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Reply to "Comment on H.C. Rosu, O. Cornejo-Perez and P. Chen, "Nonsingular parametric oscillators Darboux related to the classical harmonic oscillator"" by A. Schulze-Halberg

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Abstract –*** Missing author ***

The comment of Schulze-Halberg [1] on our work [2] stems from the formalism of Darboux covariance of second-order differential operators [3]. We thus recall that two second-order linear differential operators D_1 and D_2 are Darboux covariant, or isospectral, if: (i) their factorizations are of the type $D_1 = L_1L_2$ and $D_2 = L_2L_1$, respectively, and (ii) their eigenfunctions are related through $\psi_2 = L\psi_1$, where L is known as the intertwining operator. The $L_{1,2}$ operators are first order and their non-operatorial part is the logarithmic derivative of an eigenfunction ψ_1, or a linear combination of eigenfunctions of D_1. In the simplest constructions of this type, the nodeless ground state of D_1 is employed. The non-operatorial part of the intertwiner L is also a logarithmic derivative of a function that should be linearly independent with respect to ψ_1. The Darboux covariance led in the 1980s to the development of the rich area of supersymmetric quantum mechanics [4].

On the other hand, our application is in the area of classical mechanics where the Darboux covariance should be applied at fixed *zero* 'spectral parameter', where, as shown by Schulze-Halberg, there is a direct relationship between the function used in the intertwiner and ψ_1. Then, at the formal level, Schulze-Halberg obtains closed form expressions of the solutions of the two Darboux partner equations in terms of the factorization coefficients α and β. This is an important formal result. However, although it might look like a simplification, we want to point out that one still should substitute into formulas (10) and (11) our expressions of α and β in order to show that the solutions are the ones we obtained. In other words, for equations of given structure, α and β can be obtained only through the method explained in our papers [2, 5, 6] in which the factorizations have been used for convenience since it is not easy to disentangle their expressions directly from the equation. Thus, the results of Schulze-Halberg are of relevance once the two factorization coefficients have been determined and for designed differential equations by giving the expressions of α and β.

Suppose, for example, that we give $\alpha = x$ and $\beta = 1$, then the $\alpha\beta$ designed equation is

$$\psi'' + \left(x + \frac{2}{x}\right)\psi' + \psi = 0 \tag{1}$$

whose general solution calculated with the formula (10) of Schulze-Halberg is

$$\psi = \frac{\sqrt{2\pi}}{2x}\left[1 + \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right] . \tag{2}$$

On the other hand, the Darboux partner equation is

$$\phi'' + x\phi' + \phi = 0 \tag{3}$$

whose general solution calculated with formula (11) of the Comment is

$$\phi = \exp\left(-\frac{x^2}{2}\right)\int^x \exp\left(\frac{t^2}{2}\right) dt\ . \tag{4}$$

However, if the equation to be analysed is

$$y'' + \frac{4\omega_0}{\sin 2\omega_0 t} y' + \omega_0^2 y = 0\ , \tag{5}$$

and one knows a priori it is an $\alpha\beta$-designed equation but without knowing α and β, it will not be easy to find α and β and one should use standard methods such as those in our papers. In fact, equation (5) corresponds to the case $\lambda = 1$ and therefore it is still a singular oscillator equation but we use it herein for illustrative purposes. Then, it is easy to check that

$$y = \frac{\omega_0}{\sin \omega_0 t} \tag{6}$$

is a particular solution of (5) that we extracted from the expression of the general solution given in our paper [2] for $C_1 = 0$ and $C_2 = \omega_0$.

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