

FRW Barotropic Zero Modes: Dynamical Systems Observability

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Abstract

The *dynamical systems observability* properties of barotropic bosonic and fermionic FRW cosmological oscillators are investigated. Nonlinear techniques for dynamical analysis have been recently developed in many engineering areas but their application has not been extended beyond their standard field. This paper is a small contribution to an extension of this type of dynamical systems analysis to FRW barotropic cosmologies. We find that determining the Hubble parameter of barotropic FRW universes does not allow the *observability*, i.e., the determination of neither the barotropic FRW zero mode nor of its derivative as dynamical cosmological states. Only knowing the latter ones correspond to a rigorous dynamical observability in barotropic cosmology.

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1 Introduction

Over the years, modern dynamical systems theory has been applied with considerable success to the evolution of cosmological models. Many results concerning the possible asymptotic cosmological states (at both early and late times) have been obtained and compared with Hamiltonian methods and numerical studies in the authoritative book of Wainwright and Ellis [6].

Dynamical systems observability (DSO henceforth) is a rigorous mathematical concept used to investigate if it is possible to know the internal functioning of a given dynamical system. In the case of cosmology, DSO can be considered a criterion of whether the possible states of the dynamical universe as defined by some intelligent observers can be indeed estimated. More precisely, one says that a dynamical system is *observable* if it allows the usage of the available information about the input $u(t)$ and the output $y(t)$ of the system to estimate the states $X(t)$ of the system. Thus, the main idea behind (engineering) observability is to find the dynamical states of a system based on the knowledge of its output.

In this paper we will investigate a cosmological application of this type of observability analysis. In the context of FRW barotropic cosmologies one important measured output is the Hubble parameter and an interesting problem is if we can determine the so-called barotropic zero modes since these modes can be identified as the normal dynamical cosmological states. In the conformal time variable, the logderivative of these modes determine the Hubble parameter and thus they are equivalent to the comoving time scale factors. The answer that we get according to the (engineering) observability analysis is that we cannot determine the FRW zero modes and therefore the scale factor if the measurable output is the conformal Hubble parameter. However, knowing the latter ones or their derivatives leads to a rigorous dynamical observability in barotropic cosmology. We choose to present in the first sections of the paper the physical part of the problem and the direct mathematical application, whereas in the last section we collect the required mathematical results. We end up with a short conclusion section.

equations:

2 FRW barotropic cosmology

Barotropic cosmological zero modes are simple trigonometric and/or hyperbolic solutions of second order differential equations of the oscillator type to which the FRW system of equations can be reduced when it is passed to the conformal time variable. They have been discussed in a pedagogical way by Faraoni [1] and also were the subject of several recent papers [4]. In this section, we briefly review their mathematical scheme.

Barotropic FRW cosmologies in comoving time t obey the Einstein-Friedmann dynamical equations for the scale factor $a(t)$ of the universe supplemented by the (barotropic) equation of state of the cosmological fluid

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) , \\ H_0^2(t) &\equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{\kappa}{a^2} , \\ p &= (\gamma - 1)\rho , \end{aligned}$$

where ρ and p are the energy density and the pressure, respectively, of the perfect fluid of which a classical universe is usually assumed to be made of, $\kappa = 0, \pm 1$ is the curvature index of the flat, closed, open universe, respectively, and γ is the constant adiabatic index of the cosmological fluid.

2.1 FRW barotropic bosonic zero modes

Passing to the conformal time variable η , defined through $dt = a(\eta)d\eta$, one can combine the three equations in a single Riccati equation for the Hubble parameter $H_0(\eta)$ (we shall use either $\frac{d}{d\eta}$ or $'$ for the derivative with respect to η in the following)

$$H_0' + cH_0^2 + \kappa c = 0, \quad (1)$$

where $c = \frac{3}{2}\gamma - 1$.

Employing now $H_0(\eta) = \frac{1}{c} \frac{w'}{w}$ one gets the very simple (harmonic oscillator) second order differential equation

$$w'' - c \cdot c_{\kappa,b} w = 0, \quad (2)$$

where $c_{\kappa,b} = -\kappa c$. The solutions are the following:

$$\text{For } \kappa = 1 : \quad w_{1,b} \sim \cos(c\eta + \phi) \quad \rightarrow \quad w_{1,b} \sim [a_{1,b}(\eta)]^c,$$

where ϕ is an arbitrary phase.

$$\text{For } \kappa = -1 : \quad w_{-1,b} \sim \sinh(c\eta) \quad \rightarrow \quad w_{-1,b} \sim [a_{-1,b}(\eta)]^c.$$

parameters.

Moreover, the particular Riccati solutions particular solution, i.e., $H_0^+ = -\tan c\eta$ and $H_0^- = \coth c\eta$ for $\kappa = \pm 1$, respectively, are related to the common factorizations of the equation (2) FACTORIZATION

$$\left(\frac{d}{d\eta} + cH_0 \right) \left(\frac{d}{d\eta} - cH_0 \right) w = w'' - c(H_0' + cH_0^2)w = 0. \quad (3)$$

Borrowing a terminology from supersymmetric quantum mechanics, we call the solutions w as bosonic zero modes. As one can see, they are actually some powers of the scale factors of the barotropic universes.

2.2 FRW barotropic fermion zero modes

A class of barotropic FRW cosmologies with inverse scale factors with respect to the bosonic ones can be obtained by considering the supersymmetric partner

(or fermionic) equation of Eq. (3) which is obtained by applying the factorization brackets in reverse order

$$\left(\frac{d}{d\eta} - cH_0\right) \left(\frac{d}{d\eta} + cH_0\right) w = w'' - c(-H'_0 + cH_0^2)w = 0 . \quad (4)$$

Thus, one can write

$$w'' - c \cdot c_{\kappa,f} w = 0 , \quad (5)$$

where

$$c_{\kappa,f}(\eta) = -H'_0 + cH_0^2 = \begin{cases} c(1 + 2\tan^2 c\eta) & \text{if } \kappa = 1 \\ c(-1 + 2\coth^2 c\eta) & \text{if } \kappa = -1 \end{cases}$$

denotes the supersymmetric partner adiabatic index of fermionic type associated through the mathematical scheme to the constant bosonic index. Notice that the fermionic adiabatic index is time dependent. The fermionic w solutions are

$$w_{1,f} = \frac{c}{\cos(c\eta + \phi)} \quad \rightarrow \quad a_{1,f}(\eta) \sim [\cos(c\eta + \phi)]^{-1/c} ,$$

and

$$w_{-1,f} = \frac{c}{\sinh(c\eta)} \quad \rightarrow \quad a_{-1,f}(\eta) \sim [\sinh(c\eta)]^{-1/c} ,$$

for $\kappa = 1$ and $\kappa = -1$, respectively.

We can see that the bosonic and fermionic barotropic cosmologies are reciprocal to each other, in the sense that

$$a_{\pm,b} a_{\pm,f} = \text{const} .$$

Thus, bosonic expansion corresponds to fermionic contraction and viceversa. A matrix formulation of the previous results is possible employing two Pauli matrices [5]. The following spinor zero mode is obtained:

$$W = \begin{pmatrix} w_{\kappa,f} \\ w_{\kappa,b} \end{pmatrix} ,$$

showing that the two reciprocal barotropic zero modes enter on the same footing as the two components of the spinor W .

3 DSO Analysis of the FRW Barotropic Zero Modes

The DSO analysis is based on the concept of observability space \mathcal{O} , which is a smooth codistribution defined in an algorithmic way by Isidori [3]. All the

basic mathematical material related to this concept is presented in section 5 of this paper.

Frequently, the following form is used in controlled systems

$$\Gamma : \begin{cases} \dot{X} = \mathbf{f}(X) + \mathbf{g}(X)\mathbf{u} \\ y = h(X) \end{cases} \tag{6}$$

where $\mathbf{u} \in \Omega \subset R^1$, $X \in \mathcal{S} \subset R^2$ and \mathbf{f} and \mathbf{g} are \mathcal{C}^∞ functions. Note that in general $f(X) = \mathbf{f}(x) + \mathbf{g}(X)\mathbf{u}$. For the sake of simplicity we denote by Γ the set of equations 6.

In this section, we write the conformal time oscillator equations in the above dynamical systems form and introduce the corresponding nonlinear outputs for which we apply the algorithmic observability criterion of Isidori according to the basic concepts in section 5.

BOSONIC CASE

3.1 The Bosonic Case

The dynamical system for this case is based on Eq. (2)

$$\Gamma_b : \begin{cases} \dot{X}_1 = X_2 \\ \dot{X}_2 = -c(-\kappa \cdot c) X_1 \end{cases} \tag{7}$$

In matrix form

$$\dot{X} = f_j(X) , \tag{8}$$

where $X \in \mathbb{R}^2$, with $j = 1$ when $\kappa = 1$ and $j = 2$ when $\kappa = -1$.

Suppose that we take as outputs

$$h_1 = \frac{X_2}{X_1} \equiv \frac{\dot{X}_1}{X_1} \equiv c H_0(\eta) \quad h_2 = \frac{X_1}{X_2} \equiv 1/h_1 \tag{9}$$

or just

$$h_3 = X_1 \equiv w \quad h_4 = X_2 \equiv w' . \tag{10}$$

Then, we can write the codistribution

$$\mathcal{O}_{j,i} = [h_i(X), L_{f_j}h_i(X)] \tag{11}$$

$$\Xi_{j,i} = \mathbf{d} \mathcal{O}_{j,i} \tag{12}$$

This gives for $j = 1$:

$$\Xi_{b,1,1} = - \begin{bmatrix} \left(-\frac{X_2}{X_1}\right) \frac{1}{X_1} & \frac{1}{X_1} \\ \left(-\frac{X_2}{X_1}\right) \frac{2X_2}{X_1^2} & \frac{2X_2}{X_1^2} \end{bmatrix} , \quad \Xi_{b,1,2} = - \begin{bmatrix} \frac{1}{X_2} & \left(-\frac{X_1}{X_2}\right) \frac{1}{X_2} \\ \frac{2X_1 \cdot c^2}{X_2^2} & \left(-\frac{X_1}{X_2}\right) \frac{2X_1 \cdot c^2}{X_2^2} \end{bmatrix} . \tag{13}$$

Thus, for this case there exists a linear dependence in the columns of the above matrices, therefore the rank is less than the number of states. As consequence the outputs h_1 and h_2 make the systems unobservable.

For $j = 2$:

$$\Xi_{b,2,1} = - \begin{bmatrix} \left(-\frac{X_2}{X_1}\right) \frac{1}{X_1} & \frac{1}{X_1} \\ \left(-\frac{X_2}{X_1}\right) \frac{2X_2}{X_1^2} & \frac{2X_2}{X_1^2} \end{bmatrix}, \quad \Xi_{b,2,2} = \begin{bmatrix} \frac{1}{X_2} & \left(-\frac{X_1}{X_2}\right) \frac{1}{X_2} \\ \frac{2X_1 c^2}{X_2^2} & \left(-\frac{X_1}{X_2}\right) \frac{2X_1 c^2}{X_2^2} \end{bmatrix}. \quad (14)$$

Again, it is possible to notice the linear dependence in the columns of the above matrices, therefore the rank is less than the number of states. As consequence the outputs h_1 and h_2 make the systems unobservable.

However, for the outputs h_3 and h_4 , i.e., if one can know directly the zero modes or their derivatives, respectively, then

$$\Xi_{b,1,3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Xi_{b,1,4} = \begin{bmatrix} 0 & 1 \\ +c^2 & 0 \end{bmatrix}. \quad (15)$$

and

$$\Xi_{b,2,3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Xi_{b,2,4} = \begin{bmatrix} 0 & 1 \\ -c^2 & 0 \end{bmatrix}. \quad (16)$$

for $j = 1$ and $j = 2$, respectively.

Both pairs of matrices have linear independent columns in an obvious way and therefore the cosmological system is observable.

3.2 The Fermionic Case

We write the cosmological dynamical system derived from Eq. (5) in the form

$$\Gamma_f : \begin{cases} \frac{dX_1}{d\eta} = X_2 \\ \frac{dX_2}{d\eta} = -c F_j(\eta) X_1 \end{cases} \quad (17)$$

where we defined

$$F_1(\eta) = c (1 + 2 \tan^2 c\eta) \quad (18)$$

$$F_2(\eta) = c (-1 + 2 \coth^2 c\eta) \quad (19)$$

In matrix form

$$\frac{dX}{d\eta} = f_j(X), \quad (20)$$

where $X \in \mathbf{R}^2$. Using the same outputs as in the bosonic case and defining the codistribution in the same way we get

$$\Xi_{f,1,1} = - \begin{bmatrix} \left(-\frac{X_2}{X_1}\right) \frac{1}{X_1} & \frac{1}{X_1} \\ \left(-\frac{X_2}{X_1}\right) \frac{2X_2}{X_1^2} & \frac{2X_2}{X_1^2} \end{bmatrix}, \quad \Xi_{f,1,2} = c \begin{bmatrix} \frac{1}{X_2} & \left(-\frac{X_1}{X_2}\right) \frac{1}{X_2} \\ \frac{2X_1 F_1(\eta)}{X_2^2} & \left(-\frac{X_1}{X_2}\right) \frac{2X_1 F_1(\eta)}{X_2^2} \end{bmatrix}. \quad (21)$$

The linear dependence in the columns of the above matrices is manifest and therefore the rank is less than the number of states. We conclude that the outputs h_1 and h_2 make the system unobservable.

Using now $F_2(\eta)$, the matrices

$$\Xi_{f,2,1} = - \begin{bmatrix} \left(-\frac{X_2}{X_1}\right) \frac{1}{X_1} & \frac{1}{X_1} \\ \left(-\frac{X_2}{X_1}\right) \frac{2X_2}{X_1^2} & \frac{2X_2}{X_1^2} \end{bmatrix}, \quad \Xi_{f,2,2} = c \begin{bmatrix} \frac{1}{X_2} & \left(-\frac{X_1}{X_2}\right) \frac{1}{X_2} \\ \frac{2X_1 F_2(\eta)}{X_2^2} & \left(-\frac{X_1}{X_2}\right) \frac{2X_1 F_2(\eta)}{X_2^2} \end{bmatrix} \quad (22)$$

are linear dependent at the level of their columns, so the rank is less than the number of states and the outputs h_1 and h_2 make the cosmological systems unobservable.

However, again, if one can know directly the zero modes or their derivatives, then

$$\Xi_{f,1,3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Xi_{f,1,4} = \begin{bmatrix} 0 & 1 \\ -cF_1(\eta) & 0 \end{bmatrix} \quad (23)$$

and

$$\Xi_{f,2,3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Xi_{f,2,4} = \begin{bmatrix} 0 & 1 \\ -cF_2(\eta) & 0 \end{bmatrix}. \quad (24)$$

All these four matrices have linear independent columns and therefore the cosmological system is observable.

4 Nonlinear Observability: Definitions and Basic Theorem of Observability

The strictly mathematical material of this section refers to the dynamical system of equations Γ given in Eq. (6).

Definition 4.1 (Indistinguishability and Observability) [2]

Two states $X_a, X_b \in \mathcal{S}$ are said to be indistinguishable (denoted by $X_a I X_b$) if

(Γ, X_a) and (Γ, X_b) realize the same input-output map, i.e. for every admissible input $u(t) \in \mathbf{u}$ defined in $[t_o, t_1]$

$$\Gamma_{X_a}(u(t)) = \Gamma_{X_b}(u(t))$$

Let $I(X_o)$ be the set of points indistinguishable from X_o . The system Γ is said to be observable at X_o if $I(X_o) = \{X_o\}$ and is observable if $I(X) = \{X\}$ for every $X \in \mathcal{S}$

Since the observability is a global concept distinguishing between all the points in \mathcal{S} is a difficult goal. Therefore, we need a local concept and furthermore we merely need to distinguish X_o from its neighbors. For this purpose we define the concept of local weak observability.

Definition 4.2 (Local Weak Observability) [2]

We shall say that Γ is locally weakly observable at X_o if there exist an open neighborhood \mathbf{U} of X_o such that for every open neighborhood \mathbf{V} of X_o contained in \mathbf{U} , $I_{\mathbf{V}}(X_o) = \{X_o\}$ and is locally weakly observable if $I_{\mathbf{V}}(X) = \{X\}$ for every $X \in \mathcal{S}$.

The advantage of the *local weak observability* over the other concepts is that its proof is just necessary a simple algebraic test.

4

Definition 4.3 (Lie Derivative of a Real Valued Function) [3]

Consider a smooth vector field f and a function λ , defined in an open set $U \subset \mathbb{R}^n$. The Lie derivative of λ along f is the function $L_f \lambda : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by:

$$L_f \lambda(X) = \frac{\partial \lambda}{\partial X} f(X) = \sum_{i=1}^n \frac{\partial \lambda}{\partial X_i} f_i(X) .$$

Definition 4.4 (Observability Smooth Codistribution)

The observability smooth codistribution can be generated incrementally in the Isidori algorithmic way [3] as follows

$$\begin{aligned} \Omega_0 &= \text{span}\{h(X)\} \\ \Omega_k &= \Omega_{k-1} + \sum_{i=1}^q \text{span}\{L_{\tau_i} \Omega_{k-1}\} . \end{aligned}$$

This smooth codistribution is called the observability space \mathcal{O} . $L_{\tau_i} \Omega$ is defined in **Definition 4.3**.

Definition 4.5 (Differential of f) [3]

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function. The differential of f is defined by:

$$df = \frac{\partial f}{\partial X} = \left[\frac{\partial f}{\partial X_1} \quad \frac{\partial f}{\partial X_2} \quad \cdots \quad \frac{\partial f}{\partial X_n} \right].$$

Definition 4.6 (Observability Rank Condition)

Γ is said to satisfy the observability rank condition at X_o if the dimension of $d\mathcal{O}(X_o)$ is n where n is the dimension of the system and the differential d is given by **Definition 4.5**. Moreover, $d\mathcal{O}(X)$ satisfies the observability rank condition if this is true for any $X \in \mathcal{S}$.

Theorem 4.7 (Basic Theorem of Observability)

If Γ satisfies the observability rank condition in **Definition 4.6** at X_o then Γ satisfies the observability rank condition for any $X \in \mathcal{S}$ if the dimension of $d\mathcal{O}(X)$ is complete.

The complete proof of this theorem is given in [2].

5 Conclusion

In this paper we have treated the FRW barotropic zero modes as dynamical state(s) of the universe for which we applied the rigorous techniques of non-linear observability as defined in engineering by Hermann and Krener, and further discussed by Isidori in his textbook. We have shown that these zero modes remain undetermined if only the Hubble parameter is known, which in general is an expected result for cosmologists since it appears that the Hubble parameter provides only partial though important cosmological information. In other words, the usual type of cosmological measurements relying on the Hubble parameter are not the ones providing the complete dynamical information about the universe. On the other hand, we have obtained the interesting result that having the chance to know directly the FRW zero modes gives us the opportunity to know dynamically the whole system. This, indirectly, offers the possibility that by fitting one can infer the initial conditions of the FRW universe.

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