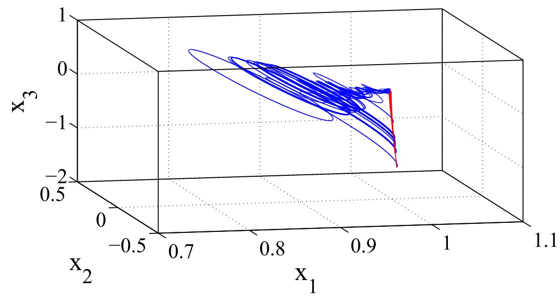

FIG. 9. The attractor generated from Eqs. (13) and (14). The blue curve corresponds to the solution when $x_1 < 1$ and the red curve when $x_1 \geq 1$.

the bifurcation diagram for $2 < a < 6700$. Figure 7 shows how the trajectory goes away along the unstable manifold $W^u_{\chi_1}$ for $a = 6800$. Figure 7 captures the segment of trajectory where the pair (A, B_2) is more repulsive than the pair (A, B_1). This provokes a very strong bounce near the switching surface and, as a consequence, the trajectory escapes away. Figure 8 shows the time series of the system, which are underdamped signals.

In order to get an explicit solution of the system given by Eqs. (13) and (14), we first recall the Fundamental Theorem for Linear Systems and a proposition about linear transformations.

Theorem 2. (Ref. 1) *[The Fundamental Theorem for Linear Systems] Let A be an $n \times n$ matrix. Then, for a given $\mathbf{x}_0 \in \mathbf{R}^n$, the initial value problem*

$$\dot{\mathbf{x}} = A\mathbf{x},$$

$$\mathbf{x}(0) = \mathbf{x}_0,$$

has a unique solution given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0.$$

Proposition 3. (Ref. 1) *If A and J are linear transformations on $\mathbf{R}^n$ and $A = PJP^{-1}$, then $e^A = Pe^JP^{-1}$.*

Now, the solution of the system given by Eqs. (13) and (14) when the flow moves into $D_1 = \{\mathbf{x} \in \mathbf{R}^3 : x_1 < 1\}$ domain is given by

$$\mathbf{x}(t) = P\begin{pmatrix} e^{0.014977232285795t} & 0 & 0 \\ 0 & \Theta_1 & -\Theta_2 \\ 0 & \Theta_2 & \Theta_1 \end{pmatrix}P^{-1}\mathbf{x}(0) \qquad (15)$$

where $\Theta_1 = e^{-0.507488616142898t}\cos(3.123724836514592t)$ and $\Theta_2 = e^{-0.507488616142898t}\sin(3.123724836514592t)$.

For the case that the flow moves into $D_2 = \{\mathbf{x} \in \mathbf{R}^3 : x_1 \geq 1\}$ domain, the solution of the system given by Eqs. (13) and (14) is found by means of a change of variables, as follows: $z_1 = x_1 - (1000/15)$, $z_2 = x_2$ and $z_3 = x_3$. Thus, the system is given by

$$\dot{\mathbf{z}} = A\mathbf{z}, \quad \text{if } \mathbf{x} \in D_2,$$

where $\mathbf{z} = (z_1, z_2, z_3)^T$. Now, using the change of variables it is possible to find the solutions of $x_1(t)$, $x_2(t)$ and $x_3(t)$ when the flow belongs to D_2 domain. Figure 9 shows the exact solution of the attractor generated by the *SW* (Eqs. (13) and (14)) with initial condition $\mathbf{x}(0) = (0.9, 0, 0)$. The attractor oscillates between two UDS *Type II*, near the switching surface $x_1 = 1$, and its equilibria are located at $\chi_1^* = (0, 0, 0)$ and $\chi_2^* = (66.6667, 0, 0)$.

Figure 10 shows three different qualitative trajectories that can possibly be obtained by moving the equilibrium point χ_2^* into domain D_2 while another equilibrium point χ_1^* at the origin into domain D_1. The switching surface *SW* is given by $\cap_{i=1}^2 cl(D_i)$. Let d_1 and d_2 denote the minimum distances from the equilibrium points χ_1^* and χ_2^* to *SW*, respectively. When $d_1 = d_2$, the trajectory is trapped into a limit cycle and if $d_1 < d_2$ this asymmetry can break periodic oscillations so that the trajectory is trapped, generating an attractor between two saddle hyperbolic equilibrium points (see Figure 10(a)). But d_2 can increase until a certain value $d_2 < d_H$ because if $d_2 > d_H$ the trajectory escapes along $W^u_{\chi_1^*}$ (see Figure 10(c)). If the distance $d_2 = d_H$, it is possible to induce that the

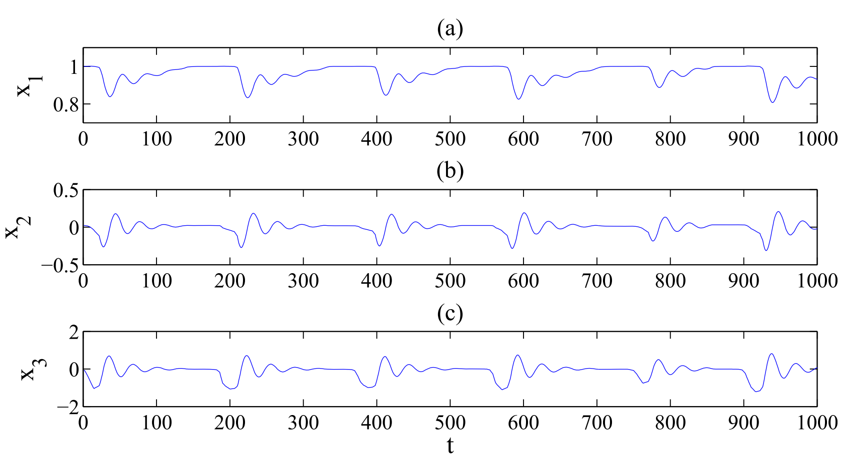

FIG. 8. Time series of the attractor shown in Figure 4. (a) x_1 state, (b) x_2 state, (c) x_3 state.

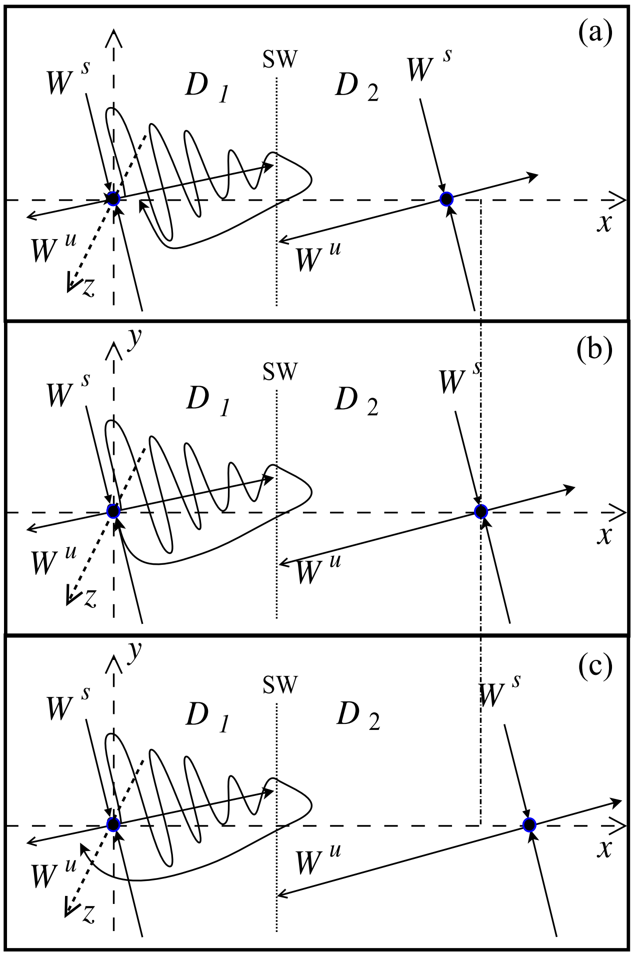

FIG. 10. Qualitative trajectories between two saddle hyperbolic equilibrium points. (a) Trajectory is trapped. (b) Homoclinic orbit is generated. (c) Trajectory is escaped along W^u.

trajectory goes to equilibrium point located at χ_1^* (see Figure 10(b)). It is worth mentioning that the stable manifold W^s of the saddle-focus χ_1^* is $2D$, whereas the unstable one W^u is $1D$. The manifold W^u is the union of χ_1 and two separatrices that tend to χ_1 as $t \to -\infty$. A homoclinic loop Γ of the saddle-focus is a trajectory bi-asymptotic to χ_1 as $t \to \pm\infty$. The construction of homoclinic orbits in order to demonstrate chaotic attractors is beyond the scope of this paper, so we have only included the largest Lyapunov exponent.

A. Scroll attractor for 3 subdomains

Under the condition given for UDS *type II* and when the trajectory escapes from $d_2 > d_H$, a natural question is: Is it possible to trap the trajectory again between the stable manifolds $W^s_{\chi_1}$ and $W^s_{\chi_2}$? The answer to this question depends on the number of subdomains. When the trajectory escapes it can be bounced again, defining another equilibrium point symmetrically, that is, $\chi_3^* = -\chi_2^*$. The switching law Eq. (14) is replaced by

$$SW = \begin{cases} B_2 = (0,0,1700), & \text{if } x_1 \geq 1; \\ B_1 = (0,0,0), & \text{if } \mid x_1 \mid < 1; \\ B_3(0,0,-1700), & \text{if } x_1 \leq -1. \end{cases} \quad (16)$$

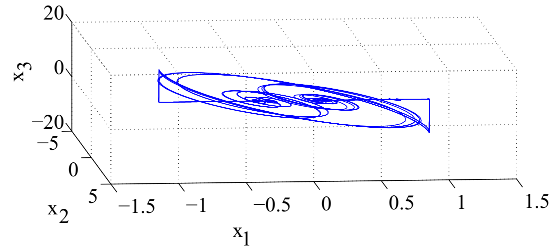

FIG. 11. The attractor generated from Eqs. (4) and (16).

The matrices $A_1 = A_2 = A_3$ and the values are determined by Eq. (4). Figure 11 shows some numerical results of the attractor generated by SW (Eqs. (4) and (16)). The attractor oscillates between the two switching surfaces $x_1 = -1$ and $x_1 = 1$, near the stable manifold $W^s_{\chi_1^*}$, and its equilibria are located at $\chi_1^* = (0,0,0)$, $\chi_2^* = (8000,0,0)$ and $\chi_3^* = (-8000,0,0)$. Figure 12 shows the time series of the system: (a) x_1 state, (b) x_2 state, (c) x_3 state.

B. Different *A* matrices

A precise location of the basin of attraction can be described by defining the equilibria χ_i^* and their stable manifolds $W^s_{\chi_i}$, $i = 1,2$. Here, we show an example in which both matrices A_i and vectors B_i are changed as follows:

$$SW = \begin{cases} A_1, B_1 = (-5,-12,19)^T, & \text{if } x_1 < 9, \\ A_2, B_2 = (-5,-12,-20)^T, & \text{otherwise}, \end{cases} \quad (17)$$

where

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.5 & -4 & -0.12 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 15 & -25 & -0.5 \end{pmatrix}.$$

Figure 13 shows some numerical results of the attractor generated by the *SW* (Eq. (17)). The attractor oscillates near

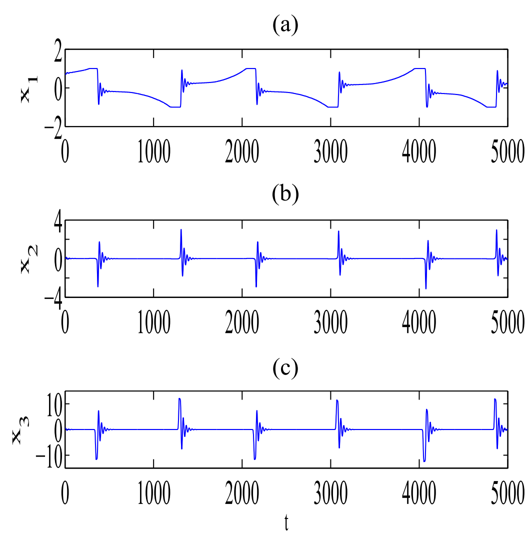

FIG. 12. Time series of the attractor shown in Figure 11. (a) x_1 state, (b) x_2 state, (c) x_3 state.

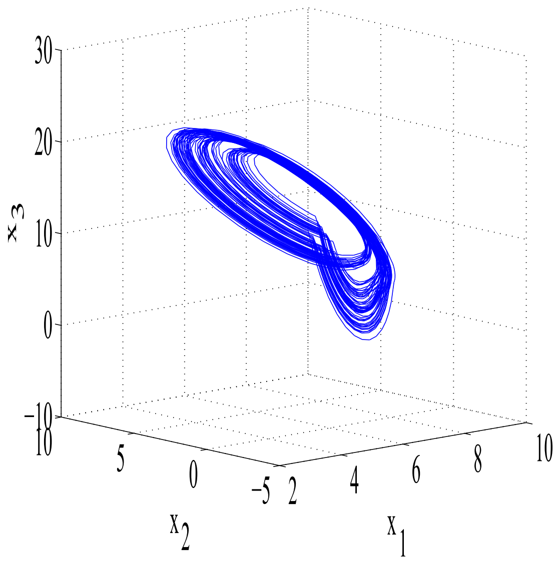

FIG. 13. The attractor generated from Eq. (17).

the switching surface $x_1 = 9$, and its equilibria are located at $\chi_1^* = (4.88, 5.00, 12.00)$ and $\chi_2^* = (10.0667, 5.00, 12.00)$, respectively.

IV. CONCLUSIONS

This paper has studied a mechanism of constructing chaos-generating systems based on two piece-wise linear systems. Particularly, it deals with UDS *Type II*. It derives conditions to generate chaotic attractors. The attractor arises from a switching system having at least two UDS *Type II*. Two examples are shown by means of considering a system in which the A matrix is the same in both domains D_1 and D_2, and the difference lies only in the B matrices which determine the locations of the equilibrium points. The other example considers a system in which both matrices A and B are different in both domains D_1 and D_2. This result was extended to yield a system with three domains and a precise location of the basin of attraction can be given when the system is comprised by UDS's *Type II*.

This approach may be further extended to generate: (1) chaotic systems with multistabilities by adding more UDS *Type II*, and (2) multivibrators in a similar way that Ref. 14 where local Lyapunov exponents were changed to control the stability in order to generate the multivibrator.

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