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QUANTUM HAMILTONIANS AND PRIME NUMBERS

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A short review of Schrödinger hamiltonians for which the spectral problem has been related in the literature to the distribution of the prime numbers is presented here. We notice a possible connection between prime numbers and centrifugal inversions in black holes and suggest that this remarkable link could be directly studied within trapped Bose-Einstein condensates. In addition, when referring to the factorizing operators of Pitkanen and Castro and collaborators, we perform a mathematical extension allowing a more standard supersymmetric approach.

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1. Introduction

The problem of the nontrivial zeros of Riemann's zeta function, i.e., whether they all lie on the line $z = \frac{1}{2} + it$ in the complex plane or not, is a famous unsolved mathematical problem in which physics, especially quantum mechanics and chaos theory, could have a substantial and rewarding contribution in view of its direct connection with the distribution of prime numbers.¹ The phase of Riemann's zeta function could be considered the main mathematical object able to trace this important arithmetic distribution.

In this work, my first goal is to provide a short survey of the hamiltonians that so far have been proposed to give hints for a spectral solution (also known as the Hilbert-Polya conjecture) of the location of the zeta zeros on the critical line. Along the review the reader can encounter several relevant novel implications in this highly interesting topic, especially in sections **2** and **4**. Since the logderivative of the phase shift of the Coulomb repulsive potential with reversed centrifugal barrier is a function that approximates very well the Riemann phase for the negative orbital number $l = -1/4$ and because centrifugal inversions have been conjectured in black holes a promising connection between prime numbers and black hole solutions in general relativity is suggested. Moreover, this could be under direct experimental investigation in the not quite far future if sonic black hole configurations could be produced in trapped Bose-Einstein condensates. In addition, when addressing the factorizing operators of Pitkanen and Castro and collaborators, a simple mathematical extension of their work is dealt that brings their approach closer to the standard supersymmetric methods of quantum mechanics.

2. Bhaduri, Khare, Low (1995) ²

BKL showed that the density of zeros of Riemann's ζ function is determined by its phase $\theta(t)$

$$\exp[2i\theta(t)] = \exp(-it \ln \pi) R_{\Gamma}(t) , \quad R_{\Gamma}(t) = \frac{\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{it}{2}\right)} , \quad (1)$$

where the convention $\theta(0) = \pi$ has been introduced. Although this phase is smooth, i.e., it does not include the jumps by π due to the zeros of the modulus of ζ , it counts well the zeros on the critical line. It can also be expressed as the logderivative of the $l = -\frac{1}{4}$ phase shift ($\arg\Gamma\left(\frac{3}{4} + \frac{it}{2}\right)$) with respect to the distorted Coulomb wave of the repulsive Coulomb potential

$$H_{bkl} = \frac{d^2}{dy^2} - \frac{l(l+1)}{y^2} - \frac{\epsilon}{y} + k^2 , \quad (2)$$

where $l = -\frac{1}{4}$, $\epsilon = -\frac{mE}{2\hbar^2}$, and $k = \frac{m\omega}{2\hbar}$.

The potential in Eq. (2) is also equivalent to an inverted harmonic half-oscillator. Of course, such a negative fractional phase shift does not look physical at first glance as it points to an *attractive* centrifugal barrier (i.e., well) but this is precisely the case for its application to prime numbers. Even more, I hail this as a very appealing feature because of a paper of Cirone *et al.*,³ where the $-1/4$ correction is called the *quantum anti-centrifugal force* being a metric effect related to the radial derivatives in the Laplacian. Furthermore it is well established that a 'centrifugal force reversal' occurs for $r < 3m$ in a Schwarzschild spacetime.⁴ The connection between prime numbers and black holes is of course a distinguished one and of much transcendence.

It is not hopeless to succeed in touching experimentally the promising connections alluded above. Especially suitable is the BEC physics. According to Garay, sonic black holes could be created in BECs trapped by means of sufficiently tight ring-shaped external potentials produced with state-of-the-art or planned technology.⁵ In this case, the sonic black holes are stationary solutions of the Gross-Pitaevskii nonlinear equation displaying a boundary from which only a tiny thermal quantum sound radiation can escape. Of course, a detailed study of the instability of these sonic horizons is needed as recently emphasized by Leonhardt and collaborators.⁶

But most promising, in 2001, Ott *et al.*,⁷ reported the production of a BE condensate in a microstructured magnetic surface trap. This is remarkable because the booming realm of surface-patterning technological processes opens up for the beautiful physical phase-shift approach to prime numbers. Thus, by means of micro-fabricated electrical circuits, precision measurement of scattering properties becomes feasible and interferometric measurements should allow soon a direct access of the phase shifts that was never available in the past.⁸

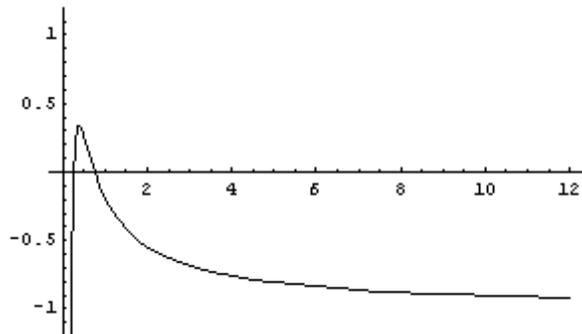


Fig. 1 The BKL Coulomb potential with $-1/4$ centrifugal correction and the other parameters scaled to unity.

3. Berry and Keating (1999) ⁹

This is in the class of Berry and Keating contributions, who were stimulated by ideas and remarks of Connes. According to Berry and Keating, the hamiltonian

$$H_{bk} = -i\hbar \left(x\partial_x + \frac{1}{2} \right) \quad (3)$$

is the simplest hermitian operator connected to Riemann's ζ , though it is only a canonically rotated form of the upturned harmonic oscillator $p^2 - x^2$. The simplicity results from avoiding the complications of the parabolic cylinder eigenfunctions. The linear independent wave functions in the x representation are of the type

$$\psi_{\pm}(x; E) = \frac{A\theta(\pm x)}{|x|^{1/2 - iE/\hbar}}, \quad (4)$$

where θ is the Heaviside step function. One can also write them in terms of the smoothed counting function of the Riemann zeros

$$\psi_{\pm}(x; E) = \frac{\exp[-i\pi(N_{sm}(E) - 1)]}{|x|^{1/2 - iE/\hbar}}, \quad (5)$$

where Berry and Keating define $N_{sm}(E) = \theta_R(E)/\pi + 1$, and

$$\theta_R(E) = -\frac{E}{2} \log \pi + \text{Im} \log \Gamma \left(\frac{1}{4} + \frac{1}{2}iE \right). \quad (6)$$

In addition, Aneva,¹⁰ considered chaos quantization conditions for H_{bk} and their geometrical (group) interpretation in an effort to get a discrete spectrum for H_{bk} .

4. Castro et al (2001, 2002) ¹¹

Castro and collaborators, based on works of Pitkanen,¹² introduced the simple and appealing nonrelativistic supersymmetric quantum mechanical (SUSYQM) schemes for the Riemann zeros problem. They also elaborated on p-adic oscillators for the same problem. Their first order ‘‘ladder’’ operators are

$$D_1 = -\frac{d}{d \ln t} + \frac{dV(t)}{d \ln t} + k_c \quad \text{and} \quad D_2 = \frac{d}{d \ln t} + \frac{dV(1/t)}{d \ln t} + k_c, \quad (7)$$

where k_c is a constant parameter and $V(t)$ is given by the logarithm of the Gauss-Jacobi theta series $G(t^l)$, $e^{2V(t)} = G(t^l)$, $l = \text{const}$

$$G(t^l) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t^l} \quad (8)$$

and $V(1/t) \neq V(t)$. The resulting pair of susy partner hamiltonians have the same *real* spectrum $s(1-s)$ on both the real line (trivial zeros) and Riemann’s critical line (nontrivial zeros). The ‘coherent’ eigenfunctions are written $\Psi_s(t)$ and $\Psi_s(1/t)$, respectively, where

$$\Psi_s(t) = t^z \left[\frac{\exp(-t)}{1 - \exp(-t)} \right]^{\frac{1}{2}}. \quad (9)$$

The Mellin transform of the Gauss-Jacobi theta series is the zeta function $\zeta(s)$ and the inner product of two eigenfunctions, say Ψ_{s_1} and Ψ_{s_2} , can be written in terms of $\zeta(a(s_1^* + s_2) + b - \frac{a}{2})$, where a and b can be expressed in terms of k_1 , k_2 and l . The zeta zeros are in one-to-one correspondence with the ‘orthogonality’ conditions.

Here I will show that there is a standard way of doing SUSYQM for this important case. Using a relationship of Gauss that was mentioned by Castro

$$G(1/x) = x^{1/2} G(x), \quad (10)$$

one can write the above ladder operators in the form

$$D_1 = -\frac{d}{d \ln t} + \frac{dV(t)}{d \ln t} + k_1 \quad \text{and} \quad D_2 = \frac{d}{d \ln t} + \frac{dV(t)}{d \ln t} + k_2, \quad (11)$$

where $k_2 = k_1 + \frac{1}{4}$.

This leads to

$$H_A \Psi_A = D_2 D_1 \Psi_A = [-d_t^2 + (k_1 - k_2)d_t + d_t V + V^2 + (k_1 + k_2)V + k_1 k_2] \Psi_A \quad (12)$$

and

$$H_B \Psi_B = D_1 D_2 \Psi_B = [-d_t^2 + (k_1 - k_2)d_t - d_t V + V^2 + (k_1 + k_2)V + k_1 k_2] \Psi_B, \quad (13)$$

where $d_t = \frac{d}{d \ln t}$ and $V = \frac{1}{2} \ln G(t^l)$.

Using now the gauge transformation $\Psi_{A,B} = \psi_{A,B}t^{l/4}$ one can eliminate the first derivative d_t and get the Schroedinger equations

$$d_t^2 \psi_{A,B} + S_{A,B} \psi_{A,B} = \lambda_{k_1, k_2} \psi_{A,B} , \quad (14)$$

where $\lambda_{k_1, k_2} = (\frac{k_1 - k_2}{2})^2 - k_1 k_2$ plays the role of eigenvalue and $S_{A,B}$ are in the position of Schroedinger susy partner potentials

$$S_{A,B} = \pm d_t V + V^2 + (k_1 + k_2)V . \quad (15)$$

5. Chadan and Musette (1993) ¹³

These authors used

$$H_{cm} = -\frac{d^2}{dx^2} - \frac{[l(l+1) - f_{cm}(x; g, \alpha, \beta)]}{x^2} , \quad (16)$$

where the centrifugal correction of Chadan and Musette is

$$F_{cm} = \frac{f_{cm}(x; g, \alpha, \beta)}{x^2} = \frac{\frac{g}{L^2(x)} - \frac{\alpha}{L(x)} - \frac{\beta}{L(x)LL(x)}}{x^2} . \quad (17)$$

In Eq. (17) $L = \log(1/x)$, $LL = \log \log(1/x)$, $l = 1$, $\alpha = -27/4$, $\beta = -3/2$ and g is a coupling constant.

Chadan and Musette proposed the above rather complicated singular potential in a closed interval $[0, R]$ and Dirichlet boundary conditions at both ends. They gave arguments that the spectrum in the coupling constant $g = \frac{1}{4} + \frac{t^2}{4}$ (t is the imaginary part on the critical line), which is real and discrete, with $g_n > 1/4$, coincides approximately with the nontrivial Riemann zeros when $R = e^{-4\pi/3}$.

We note that this is a so-called Sturmian quantum problem, i.e., a quantization problem in the coupling constant of the potential. A very detailed analysis of this singular hamiltonian and a generalization thereof from the point of view of inverse scattering and s -wave Jost functions has been performed in the important work of Khuri.¹⁴

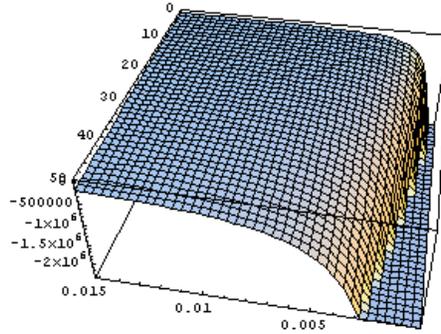


Fig. 2 The centrifugal correction F_{cm} of Chadan and Musette for $R \leq 0.015$ and $g \leq 50$.

6. Joffily (2003): zeros of s-wave Jost functions ¹⁵

The first to relate the nontrivial zeta zeros to the complex poles of a scattering matrix were Pavlov and Fadeev,¹⁶ for the case of a particle on a surface of negative curvature.

Joffily, on the other hand, suggested a ‘potential scattering’ Hilbert-Polya conjecture. He argued that the zeros of Riemann’s zeta function could be put in one-to-one correspondence with the zeros of the s-wave Jost function for cutoffted potentials in the complex momenta plane. Joffily considered the s-wave Jost function $f_+(\gamma)$ in the dimensionless argument $\gamma = kR$ (R is the cutoff of the potential) and the transformation

$$z = -i \frac{\gamma}{2\text{Im}\gamma} \quad (18)$$

by which the lower half of complex γ -plane is mapped onto the critical zeta line.

7. The fractal potential of Wu and Sprung (1993) ¹⁷

Wu and Sprung obtained a local potential given by an Abel integral equation based on the smooth number of Riemann zeros below E , $N(E) = \frac{E}{2\pi} \ln \frac{E}{2\pi e} + \frac{7}{8}$, considered as the semiclassical WKB density of states below E of a potential. Their potential $y(x) = \frac{V}{V_0}$ scaled with respect to $V(0) = V_0$ (Wu and Sprung chose $V_0 = 3.10073\pi$) is given by

$$x = \frac{\sqrt{V_0}}{\pi} \left[\sqrt{y-1} \ln \frac{V_0}{2\pi e^2} + \sqrt{y} \ln \frac{\sqrt{y} + \sqrt{y-1}}{\sqrt{y} - \sqrt{y-1}} \right]. \quad (19)$$

As a function of the fitted Riemann zeros, $y(x; N)$ shows fluctuations with respect to the smooth potential $y(x)$ and therefore $y(x; N)$ is a fractal curve. Using a box-counting-method analysis, Wu and Sprung obtained the result that the first 500 Riemann nontrivial zeros can be fitted with the fractal potential $y(x; 500)$ of fractal dimension 1.5.

8. Okubo’s two-dimensional potential (1998) ¹⁸

Okubo introduced

$$H_{\text{ok}}(x, y; \beta_{\text{ok}}) = \frac{\partial^2}{\partial x \partial y} + i\beta_{\text{ok}} y \frac{\partial}{\partial y} + i(1 - \beta_{\text{ok}}) x \frac{\partial}{\partial x} + \frac{i}{2}, \quad (20)$$

where β_{ok} is a real parameter. He gave arguments that this two-dimensional Lorentz-invariant Hamiltonian may be relevant to the RH. Some eigenfunctions of his two-dimensional Hamiltonian corresponding to infinite-dimensional representation of the Lorentz group have many interesting properties. Especially, a relationship exists between the zero zeta function condition (ZZFC) that should be satisfied by any nontrivial zeta zero

$$\int_0^\infty d\tau \frac{\tau^{z-1}}{1 + \exp \tau} = 0, \quad (21)$$

for $\text{Re } z \in (0, 1)$ and the absence of trivial representations in the wave function. The eigenfunctions of $H_{\text{ok}}\phi = \lambda\phi$ with $z = \frac{1}{2} + i\lambda$ are of the form

$$\phi(x, y) = \int_0^\infty d\tau \tau^{z-1} \exp(ix\tau^{1-\beta_{\text{ok}}})g(\tau + y\tau^{\beta_{\text{ok}}}) , \quad (22)$$

where g is an arbitrary function which vanishes as its argument goes to ∞ . The choice

$$g_0(\tau + y\tau^{\beta_{\text{ok}}}) = \frac{1}{1 + \exp(\tau + y\tau^{\beta_{\text{ok}}})} \quad (23)$$

yields a function ϕ_0 which is an infinite-dimensional *unitary* realization of the Lorentz group $\text{SO}(1,1)$. If the so-called ZZFC condition is applied to ϕ_0 the representation space of $\text{SO}(1,1)$ does not contain any singlet representation of the group.

Berry and Keating commented that Okubo's hamiltonian is a more general, relativistic oscillator than their own quantized xp .

9. Mussardo's potential (1997) ¹⁹

Mussardo's potential W is given by

$$x(W) = \sum_{m=1}^{\infty} \frac{\mu(m)}{m} \int_{E_0}^W \frac{\epsilon^{\frac{1-m}{m}}}{\ln \epsilon \sqrt{W - \epsilon}} d\epsilon , \quad (24)$$

where μ is the Möbius function.

Since Mussardo applies, like Wu and Sprung, the semiclassical method, his work is based on Abel's integral equation. All that we said with respect to the work of Wu and Sprung applies here as well.

Mussardo also suggested a resonance experiment to implement the primality test up to a maximum number N depending on a cutoff ϵ_0 . Sharp resonances in the transmission amplitude $T_N(E)$ of plane waves impinging on the potential $W(x; \epsilon_0(N))$ would indicate that the numbers N are prime.

10. de Oliveira and Pellegrino (2001) ²⁰

These authors considered discrete Schrödinger operators of the type

$$(H_V u)_n = u_{n+1} + u_{n-1} + \lambda V_n u_n \quad (25)$$

defining $V_0 = 0$, $V_n = 1$ if $n + 1$ is a prime number, and $V_n = 0$ if not. They have found a localization-delocalization transition as a function of the potential strength λ .

11. Many-body hamiltonians: Boos and Korepin (2001, 2002) ²¹

Many-body hamiltonians are interesting more from the statistical physics point of view than the spectral one. Boos and Korepin,²¹ showed that particular correlation functions in quantum spin chains can be expressed in terms of the values of Riemann

zeta function at odd arguments (for even arguments, it is known that $\zeta(2n) \sim B_{2n}$, the even Bernoulli numbers). They put forward the following conjecture

Arbitrary correlators of the XXX antiferromagnet are described as certain combinations of $\ln 2$, the Riemann zeta function with odd arguments and rational coefficients.

An extension of this result to complex arguments could be interpreted as decay (decoherence) of the spin chain correlators.

We recall that the XXX finite N chain antiferromagnet is described since 1928 by the following Heisenberg hamiltonian

$$H_{XXX} = \sum_{i=1}^N (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z - 1), \quad (26)$$

where the σ 's are Pauli matrices.

12. Crehan (1995) ²²

Crehan used a classical theorem of Hardy and Littlewood stating that the position of the n th prime on the critical line is limited by $t_n < a^{-1}n$ ($a = \text{const}$) to get a theorem asserting that there is an infinite family of classically integrable nonlinear oscillators whose quantum spectrum is given by the imaginary part of the sequence of zeros on the critical line of the Riemann zeta function. In addition, Forrester and Odlyzko,²³ related the distribution of zeta critical zeros to generalized Painleve fifth differential equation.

13. Conclusion

The spectral interpretation of the imaginary parts of the nontrivial Riemann zeros stimulated mathematical physicists to propose several quantum hamiltonians with spectra that could be useful in tackling with this century-old problem. Here the suggestions made in this area are gathered together in order to catch a global glimpse of these research facts. I provide a number of useful comments that hopefully will be taken into account in future possible experiments. The most exciting issue is the possibility of direct experimental access to negative phase shifts in the physics of trapped BECs. This cutting-edge research is of direct relevance for the link between prime numbers and black holes. I also hint that the rapidly developing **PT** quantum mechanics,²⁴ a specific type of non-hermitic hamiltonians symmetric with respect to parity and time reversal discrete operations that display real spectra, will have an important contribution in this field. In fact, the susy hamiltonian pair of Pitkanen and Castro and collaborators are not hermitic but have real spectra.

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