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An Amplitude-Phase (Ermakov-Lewis) Approach for the Jackiw-Pi Model of Bilayer Graphene

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Abstract

In the context of bilayer graphene we use the simple gauge model of Jackiw and Pi to construct its numerical solutions in powers of the bias potential V according to a general scheme due to Kravchenko. Next, using this numerical solutions, we develop the Ermakov-Lewis approach for the same model. This leads us to numerical calculations of the Lewis-Riesenfeld phases that could be of forthcoming experimental interest for bilayer graphene. We also present a generalization of the Ioffe-Korsch nonlinear Darboux transformation.

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1 Introduction

The recent discovery by Novoselov et al [1] that a single atomic layer of graphite usually known as graphene can be stable under some special experimental conditions opened widely the realm of a soon-to-come technology of graphene-based electronic devices. The main transport mechanism of the charged carriers is in this case an enhanced quantum Hall effect accompanied by a non-zero Berry's geometric phase that has been experimentally observed [2]. On the other hand, the quantum Hall effect and the Berry phase have been also discussed for graphene bilayers [3] that technologically could be even more interesting than graphene itself. Motivated by these works, in this paper, we introduce the more general Lewis-Riesenfeld phases for the case of a graphene bilayer. These phases are defined in a mathematical formalism called the Ermakov-Lewis approach, which

is shortly presented. For the bilayer, we will make usage of the recent gauged Dirac model of Jackiw and Pi [4]. With the aid of a mathematical procedure due to Kravchenko [5] we present the results in a series expansion in the bias potential V applied to the bilayer graphene for any gauge potential \mathcal{A} . We give detailed numerical results for a particular case of \mathcal{A} .

2 Jackiw-Pi model for bilayer graphene

Recently Jackiw and Pi introduced a simple gauged Dirac model of bilayer graphene that can be reduced to the following coupled system of first-order differential equations [4]

$$(D_r - \Phi(r))u(r) = Vv(r), \quad (1)$$

$$(D_r + \Phi(r))v(r) = -Vu(r), \quad (2)$$

where $D_r = \frac{d}{dr}$, V is a constant representing the external bias voltage and

$$\Phi(r) = \frac{k/2}{r} - \frac{k\mathcal{A}(r)}{r}. \quad (3)$$

Physically, function Φ has two components:

- a scalar field $\varphi = \frac{k/2}{r}$ that describes a particular dimerization called Kekulé distortion [6] which can be also interpreted as characterizing the condensate arising from states bound by interlayer Coulomb forces between particles in one layer and holes in the other; the fingerprint of φ in Φ is the parameter k which should be an odd integer number.
- a vector potential, \mathcal{A} , which was first introduced by Jackiw and Pi for a monolayer graphene [7] to unpin the vortices and then studied in the context of bilayer graphene as well [4]. \mathcal{A} is a gauge field satisfying some appropriate conditions in the origin and at infinity. In [4] it was supposed that

$$\mathcal{A}(0) = 0 \text{ and } \mathcal{A}(\infty) = \frac{1}{2}. \quad (4)$$

It is easy to see that the spinor components u and v are necessarily solutions of the following second-order differential equations obtained directly from system (1)-(2)

$$-(D_r + \Phi(r))(D_r - \Phi(r))u = V^2u, \quad (5)$$

$$-(D_r - \Phi(r))(D_r + \Phi(r))v = V^2v. \quad (6)$$

This couple of equations can be rewritten in the following way

$$-D_r^2u + q_1(r, V)u = 0, \text{ where } q_1(r, V) = \Phi^2 + \Phi' - V^2, \quad (7)$$

$$-D_r^2v + q_2(r, V)v = 0, \text{ where } q_2(r, V) = \Phi^2 - \Phi' - V^2. \quad (8)$$

The latter form of the uncoupled system shows that the two functions u and v in the Jackiw-Pi model are supersymmetric partners in supersymmetric quantum mechanics and therefore many known supersymmetric results can be directly applied.

3 Kravchenko's representation for the general Schrödinger solution

Consider the Sturm-Liouville equation

$$D_r(pD_r u) + qu = \omega^2 u, \quad (9)$$

where p and q are complex-valued functions of a real variable $r \in [a, b]$, $p \in C^1(a, b)$ is bounded and non-vanishing on $[a, b]$ and ω is an arbitrary complex constant. Suppose that there exists a solution g_0 of the equation

$$D_r(pD_r g_0) + qg_0 = 0$$

on (a, b) such that $g_0 \in C^2(a, b)$ together with $\frac{1}{g_0}$ are bounded on $[a, b]$.

Kravchenko [5] proved that the general solution of (9) has the form

$$u = c_1 u_1 + c_2 u_2 \quad (10)$$

where c_1, c_2 are arbitrary complex constants and

$$u_1 = g_0 \sum_{\text{even } n=0}^{\infty} \frac{\omega^n}{n!} \tilde{X}^{(n)}, \quad u_2 = g_0 \sum_{\text{odd } n=1}^{\infty} \frac{\omega^{n-1}}{n!} X^{(n)} \quad (11)$$

with $\tilde{X}^{(n)}$ and $X^{(n)}$ being defined by the following recursive relations

$$\tilde{X}^{(0)} \equiv 1, \quad X^{(0)} \equiv 1,$$

$$\tilde{X}^{(n)}(r) = \begin{cases} n \int_a^r \tilde{X}^{(n-1)}(\xi) g_0^2(\xi) d\xi & \text{for an odd } n \\ n \int_a^r \tilde{X}^{(n-1)}(\xi) \frac{d\xi}{p(\xi)g_0^2(\xi)} & \text{for an even } n \end{cases} \quad (12)$$

$$X^{(n)}(r) = \begin{cases} n \int_a^r X^{(n-1)}(\xi) \frac{d\xi}{p(\xi)g_0^2(\xi)} & \text{for an odd } n \\ n \int_a^r X^{(n-1)}(\xi) g_0^2(\xi) d\xi & \text{for an even } n. \end{cases} \quad (13)$$

This iterative scheme is appropriate for an easy implementation of numerical solutions and therefore for not exactly-solvable problems.

4 Application to the Jackiw-Pi uncoupled system

To apply Kravchenko's procedure to the Jackiw-Pi uncoupled system we start with the equation for u . It is easy to find a particular solution g_0 from the factorized form of the u equation (5). We immediately get

$$\frac{g_0'}{g_0} = \frac{k}{r} \left(\frac{1}{2} - A(r) \right), \quad \longrightarrow g_0(r) = r^{\frac{k}{2}} e^{-k \int \frac{A(r)}{r} dr},$$

and the general solution u has the form $u = C_1 u_1 + C_2 u_2$, with

$$u_1 = \frac{g_0(r)}{g_0(a)} \sum_{\text{even } n=0}^{\infty} \frac{\tilde{X}^{(n)}}{n!} V^n, \quad u_2 = -g_0(a)g_0(r) \sum_{\text{odd } n=1}^{\infty} \frac{X^{(n)}}{n!} V^{n-1}, \quad (14)$$

where $\tilde{X}^{(n)}$ and $X^{(n)}$ are as in (12) and (13), respectively, for $p \equiv -1$.

The general solution $v(r)$ of equation (6) can be obtained now by means of the Darboux transformation $v = C_1 v_1 + C_2 v_2 = \frac{1}{V} g_0 D_r g_0^{-1} (C_1 u_1 + C_2 u_2)$

$$v_1 = -\frac{1}{g_0(a)g_0(r)} \sum_{\text{odd } n=1}^{\infty} \frac{\tilde{X}^{(n)}}{n!} V^n, \quad v_2 = \frac{g_0(a)}{Vg_0(r)} \sum_{\text{even } n=0}^{\infty} \frac{X^{(n)}}{n!} V^n. \quad (15)$$

By construction, u_1 and u_2 satisfy the following initial conditions

$$u_1(a) = 1, \quad u_1'(a) = \Phi(a), \quad u_2(a) = 0, \quad u_2'(a) = 1, \quad (16)$$

with their Wronskian being equal to $W = 1$. Consequently, the functions v_1 and v_2 take the initial values

$$v_1(a) = 0, \quad v_1'(a) = -V, \quad v_2(a) = \frac{1}{V}, \quad v_2'(a) = -\frac{\Phi(a)}{V}. \quad (17)$$

If one takes $V = 0$ in (5), the formulas (14) give

$$u_1(r) = c_1 g_0(r) \quad \text{and} \quad u_2(r) = c_2 g_0(r) \int_a^r \frac{d\xi}{g_0^2(\xi)},$$

where the expression in u_2 represents the well known construction of a second linearly independent solution of a Schrödinger equation.

When $r \rightarrow \infty$, $\mathcal{A} \rightarrow \frac{1}{2}$, the system (1)-(2) reads as

$$D_r u - Vv = 0, \quad D_r v + Vu = 0.$$

A solution g_0 in this case is an arbitrary constant, e.g., $g_0 \equiv 1$. Calculating $\tilde{X}^{(n)}$ and $X^{(n)}$ with the aid of (12)-(13) we obtain

$$\tilde{X}^{(n)}(r) = \begin{cases} (-1)^{\frac{n-1}{2}} r^n, & \text{for an odd } n \\ (-1)^{\frac{n}{2}} r^n, & \text{for an even } n \end{cases}, \quad X^{(n)}(r) = \begin{cases} (-1)^{\frac{n+1}{2}} r^n, & \text{for an odd } n \\ (-1)^{\frac{n}{2}} r^n, & \text{for an even } n. \end{cases}$$

Now substitution into (14) and (15) gives us the following solutions (cf [4])

$$u_1 = \cos Vr, \quad u_2 = \frac{1}{V} \sin Vr, \quad v_1 = -\sin Vr, \quad v_2 = \frac{1}{V} \cos Vr.$$

When $r \rightarrow 0$, and consequently $\mathcal{A} \rightarrow 0$, the superpotential Φ takes the form $\Phi = \frac{k}{2r}$. For system (1), (2) with this particular potential, in [4] the following solution is presented

$$u = \sqrt{r} J_{(k-1)/2}(Vr), \quad v = -\sqrt{r} J_{(k+1)/2}(Vr), \quad (18)$$

which is easily obtained from (14) and (15) choosing $g_0 = r^{\frac{k}{2}}$ and calculating u_1 and v_1 . Indeed, for $\tilde{X}^{(n)}$ we obtain the formulas

$$\tilde{X}^{(n)}(r) = \begin{cases} (-1)^{\frac{n-1}{2}} \frac{n!!(k-1)!!}{(k+n)!!} r^{k+n}, & \text{for an odd } n \\ (-1)^{\frac{n}{2}} \frac{(n-1)!!(k-1)!!}{(k+n-1)!!} r^n, & \text{for an even } n \end{cases}$$

leading us to the solution (18) multiplied by a constant.

We have now all the elements to proceed to specific calculations. Let us consider the Dirac system (1)-(2) choosing in (3) the vector potential $\mathcal{A}(r) = \frac{1}{2} - \frac{1}{2} e^{-\mu r}$ which is in agreement with conditions (4). The parameter μ^{-1} plays the role of a screening length. The superpotential (3) then becomes $\Phi(r) = \frac{k/2}{r} e^{-\mu r}$ and in this case the system (7)-(8) has the following form

$$-D_r^2 u + \frac{k}{2r} e^{-\mu r} \left(\frac{k}{2r} e^{-\mu r} - \frac{1}{r} - \mu \right) u = V^2 u, \quad (19)$$

$$-D_r^2 v + \frac{k}{2r} e^{-\mu r} \left(\frac{k}{2r} e^{-\mu r} + \frac{1}{r} + \mu \right) v = V^2 v, \quad (20)$$

The plots of u and v calculated within Kravchenko's scheme are presented in Fig.1.

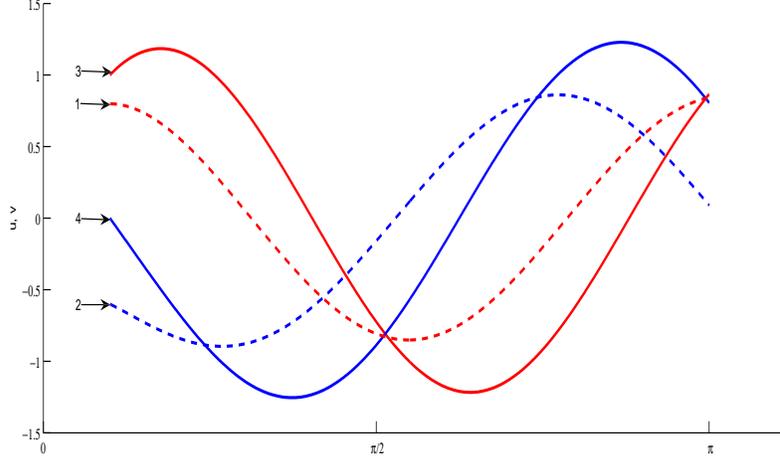


Fig.1. The solutions $u(r)$ (red) and $v(r)$ (blue) of the supersymmetric system (19)-(20) for $k = 1$ and two cases of the arbitrary constants C_1 and C_2 : — is for $C_1 = 1, C_2 = 0$, and - - - is for $C_1 = \frac{1}{\sqrt{\Phi(a)}}, C_2 = -\Phi(a)$.

5 The Ermakov-Lewis approach

We will now expound on the Ermakov-Lewis (EL) formalism for the supersymmetric system (7)-(8). If the functions u and v are written in the form [8]

$$u(r) = \alpha \rho_u(r) \sin(\varphi_u(r, V) - \beta), \quad v(r) = \gamma \rho_v(r) \sin(\varphi_v(r, V) - \delta), \quad (21)$$

where α, β, γ and δ are arbitrary constants, then both the amplitude functions ρ_u and ρ_v and the phase functions φ_u and φ_v are directly related to the Ermakov-Milne-Pinney nonlinear equations [9]

$$\rho_{u,v}''(r) + q_{1,2}(r, V) \rho_{u,v}(r) = \frac{\lambda}{\rho_{u,v}^3(r)}, \quad (22)$$

where λ is an arbitrary constant, in the following way. While ρ_u and ρ_v are solutions of (22) with a respective potential, the phase functions φ_u and φ_v are expressed in terms of ρ_u and ρ_v as follows

$$\varphi_{u,v}(r, V) = \int_a^r \frac{1}{\rho_{u,v}^2(r')} dr'. \quad (23)$$

For ρ_u and ρ_v there is an elegant representation [10] in terms of pairs of linearly independent solutions (u_1, u_2) and (v_1, v_2) of (7) and (8) respectively.

Namely, general solutions of (22) can be written in the following way

$$\rho_u = \sqrt{Au_1^2 + Bu_2^2 + 2Cu_1u_2}, \quad \rho_v = \sqrt{Av_1^2 + Bv_2^2 + 2Cv_1v_2}, \quad (24)$$

where A , B and C are constants related by the equality $AB - C^2 = \frac{\lambda}{W^2}$, with W being the Wronskian $u_1u_2' - u_2u_1' = v_1v_2' - v_2v_1'$ (which is a constant that does not change through the Darboux transformation).

The solution of the Ermakov equations allows us to obtain another important dynamic invariant quantity I_u and I_v , respectively, called the Lewis invariant [11]. I_u is constructed in terms of any solution u of (7) and the corresponding solution ρ_u of the Ermakov-Milne-Pinney nonlinear equation (22) (and analogously I_v is constructed in terms of v and ρ_v)

$$I_u = \frac{1}{2} \left(\frac{\lambda u^2}{\rho_u^2} + (\rho_u u' - \rho_u' u)^2 \right), \quad I_v = \frac{1}{2} \left(\frac{\lambda v^2}{\rho_v^2} + (\rho_v v' - \rho_v' v)^2 \right). \quad (25)$$

For the fixed interval $[a, b]$ the expressions $\frac{1}{\pi} \varphi_{u,v}(b, E)$ give us the quantum number function depending on the energy E [8],[12],[13]

$$N(E) = \frac{1}{\pi} \int_a^b \frac{1}{\rho^2(r)} dr, \quad (26)$$

used in standard quantum mechanics to identify bound states since at a bound state energy E_n the function $N(E)$ takes integer values $N(E_n) = n + 1$, $n = 0, 1, \dots$ and this does not depend on the choice of the constants A, B , and C . However, if the energy is different from E_n then (26) is not uniquely determined, i.e., for $E_n < E < E_{n+1}$ the function $N(E)$ depends on the boundary conditions imposed to ρ .

An important point of the EL formalism is that the phase function (23) also known as Lewis-Riesenfeld phase [14], has two components: the usual dynamic phase and a geometric phase that are given by the following formulas ([15])

$$\Delta \varphi_{u,v}^{dyn}(r) = \int_a^r \left[\frac{1}{\rho_{u,v}^2(r')} - \frac{1}{2} \frac{d}{dr'} \left(\rho_{u,v}(r') \frac{d\rho_{u,v}(r')}{dr'} \right) + \left(\frac{d\rho_{u,v}(r')}{dr'} \right)^2 \right] dr' \quad (27)$$

and

$$\Delta \varphi_{u,v}^{geom}(r) = \int_a^r \left[\frac{1}{2} \frac{d}{dr'} \left(\rho_{u,v}(r') \frac{d\rho_{u,v}(r')}{dr'} \right) - \left(\frac{d\rho_{u,v}(r')}{dr'} \right)^2 \right] dr', \quad (28)$$

Any of these phases can be calculated by employing Kravchenko's iterative series (14) and (15) for the ρ functions (24). Denoting

$$\begin{aligned}\tilde{Z}_u &= \sum_{\text{even}} \sum_{n=0}^{\infty} \sum_{\text{even}} \frac{\tilde{X}^{(i)} \tilde{X}^{(n-i)}}{i!(n-i)!} V^n, \quad Z_u = \sum_{\text{odd}} \sum_{n=1}^{\infty} \sum_{\text{odd}} \frac{X^{(i)} X^{(n-i)}}{i!(n-i)!} V^{n-1}, \\ \tilde{Z}_v &= \sum_{\text{odd}} \sum_{n=1}^{\infty} \sum_{\text{even}} \frac{\tilde{X}^{(i)} X^{(n-i)}}{i!(n-i)!} V^n\end{aligned}$$

we get

$$\rho_u(r) = g_0(r) \left[\frac{A}{g_0^2(a)} \tilde{Z}_u + B g_0^2(a) Z_u - \frac{2C}{V} \tilde{Z}_u \right]^{\frac{1}{2}}. \quad (29)$$

Introducing the notations

$$\begin{aligned}\tilde{Z}_v &= \sum_{\text{odd}} \sum_{n=1}^{\infty} \sum_{\text{odd}} \frac{\tilde{X}^{(i)} \tilde{X}^{(n-i)}}{i!(n-i)!} V^{n+1}, \quad Z_v = \sum_{\text{even}} \sum_{n=0}^{\infty} \sum_{\text{even}} \frac{X^{(i)} X^{(n-i)}}{i!(n-i)!} V^n, \\ \tilde{Z}_v &= \sum_{\text{odd}} \sum_{n=1}^{\infty} \sum_{\text{even}} \frac{X^{(i)} \tilde{X}^{(n-i)}}{i!(n-i)!} V^n,\end{aligned}$$

the amplitude function ρ_v is calculated as follows

$$\rho_v(r) = \frac{1}{g_0(r)} \left[\frac{A}{g_0^2(a)} \tilde{Z}_v + \frac{B g_0^2(a)}{V} Z_v - \frac{2C}{V} \tilde{Z}_v \right]^{\frac{1}{2}} \quad (30)$$

The particular values of the functions ρ_u and ρ_v can be obtained by relating the constants A, B and C to (16) and (17), that is

$$\begin{aligned}\rho_u(a) &= \sqrt{A}, \quad \rho'_u(a) = \frac{A\Phi(a) + C}{\sqrt{A}}, \\ \rho_v(a) &= \frac{\sqrt{B}}{V}, \quad \rho'_v(a) = -\frac{\sqrt{B}\Phi(a)}{V} - \frac{VC}{\sqrt{B}}.\end{aligned} \quad (31)$$

Based on (29) and (30), we have calculated the Lewis-Riesenfeld angular integrals (27) and (28) for the same decaying exponential profile of $\mathcal{A}(r)$. The results for the geometric Lewis-Riesenfeld phases are plotted in Fig. 2 and Fig. 3 of the paper for $\lambda = 1$ and $\lambda = 4$, respectively, and for the same finite interval.

One can check by direct calculation that the values of the Ermakov-Lewis invariant (25) do not change under the supersymmetric transformation, that is $I_u = I_v$. Indeed, noticing that I_u and I_v are r -independent quantities, we can calculate them at one point only, e.g., at $r = a$ and with arbitrary constants C_1, C_2 and A, B, C . We have

$$\begin{aligned}I_u(a) &= \frac{1}{2} (AC_2^2 + BC_1^2 - 2CC_1C_2), \\ I_v(a) &= \frac{1}{2} \left(\frac{\lambda C_2^2}{B} + BC_1^2 - 2CC_1C_2 + \frac{C^2 C_2^2}{B} \right).\end{aligned}$$

Taking into account that $C^2 = AB - \lambda$ we obtain $I_u(a) = I_v(a)$.

6 Generalization of the Ioffe-Korsch intertwining formulas

If the functions u and v are solutions of a coupled first-order system of the Dirac type then the corresponding amplitude functions ρ_u and ρ_v are related through nonlinear Dirac-Ermakov relationships that in a particular case $\lambda = 1$ have been obtained previously by Ioffe and Korsch [13]. Consider the equality

$$\rho_u^2 = Au_1^2 + Bu_2^2 + 2Cu_1u_2.$$

Substitution of $u_1 = -\frac{1}{\sqrt{V}}(D_r + \Phi)v_1$ and $u_2 = -\frac{1}{\sqrt{V}}(D_r + \Phi)v_2$ leads to the relationship

$$V^2\rho_u^2 = \Phi^2\rho_v^2 + \Phi(\rho_v^2)' + S_v,$$

where $S_v = A(v_1')^2 + B(v_2')^2 + 2Cv_1'v_2'$. Taking into account (21), the functions v_1 and v_2 can be expressed as $v_1 = \gamma_1\rho_v \sin(\varphi_v - \delta_1)$ and $v_2 = \gamma_2\rho_v \sin(\varphi_v - \delta_2)$. Calculating the constants γ_i and δ_i according to (17) and (31) we get

$$\begin{aligned} v_1 &= -\sqrt{B}\rho_v \sin \varphi_v, \\ v_2 &= \frac{\rho_v}{\sqrt{B}}(C \sin \varphi_v + \cos \varphi_v). \end{aligned}$$

Now taking the derivatives and substituting them into the expression for S_v we obtain

$$S_v = (\rho_v')^2 + \frac{1}{\rho_v^2} + (\lambda - 1) \left(\rho_v' \sin \varphi_v + \frac{1}{\rho_v} \cos \varphi_v \right)^2,$$

where one can notice that $\left(\rho_v' \sin \varphi_v + \frac{1}{\rho_v} \cos \varphi_v \right)^2 = \frac{1}{B} (v_1')^2$.

Employing the same procedure with respect to ρ_u^2 and taking into consideration that

$$\begin{aligned} u_1 &= \frac{\rho_u}{\sqrt{A}}(-C \sin \varphi_u + \cos \varphi_u), \\ u_2 &= \sqrt{A}\rho_u \sin \varphi_u, \end{aligned}$$

we arrive at the nonlinear Dirac-Ermakov system involving the amplitudes ρ_u and ρ_v and the phases φ_u, φ_v of the Dirac spinors u and v

$$V^2\rho_u^2 = ((D_r + \Phi)\rho_v)^2 + \frac{1}{\rho_v^2} + (\lambda - 1) \left(\rho_v' \sin \varphi_v + \frac{1}{\rho_v} \cos \varphi_v \right)^2, \quad (32)$$

$$V^2\rho_v^2 = ((D_r - \Phi)\rho_u)^2 + \frac{1}{\rho_u^2} + (\lambda - 1) \left(\rho_u' \sin \varphi_u + \frac{1}{\rho_u} \cos \varphi_u \right)^2, \quad (33)$$

where again it is worth noticing that $\left(\rho_u' \sin \varphi_u + \frac{1}{\rho_u} \cos \varphi_u \right)^2 = \frac{1}{A} (u_2')^2$. Thus, there is a cross contribution to the squares of the intertwined amplitude functions of one-unit jumps in λ which is given by the squares of the derivatives of the corresponding eigenfunctions. For $\lambda = 1$ the system (32)-(33) reduces to the result in [13].

7 Application to the Jackiw-Pi model for screened-exponential form of $\mathcal{A}(r)$

To associate the spinor components u and v to the amplitude functions ρ_u and ρ_v the relationship between the constants C_1 and C_2 and A, B and C should be fixed. For instance, choosing $\rho_u(a) = u(a)$, $\rho'_u(a) = u'(a)$ we get $A = C_1^2$, $B = \frac{\lambda}{C_1^2} + C_2^2$, $C = C_1 C_2$.

The corresponding $\Delta\varphi_{u,v}^{tot} = (n+1/2)\varphi_{u,v}$ and $\Delta\varphi_{u,v}^{geom}$ for $\lambda = 1$ and $\lambda = 4$ are displayed in Fig.2, Fig.3. and Fig.4. In addition, the amplitude functions for the same values of the parameters are shown in Fig.5. Notice that since the problem is undefined at the origin because of the $\frac{1}{r}$ singularity of $\Phi(r)$ the plots are displayed for the interval $[0.1\pi, \pi]$. For all plots we use the following three sets of values of the constants A, B and C :

1. $A = 1, B = \lambda, C = 0$. These values imply $C_1 = 1, C_2 = 0$,
 $u(r) = u_1, v(r) = v_1$.
2. $A = \frac{1}{\Phi(a)}, B = (\lambda + 1)\Phi(a), C = -1$. In this case $C_1 = \frac{1}{\sqrt{\Phi(a)}}$,
 $C_2 = -\Phi(a), \rho'_u(a) = 0$.
3. $A = \frac{\lambda}{V^2}, B = V^2, C = 0$ leading to $C_1 = \frac{\sqrt{\lambda}}{\Phi(a)}, C_2 = 0$.

The values of V, μ and k are chosen as follows: $V = 2.08236, \mu = \frac{1}{20}$ and $k = 1$.

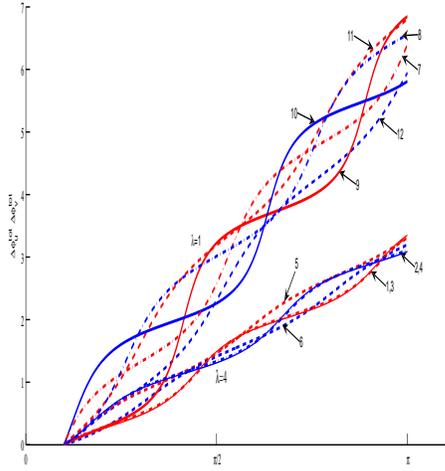


Fig.2. $\Delta\varphi_u^{tot}$ (red) and $\Delta\varphi_v^{tot}$ (blue) are plotted for three cases of the parameters A, B and C : — is for $A = 1, B = \lambda, C = 0$, - - - is for $A = \frac{1}{\Phi(a)}, B = (\lambda + 1)\Phi(a), C = -1$, and - · - is for $A = \frac{\lambda}{V^2}, B = V^2, C = 0$. Each of the cases is shown for $\lambda = 1$ and $\lambda = 4$.

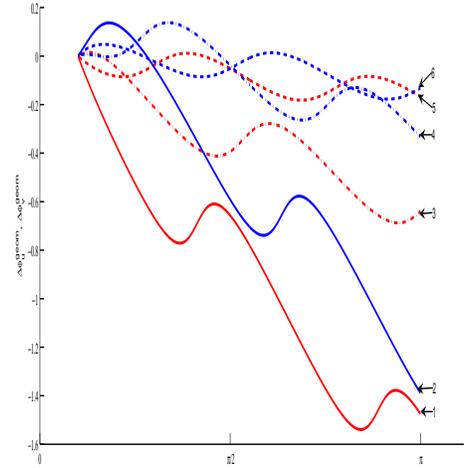


Fig. 3. Plots of $\Delta\varphi_u^{geom}$ (red) and $\Delta\varphi_v^{geom}$ (blue) given by (28) for the same three cases of the parameters A, B and C as in Fig.2. Each of the cases is shown for $\lambda = 1$.

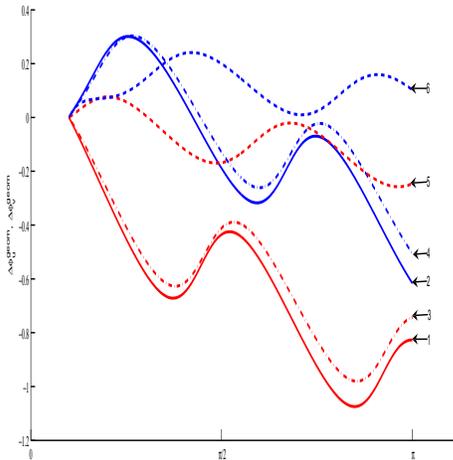


Fig. 4. Plots of $\Delta\varphi_u^{geom}$ (red) and $\Delta\varphi_v^{geom}$ (blue) given by (28) for the same three cases of the parameters A, B and C as in Fig.2. Each of the cases is shown for $\lambda = 4$.

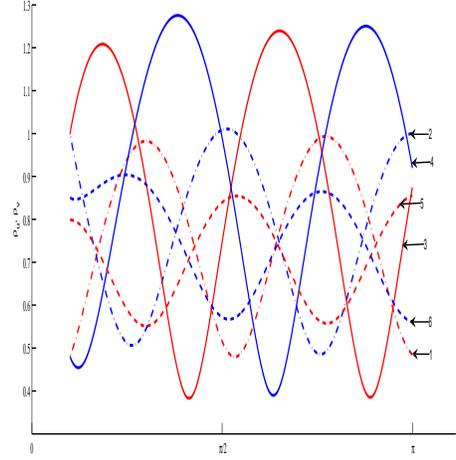


Fig.5. The amplitude functions ρ_u (red) and ρ_v (blue) given by formulas (29) and (30) for the same parameters A, B and C as in the previous plots ($\lambda = 1$).

8 Summary

After testing Kravchenko's numerical solutions in the context of Jackiw and Pi model of bilayer graphene we have developed an amplitude-phase approach of Ermakov-Lewis type for the same model and numerical solutions and for any radial profile of the gauge potential \mathcal{A} . Next, we have chosen a Debye-screened form of the profile of \mathcal{A} to illustrate our results. This choice could be of physical relevance when one considers the dynamic screening processes in the scattering of the charge carriers in the background of vortex configurations or other topological defects in the bilayer graphene.

Our technique is general and therefore can be applied to any other physical problem of similar mathematical structure. The geometric phase of Lewis-Riesenfeld type, which is more general than the Berry phase, is calculated for the first time in the graphene context. In addition, we provided a generalization of the Ioffe-Korsch nonlinear intertwining of the Milne-Pinney amplitude functions.

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