Inhomogeneous Barotropic FRW Cosmologies in Conformal Time

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Abstract

It is known that the barotropic FRW system of differential equations for zero cosmological constant can be reduced to simple harmonic oscillator (HO) differential equations in the conformal time variable. This is due to the fact that the Hubble rate parameter in conformal time is the solution of a simple Riccati equation of constant coefficients. In previous works, we have used this mathematical result to set the barotropic HO equations in the nonrelativistic supersymmetric approach by factorizing them. If a constant additive parameter, denoted by S, is added to the common Riccati solution of these supersymmetric partner cosmologies one obtains inhomogeneous barotropic cosmologies with periodic singularities in their spatial curvature indices that are counterparts of the non-shifted supersymmetric partners. The zero-mode solutions of these cyclic singular cosmologies are reviewed here as a function of real and imaginary shift parameter. We also notice the modulated zero modes obtained by using the general Riccati solution and comment on their cosmological application.

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1 Introduction

Riccati nonlinear equations are together with Bernoulli equations the oldest and the simplest nonlinear equations with many applications in the realm of physics. In recent years, their solutions under the name of superpotentials play an important role in supersymmetric quantum mechanics [1, 2]. In cosmology, the supersymmetric methods have been recently reviewed in the book of Moniz [3] and there are frequent occurrences of Riccati equations spread over the many areas of cosmology and astrophysics [4].

Riccati equations as simple as

$$R' + cR^2 + f = 0 \ , \tag{1}$$

where c is a real constant and f is a function of the independent variable play a central role in barotropic FRW cosmologies, which will be reviewed in the following. We only need to recall that particular solutions of the Riccati equation enter the factorization brackets of second order linear differential equations (usually with $c = 1$)

$$\left(\frac{d}{dt} + cR\right)\left(\frac{d}{dt} - cR\right)u = 0 \equiv u'' - c(cR^2 + R')u = 0 \ \equiv u'' + cfu = 0 \ . \tag{2}$$

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The connections $R = \frac{1}{c}\frac{u'}{u}$ or $u = e^{c\int^t R}$ between the particular solutions of the two equations are also basic results of the factorization method together with the construction of the so-called supersymmetric partner equation of equation (2) obtained by reverting the order of the factorization brackets:

$$\left(\frac{d}{dt} - cR\right)\left(\frac{d}{dt} + cR\right)v = 0 \equiv v'' - c(cR^2 - R')v = 0 \equiv v'' + c(f + 2R')v = 0 \ . \tag{3}$$

2 Cosmological Riccati equation of FRW barotropic cosmologies

As first shown by Faraoni [5], the well-known comoving Einstein-Friedmann dynamical equations of barotropic FRW cosmologies can be turned into a single Riccati equation for the Hubble parameter in conformal time $\mathcal{H}(\eta)$ that is the simpler case of (1) with $c = \tilde{\gamma}$ and $f = \kappa\tilde{\gamma}$

$$\mathcal{H}' + \tilde{\gamma}\mathcal{H}^2 + \kappa\tilde{\gamma} = 0 \ , \tag{4}$$

where henceforth the prime and also $\frac{d}{d\eta}$ stand for the derivative with respect to η, $\tilde{\gamma} = \frac{3}{2}\gamma - 1$ is related to the adiabatic index γ of the cosmological fluid and $\kappa = \pm 1$ is the curvature parameter for the closed and open universe, respectively. In the following, we will consider only the $\kappa = 1$ case since it allows us to focus on the periodic features of the problem. This conformal-time Riccati equation is valid for any barotropic fluid except for $\gamma = \frac{2}{3}$ which leads to the the simple linear equation $\mathcal{H}' = 0$. In addition, equation (4) is just the Riccati equation of the harmonic oscillator:

$$\dot{R} + \omega_0 R^2 + \omega_0 = 0 \ , \tag{5}$$

if one sets $\tilde{\gamma} \equiv \omega_0$ for the closed universe case. For the open case the analogy is with the up-side down harmonic oscillator. However, the fact that the independent variable is the conformal time and not the usual Newtonian time makes a substantial difference from the physical point of view. Since the conformal time is related to the comoving time through $t(\eta) = \int^\eta a(\eta)d\eta$, one can see that this type of time depends on the spatial scaling parameter. Therefore, the second order differential equations with which the conformal Riccati equation is connected are not really laws of force as in classical mechanics but can be treated as Schroedinger equations in quantum mechanics. We thus define the conformal Hubble parameter directly as the logarithmic derivative $\mathcal{H}_u(\eta) = \frac{1}{\tilde{\gamma}}\frac{u'}{u}$ because we know that by substituting $\mathcal{H}_u(\eta)$ in equation (4) we get the very simple harmonic oscillator equation

$$u'' + \tilde{\gamma}_u^2 u = 0 \ , \qquad \tilde{\gamma}_u^2 = \tilde{\gamma}^2 = \text{const} \ . \tag{6}$$

Due to the similarity with supersymmetric quantum mechanics, one can call the u modes as bosonic zero modes [6]. They are $\tilde{\gamma}$ powers of the scale factor parameters $a(\eta)$, i.e., $u = a^{\tilde{\gamma}}(\eta)$. Using the particular solution solutions:

$$u_1 \sim \cos\tilde{\gamma}\eta \qquad \rightarrow \qquad a_{1u}(\eta) \sim u_1^{1/\tilde{\gamma}}$$

in the definition of $\mathcal{H}_u$ one gets $\mathcal{H}_{u_1} = -\tan\tilde{\gamma}\eta$.

A supersymmetric partner equation (3) of equation (6) leads immediately to a class of cosmologies with inverse scale factors with respect to the standard barotropic ones but with a conformal-time-dependent curvature index [6]

$$v'' + \kappa_v(\eta)\tilde{\gamma}^2 v = 0 \ , \tag{7}$$

where

$$\kappa_v(\eta) = -(1 + 2\tan^2\tilde{\gamma}\eta) \tag{8}$$

denotes the conformal-time-dependent supersymmetric partner curvature index of fermionic type associated through the mathematical scheme to the constant bosonic curvature index.

A particular fermionic solution v is of the following type

$$v_1 = \frac{\tilde{\gamma}}{\cos\tilde{\gamma}\eta} \qquad \rightarrow \qquad a_{1v}(\eta) \sim v_1^{1/\tilde{\gamma}} \ .$$

We can see that the u and v barotropic cosmologies are dual to each other from the standpoint of these particular solutions, in the sense that $u_1 v_1 = \tilde{\gamma}$ and therefore the geometric mean of their scale parameters

$$a_g = (a_{1u} a_{1v})^{1/2} = (\tilde{\gamma})^{1/2\tilde{\gamma}}$$

is constant. Thus, a joint evolution of a u cosmology of constant curvature index and a v cosmology of the time-dependent curvature index (8) is stationary in conformal time from the standpoint of their geometric mean scale parameter a_g. The fermionic metric is of the form:

$$ds^2 = a_{1v}^2(\eta)\left[-d\eta^2 + \frac{dr^2}{1-\kappa_v(\eta)r^2} + r^2 d\Omega^2\right] .$$

This metric should be thought of as an averaged metric in an inhomogeneous cosmology and so it does not even have to satisfy the Einstein field equations [7]. Such metrics, with other time-dependent curvature indices have been used to mimic the backreaction of small scale density perturbations on the large scale spacetime geometry [7, 8].

3 Barotropic Cosmologies with Conformal Hubble Parameters Having Non-Zero Initial Conditions

Since the Riccati equation is a first-order differential equation, its solution is entirely determined by one initial condition. In the case of the cosmological Riccati equation (4), one cannot be sure that the initial condition is $\mathcal{H}(0) = 0$.

We thus reexamine the consequences of a constant shift $S \neq 0$ of the conformal Hubble parameter that provides a non-zero initial condition, [9]

$$\mathcal{H}_S(\eta) = \mathcal{H}_{u1}(\eta) + S \ , \quad \mathcal{H}_S(0) = S \ .$$

It is easy to obtain the corresponding second-order differential equations in this case by the factorization method

$$\left(\frac{d}{d\eta} + \tilde{\gamma}\mathcal{H}_S\right)\left(\frac{d}{d\eta} - \tilde{\gamma}\mathcal{H}_S\right)\mathcal{U} = 0 \ \equiv \ \mathcal{U}'' + \kappa_{S,u}(\eta)\tilde{\gamma}^2\mathcal{U} = 0 \ , \tag{9}$$

where

$$\kappa_{S,u}(\eta) = \left[1 - S^2 + 2S\tan\tilde{\gamma}\eta\right] \tag{10}$$

and

$$\left(\frac{d}{d\eta} - \tilde{\gamma}\mathcal{H}_S\right)\left(\frac{d}{d\eta} + \tilde{\gamma}\mathcal{H}_S\right)\mathcal{V} = 0 \ \equiv \ \mathcal{V}'' + \kappa_{S,\nu}(\eta)\tilde{\gamma}^2\mathcal{V} = 0 \ , \tag{11}$$

where

$$\kappa_{S,\nu}(\eta) = -\left[1 + S^2 - 2S\tan\tilde{\gamma}\eta + 2\tan^2\tilde{\gamma}\eta\right] \ . \tag{12}$$

One can see that when $S = 0$, the initial pair of unshifted partner equations are obtained and the corresponding spatial curvature indices are recovered

$$\kappa_{0,u}(\eta) = 1 \ , \qquad \kappa_{0,\nu}(\eta) = -\left[1 + 2\tan^2\tilde{\gamma}\eta\right] \ . \tag{13}$$

While in classical mechanics, these equations define parametric oscillators [10], in cosmology they describe two new classes of barotropic-like cosological universes that we call $\mathcal{U}$ and $\mathcal{V}$ cosmologies, respectively. One can also write averaged conformal-like metrics with the corresponding variable curvature indices. These two modified cosmologies are periodic, of period $T = \frac{\pi}{\tilde{\gamma}}$, and have the same conformal Hubble parameter given by $\mathcal{H}_S(\eta)$. Both cosmologies have periodic singularities in their time-dependent curvature indices.

The linear independent solutions $\mathcal{U}_1$ and $\mathcal{U}_2$ have the following form

$$\mathcal{U}_1(\eta) = e^{-i\Omega_S\eta}\, {}_2F_1\left(1, -iS; 2 - iS; -e^{-2i\tilde{\gamma}\eta}\right) \ , \tag{14}$$

or $\mathcal{U}_1(\eta) = e^{-S\tilde{\gamma}\eta}\left[z\, {}_2F_1\left(1, -iS; 2 - iS; -z^2\right)\right]$, where $z = e^{-i\tilde{\gamma}\eta}$, $\Omega_S = \tilde{\gamma}(1 - iS)$ and ${}_2F_1\left(a, b; c; z\right)$ is the hypergeometric function, and

$$\mathcal{U}_2(\eta) = e^{i\Omega_S\eta}(2\cos^2\tilde{\gamma}\eta - i\sin 2\tilde{\gamma}\eta) \sim e^{S\tilde{\gamma}\eta}\cos\tilde{\gamma}\eta \ . \tag{15}$$

We also notice that the simple convergence condition $\Re(a+b-c) < 0$ for the hypergeometric series is fulfilled for all real values of η.

On the other hand, the linear independent solutions $\mathcal{V}_1$ and $\mathcal{V}_2$ are given by:

$$\mathcal{V}_1(\eta) = \frac{e^{-i\Omega_S\eta}}{(2\cos^2\tilde{\gamma}\eta - i\sin 2\tilde{\gamma}\eta)} \ , \tag{16}$$

or $\mathcal{V}_1(\eta) = \frac{e^{-S\tilde{\gamma}\eta}}{2\cos\tilde{\gamma}\eta}$, and

$$\mathcal{V}_2(\eta) = \tilde{\gamma}^2 e^{i\Omega_S\eta}\left[(1 - i\tan\tilde{\gamma}\eta\,) + iS(1-iS)(2\cos^2\tilde{\gamma}\eta - i\sin 2\tilde{\gamma}\eta) - 2iS\right] \ , \tag{17}$$

or $\mathcal{V}_2(\eta) = 2\tilde{\gamma}^2 e^{S\tilde{\gamma}\eta}\left[\frac{1}{2\cos\tilde{\gamma}\eta} + S(S\cos\tilde{\gamma}\eta + \sin\tilde{\gamma}\eta)\right]$.

However, the more complicated formof these zero-mode solutions is directly their Floquet-Bloch form that leads to the following considerations. The parameter S affects only the period of the phases $e^{\pm i\Omega_S\eta}$ of the solutions but not that of their periodic part. The solutions (14),(15) and the Bloch factors $e^{\pm i\Omega_S\eta}$ in (16),(17) are bounded if and only if the "quasifrequency" Ω_S has a real value, or equivalently

$$\tilde{\gamma}(1 - iS) \in \Re \ . \tag{18}$$

Taking into account that $\tilde{\gamma} \in \Re$, the last condition reads as $S = is$, $s \in \Re$. Thus, equation (9) has bounded solutions $\forall s \in \Re$. Notice that for a purely imaginary shift parameter, the curvature indices $\mathcal{K}_u$ and $\mathcal{K}_\nu$ are related through

$$\kappa_{S,u}(-\eta) = \kappa^*_{S,u}(\eta) \ , \qquad \kappa_{S,v}(-\eta) = \kappa^*_{S,v}(\eta) \ ,$$

where * denotes the complex conjugation operation. This means that we can have in this cosmological context the parity-conformal time (PT) symmetry [11]. Additionally, by inspecting the solutions (14) and (15) and (16) and (17), we note that they are periodic for $s_p = (2m-1)$, $m = 0,1,2,...$ and antiperiodic for $s_a = 2m$, $m = \pm 1, \pm 2, ...$ For $\tilde{\gamma} = 1$ (radiation-filled universe), the solutions in the periodic case $m = 2$, i.e.,

$$\mathcal{U}_1(\eta) = \frac{e^{-i3\eta}}{2\cos\eta}\,{}_2F_1\left(1,2;5;\frac{1}{1+e^{2i\eta}}\right) \ , \quad \mathcal{U}_2(\eta) = e^{i3\eta}\cos\eta \tag{19}$$

and

$$\mathcal{V}_1(\eta) = \frac{e^{-i3\eta}}{2\cos\eta} \ , \quad \mathcal{V}_2(\eta) = \left[\frac{e^{i3\eta}}{2\cos\eta} - 12e^{i3\eta}\cos\eta - 6e^{i2\eta}\right] \tag{20}$$

are displayed in Figs. 1 and 2, respectively, whereas the zero-mode solutions in the antiperiodic case $m = 2$,

$$\mathcal{U}_1(\eta) = \frac{e^{-i4\eta}}{2\cos\eta}\,{}_2F_1\left(1,2;6;\frac{1}{1+e^{2i\eta}}\right) \ , \quad \mathcal{U}_2(\eta) = e^{i4\eta}\cos\eta \tag{21}$$

and

$$\mathcal{V}_1(\eta) = \frac{e^{-i4\eta}}{2\cos\eta} \ , \quad \mathcal{V}_2(\eta) = \left[\frac{e^{i4\eta}}{2\cos\eta} - 24e^{i4\eta}\cos\eta - 8e^{i3\eta}\right] \tag{22}$$

can be found in Figs. 3 and 4, respectively. In general, the $\mathcal{U}$ modes are regular indicating that these shifted cosmologies are not sensitive to the singularities of their curvature indices at the level of their zero modes. On the other hand, the $\mathcal{V}$ cosmologies have periodic singularities in their imaginary parts, but not in their real parts. In addition, the duality property is maintained for the pair of zero-modes $\mathcal{U}_2\mathcal{V}_1 = \text{const}$.

The dualities introduced by the supersymmetric approach have certain similarities with the superstring dualities, [12] and the phantom duality [13]. In the first case, there is an invariance of the action with respect to an inversion of the cosmological scale factor and special shifts of the value of the dilaton field ($a \to a^{-1}$ and $\Phi \to \Phi - 6\ln a$, respectively). In our case, the scale factor duality is the same but the shift is done in the conformal Hubble parameter. Moreover, the string action with such properties corresponds to cosmologies that are spatially flat and homogeneous. Thus, our scale factor duality connecting homogeneous and inhomogeneous nonflat cosmologies look more general. On the other hand, Dąbrowski et al. [13] discussed the symmetry $\gamma \to -\gamma$ of a nonlinear oscillator equation while in our case, the $\tilde{\gamma}$ parameter occurs mostly trigonometrically. For the unshifted bosonic and fermionic cosmologies, we see readily that $\tilde{\gamma} \to -\tilde{\gamma}$

is a symmetry preserving their curvature indices, although we have now $u_1v_1 = -\tilde{\gamma}$ and $a_g = (-\tilde{\gamma})^{=1/2\tilde{\gamma}}$, a different constant. In the case of the shifted cosmologies, the sign changes $\tilde{\gamma} \to -\tilde{\gamma}$, $\kappa_{S,u}(\eta) \to \kappa_{-S,u}(\eta)$, $\kappa_{S,v}(\eta) \to \kappa_{-S,v}(\eta)$ leave the equations unchanged. We also draw the attention of the interested reader to a recent paper of Faraoni [14] for a discussion of scale factor dualities of the spatially flat Friedmann equations with barotropic fluids, where also many references on such dualities are provided.

If we move now to real S values, we see that $\mathcal{U}_2(\eta)$ zero mode is bound and nonsingular for negative real values of S. So, we can obtain damped cyclic behavior of the universe mimicking viscous effects directly by tuning the S parameter in the underdamped regime. Plots of such underdamped zero modes are given in Fig. 5.

In addition, damped cyclic behavior can be also obtained using the general Riccati solution of the shifted cosmologies, which introduces one-parameter zero modes of the form, [2, 15]

$$\mathcal{U}_{2;\lambda}(\eta) = \frac{\mathcal{U}_2(\eta)}{\lambda + \int^{\eta} \mathcal{U}_2^2(\eta)} \tag{23}$$

In this case, λ plays the role of a damping lag in conformal time after which the weighting effect of the integral starts to dominate, see Fig. 6. The advantage of this deformed zero mode is that one can give a physical interpretation to the parameter λ, which is related to the introduction of finite-interval boundaries on the conformal time axis. This is very well described in Section II A of a paper by Monthus *et al.* [16] in the case of quantum mechanics where it is clearly shown how the introduction of boundary conditions at certain points on the axis generates a modulation of the particular solution as given by (23) and the λ parameter can be fixed through the boundary conditions. This can be used to simulate the effects of voids directly on the cosmological zero modes [17].

4 Summary and Final Remarks

We have reviewed here the procedures that led us to introduce inhomogeneous supersymmetric-partner classes of barotropic cosmologies of variable spatial curvature index, by considering non zero initial conditions of the conformal Hubble parameter of FRW barotropic cosmologies. These results have been obtained in a supersymmetric context similar to supersymmetric quantum mechanics and can be applied to any type of cosmological fluid except for $\gamma = 2/3$. In other words, these classes of inhomogeneous cosmologies together with the unshifted supersymmetric partner cosmology can be associated to any barotropic cosmological fluid with the only one exception of the coasting (non-accelerating and non-decelerating) universe. Interestingly, we have found that even purely imaginary initial conditions can be considered. It is known that such 'unnatural' initial conditions are required to explain why a thermodynamic arrow of time exists. Cyclic behavior in conformal time of the cosmological zero modes can be obtained in addition to that of their curvature indices that in the pure imaginary case are also related through the parity-time (PT) property. Both inhomogeneity [18] and cyclicity [19] are debated issues since the accelerated expansion of the universe got experimental evidence at the end of the past century. On the other hand, if we forget about the topology of the universe, the same results can be interpreted as due to the time dependence of the adiabatic indices of the cosmological fluids [20]. Along this line, Dąbrowski and Denkievicz [21] have provided a discussion of an explicit barotropic model in which the cosmological singularity occurs only in the singular time-dependent barotropic index. They assert that physical examples of such singularities appear in f(R), scalar field, and brane cosmologies [22]. We think that the barotropic inhomogeneous cosmologies of supersymmetric type with periodic singularities in the curvature index could be related to the same physical examples. Moreover, the periodic singularities are not an impediment to build appropriate cosmological zero modes along the whole conformal time axis [23] which can be used to define novel classes of scale factors corresponding to these generalized barotropic universes within any chosen period of the curvature index.

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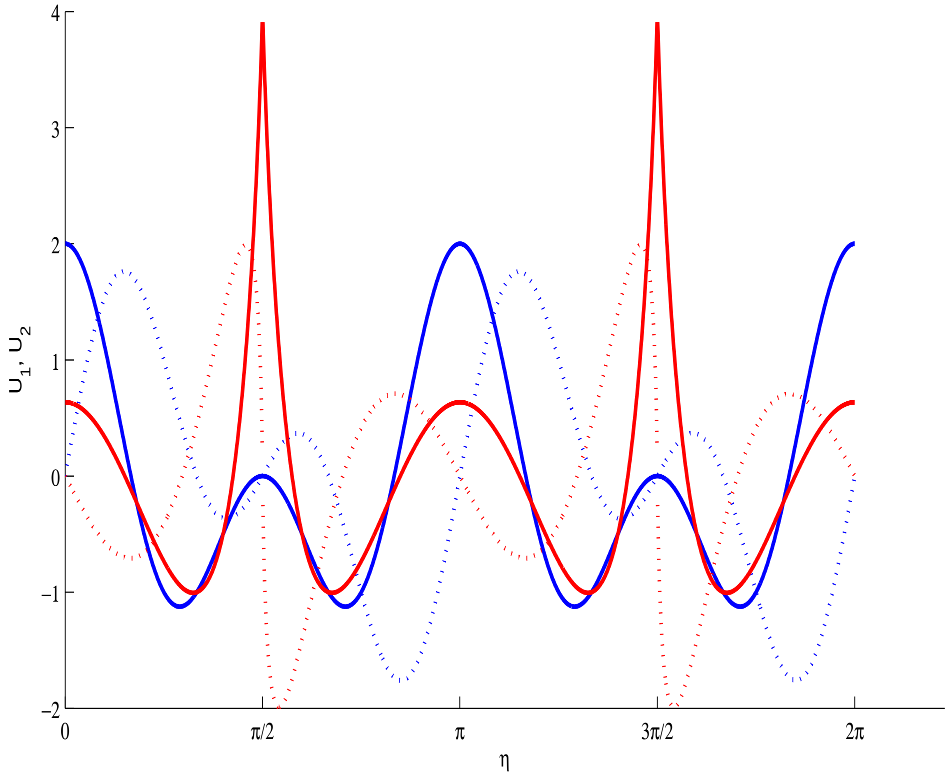

Fig. 1: The real (solid lines) and imaginary (dotted lines) parts of the periodic zero modes $\mathcal{U}_1(\eta)$ (in red) and $\mathcal{U}_2(\eta)$ (in blue) for the shift parameter $S = 3i$.

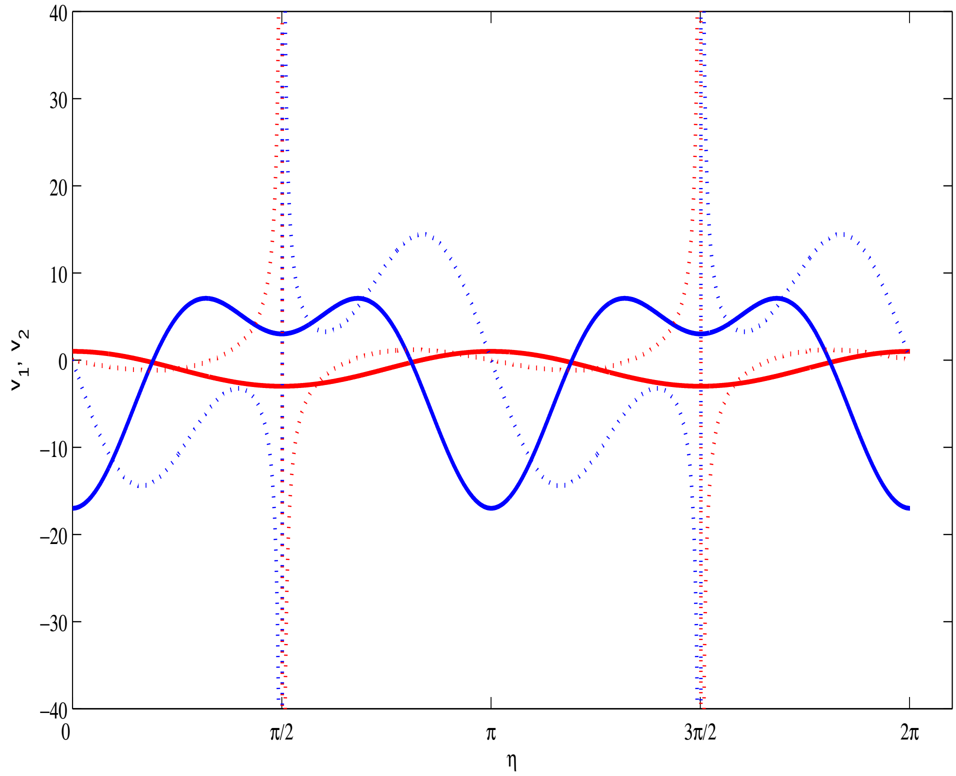

Fig. 2: The real (solid line) and imaginary (dotted lines) parts of the periodic zero modes $\mathcal{V}_1(\eta)$ (in red) and $\mathcal{V}_2(\eta)$ (in blue) for the shift parameter $S = 3i$.

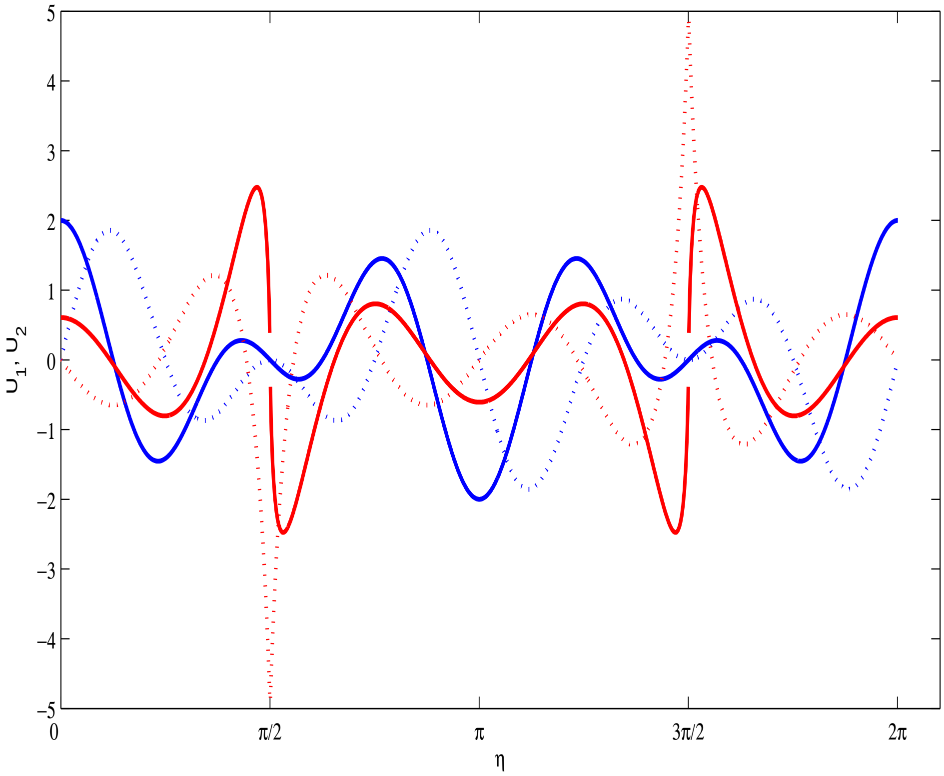

Fig. 3: The real (solid lines) and imaginary (dotted lines) parts of the antiperiodic zero modes $\mathcal{U}_1(\eta)$ (in red) and $\mathcal{U}_2(\eta)$ (in blue) for the shift parameter $S = 4i$.

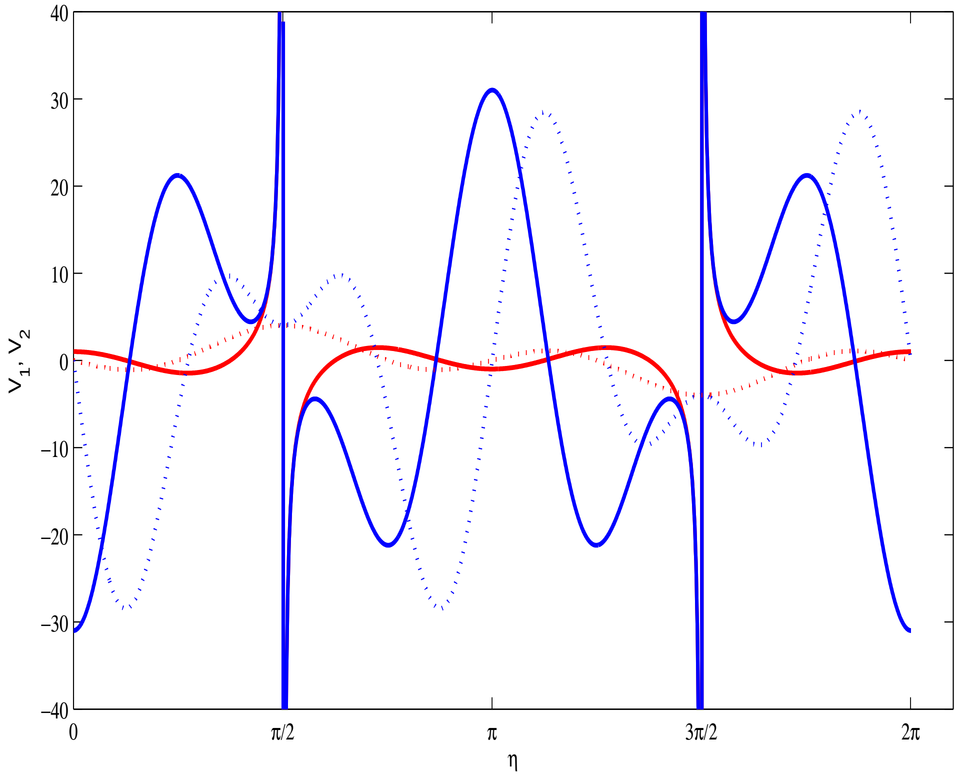

Fig. 4: The real (solid lines) and imaginary (dotted lines) parts of the antiperiodic zero modes $\mathcal{V}_1(\eta)$ (in red) and $\mathcal{V}_2(\eta)$ (in blue) for the shift parameter $S = 4i$.

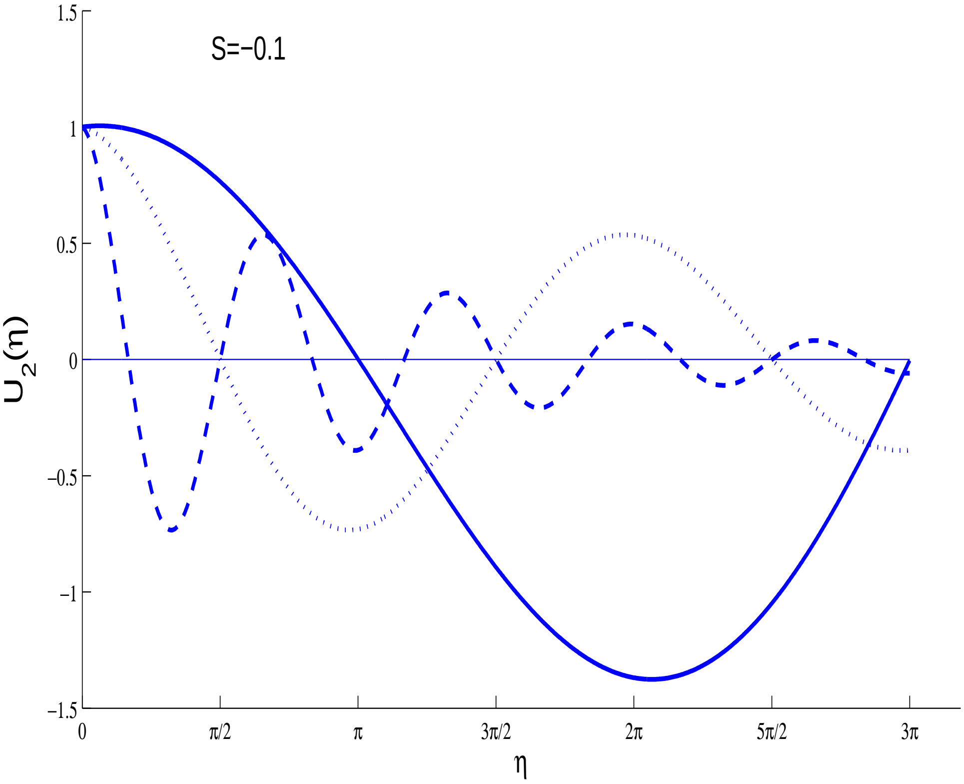

Fig. 5: The $\mathcal{U}_2(\eta)$ zero mode for $S = -0.1$ in the cases $\tilde{\gamma} = -1/2$ (solid line) corresponding to the vacuum, $\tilde{\gamma} = 1$ (dotted line) corresponding to the radiation case, and $\tilde{\gamma} = 3$ (dashed line) corresponding to a stiff fluid.

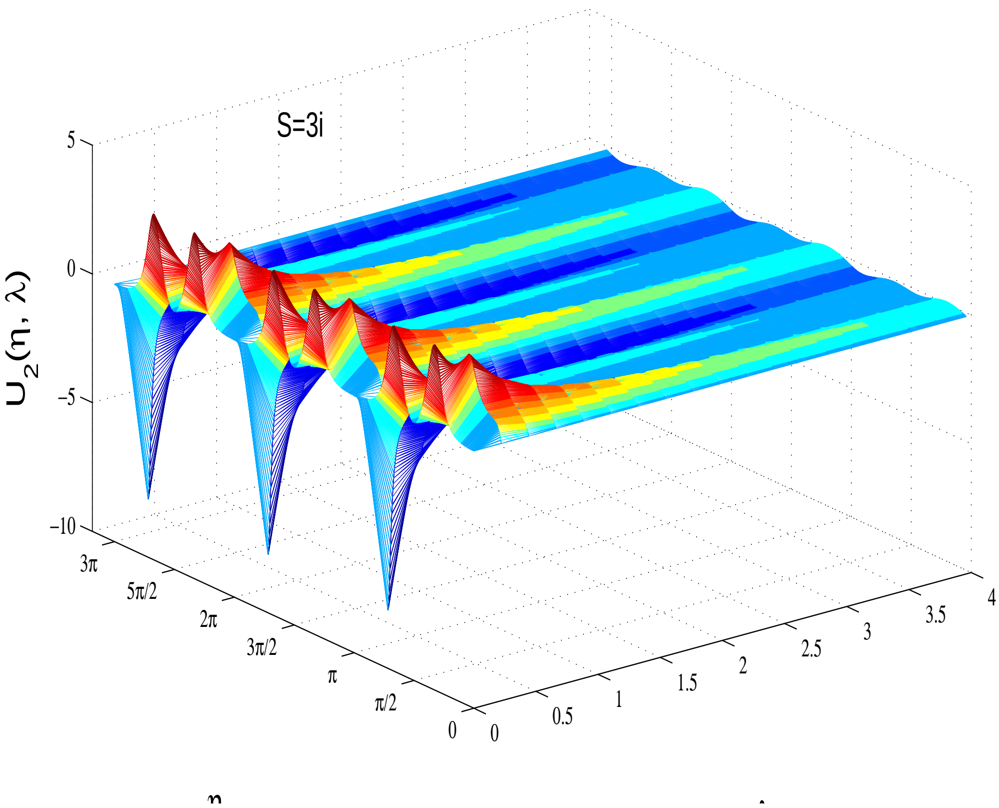

Fig. 6: The imaginary part of the $\mathcal{U}_{2;\lambda}(\eta)$ zero mode for $S = 3i$ and $\tilde{\gamma} = 1$ showing damped oscillations at small values of λ.