Exponential Stability of Integral Delay Systems with a Class of Analytic Kernels

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***Abstract*— The exponential stability of a class of integral delay systems with analytic kernels is investigated by using the Lyapunov-Krasovskii functional approach. Sufficient delay-dependent stability conditions and exponential estimates for the solutions are derived. Special attention is paid to the particular cases of polynomial and exponential kernels.**

***Index Terms*— Integral delay system, exponential stability, Lyapunov-Krasovskii functionals.**

I. Introduction

Integral delay systems play an important role in several stability problems of time-delay systems. This class of systems is found as delay approximations of the partial differential equations for describing the propagation phenomena in excitable media [18], in the stability analysis of additional dynamics introduced by some system transformations [3], [5], [6], [7], in the internal stability problem of controllers used for finite spectrum assignment of time-delay systems [9], as well as in the stability analysis of some difference operators in neutral type functional differential equations [4], [8].

Lyapunov-Krasovskii theorems for integral delay systems have been recently introduced in [12]. It has been shown there that new type of Lyapunov-Krasovskii conditions are required in order to properly address the dynamics of this class of systems. General expressions of quadratic functionals with a given time derivative are provided. The functionals are shown to be useful in solutions of such problems as the estimations of robustness bounds and calculations of exponential estimates for the solutions of exponentially stable integral delay systems.

However, there are still some technical difficulties, associated with the positivity check of such functionals, limiting their practical application to the stability analysis of integral delay systems. This motivated the work [13] where it was shown that, based on the general expressions of functionals presented in [12], various reduced type functionals can be constructed to obtain stability conditions formulated directly in terms of the coefficients of some classes of integral delay systems.

In the present paper, we address the exponential stability of a special class of integral delay systems with analytic kernels. For such integral systems, we derive sufficient delay-dependent stability conditions, expressed in terms of linear matrix inequalities, by using the Lyapunov-Krasovskii functional approach.

This work was supported by CONACYT under Grants 131587 and 61076.

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The special class of integral delay systems under consideration includes those with polynomial and exponential kernels, which found important applications in the internal stability problem of controllers used for the finite spectrum assignment of input time-delay systems, a topic that has received a sustained attention in the past few years, see for instance [2], [14], [15], [16], [21], [22], and that was proposed as an interesting open problem in the survey paper [20]. We here address such a problem from a Lyapunov-Krasovskii framework and develop a linear matrix inequalities formulation for it. More explicitly, a design method guaranteeing a numerically safe implementation of the controllers and a prescribed decay rate of the closed-loop system solutions is proposed. We believe that such an approach to the problem is completely new.

The paper is structured as follows: In section II, preliminaries concerning the class of integral delay systems to be considered, basic facts about the solutions and Lyapunov-Krasovskii stability conditions are introduced. The main results are given in section III. First, we consider integral delay systems with a general class of analytic kernels. Then, the special cases of integral delay systems with polynomial and exponential kernels are addressed. In all cases, exponential estimates and delay-dependent conditions for the exponential stability are expressed in terms of linear matrix inequalities. In section IV, we apply the obtained results to the internal stability problem of controllers used for the finite spectrum assignment of time-delay systems. Several concluding remarks end the paper.

Notation: Throughout the paper we will use the Euclidean norm for vectors and the induced norm for matrices, both denoted by $\|\cdot\|$. We denote by A^T the transpose of A, I_n and 0_n stand for the identity and zero matrices of $n\times n$ dimension, while $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalues of a symmetric matrix A, respectively.

II. Preliminaries

In this contribution we focus on the exponential stability of the class of integral delay systems with analytic kernels described by

$$x(t)=\int_{-h}^{0}G^T\mathcal{B}(\theta)x(t+\theta)d\theta, \forall t\geq 0, \tag{1}$$

where

$$G^T=\left[\begin{array}{cccc} G_0^T & G_1^T & \cdots & G_N^T \end{array}\right]$$

and

$$\mathcal{B}(\theta)^T=\left[\begin{array}{cccc} \mathcal{B}_0(\theta)^T & \mathcal{B}_1(\theta)^T & \cdots & \mathcal{B}_N(\theta)^T \end{array}\right],$$

with $G \in \mathbb{R}^{n(N+1)\times m}$ and $\mathcal{B}(\theta) \in \mathbb{R}^{n(N+1)\times m}$ a continuously differentiable matrix function defined for $\theta \in [-h, 0]$ which satisfies the following assumptions:

- There exists a matrix $M \in \mathbb{R}^{n(N+1)\times n(N+1)}$ such that
$$\dot{\mathcal{B}}(\theta) = M\mathcal{B}(\theta), \forall\theta \in [-h, 0]. \tag{2}$$
- There exists $\gamma > 0$ such that for all $\theta \in [-h, 0]$
$$\gamma < \lambda_{\min}\left\{\mathcal{B}^T(\theta)\mathcal{B}(\theta)\right\}. \tag{3}$$

Remark 1: Classes of kernels that satisfy the assumptions (2) and (3) include those written in a polynomial basis as well as transcendental kernels such as exponential ones.

Remark 2: It is important to point out that the special class of functional difference equations (1), with analytic kernels satisfying (2) and (3), belongs to the class of retarded type delay system, see [12] for discussions and the section below concerning solutions and stability concept. For the case of functional difference equations of the neutral type, the approach developed in [12] cannot be directly applied and Lyapunov stability conditions as those proposed in [19] are more appropriate.

A. Solutions and stability concept

In order to define a particular solution of (1) an initial vector function $\varphi(\theta)$, $\theta \in [-h, 0)$ should be given. We assume that φ belongs to the space of piecewise continuous bounded functions $\mathcal{PC}([-h, 0), \mathbb{R}^m)$, equipped with the uniform norm $\|\varphi\|_h = \sup_{\theta\in[-h,0)} \|\varphi(\theta)\|$.

Given any initial function $\varphi \in \mathcal{PC}([-h, 0), \mathbb{R}^m)$, there exists a unique solution $x(t, \varphi)$ of (1) which is defined for all $t \in [-h, \infty)$ with $x(t, \varphi) = \varphi(t), t \in [-h, 0)$. This solution is continuous and differentiable for all $t \in (0, \infty)$, and it suffers a jump discontinuity given by

$$\begin{aligned}\Delta x(0, \varphi) &= x(0, \varphi) - x(-0, \varphi)\\ &= \int_{-h}^{0} G^T\mathcal{B}(\theta)\varphi(\theta)d\theta - \varphi(-0).\end{aligned}$$

Definition 3: [4] System (1) is said to be *exponentially stable* if there exist $\mu > 0$ and $\alpha > 0$ such that every solution of (1) satisfies the inequality

$$\|x(t, \varphi)\| \le \mu \|\varphi\|_h e^{-\alpha t}, \quad \forall t \ge 0.$$

Remark 4: Notice that differentiation of the integral equation (1) is not an option for investigating its exponential stability since the resulting differential system with distributed delay

$$\begin{aligned}\dot{x}(t) &= G^T\mathcal{B}(0)x(t) - G^T\mathcal{B}(-h)x(t-h)\\ &\quad - \int_{-h}^{0} G^T\dot{\mathcal{B}}(\theta)x(t+\theta)d\theta,\end{aligned}$$

is not exponentially stable as it admits any constant vector as a solution, see [12] for a detailed analysis.

For any $t \ge 0$ we denote the restriction of the solution $x(t, \varphi)$ on the interval $[t-h, t)$ by $x_t(\varphi) = \{x(t+\theta, \varphi), \theta \in [-h, 0)\}$. When the initial function is irrelevant, we simply write $x(t)$ and x_t instead of $x(t, \varphi)$ and $x_t(\varphi)$.

B. Lyapunov stability conditions

The observation in Remark 4 motivated the authors of the work [12] to derive new Lyapunov stability conditions for integral delay systems of the form in (1). In particular, it was there shown that integral type quadratic functions are the natural lower bounds for the Lyapunov functionals in contrast with quadratic lower bounds of Lyapunov functionals for differential delay systems.

We here present a new formulation of the general result on Lyapunov type conditions for the exponential stability of integral delay systems given in [12]. This formulation, in the spirit of the one introduced in [17] for retarded differential delay systems, enhances the role of the exponential decay rate.

Following [12], by noting that for $t \in [0, h)$, $x_t(\varphi)$ belongs to $\mathcal{PC}([-h, 0), \mathbb{R}^m)$, and that for $t \ge h$, $x_t(\varphi)$ belongs to the space of continuous vector functions $\mathcal{C}([-h, 0), \mathbb{R}^m)$, it then follows that, in a Lyapunov-Krasovskii functional framework, the functionals should be defined on the infinite-dimensional space $\mathcal{PC}([-h, 0), \mathbb{R}^m)$.

Theorem 5: [12] Let there exists a functional $v : \mathcal{PC}([-h, 0), \mathbb{R}^m) \to \mathbb{R}$ such that

$$\alpha_1 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta \le v(\varphi) \le \alpha_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta, \tag{4}$$

for some $0 < \alpha_1 \le \alpha_2$. Given $\beta > 0$, if along solutions of (1) the inequality

$$\frac{d}{dt}v(x_t(\varphi)) + 2\beta v(x_t(\varphi)) \le 0, \quad \forall t \ge 0, \tag{5}$$

holds, then

$$\|x(t, \varphi)\| \le \mu \|\varphi\|_h e^{-\beta t}, \quad \forall t \ge 0, \tag{6}$$

where

$$\mu = \left(\max_{\theta\in[-h,0]} \left\|G^T\mathcal{B}(\theta)\right\|\right) h\sqrt{\frac{\alpha_2}{\alpha_1}}. \tag{7}$$

Based on this fundamental result, the following sufficient stability condition is obtained in [13] by using a particular Lyapunov functional satisfying the Theorem conditions.

Lemma 6: [13] System (1) is exponentially stable if

$$\left(\max_{\theta\in[-h,0]} \left\|G^T\mathcal{B}(\theta)\right\|\right) h < 1. \tag{8}$$

This inequality is valid for integral delay systems with more general kernels than those satisfying the assumptions (2) and (3). Indeed, the only assumption needed on the matrix function $\mathcal{B}(\theta)$ is continuity on the interval $[-h, 0]$. However, inequality (8) may be very conservative for kernels satisfying the assumptions (2) and (3) as it has been illustrated in [13] for the case of constant kernels. It is worth mentioning that (8) can also be obtained via frequency-domain techniques, see [10] and [11].

III. Main results

In this section, sufficient delay-dependent conditions for the exponential stability of integral delay systems of the form of (1), with kernels satisfying (2) and (3), are obtained by presenting a particular functional satisfying the conditions of Theorem 5.

Consider the functional candidate

$$v(\varphi)=\int_{-h}^{0}\varphi^T(\theta)\mathcal{B}^T(\theta)e^{2\beta\theta}\left[P+(\theta+h)\,Q\right]\mathcal{B}(\theta)\varphi(\theta)d\theta, \tag{9}$$

where $P,Q\in\mathbb{R}^{n(N+1)\times n(N+1)}$ are positive definite matrices and β is a positive constant.

From (9) it follows that

$$\begin{aligned} v(\varphi) \;&\le\; \lambda_{\max}(P+hQ)\int_{-h}^{0}\varphi^T(\theta)\mathcal{B}^T(\theta)\mathcal{B}(\theta)\varphi(\theta)d\theta \\ &\le\; \lambda_{\max}(P+hQ)\int_{-h}^{0}\lambda_{\max}(\mathcal{B}^T(\theta)\mathcal{B}(\theta))\left\|\varphi(\theta)\right\|^2 d\theta, \end{aligned}$$

and

$$\begin{aligned} v(\varphi) \;&\ge\; e^{-2\beta h}\lambda_{\min}(P)\int_{-h}^{0}\varphi^T(\theta)\mathcal{B}^T(\theta)\mathcal{B}(\theta)\varphi(\theta)d\theta \\ &\ge\; e^{-2\beta h}\lambda_{\min}(P)\int_{-h}^{0}\lambda_{\min}(\mathcal{B}^T(\theta)\mathcal{B}(\theta))\left\|\varphi(\theta)\right\|^2 d\theta, \end{aligned}$$

hence the functional (9) satisfies the following inequalities:

$$\alpha_1\int_{-h}^{0}\left\|\varphi(\theta)\right\|^2 d\theta\le v(\varphi)\le\alpha_2\int_{-h}^{0}\left\|\varphi(\theta)\right\|^2 d\theta,$$

with

$$\alpha_1 \;=\; e^{-2\beta h}\lambda_{\min}\left(P\right)\left[\min_{\theta\in[-h,0]}\lambda_{\min}(\mathcal{B}^T(\theta)\mathcal{B}(\theta))\right], \tag{10}$$

$$\alpha_2 \;=\; \lambda_{\max}\left(P+hQ\right)\left[\max_{\theta\in[-h,0]}\lambda_{\max}(\mathcal{B}^T(\theta)\mathcal{B}(\theta))\right]. \tag{11}$$

The time derivative of functional (9) along the trajectories of system (1) is such that

$$\begin{aligned} &\frac{d}{dt}v(x_t)+2\beta v(x_t)=\left(\int_{-h}^{0}G^T\mathcal{B}(\theta)x(t+\theta)d\theta\right)^T \\ &\times\mathcal{B}^T(0)\left[P+hQ\right]\mathcal{B}(0)\left(\int_{-h}^{0}G^T\mathcal{B}(\theta)x(t+\theta)d\theta\right) \\ &-\int_{-h}^{0}x^T(t+\theta)\mathcal{B}^T(\theta)e^{2\beta\theta}\left\{Q+\left[P+(\theta+h)\,Q\right]M\right) \\ &+M^T\left[P+(\theta+h)\,Q\right]\}\mathcal{B}(\theta)x(t+\theta)d\theta \\ &-x^T(t-h)e^{-2\beta h}\mathcal{B}^T(-h)P\mathcal{B}(-h)x(t-h). \end{aligned}$$

By using the Jensen integral inequality we have

$$\begin{aligned} &\left(\int_{-h}^{0}G^T\mathcal{B}(\theta)x(t+\theta)d\theta\right)^T\mathcal{B}^T(0)\left[P+hQ\right] \\ &\qquad\times\mathcal{B}(0)\left(\int_{-h}^{0}G^T\mathcal{B}(\theta)x(t+\theta)d\theta\right) \\ &\le h\int_{-h}^{0}\left\{x^T(t+\theta)\mathcal{B}(\theta)^TG\mathcal{B}^T(0)\left[P+hQ\right]\right. \\ &\qquad\left.\times\mathcal{B}(0)G^T\mathcal{B}(\theta)x(t+\theta)\right\}d\theta. \end{aligned}$$

We therefore get the following inequality for the derivative:

$$\begin{aligned} &\frac{d}{dt}v(x_t)+2\beta v(x_t) \\ \le\;& -\int_{-h}^{0}e^{2\beta\theta}x^T(t+\theta)\mathcal{B}^T(\theta)\Gamma(\theta)\mathcal{B}(\theta)x(t+\theta)d\theta, \end{aligned}$$

where

$$\begin{aligned} \Gamma(\theta) \;=\;& Q+M^T\left[P+(\theta+h)\,Q\right]+\left[P+(\theta+h)\,Q\right]M \\ & -he^{-2\beta\theta}G\mathcal{B}^T(0)\left[P+hQ\right]\mathcal{B}(0)G^T. \end{aligned}$$

Thus, if $\Gamma(\theta)>0,\forall\theta\in[-h,0]$, then the exponential stability of (1) follows. Observing that

$$\begin{aligned} &\frac{-\theta}{h}\Gamma(-h)+\frac{(\theta+h)}{h}\Gamma(0) \\ &=Q+M^T\left[P+(\theta+h)\,Q\right]+\left[P+(\theta+h)\,Q\right]M \\ &-\left[\frac{-\theta}{h}e^{2\beta h}+\frac{(\theta+h)}{h}\right]hG\mathcal{B}^T(0)\left[P+hQ\right]\mathcal{B}(0)G^T, \end{aligned}$$

and taking into account the convexity of the exponential function

$$\frac{-\theta}{h}e^{2\beta h}+\frac{(\theta+h)}{h}\ge e^{-2\beta\theta},\forall\theta\in[-h,0],$$

one has that $\Gamma(\theta)>0,\forall\theta\in[-h,0]$, if and only if $\Gamma(0)>0$ and $\Gamma(-h)>0$.

Summarizing the above, we can state the following result.

Theorem 7: An integral delay system (1), with a kernel satisfying (2) and (3), is exponentially stable with a decay rate β if there exist positive definite matrices $P,Q\in\mathbb{R}^{n(N+1)\times n(N+1)}$ such that

$$\begin{aligned} &Q+M^T\left[P+hQ\right]+\left[P+hQ\right]M \\ &\qquad -hG\mathcal{B}^T(0)\left[P+hQ\right]\mathcal{B}(0)G^T>0, \end{aligned} \tag{12}$$

and

$$Q+M^TP+PM-he^{2\beta h}G\mathcal{B}^T(0)\left[P+hQ\right]\mathcal{B}(0)G^T>0, \tag{13}$$

Furthermore, for any initial condition $\varphi\in\mathcal{PC}([-h,0),\mathbb{R}^m)$, the solution $x(t,\varphi)$ of (1) satisfies the exponential upper bound (6), where μ,α_1 and α_2 are determined by (7), (10) and (11), respectively.

A. Integral delay systems with polynomial and exponential kernels

For $n=m$ and matrix functions $\mathcal{B}(\theta)$ of the form

$$\mathcal{B}(\theta)^T=\left[\begin{array}{cccc} I_n & I_n\theta & \cdots & I_n\theta^N \end{array}\right], \tag{14}$$

we get the following result.

Corollary 8: System (1) with a matrix function $\mathcal{B}(\theta)$ of the form (14) is exponentially stable with a decay rate β if there exists positive definite matrices $P,Q\in\mathbb{R}^{n(N+1)\times n(N+1)}$ such that (12) and (13) holds, where the matrix M is given by

$$M=\left[\begin{array}{ccccc} 0_n & 0_n & \cdots & \cdots & 0_n \\ I_n & 0_n & \cdots & \cdots & 0_n \\ 0_n & 2I_n & \ddots & & 0_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_n & \cdots & 0_n & NI_n & 0_n \end{array}\right].$$

Note that in this particular case of polynomial kernels the positive constants α_1 and α_2, respectively defined by (10) and (11), take the form

$$\begin{aligned} \alpha_1 \;&=\; e^{-2\beta h}\lambda_{\min}(P), \\ \alpha_2 \;&=\; (1+h^2+h^4+...+h^{2N})\lambda_{\max}(P+hQ). \end{aligned}$$

TABLE I: Maximum delay value for different decays β

β	0	1	2	3
Corollary 9	1.99	0.7	0.426	0.349
Norm condition (8)	0.988	–	–	–

For $m = n$ and kernels of the form $G^T\mathcal{B}(\theta) = G_0^T e^{A\theta}$, the following result is obtained.

Corollary 9: System (1) with a matrix function $\mathcal{B}(\theta)$ of the form (14) is exponentially stable if there exist positive definite matrices $P, Q \in \mathbb{R}^{n\times n}$ such that (12) and (13) hold with $M = A$.

Remark 10: In the case of integral delay systems with constant kernels $G^T\mathcal{B}(\theta) = G_0^T$, sufficient conditions for the exponential stability can be derived either from Corollary 8 ($N = 0$) or Corollary 9 (A is the null matrix). In this case, the conditions of Theorem 7 reduce to

$$Q - he^{2\beta h} G_0 \left[P + hQ\right] G_0^T > 0.$$

For $\beta = 0$, these conditions are the same as those obtained in [13].

Now we illustrate the main results by some examples.

Example 11: Consider the integral delay system

$$x(t) = \int_{-h}^{0} e^{A\theta} x(t+\theta) d\theta, \tag{15}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}.$$

For $\beta = 0$, by using Corollary 9, we found a feasible solution of the matrix inequalities (12)-(13) for delay values $0 \le h \le 1.99$. For instance, for $h = 1.99$ we obtain the following numerical values for matrices P and Q:

$$P = \begin{pmatrix} 16.5037 & -9.3259 \\ -9.3259 & 5.7783 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0.5736 & -0.5738 \\ -0.5738 & 0.5779 \end{pmatrix}.$$

In Table I, we present the maximum delay value computed by using Corollary 9, for different β. The results are less conservative than the norm condition (8) which, in the case of exponential kernels, provides always maximum delay values less than one.

Example 12: Consider now the system (15) with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Because of the particular form of matrix A, the critical delay value can be computed analytically by using frequency-domain techniques. Simple calculations based on the characteristic function

$$f(s) = \left(\frac{s + e^{-hs} - 1}{s}\right)^2$$

show that (15) is exponentially stable for all delay values $h < 1$. For this particular matrix A, $e^{A\theta} = I_2 + A\theta$, this integral system can also be viewed as one having a polynomial kernel $G^T\mathcal{B}(\theta)$ with

$$G^T = \begin{bmatrix} I_2 & A \end{bmatrix} \text{ and } \mathcal{B}(\theta)^T = \begin{bmatrix} I_2 & \theta I_2 \end{bmatrix}^T.$$

TABLE II: Maximum delay value for different decays β

β	0	1	2	3
Corollary 8	0.999	0.628	0.471	0.384
Corollary 9	0.999	0.567	0.426	0.349
Norm condition (8)	0.7070	–	–	–

For different β, by applying Corollaries 8 and 9, we obtain the maximum delay value given in Table II. For $\beta = 0$, Corollaries 8 and 9 both give $h_{\max} = 0.999$ which coincides with the exact critical delay value and improves the maximum $h_{\max} = 0.7070$ obtained from the norm condition (8). We also see that in this case for $\beta > 0$, Corollary 8 provides less conservative results than Corollary 9.

In order to illustrate the computation of the μ-factor involved in the exponential upper bound (6), let us consider, in this numerical example, that $\beta = 1$ and $h = 0.5$. Direct calculations derived from (7), (10) and (11) yield $\mu = 8.7696$, $\alpha_1 = 0.0129$ and $\alpha_2 = 2.4164$, respectively.

Then, it follows that every solution $x(t,\varphi)$ of (15), $\varphi \in \mathcal{PC}([-0.5, 0), \mathbb{R}^2)$, admits the exponential upper bound

$$\|x(t,\varphi)\| \le 8.7696 \,\|\varphi\|_{0.5}\, e^{-t}, \forall t \ge 0.$$

IV. Application to the Finite Spectrum Assignment problem

In this section, we apply the obtained results in developing a design method for tackling the numerical implementation problem of control laws with distributed delay which assign a finite spectrum to input delay systems.

This problem concerns systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t-h), \tag{16}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $h > 0$ is the input delay.

The predictor like control law

$$u(t) = K\left(e^{Ah}x(t) + \int_{-h}^{0} e^{-A\theta} Bu(t+\theta)d\theta\right) \tag{17}$$

assigns a finite spectrum to the delay free closed-loop system (16)-(17) which coincides with the spectrum of the matrix $A + BK$, see [9].

The practical value of such a result is limited by the instability phenomenon linked to the numerical implementation of the integral term in (17). It has been shown in [2], [15], [21] that if the integral is approximated by a finite sum then the resulting closed-loop delay system may become unstable if the ideal controller is not internally stable.

A definite result has been demonstrated in [14]: a necessary and sufficient condition for a numerically safe implementation (by using any type of numerical integration rule) of the controller (17) is both the stability of the ideal closed-loop system (16)-(17), i.e., $A+BK$ is Hurwitz, and the stability of the internal dynamics of the controller (17), i.e., the stability of the integral delay system

$$z(t) = \int_{-h}^{0} Ke^{-A\theta} Bz(t+\theta)d\theta. \tag{18}$$

Based on this observation, by combining known results on matrix inequalities for guaranteeing that all eigenvalues of the matrix $A+BK$ have real parts less than a certain prescribed $-\sigma$ [1] with Theorem 7 applied to the integral delay system (18), with $G^T = K, B(\theta) = e^{-A\theta}B$ and $M = -A$, we easily arrive at the following result.

Proposition 13: The input delay system (16), where without loss of generality we can assume that the matrix B is full column rank, admits a numerically safe implementation of the control law (17) with a guaranteed decay rate σ for the closed-loop system solutions, if there exists a matrix $K \in \mathbb{R}^{m\times n}$ and positive definite matrices $R, P, Q \in \mathbb{R}^{n\times n}$ such that

$$(A+BK+\sigma I)^T R + R(A+BK+\sigma I) < 0, \tag{19}$$

$$Q - A^T[P+hQ] - [P+hQ]A - hK^TB^T[P+hQ]BK > 0, \tag{20}$$

$$Q - A^TP - PA - he^{2\sigma h}K^TB^T[P+hQ]BK > 0. \tag{21}$$

The controller synthesis conditions of Proposition 13 involve nonlinear terms. Of course, condition (19) can be expressed as a linear matrix inequality: pre- and post multiplying inequality (19) by the matrix $S = R^{-1}$ and then setting $\bar{K} = KS$ yields

$$SA^T + AS + \bar{K}^TB^T + B\bar{K} + 2\sigma S < 0, \tag{22}$$

which is linear in the variables S and $\bar{K}$.

As a consequence, an easy way to solve the synthesis problem is first to find a feasible set $(S, \bar{K})$ satisfying inequality (22). Then set $K = \bar{K}S^{-1}$ and solve the linear matrix inequalities (20) and (21) for the variables P and Q. However, with additional computational effort, the synthesis conditions can be expressed as a set of linear matrix inequalities to be solved simultaneously and for which better results can be obtained as presented next.

First, by using the elimination procedure along with Finsler's Lemma, the inequality (22) can be expressed as

$$SA^T + AS + 2\sigma S - \alpha BB^T < 0, \tag{23}$$

for some scalar α, see [1], and if there exist positive definite matrix S and a scalar α such that (23) holds, then a feedback gain is given by

$$K = -\frac{\alpha}{2}B^TS^{-1}. \tag{24}$$

Because of the homogeneity of (23) in S and α, we can assume that α is positive and select $\alpha = 2$. Substituting (24) into inequalities (20) and (21) one obtains

$$Q - A^T[P+hQ] - [P+hQ]A - hS^{-1}BB^T[P+hQ]BB^TS^{-1} > 0.$$

$$Q - A^TP - PA - he^{2\sigma h}S^{-1}BB^T[P+hQ]BB^TS^{-1} > 0.$$

Then, by introducing a new variable Y and a positive scalar λ such that $Y > S^{-1}$ and $(P+hQ)^{-1} > \lambda I$, respectively, the

TABLE III: Maximum delay value, feedback gain and ideal closed-loop eigenvalues.

σ	0	0.5	1	
$h_{\max}$	∞	0.781	0.366	
K	$(-0.0005, -0.0267)$	$(-0.5004, -1.004)$	$(-2.0015, -2.009)$	$(-8.0$
$s_{1,2}$	$-0.0134 \pm 0.01762i$	$-0.5002 \pm 0.5002i$	$-1.0004 \pm 1.0003i$	-2.00

above two inequalities can be replaced by

$$\begin{bmatrix} Q - A^T[P+hQ] - [P+hQ]A & YBB^T \\ BB^TY & \frac{\lambda}{h}I \end{bmatrix} > 0, \tag{25}$$

$$\begin{bmatrix} Q - A^TP - PA & YBB^T \\ BB^TY & \frac{\lambda}{h}e^{-2\sigma h}I \end{bmatrix} > 0, \tag{26}$$

$$\begin{bmatrix} Y & I \\ I & S \end{bmatrix} > 0, \tag{27}$$

$$\lambda(P+hQ) < I, \tag{28}$$

where Schur complement has been applied to obtain the first three inequalities while the last one directly follows by noting that $(P+hQ)^{-1} > \lambda I$ is equivalent to $\lambda(P+hQ) < I$.

Summarizing the above, if there exist positive definite matrices P, Q, S, Y and a positive scalar λ such that the inequalities (23), (25)-(28) hold, then the feedback gain $K = -B^TS^{-1}$ achieves a successful numerical implementation of the controller (17), with guaranteed decay rate σ for the closed-loop system solutions.

We use this linear inequality formulation in the following illustrative examples.

Example 14: Let us consider the input delay system (16) with system matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The corresponding linear system represents a double integrator with delayed input, which is commonly found in mechanical systems.

The maximum delay value $h_{\max}$ and the corresponding feedback gain K, computed by solving (23), (25)-(28) for different decays σ, and the ideal closed-loop eigenvalues $s_{1,2}$ are displayed in Table 3. We see that for $\sigma = 0$, a numerical safe implementation of the controller (17) can be achieved for any h satisfying $0 \le h < \infty$. In this particular case, in order to illustrate that a feasible solution can be found for an arbitrarily large delay, the feedback gain K and the corresponding ideal closed-loop eigenvalues are computed for a delay value $h = 10$ in Table III.

The example that follows represents the linearization of a fluid-flow model for describing the behavior of some classes of TCP/AQM networks, see [11] and the references therein for details. For such a system, the application of the controller (17) has been recently proposed in [11]. It was shown there that for any given set of network parameters there always exists a feedback gain such that a numerically safe implementation of the controller (17) can be achieved. Such a result was proved by using the sufficient condition (8) which, unlike the present approach, does not provide an estimate of the decay rate for the closed-loop system solutions.

TABLE IV: Feedback gain and ideal closed-loop eigenvalues for different σ

σ	0	0.5	0.848
K	$(0.0147, 0.0001)$	$(0.0328, 0.0003)$	$(0.0585, 0.0009)$
$s_{1,2}$	$-0.6479, -0.1856$	$-0.5949 \pm 0.03945i$	$-0.848 \pm 0.5076i$

Example 15: The linearized TCP/AQM network model of the form (16) has system matrices

$$A = \begin{pmatrix} -\frac{2n}{\tau^2 c} & 0 \\ \frac{n}{\tau c} & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} -\frac{\tau c^2}{2n^2} \\ 0 \end{pmatrix}.$$

where n represents the number of TCP flows, c is the link capacity and τ denotes the round-trip time (delay).

We here apply the results of this section in computing the feedback gain for guaranteeing a numerical implementation of the controller along with a certain prescribed decay rate σ. For network parameters $n = 40$, $\tau = 0.7$ and $c = 300$, the inequalities (23), (25)-(28) are feasible for a maximum decay rate $\sigma_{\max} = 0.848$. The computed feedback gain and the corresponding ideal closed-loop eigenvalues for different $\sigma \leq \sigma_{\max}$ are displayed in Table IV.

V. Concluding Remarks

In this paper, delay-dependent sufficient conditions for the exponential stability of some classes of integral delay systems with analytic kernels, which found important applications in several stability problems of time-delay systems, are given. The results are obtained by using the Lyapunov-Krasovskii functional approach recently developed in [12] and [13] for integral delay systems. An application to the finite spectrum assignment problem of input delay systems is also presented. As a result, a novel linear matrix inequality formulation for guaranteeing a numerically safe implementation of the controllers and a prescribed decay rate for the closed-loop system solutions is proposed.

VI. Acknowledgment

The authors wish to thank the anonymous Reviewers and Associate Editor P. Pepe for their careful reading and valuable suggestions that helped us to improve the work.

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