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Forced synchronization of autonomous dynamical Boolean networks

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We present the design of an autonomous time-delay Boolean network realized with readily available electronic components. Through simulations and experiments that account for the detailed nonlinear response of each circuit element, we demonstrate that a network with five Boolean nodes displays complex behavior. Furthermore, we show that the dynamics of two identical networks display near-instantaneous synchronization to a periodic state when forced by a common periodic Boolean signal. A theoretical analysis of the network reveals the conditions under which complex behavior is expected in an individual network and the occurrence of synchronization in the forced networks. This research will enable future experiments on autonomous time-delay networks using readily available electronic components with dynamics on a slow enough time-scale so that inexpensive data collection systems can faithfully record the dynamics. © 2015 AIP Publishing LLC.

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In real-world, there exist many situations in which simple objects are interconnected in a network, such as power grids, ecosystems, communication systems, human relationships, economic trading, and biological neural networks. In the network approach for analyzing these systems, the individual simple systems are represented as nodes and their interactions are represented as links. Mathematically, a network is represented as a collection of points and ordered pairs describing their connections, this mathematical object is called a graph. However, it is often difficult to find a graph that represents a complex real-world system, especially when signals travelling along the links experience a delay time that is long in comparison to the response time of the nodes. To simplify the problem for the case when the nodes display a switching-like behavior, a Boolean approximation for the nodes is appropriate and allows for progress on a theoretical analysis of the dynamics. Boolean networks are a particularly simple form of a complex system and have been found to be useful in the research of a wide range of systems, including gene regulatory systems, circuit theory, and computer science. A particularly interesting behavior is when networks synchronize their behavior. Here, we present the design of a Boolean network that displays complex behavior and shows that the networks can synchronize their behavior when they are forced by a common periodic Boolean signal. This research lays the foundation for future experimental research on time-delay Boolean networks that will help to guide theoretical research on these systems difficult to analyze.

I. INTRODUCTION

Boolean networks were first proposed by Kauffman in 1969 as a mathematical framework for studying gene regulatory networks.¹ Boolean networks have received a great deal of attention across many disciplines because they are an approximate model of any network system where the network nodes display switch-like behavior. Examples include electronic logic circuits or gene regulatory networks, and it is often useful to assume that for these systems, the state variables take only two values (e.g., “high” and “low”), updated according to specified Boolean functions.^{2–6}

There are three ways in which the state variables can be updated in Boolean networks: synchronously, asynchronously, or autonomously. Synchronous update rules assume that an external process, such as a clock, synchronizes all the updates or a device that selects a particular order of individual gate updates.⁷ Because there are only a finite number of total states, the network must eventually visit a state that it has been before, and because the update rules are deterministic, the network settles into a periodic attractor or fixed point. For asynchronous updating rules, the Boolean states of the nodes are updated according to their logic functions simultaneously successively with randomly chosen updating order. In autonomous updating, the future behavior is determined by the history of the past network switching events, and the time-delays along the network links must be taken into account. In other words, a node in an autonomous Boolean network (ABN) updates its Boolean state whenever Boolean transitions are present at its inputs. The mathematics describing autonomous time-delay Boolean networks is much less developed, although it is known that they can display aperiodic patterns if the logic elements have instantaneous response times, the link time delays are incommensurate,

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and the nodes predominantly perform the exclusive-OR (XOR) Boolean operation.^{8–10} The importance of studying ABNs lies on the fact that several processes in nature, for example, genetic and metabolic regulation networks, update their states in continuously way, thus, ABNs can approach in a realistic way the qualitative behavior of this real-world network. Another important point of ABNs is their possible usefulness to generate true random signals applicable to radars or cryptographic systems. A practical example of an ABN consists on modelling neurological networks in which it is possible to see recurring topological structures of node assembled in loops with directed links as occurs with *C. elegans*. At nodes that have multiple inputs, signals are combined and the propagation times from their output to their input via the different loops are given by the loop sizes and are affected by heterogeneity in link time delays.¹¹

Recently, Zhang *et al.*¹² studied experimentally the dynamics of a three-node autonomous network where the nodes consisted of commercially-available discrete high-speed electronic logic gates. The network links were implemented by cascading an even number of inverted gates, which effectively delayed the signal. For most situations, this network displayed so-called Boolean chaos even though the nodes did not have an instantaneous response time assumed in the earlier theoretical studies described above.¹³ More recent work has demonstrated that large-scale autonomous time-delay Boolean networks can be realized experimentally using field programmable gate arrays.¹⁴

In nature, synchronization phenomena occur when two or more systems are coupled or are driven by a common signal. For the case of coupled systems, synchronization can be observed for either unidirectional coupling in a master-slave configuration or bidirectional coupling where signals are shared between both systems. Synchronization phenomena in coupled systems have been extensively studied during the last decades and several concepts describing synchronized dynamics have been introduced in Refs. 15 and 16. Observed behaviors include frequency entrainment,¹⁷ phase synchronization,¹⁴ lag synchronization,¹⁸ multimodal synchronization,¹⁹ complete (or full) synchronization,⁷ and generalized synchronization.²⁰

Different cases of forced synchronization have been presented, such as antisymmetric, lag, phase, and identical synchronization. In Ref. 21, forced synchronization phenomenon was explored using two identical slave systems forced harmonically, thus different modes of synchronized behaviors were found due to the high sensibility to changes in the parameters values of the external driving, for example, identical, lag, and antisymmetric forced synchronization appear when the two systems oscillate into a limit cycle and phase locking forced synchronization when one of the attractors is a torus and the other one is a limit cycle.

The purpose of this article is to introduce the design of an ABN that can be realized with readily-available, low-speed, and inexpensive electronic components. The network consists of five nodes realized using a novel reconfigurable logic gate (numerical simulation) and commercially-available discrete electronic logic gates (experimentally). In numerical simulation, the time-delay network links are realized by a low-pass resistor-capacitor (RC) circuit, which

allows for delays even with slower-speed signals, although it causes some signal degradation. Experimentally, the network links are implemented by cascading an even number of inverted gates, which effectively delayed the signal. We study forced synchronization in two identical ABNs by driving two nodes in each network with a periodic Boolean signal. The dynamics of an individual ABN and the two forced networks are studied through numerical simulations using SPICE (Simulation Program with Integrated Circuit Emphasis) environment and experimentally using commercial electronic logic gates. We focused this work in forced synchronization scheme because it has application in chaotic communication systems, where there are at least two forced systems involved: (a) the transmitter, which has the information signal as its external driving and (b) the receiver, which is forced by the incoming carrier (delivered by the transmitter). In this kind of communication systems, the goal is achieved when the transmitter forces the receiver to synchronize. As a result, the receiver may be seen as a filter because its response depends on the transmitter output, losing its autonomy. On the other hand, the transmitter is forced by the information signal, but it must preserve its autonomy and its chaotic behavior thus it cannot behave as a filter.

Our work has important implications for research on ABNs. From a fundamental perspective, we elucidate the conditions under which an external drive signal can force the synchronization of ABNs. From a practical perspective, our ABN design can be realized in experiments using inexpensive and readily available commercial components and the time scale of the dynamics is slow enough that inexpensive waveform capture hardware can be used. Furthermore, all locations within the circuit can be probed so that it is possible to fully characterize the non-ideal behavior of the circuit, which may be important for observing Boolean chaos,¹³ and which is not possible when using field programmable gate arrays.¹⁴

The remainder of this article is as follows. In Sec. II, some concepts about Boolean networks and forced synchronization are given. In Sec. III, the design of the five-node ABN is presented. Section IV describes our results on the complex behavior observed in the ABN as well as the prediction of forced synchronization. Section V contains the experimental results. We discuss our results and conclude in Sec. VI.

II. PRELIMINARIES

A Boolean network consists of a number of nodes connected to other ones through directed or undirected links. Each node is associated with a binary state variable and its value is determined by a Boolean logic function that evaluates input arguments of such node. Based on this, a Boolean network is defined in a general form as a triplet

$$BN = (\mathbb{G} = (V, E), \mathbb{B} = \{0, 1\}, \mathbb{F} = \{f_1, \dots, f_n\}), \quad (1)$$

where \mathbb{G} is a directed or undirected graph comprised by the set of vertices V and the set of edges E , \mathbb{B} is the set of logic states that can be associated with each vertex, and \mathbb{F} is the

set of local activation functions also known as truth tables. It is worth mentioning that the set of Boolean variables of the *BN* is given by its vertices $V = \{x_1, x_2, \dots, x_n\}$. Thus, the phase space is given by \mathbb{B}^n .

In many cases of interest, signals propagate along the network links at slow enough speed so that the time delay along the link is comparable to or larger than the characteristic response time of the node. In this case, the time delay must be accounted for in the model and can be introduced by prescribing a set of delays $\{\tau_{ij}\}$, $i, j = 1, \dots, n$, such that $\tau_{ij} \geq 0$, where τ_{ij} is the time it takes for x_j to have an effect on x_i , i.e., the time that takes a signal to propagate to node i from node j . Notice that it is not necessary to have $\tau_{ij} = \tau_{ji}$. Thus, the feedbacks among the Boolean variables are described by the set of local activation functions $f_i : \mathbb{B}^n \rightarrow \mathbb{B}$, $i = 1, \dots, n$, as follows:

$$\begin{aligned} x_1(t) &= f_1(x_1(t - \tau_{11}), x_2(t - \tau_{12}), \dots, x_n(t - \tau_{1n})), \\ x_2(t) &= f_2(x_1(t - \tau_{21}), x_2(t - \tau_{22}), \dots, x_n(t - \tau_{2n})), \\ &\vdots \\ x_n(t) &= f_n(x_1(t - \tau_{n1}), x_2(t - \tau_{n2}), \dots, x_n(t - \tau_{nn})). \end{aligned} \quad (2)$$

The set of Boolean difference equations given by Eq. (2) determines the dynamics of the *BN* considering time delays, thereby defining an ABN. The dynamics of the ABN specified by Eq. (2) can be numerically solved^{8–10} once the Boolean functions of the nodes are defined and the initial condition functions on an interval

$$x_i(t) = x_{i_0}(t) \quad \text{for } t_0 - \tau \leq t \leq t_0, \quad i = 1, \dots, n \quad (3)$$

are defined, where $\tau = \max\{\tau_{ij}\}$ is the length of the memory of the system.

Equation (2) is still a highly idealized description of the electronic circuit design presented below. It does not account for the fact that the rise and fall times of the Boolean signals propagating along the links are increased by the filtering effect of the *RC*-circuit, the finite response time of the logic elements, or electronic noise in the system.^{13,14} Nevertheless, it provides a good starting point for understanding real-world ABNs.

The general idea of forced synchronization is accomplished by considering slave systems which are forced by an external signal as it is shown in Figure 1. Under this scheme, m autonomous Boolean networks driven by an external signal S_e can be given as follows:

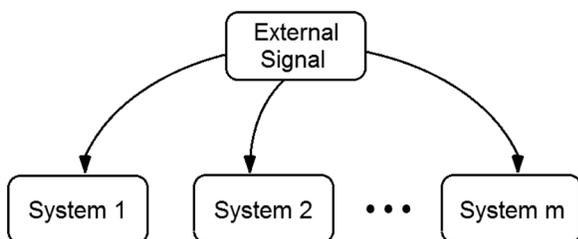


FIG. 1. Topology of m ABNs for the forced synchronization scheme.

$$\begin{aligned} \mathbf{x}^{(1)}(t) &= \mathbb{F}(x_1^{(1)}(t), x_2^{(1)}(t), \dots, x_n^{(1)}(t), S_e), \\ \mathbf{x}^{(2)}(t) &= \mathbb{F}(x_1^{(2)}(t), x_2^{(2)}(t), \dots, x_n^{(2)}(t), S_e), \\ &\vdots \\ \mathbf{x}^{(m)}(t) &= \mathbb{F}(x_1^{(m)}(t), x_2^{(m)}(t), \dots, x_n^{(m)}(t), S_e), \end{aligned} \quad (4)$$

where $\mathbf{x}^{(i)}(t) \in \mathbb{B}^n$, for $i = 1, \dots, m$. The forced synchronization phenomenon occurs when m autonomous Boolean networks given by (4) show correlated behavior because of an external signal S_e .

Definition II.1. Autonomous Boolean networks $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$, $i, j = 1, 2, \dots, m$, achieve complete forced synchronization, if for any $\mathbf{x}^{(i)}(0), \mathbf{x}^{(j)}(0) \in \mathbb{B}^n$, there is a time $t_k \geq 0$, such that $\mathbf{x}^{(i)}(t) = \mathbf{x}^{(j)}(t)$, for $t \geq t_k$.

III. AUTONOMOUS BOOLEAN NETWORKS

In the spirit of Ref. 12, we propose a topology of a Boolean network that consists of five nodes as is shown in Fig. 2. Each node has two inputs and one output that propagates to two different nodes. Thus, this particular Boolean network is described approximately by the set of local activation functions $f_i : \mathbb{B}^5 \rightarrow \mathbb{B}$, $i = 1, \dots, 5$, as follows:

$$\begin{aligned} x_1(t) &= f_1(x_u(t - \tau_{iu}), x_v(t - \tau_{jv})), \\ x_2(t) &= f_2(x_u(t - \tau_{iu}), x_v(t - \tau_{jv})), \\ &\vdots \\ x_5(t) &= f_5(x_u(t - \tau_{iu}), x_v(t - \tau_{jv})), \end{aligned} \quad (5)$$

where $i, j, u, v = 1, \dots, 5$ and $x_{i_0}(t) = 0$ for all the length of the memory of the system. Nodes 1 and 2 execute the XOR logic operation, node 3 executes the XNOR logic operation, while nodes 4 and 5 execute the OR logic operation. For this first approach, notice that a logic zero is introduced to each OR gate.

The Boolean delay equations that describe this Boolean network are

$$x_1(t) = x_4(t - \tau_{14}) \oplus x_5(t - \tau_{15}), \quad (6)$$

$$x_2(t) = x_1(t - \tau_{21}) \oplus x_4(t - \tau_{24}), \quad (7)$$

$$x_3(t) = x_1(t - \tau_{31}) \oplus x_5(t - \tau_{35}) \oplus 1, \quad (8)$$

$$x_4(t) = x_2(t - \tau_{42}) + 0, \quad (9)$$

$$x_5(t) = x_3(t - \tau_{53}) + 0. \quad (10)$$

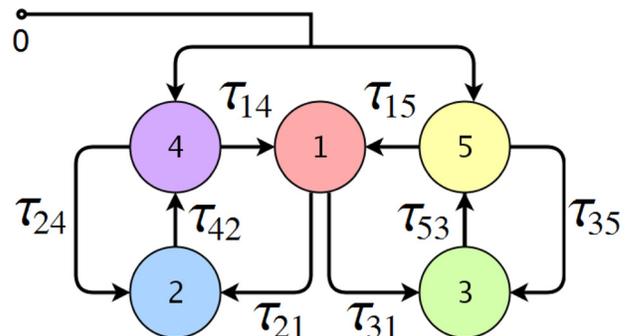


FIG. 2. Proposed design of the five-node Boolean network that displays complex dynamics.

Theorem III.1 For an autonomous Boolean network given by the set of equations (6)–(10), the orbits are always oscillating.

Proof. To prove that this autonomous Boolean network always generates oscillations in an infinite-dimensional phase space, we must show that the Boolean network does not have a fixed point to which the orbit can converge. If a network with finite states has a fixed point then some of the orbits are not always oscillating due to some orbits would be eventually fixed. By contradiction, we demonstrate this statement for autonomous Boolean network. We assume that there exists a $(x_1^*, x_2^*, \dots, x_5^*)$, fixed point of the system, such that

$$x_1^*(t) = x_1(t - \tau)$$

$$x_2^*(t) = x_2(t - \tau)$$

$$x_3^*(t) = x_3(t - \tau)$$

$$x_4^*(t) = x_4(t - \tau)$$

$$x_5^*(t) = x_5(t - \tau)$$

for $t \gg \tau = \max\{\tau_{14}, \tau_{15}, \tau_{21}, \tau_{24}, \tau_{31}, \tau_{35}, \tau_{42}, \tau_{53}\}$.

Thus, Eqs. (6)–(10) can be rewritten as

$$x_1(t) = x_4(t) \oplus x_5(t), \quad (11)$$

$$x_2(t) = x_1(t) \oplus x_4(t), \quad (12)$$

$$x_3(t) = x_1(t) \oplus x_5(t) \oplus 1, \quad (13)$$

$$x_4(t) = x_2(t), \quad (14)$$

$$x_5(t) = x_3(t). \quad (15)$$

Equation (14) implies that $x_4(t)$ has the same logic value than $x_2(t)$, and Eq. (15) implies that $x_5(t)$ has the same logic value than $x_3(t)$. Using this result, we can rewrite Eqs. (11)–(13) as

$$x_1(t) = x_2(t) \oplus x_3(t), \quad (16)$$

$$x_2(t) = x_1(t) \oplus x_2(t), \quad (17)$$

$$x_3(t) = x_1(t) \oplus x_3(t) \oplus 1. \quad (18)$$

If we insert Eq. (16) into (17) and (18), we obtain

$$x_2(t) = x_2(t) \oplus x_3(t) \oplus x_2(t), \quad (19)$$

$$x_3(t) = x_2(t) \oplus x_3(t) \oplus x_3(t) \oplus 1. \quad (20)$$

Equation (19) implies $x_2(t) = x_3(t)$, but Eq. (20) implies $x_3(t) = x_2(t)$. Therefore, we have a contradiction, which leads us to conclude that the Boolean network has no fixed point and it will oscillate permanently. \square

Corollary III. 2. A Boolean network without fixed points presents always periodic behavior if its delays are commensurate.

To verify these predictions, we design an electronic circuit through numerical simulation using SPICE with the topology shown in Fig. 2. Each node of the Boolean network is a reconfigurable logic cell discussed in Ref. 26. The time-delay links are realized using a RC -circuit in low-pass filter

configuration. With this circuit, we can generate continuously adjustable time delay set by the values of R and C , although signal distortions become more pronounced for larger delays. For a logic threshold equal to half of the difference between the “high” and “low” voltages, the time delay is given by $\tau = (\ln 2)RC$. The values of R and C used in to generate the different delays our simulations are shown in Table I. Last row indicates values of τ_{ij} . The delays τ_{42} and τ_{53} are considered as an intrinsic delay due to the gate response ~ 60 ns.

In order to build the ABN previously described, we consider a reconfigurable structure for each node and we refer to it as an autonomous dynamical Boolean network. Thus, the Boolean network illustrated in Fig. 2 can be realized using several programmable logic cells. Figure 3 shows the temporal evolution of the voltages of the ABN, which displays a complex, non-repeating behavior, likely a manifestation of Boolean chaos.¹² As nodes 4 and 5 execute the OR logic operation and one of its inputs is a zero logic state, then these nodes behave as time-delay buffers, i.e., outputs of the nodes 4 and 5 are the same to that of the outputs of nodes 2 and 3, respectively, but delayed a certain time for the network shown in Fig. 2.

IV. FORCED SYNCHRONIZATION OF BOOLEAN NETWORKS

Forced synchronization is accomplished by considering two slave systems given by Eqs. (6)–(10) with the addition of an external Boolean driving signal. Under this scheme, forced synchronization phenomenon occurs when oscillations of these two slave systems $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ show correlated behavior because of the external signal S_e . Complete forced synchronization can be detected by determining when the asymptotic behavior of slave systems is given by

$$\lim_{t \rightarrow \infty} |\mathbf{x}^{(1)}(t) \oplus \mathbf{x}^{(2)}(t)| = 0. \quad (21)$$

Thus, complete forced synchronization occurs when orbits $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ satisfy Eq. (21) with $\mathbf{x}^{(1)}(0) \neq \mathbf{x}^{(2)}(0)$, i.e., after a transient time, the trajectories of the two orbits are identical and independent of the initial conditions. Note that complete forced synchronization is just a type of synchronization but there exist others, such as phase or generalized synchronization.^{22,23}

We investigate the dynamics of the forced synchronization scheme shown in Fig. 1 with two ABNs using our SPICE-based simulation software. We coupled two slave Boolean networks by forcing nodes 4 and 5 of each network

TABLE I. Parameter values to generate the different link delays τ_{ij} .

	τ_{14}	τ_{15}	τ_{21}	τ_{24}	τ_{31}	τ_{35}
R	145 Ω	3.1 k Ω	1.1 k Ω	1.3 k Ω	2.3 k Ω	7.7 k Ω
C	22 pF	22 pF	22 pF	22 pF	22 pF	22 pF
τ	2.21 ns	47.27 ns	7.28 ns	19.82 ns	35.07 ns	117.41 ns

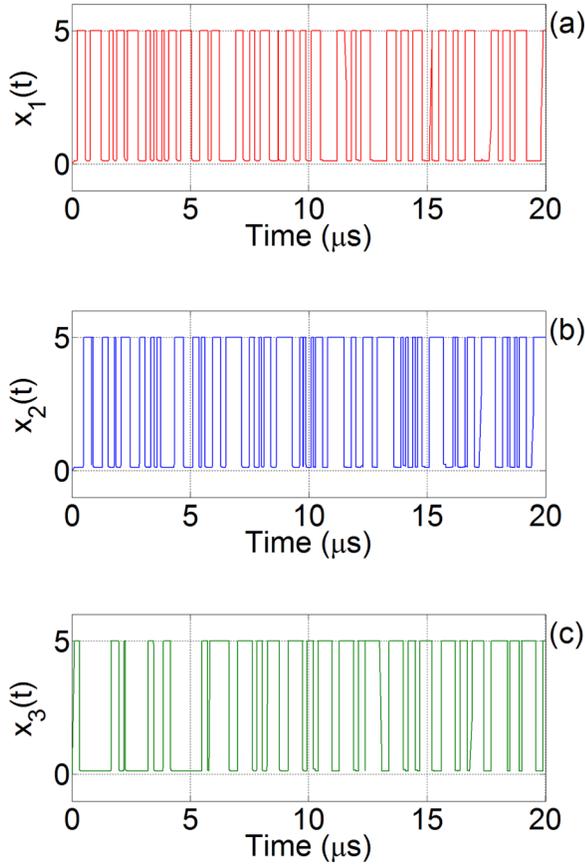


FIG. 3. Temporal evolution in the simulated ABN measured at the output of three nodes. (a) Node 1 temporal evolution. (b) Node 2 temporal evolution. (c) Node 3 temporal evolution.

with an external Boolean signal with a frequency of 700 kHz, as shown in Fig. 4.

The forced autonomous Boolean networks are described as follows:

$$x_1^i(t) = x_4^i(t - \tau_{14}) \oplus x_5^i(t - \tau_{15}), \quad (22)$$

$$x_2^i(t) = x_1^i(t - \tau_{21}) \oplus x_4^i(t - \tau_{24}), \quad (23)$$

$$x_3^i(t) = x_1^i(t - \tau_{31}) \oplus x_5^i(t - \tau_{35}) \oplus 1, \quad (24)$$

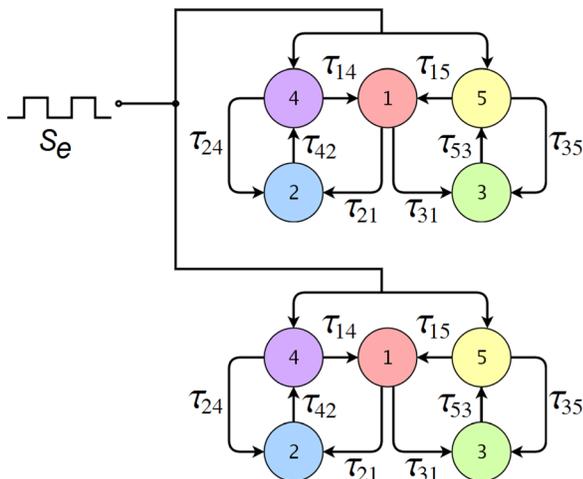


FIG. 4. Two Boolean networks driven by an external signal.

$$x_4^i(t) = x_2^i(t - \tau_{42}) + S_e, \quad (25)$$

$$x_5^i(t) = x_3^i(t - \tau_{53}) + S_e, \quad (26)$$

where $i = 1, 2$.

Theorem IV.1 For a forced autonomous Boolean network given by the set of equations (22)–(26), the external signal determines the periods of time in which orbits are oscillating or converging to a fixed point.

Proof. The external signal is oscillating between zero and one logic, then it has two different effects over the Boolean network. When the external signal is zero logic, then it leaves nodes 4 and 5 to behave as it was explained in Theorem III.1. Thus, the orbits are always oscillating. However, when the external signal is one logic, we need to prove that the Boolean network has a fixed point. So we start by describing the system as follows:

$$x_1(t) = x_4(t - \tau_{14}) \oplus x_5(t - \tau_{15}), \quad (27)$$

$$x_2(t) = x_1(t - \tau_{21}) \oplus x_4(t - \tau_{24}), \quad (28)$$

$$x_3(t) = x_1(t - \tau_{31}) \oplus x_5(t - \tau_{35}) \oplus 1, \quad (29)$$

$$x_4(t) = x_2(t - \tau_{42}) + 1, \quad (30)$$

$$x_5(t) = x_3(t - \tau_{53}) + 1. \quad (31)$$

Again, we assume that $(x_1^*, x_2^*, \dots, x_5^*)$ is a fixed point of the system, such that

$$x_1^*(t) = x_1(t - \tau)$$

$$x_2^*(t) = x_2(t - \tau)$$

$$x_3^*(t) = x_3(t - \tau)$$

$$x_4^*(t) = x_4(t - \tau)$$

$$x_5^*(t) = x_5(t - \tau)$$

for $t > \tau = \max\{\tau_{14}, \tau_{15}, \tau_{21}, \tau_{24}, \tau_{31}, \tau_{35}, \tau_{42}, \tau_{53}\}$.

Thus, Eqs. (27)–(31) can be rewritten as follows:

$$x_1(t) = x_4(t) \oplus x_5(t), \quad (32)$$

$$x_2(t) = x_1(t) \oplus x_4(t), \quad (33)$$

$$x_3(t) = x_1(t) \oplus x_5(t) \oplus 1, \quad (34)$$

$$x_4(t) = 1, \quad (35)$$

$$x_5(t) = 1. \quad (36)$$

Equations (35) and (36) imply that $x_4(t) = 1$, $x_5(t) = 1$, so equations (32)–(36) can be rewritten as follows:

$$x_1(t) = 1 \oplus 1, \quad (37)$$

$$x_2(t) = 1 \oplus 1 \oplus 1, \quad (38)$$

$$x_3(t) = 1 \oplus 1 \oplus 1 \oplus 1. \quad (39)$$

Therefore, when the external signal is one, there exists an unique fixed point in $x_1(t) = 0$, $x_2(t) = 1$, $x_3(t) = 0$, $x_4(t) = 1$, $x_5(t) = 1$. \square

Corollary IV.2. Two forced Boolean networks given by the set of equations (22)–(26) present complete forced

synchronization when they are forced by a Boolean external signal.

When the external signal is logic one (5 V), it forces nodes 4 and 5 to update their outputs to one logic. Each time the external signal changes to one, the logic states of the rest of nodes of both Boolean networks try to get closer until the time when synchronization occurs and all states of both Boolean networks are equal.

Figure 5 shows numerical simulations of the temporal evolution of the voltages of the two Boolean networks measured at the output of three of the nodes. It is seen that the complex behavior displayed by each node gives way to nearly identical periodic behavior in both ABNs essentially immediately after the forcing signal is applied. Furthermore, note that the duty cycle of the waveform measured at each node can be different from the duty cycle of the applied forcing function (50%), demonstrating that the response of each node of the ABN is not just equal to the drive signal for all initial conditions.

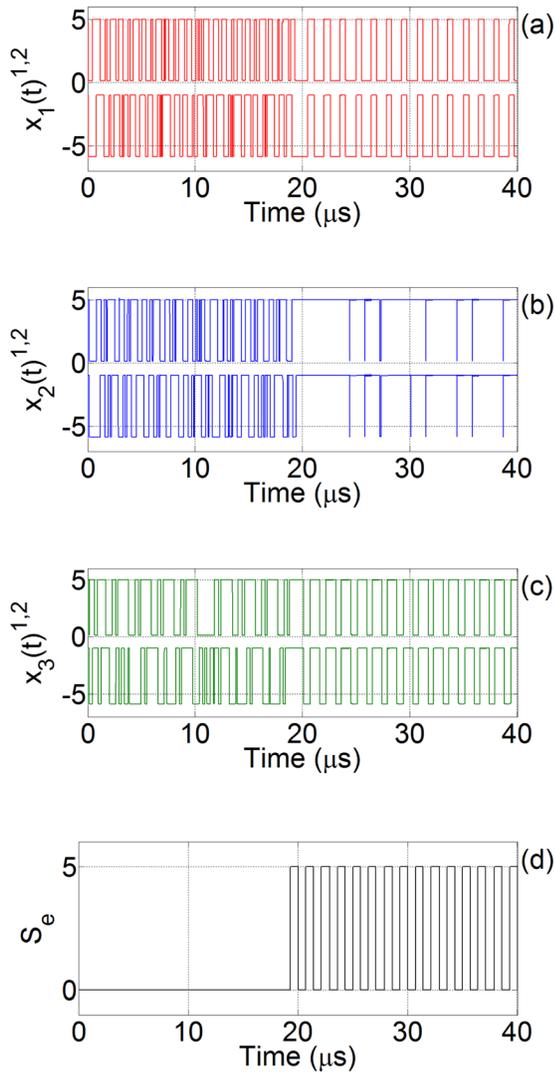


FIG. 5. Temporal evolution of the voltages of the two driven ABNs realized numerically. (a) Outputs of nodes 1. (b) Outputs of nodes 2. (c) Outputs of nodes 3. In three cases, upper signal corresponds to Boolean network 1, while lower signal corresponds to Boolean network 2 and this signal has an offset of -6 V. (d) External signal.

To detect forced synchronization of the two slave ABNs, we use auxiliary XOR gates with inputs given by the same node of each network. When complete synchronization is achieved, the XOR output is logic low (0 V). Figure 6 shows the simulated temporal evolution of the outputs of the auxiliary XOR Boolean gates, where we can corroborate complete synchronization after a transient time. Note that there exist two brief desynchronization events (short spikes) due to the external signal forces directly nodes 4 and 5 to take a high level state, and forces indirectly the rest of nodes of both networks to get nearly a common state. After a transient time, synchronization occurs and all states of both Boolean networks are equal.

V. EXPERIMENTAL RESULTS

The Boolean network illustrated in Fig. 2 can be experimentally realized using commercial electronic logic gates, in

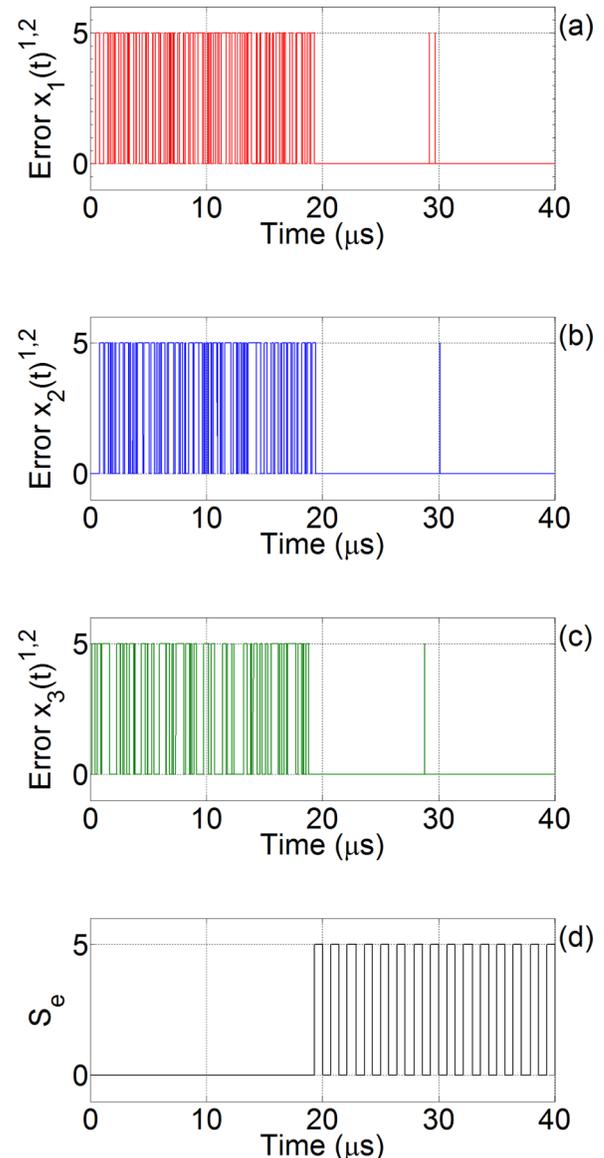


FIG. 6. Temporal evolution of the errors between corresponding nodes in the two forced ABNs realized numerically. (a) Error between nodes 1. (b) Error between nodes 2. (c) Error between nodes 3. (d) External signal used to force both ABNs.

our case we used logic gates from the family HD74LSXX. The time delays come about from a combination of an even number of NOT gates or Schmitt triggers wired in series, which acts effectively as a time-delay buffer. Figure 7 shows the temporal evolution of the voltages of the experimental ABN.

The values of τ_{ij} are given in the last two lines of Table II, the labels τ^1 and τ^2 correspond to the first and second response ABNs, respectively.

Chaos is characterized by the exponential divergence of trajectories with nearby initial conditions which is indicated by a positive Lyapunov exponent. There are several approaches to estimate the Lyapunov exponents of a system, for example, Ref. 25 used an approach for computing Lyapunov exponents of a dynamical system described by differential equations. In Ref. 12, a method to estimate the Lyapunov exponent for dynamical systems based on difference equations was proposed, the metric used to measure divergence of trajectories was proposed in Ref. 9 as follows:

$$d(s) = \frac{1}{T} \int_s^{s+T} x(t' + t_a) \oplus y(t' + t_b) dt', \quad (40)$$

where \oplus is the XOR operation, and the Boolean distance $d(s)$ is computed over the interval T . In this work, this second method was used to estimate the largest Lyapunov exponent of an experimental ABN which is described by the following algorithm:

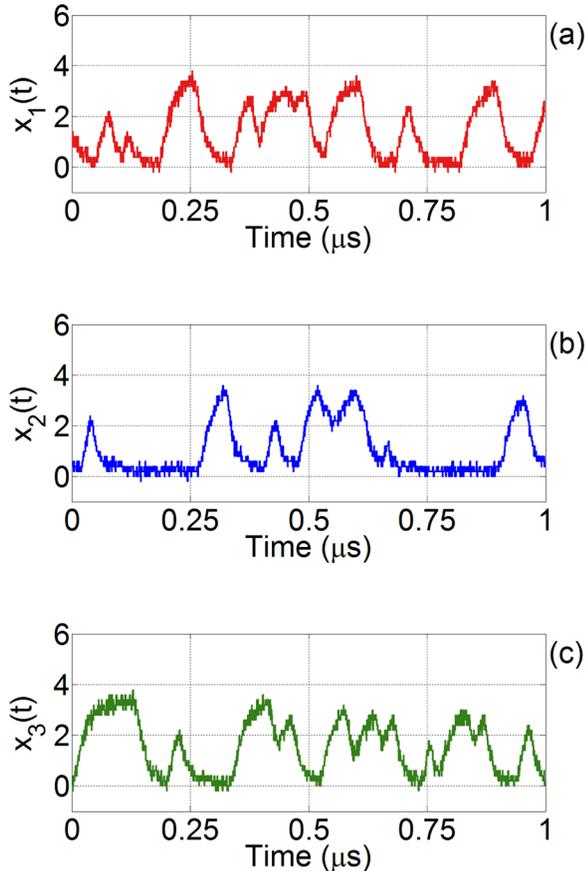


FIG. 7. Temporal evolution of the experimental ABN measured at the output of three of its nodes. (a) Output of node 1. (b) Output of node 2. (c) Output of node 3.

TABLE II. Parameter values to generate the different link delays τ_{ij} for the two experimental networks.

	τ_{14}	τ_{15}	τ_{21}	τ_{24}	τ_{31}	τ_{35}	τ_{42}	τ_{53}
τ^1	110 ns	232 ns	340 ns	150 ns	437 ns	220 ns	47 ns	51 ns
τ^2	109 ns	223 ns	352 ns	145 ns	419 ns	224 ns	40 ns	47 ns

- Acquire a long time series V of voltage of the experimental ABN.
- Transform the time series V into a Boolean time series $x(t)$.
- For a given $\delta > 0$ and $T = T_0$, search segments of $x(t)$ starting at times t_a and t_b such that $d_0 = d(s) < \delta$.
- Compute $d(s)$ for $s \in [0, k * T_0]$, using the segments found in the previous step, for $k > 0$.
- Compute $\langle \ln d(s) \rangle$, where $\langle \rangle$ denotes an average over all matching (t_a, t_b) pairs.
- Estimate the Lyapunov exponent $\lambda_{ab} = (\ln d(s) - \ln d_0) / s$.

For our experiment, $T_0 = 200$ ns and $\delta = 0.01$. We acquired a 0.5 ms time series of the voltage in node 1 output, and found ~ 3600 matching pairs (t_a, t_b) that satisfy $d_0 = d(s) < 0.01$, $\ln d_0 = -4.6$. Figure 8 shows the associated Boolean variables, $x(s + t_a)$ (top figure) and $x(s + t_b)$ (bottom figure), for the typical segments of $V(s + t_a)$ and $V(s + t_b)$, when $t_a = 2.10401 \times 10^{-4}$ s and $t_b = 3.92001 \times 10^{-4}$ s. Figure 9 shows the time evolution of $\langle d(s) \rangle$. The average of λ_{ab} over all pairs of similar segments is our estimate of the largest Lyapunov exponent λ of the system. We find $\lambda = 0.029$ ns $^{-1}$, which demonstrates that the network is chaotic.

Also, we investigate experimentally the dynamics of the forced synchronization scheme shown in Fig. 1 with two experimental ABNs. We coupled two slave Boolean networks by forcing nodes 4 and 5 of each network with an external Boolean signal with a frequency of 700 kHz, as shown in Fig. 4.

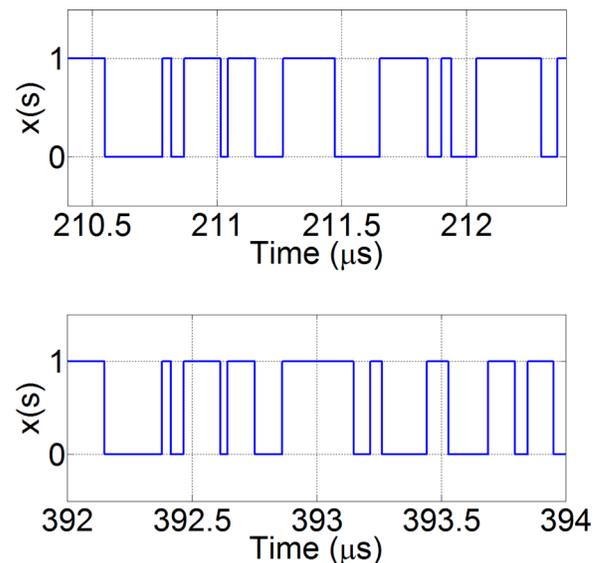


FIG. 8. The resulting Boolean variables associated with V corresponding to the matching pair $(t_a = 2.10401 \times 10^{-4}$, $t_b = 3.92001 \times 10^{-4}$) s.

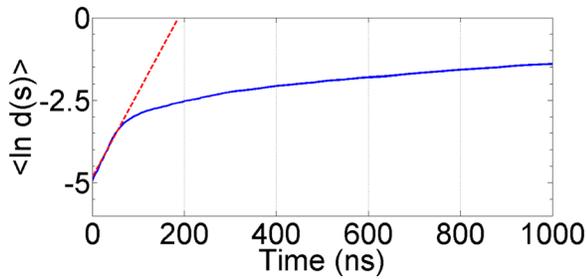


FIG. 9. Logarithm of the Boolean distance as a function of time, averaged over the phase-space attractor.

When the external signal of 700 kHz with a 90% duty cycle is applied to both Boolean networks, there are two clearly distinguishable behaviors shown in Fig. 10. The first one (no transitions period) occurs when external signal is “high,” it forces directly to nodes 4 and 5 to have the same logic value, and indirectly forces to the remaining nodes. The second one (short spikes) occurs when external signal is “low” and each Boolean network has a free dynamics given by its proper conditions. Furthermore, note that the duty cycle of the applied forcing function (50%) in simulation is different from the duty cycle of the applied forcing function (90%) in electronic implementation, due to in simulations, the ABNs are identical but experimentally they are different.

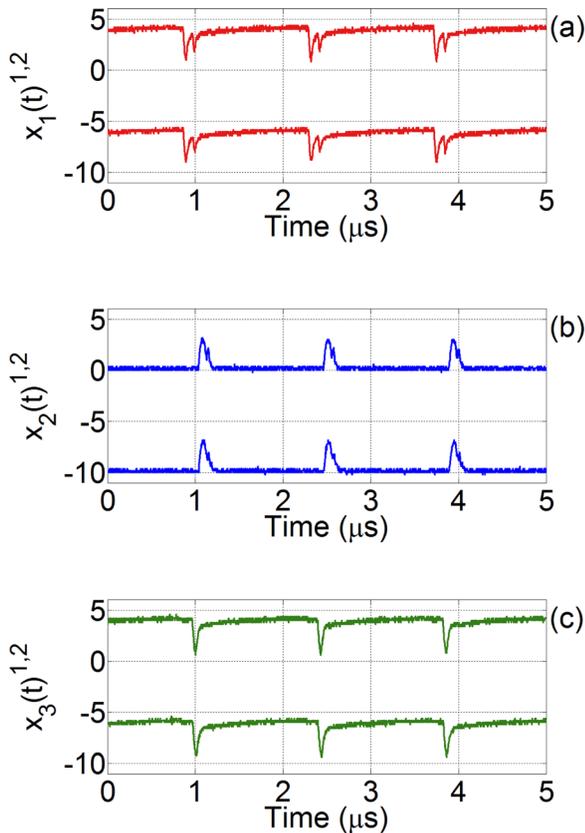


FIG. 10. Temporal evolution of the voltages of the two driven ABNs realized experimentally. (a) Experimental measures of nodes 1 outputs. (b) Experimental measures of nodes 2 outputs. (c) Experimental measures of nodes 3 outputs. In three cases, upper signal corresponds to Boolean network 1, while lower signal corresponds to Boolean network 2 and this signal has an offset of -10 V.

To determine if both Boolean networks achieve synchronization, we introduced corresponding nodes voltages to auxiliary XOR gates. Figure 11 shows the temporal evolution of the error between three corresponding nodes after the external signal is applied, from the figure, it is possible to see that the error practically tends to zero and we can see essentially complete synchronization after a transient time.

VI. CONCLUSIONS

We presented the design of an autonomous Boolean network with five reconfigurable nodes and time delays along the links realized of two different ways: in numerical simulation with a continuously adjustable RC filter and experimentally with a cascading configuration of an even number of inverters logic gates. We show that the autonomous Boolean network does not have a fixed point so the dynamics always must oscillate and present a chaotic behavior. We also demonstrate forced synchronization of two networks, where a common periodic Boolean signal drives each network. Under this scheme, complete synchronization is obtained, when the dynamics of the two networks is identical, numerical and experimental results were presented. In a future study, we will use this system to investigate other generalized forms of synchronization, such as phase¹¹ or lag synchronization.¹⁸ Finally, by using the reconfigurability of

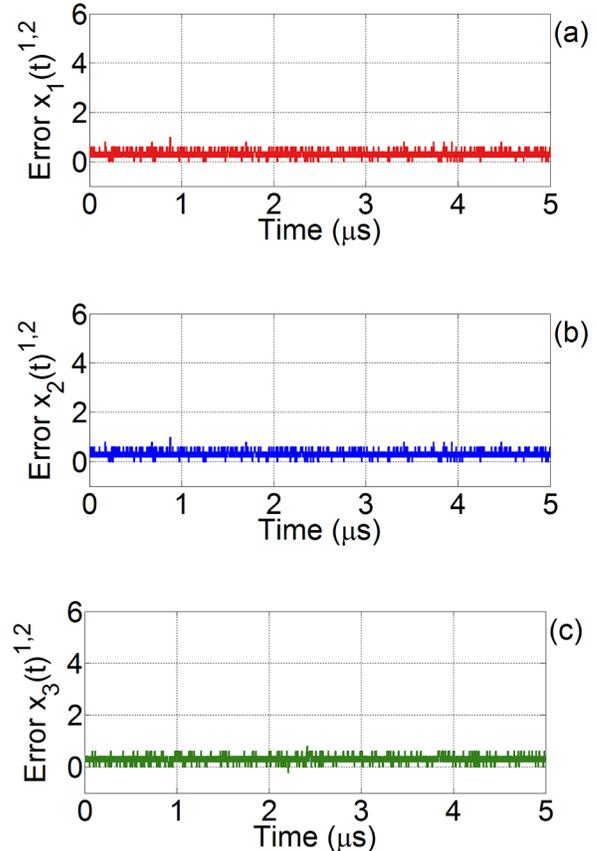


FIG. 11. Temporal evolution of the errors between corresponding nodes in the two forced ABNs realized experimentally. (a) Error between nodes 1. (b) Error between nodes 2. (c) Error between nodes 3.

the network nodes, we will explore evolution of networks by changing the logic based on the state of the network.²⁴ Such evolutionary networks have application to neuronal and social networks, for example.

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