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## Computing non-fragile PI controllers for delay models of TCP/AQM networks

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This paper focuses on the stabilization problem of fluid-flow delay models of Transmission Control Protocol/Active Queue Management (TCP/AQM) networks by using a proportional-integral (PI) controller as AQM strategy. **More precisely**, the complete set of PI controllers that exponentially stabilizes the corresponding linear time-delay system is derived. Using the particular geometric properties of this set of the controller parameters, **the issues of robustness** to uncertainty in the network parameters and to perturbation in the controller coefficients are addressed. Then, a methodology to compute a non-fragile PI AQM controller is provided. Finally, exponential estimates for the closed-loop system solutions, **allowing to evaluate** the performance of the corresponding PI-controlled closed-loop system, are **proposed** by using a Lyapunov-Krasovskii functional approach. An illustrative example completes the presentation.

### 1 Introduction

In computer networks, the AQM mechanisms are employed by the routers to assist the TCP management for congestion avoidance. Recently, based on the fluid-flow delay model representation introduced by Hollot *et al.* (2002) for describing the behaviour of TCP in computer networks, several control structures have been proposed as AQM strategies. Thus, for example, P, PI and  $\mathcal{H}^\infty$  controllers have been proposed by Hollot *et al.* (2002) and Quet and Özbay (2004). Furthermore, it was shown **in these studies** that such controllers improve the performance obtained with standard AQM schemes (e.g. based on Random Early Detection (RED)).

Due to their simplicity, the P and PI controllers proposed by Hollot *et al.* (2002) have become a reference for the development of new AQM controllers as they are currently implemented in the Network Simulator (*ns-2*). However, the design of such controllers is based only on *sufficient conditions* for guaranteeing the closed-loop stability of the linearization and, therefore, they do *not provide* the set of *all stabilizing P and PI gain values*.

More recently, in Michiels *et al.* (2006), the complete set of P controllers stabilizing the linearized system of a simplified version of the model was derived. Despite this result and to the best of the authors' knowledge, there are no specific results for the problem of finding ***all stabilizing PI controllers*** for the linearized model introduced by Hollot *et al.* (2002). This fact motivated us for searching a complete solution **to this** problem. More precisely, for a given set of network parameters (round-trip time, number of TCP loads and link capacity), we propose a *complete characterization* of the set of all PI controllers that exponentially stabilize the linearized model. Some preliminary results in this direction were announced in Melchor-Aguilar (2007) and Melchor-Aguilar and Castillo-Torres (2007) for simplified versions of the model.

It is worth to mention that one of the most important AQM objectives is the *robustness* with respect to *network parameters uncertainty*. This problem has been addressed by Hollot *et al.* (2002) and Quet

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and Özbay (2004), where some sufficient conditions for robust stabilization have been provided. On the other hand, the implementation of AQM schemes is subject to round-off errors and finite word length. Furthermore, it could be necessary to make an appropriate tuning of the controller's coefficients around a nominal design to obtain a desired closed-loop performance. Hence, the practical implementation of such AQM controllers demands to be robust not only to network parameters uncertainty but also to perturbations of the controller coefficients. A controller whose closed-loop system is destabilized by small perturbations in the controller coefficients is said to be *fragile*, see, e.g. Keel and Bhattacharyya (1997). As it has been widely discussed in Keel and Bhattacharyya (1997) and Mäkilä (1998), in general, it is quite difficult to analyze the robustness properties of the methods for designing stabilizing controllers with respect to both parameters uncertainty and controller coefficients perturbations.

In this context, and to the best of the authors' knowledge, the *fragility* study of *AQM controllers* has not received sufficient attention in the literature. However, a first attempt to tackle this problem can be found in Üstebay and Özbay (2007), where a method to compute the largest available intervals for the PI controller parameters, based on the linearization of a simplified version of the model, has been developed. The approach proposed in this paper is quite different. More precisely, using the complete characterization of the set of all stabilizing controllers and its corresponding geometry, we first develop a robust stability analysis of the controllers with respect to network parameters uncertainty. The geometric properties of the region defined by the stabilizing control parameters will give **also** a better insight on the **existing** connections between delay, link capacity and TCP loads. We believe that such a result is completely new. Next, we present a methodology to examine the fragility of a given stabilizing controller and give a procedure to determine the controller coefficients providing the maximum  $\ell_2$  parametric stability margin in the controller's gains space. This procedure will allow us to obtain a ***non-fragile PI controller guaranteeing simultaneously the robustness with respect to the uncertainty in the network parameters, but also the robustness to the controller gains perturbation.***

Finally, we consider the performance issue of **the corresponding** PI-controlled closed-loop system. To address this, we characterize the exponential behaviour of the closed-loop system solutions from an analytical approach of stability theory for time-delay systems. More explicitly, we compute exponential estimates for the closed-loop system solutions by using the Lyapunov-Krasovskii functional approach to compute exponential upper bounds of the solutions of exponentially stable linear time-delay systems developed by Kharitonov and Hinrichsen (2004).

The remaining part of the paper is organized as follows. Section 2 introduces the mathematical model and controller design via linearization in a time-domain setting. The complete characterization of PI stabilizing controllers is given in Section 3. Sections 4 and 5 present the robust and fragility analysis, respectively. Exponential upper bounds for the closed-loop system solutions are derived in Section 6. A numerical example illustrates the results in Section 7. Concluding remarks end the paper.

## 2 Mathematical model and controller design via linearization

We consider the dynamic fluid-flow model introduced by Hollot *et al.* (2002) for describing the behaviour of TCP/AQM networks. Such a model, relating the average values of key network variables of  $n$  homogeneous TCP-controlled sources and a single router, is described by the following coupled nonlinear differential equations including time-varying delays:

$$\begin{cases} \dot{w}(t) = \frac{1}{\tau(t)} - \frac{1}{2} \frac{w(t)w(t-\tau(t))}{\tau(t-\tau(t))} p(t-\tau(t)), \\ \dot{q}(t) = n(t) \frac{w(t)}{\tau(t)} - c, \end{cases} \quad (1)$$

where  $w(t)$  denotes the average of TCP windows size (packets),  $q(t)$  is the average queue length (packets) in the router,  $\tau(t) = \frac{q(t)}{c} + \tau_p$  is the round-trip time (secs) with  $\tau_p$  representing the propagation delay,  $c$  is the link capacity (packets/sec),  $n(t)$  is the number of TCP sessions and  $p(\cdot)$  is the probability of marking packets function which represents the AQM control strategy.

As discussed in the literature, the system (1) describes accurately the congestion avoidance algorithm

of the TCP/AQM scheme. More precisely, the first differential equation characterizes the TCP window control dynamics as follows: the first term in the right-hand side corresponds to the window's additive-increase behaviour, and the second term to the multiplicative-decrease behaviour. The second equation describes the dynamics of the bottleneck queue length as the difference between the packet arrival rate and the link capacity, assuming that there is no internal dynamics in the bottleneck.

Following similar arguments to the ones proposed by Holot *et al.* (2002), we assume that the number of TCP sessions and the link capacity are constant, i.e.,  $n(t) \equiv n$  and  $c(t) \equiv c$ . Then, for a desired equilibrium  $q_0$ , the unique equilibrium point  $(w_0, q_0, p_0)$  of (1) is defined by

$$w_0^2 p_0 = 2, w_0 = \frac{\tau c}{n} \text{ and } \tau = \frac{q_0}{c} + \tau_p.$$

We will consider network parameters  $(n, \tau, c)$  such that  $w_0 > 2$ . Since (1) describes the congestion avoidance phase of TCP, the assumption on the equilibrium windows  $w_0$  can be made without any loss of generality.

In order to design a stabilizing PI controller via the linearization of (1), we assume, as in Holot *et al.* (2002), that the time-delay argument on the queue length  $q$  is fixed to  $t - \tau$ , and introduce

$$\sigma(t) = \int_0^t (q(s) - q_0) ds.$$

We thus arrive at the following augmented system:

$$\begin{cases} \dot{w}(t) = \frac{1}{\frac{q(t)}{c} + \tau_p} - \frac{1}{2} \frac{w(t)w(t-\tau)}{\frac{q(t-\tau)}{c} + \tau_p} p(t - \tau), \\ \dot{q}(t) = \frac{nw(t)}{\frac{q(t)}{c} + \tau_p} - c, \\ \dot{\sigma}(t) = q(t) - q_0. \end{cases} \quad (2)$$

Now consider a PI controller of the form

$$p(t) = k_p q(t) + \frac{k_p}{T_i} \sigma(t), \quad (3)$$

where  $\frac{k_p}{T_i} \neq 0$ . It can be easily verified that the closed-loop system (2)-(3) has a unique equilibrium point  $(w_0, q_0, \sigma_0)$ , where  $\sigma_0 = \frac{T_i}{k_p} (p_0 - k_p q_0)$ .

The linearization of the closed-loop system (2)-(3) around the equilibrium  $(w_0, q_0, \sigma_0)$  is of the form:

$$\dot{\xi}(t) = A\xi(t) + B\xi(t - \tau), \quad (4)$$

where  $\xi(t) = \begin{pmatrix} \tilde{w}(t) \\ \tilde{q}(t) \\ \tilde{\sigma}(t) \end{pmatrix}$ ,  $A = \begin{pmatrix} -\frac{n}{\tau^2 c} & -\frac{1}{c\tau^2} & 0 \\ \frac{n}{\tau} & -\frac{1}{\tau} & 0 \\ 0 & 1 & 0 \end{pmatrix}$ ,  $B = \begin{pmatrix} -\frac{n}{\tau^2 c} & \frac{1}{c\tau^2} & -\frac{\tau c^2}{2n^2} k_p - \frac{\tau c^2}{2n^2} \frac{k_p}{T_i} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $\tilde{w}(t) = w(t) - w_0$ ,  $\tilde{q}(t) = q(t) - q_0$ , and  $\tilde{\sigma}(t) = \sigma(t) - \sigma_0$ .

Assume for the moment that it is possible to find controller's gains that make (4) exponentially stable. Then, it follows that all solutions of (2) starting sufficiently close to  $(w_0, q_0, \sigma_0)$  will exponentially approach it as  $t \rightarrow \infty$ , see, for instance, the arguments proposed by Melchor-Aguilar and Niculescu (2007) concerning an explicit construction for the Lyapunov-Poincaré method in the case of time-delay systems.

*Remark 1* It is important to point out that it is not possible to investigate directly the stability of (4) for the delay-free case ( $\tau = 0$ ) since the matrices  $A$  and  $B$  depend explicitly on the parameter  $1/\tau$ , for which a *singular perturbation approach* seems more appropriate (see, for instance, Niculescu (2008) for some ideas in this direction). We will not focus on such a methodology in the sequel.

*Remark 2* It is worth to mention that the approach developed by Silva *et al.* (2005), which is based on first determining the set of PI stabilizing controllers for the delay-free case, cannot be directly applied to

determine the complete set of PI stabilizing controllers for (4). In the context of some simplified models for describing the behaviour of AQM networks, such a method was exploited by Al-Hammouri *et al.* (2006). In the sequel, we will consider the problem under a different angle and we will try to better exploit the properties of the system.

### 3 Complete characterization of PI stabilizing controllers

It is well known that (4) is exponentially stable if and only if the characteristic function (quasipolynomial)

$$f(s) = s^3 + \frac{1}{\tau} \left(1 + \frac{n}{\tau c}\right) s^2 + \frac{2n}{\tau^3 c} s + \left[ \frac{n}{\tau^2 c} s^2 + \frac{c^2}{2n} k_p \left(s + \frac{1}{T_i}\right) \right] e^{-\tau s},$$

has no zeros with nonnegative real parts, see, e.g. Hale and Verduyn-Lunel (1993). It is worth to mention that  $f$  has an infinite (countable) number of roots.

#### 3.1 Boundary parametrization

The following result provides the complete characterization of the set of controller's gains  $(T_i, k_p)$  for which (4) is exponentially stable.

**THEOREM 3.1** *Given network parameters  $(n, \tau, c)$ , the system (4) is exponentially stable if and only if the controller's gains  $(T_i, k_p)$  belong to the stability region  $\Phi_{(n, \tau, c)}$ , plotted in Fig. 1, whose boundary in the controller's gains space  $(T_i, k_p)$  is described by*

$$\begin{aligned} T_i(\omega, n, \tau, c) &= \frac{(\omega^2 - \frac{2n}{\tau^3 c}) \cos(\omega\tau) + \frac{\omega}{\tau} (1 + \frac{n}{\tau c}) \sin(\omega\tau)}{\omega \left[ \frac{\omega}{\tau} (1 + \frac{n}{\tau c}) \cos(\omega\tau) + (\frac{2n}{\tau^3 c} - \omega^2) \sin(\omega\tau) + \frac{n\omega}{\tau^2 c} \right]}, \\ k_p(\omega, n, \tau, c) &= \frac{2n}{c^2} \left[ \left( \omega^2 - \frac{2n}{\tau^3 c} \right) \cos(\omega\tau) + \frac{\omega}{\tau} \left( 1 + \frac{n}{\tau c} \right) \sin(\omega\tau) \right], \quad \omega \in (0, \omega^*), \end{aligned} \quad (5)$$

where  $\omega^*$  is the solution of

$$\frac{n}{\tau^2 c \omega} = h(\omega, \tau), \quad \omega \in \left(0, \frac{\pi}{2\tau}\right), \quad (6)$$

where

$$h(\omega, \tau) = \frac{\tau\omega \sin(\omega\tau) - \cos(\omega\tau)}{\tau\omega (1 + \cos(\omega\tau)) + 2 \sin(\omega\tau)}.$$

*Proof* First, observe that since  $\frac{k_p}{T_i} \neq 0$ ,  $s = 0$  is not a zero of  $f(s)$ . Assume now that  $f(s)$  has a pure imaginary zero  $s = i\omega \neq 0$ . Then, a direct calculation yields (5). The parametrization (5) defines a countable number of curves in the controller's gains space  $(T_i, k_p)$  and each one of them is defined by varying  $\omega$  in the intervals  $(\omega_k^*, \omega_{k+1}^*)$ ,  $k = 0, 1, 2, \dots$ , where  $\omega_0^* = 0$  and  $\omega_k^*$ ,  $k = 1, 2, 3, \dots$  are solutions of

$$\frac{n}{\tau^2 c \omega} = h(\omega, \tau), \quad \omega \in \left(\frac{k\pi}{\tau}, \frac{(k+1)\pi}{\tau}\right), \quad k = 0, 1, \dots \quad (7)$$

Since (7) is a transcendental equation we look directly for a numerical solution. This can be found by plotting the two functions  $\frac{n}{\tau^2 c \omega}$  and  $h(\omega, \tau)$ , see Fig. 2. The curves divide the plane  $(T_i, k_p)$  into a set of connected domains. From the argument principle (see, e.g. Ahlfors (1979)), it is easy to show that for all  $(T_i, k_p)$  values inside the open domain  $\Phi_{(n, \tau, c)}$ , bounded by the curve obtained by varying  $\omega$  in the interval  $(0, \omega_1^*)$  and the coordinate axis  $k_p = 0$ , the function  $f(s)$  has no zeros with strictly positive real part.

Taking into account that  $w_0 > 2$ , a simple inspection of the numerical solution of (7) shows that in fact  $\omega_1^* \in (0, \frac{\pi}{2\tau})$ , which ends the proof. ■

*Remark 1* The procedure above sends back to the *D-decomposition method* suggested by Neimark (1949) in the 40s or to the so-called *parameter space approach* (see, for instance, Bhattacharyya *et al.* (1995); Ackermann *et al.* (2002) and the references therein).

### 3.2 Geometric properties of the stability regions

If the ideas above for characterizing the boundary of the stability regions in closed-loop have been used in the literature, however the geometric properties of such stability regions have not been fully exploited.

In the sequel, with the characterization above, we have the following result:

**PROPOSITION 3.2** *The boundary of the stability region  $\Phi_{(n,\tau,c)}$  has the following geometric properties:*

- (i) *The functions  $T_i(\omega, n, \tau, c)$  and  $k_p(\omega, n, \tau, c)$  are monotonically increasing functions of  $\omega$  in the interval  $(0, \omega^*)$ . Furthermore, it holds that  $T_i(\omega, n, \tau, c) \rightarrow -\infty$  and  $k_p(\omega, n, \tau, c) \rightarrow k_p(0, n, \tau, c) = -\frac{4n}{\tau^3 c^3}$  when  $\omega \rightarrow +0$ , and  $T_i(\omega, n, \tau, c) \rightarrow +\infty$  and  $k_p(\omega, n, \tau, c) \rightarrow k_p(\omega^*, n, \tau, c)$  when  $\omega \rightarrow -\omega^*$ , where  $0 < k_p(\omega^*, n, \tau, c) < \infty$ .*
- (ii) *The function  $\frac{k_p(\omega, n, \tau, c)}{T_i(\omega, n, \tau, c)}$  is strictly positive for all  $\omega \in (0, \omega^*)$ .*

*Proof*

- (i) The first part of the assertion can be shown to be true by checking that the derivatives of the functions w.r.t  $\omega$  are strictly positive for all  $\omega \in (0, \omega^*)$ . For the sake of brevity we omit here the technical details. The second part of the assertion can be directly obtained from (5).
- (ii) From (5) we get

$$\frac{k_p(\omega, n, \tau, c)}{T_i(\omega, n, \tau, c)} = \frac{2n\omega}{c^2} \left[ \frac{\omega}{\tau} \left( 1 + \frac{n}{\tau c} \right) \cos(\omega\tau) + \left( \frac{2n}{\tau^3 c} - \omega^2 \right) \sin(\omega\tau) + \frac{n\omega}{\tau^2 c} \right]$$

that can be written as

$$\frac{k_p(\omega, n, \tau, c)}{T_i(\omega, n, \tau, c)} = \frac{2n\omega^2}{\tau c^2} [\omega\tau (1 + \cos(\omega\tau)) + 2 \sin(\omega\tau)] \left[ \frac{n}{\tau^2 c\omega} - h(\omega, \tau) \right]. \quad (8)$$

Since  $\frac{n}{\tau^2 c\omega} > h(\omega, \tau), \forall \omega \in (0, \omega^*)$ , see Fig. 2, and  $\omega\tau (1 + \cos(\omega\tau)) + 2 \sin(\omega\tau) > 0$  for all  $\omega \in (0, \omega^*)$ , then the assertion follows.

■

*Remark 2* When  $\tau \rightarrow +0$ , the stability region  $\Phi_{(n,\tau,c)}$  tends to the whole first and third quadrants of the plane  $(T_i, k_p)$ . In other words, for small round-trip time (delay), arbitrarily controller's gains having the same sign stabilizes the closed-loop system (4). Indeed, it is easy to see that such a conclusion follows from the assertion (i) of proposition 3.2 and the fact that  $\omega^* \rightarrow +\infty$  when  $\tau \rightarrow +0$ , see Fig. 2.

## 4 Robust Stability Analysis

In this section, we address the robustness of the controller to uncertainties in the network parameters. Let us consider that the unknown parameters  $(n, \tau, c)$  satisfy the following condition:

$$n \geq n_0, \tau \leq \tau_0 \text{ and } c \leq c_0. \quad (9)$$

Our goal here is to determine the complete set of PI controllers that exponentially stabilizes (4) for all network parameters  $(n, \tau, c)$  satisfying (9).

Let  $\omega_0^*$  and  $\omega^*$  be the solutions of (6) corresponding to the parameters  $(n_0, \tau_0, c_0)$  and  $(n, \tau, c)$  respectively.

*Remark 1* For the network parameters  $(n_0, \tau_0, c_0)$  and  $(n, \tau, c)$  satisfying (9), then the following inequality

$$\omega_0^* \leq \omega^*$$

holds. Indeed, it is easy to see the validity of this inequality by simply inspecting the plots of the functions involved in (6) corresponding to the parameters  $(n_0, \tau_0, c_0)$  and  $(n, \tau, c)$  respectively, see Fig. 2.

The following result will play an essential role on deriving the robust stability conditions to network parameters uncertainty.

**PROPOSITION 4.1** *For network parameters  $(n_0, \tau_0, c_0)$  and  $(n, \tau, c)$  satisfying (9), the following inequality*

$$\frac{k_p(\omega, n, \tau, c)}{T_i(\omega, n, \tau, c)} \geq \frac{k_p(\omega, n_0, \tau_0, c_0)}{T_i(\omega, n_0, \tau_0, c_0)} \tag{10}$$

holds for all  $\omega \in (0, \omega_0^*)$ .

*Proof* From Fig. 2, we obtain the following inequalities:  $h(\omega, \tau) \leq h(\omega, \tau_0), \forall \omega \in \left(0, \frac{\pi}{2\tau_0}\right)$  and  $\frac{n}{\tau^2 c \omega} \geq \frac{n_0}{\tau_0^2 c_0 \omega}, \forall \omega > 0$ . Taking into account these inequalities in (8) we get that for all  $\omega \in (0, \omega_0^*)$

$$\frac{k_p(\omega, n, \tau, c)}{T_i(\omega, n, \tau, c)} \geq \frac{2n\omega^2}{\tau c^2} [\omega\tau (1 + \cos(\omega\tau)) + 2 \sin(\omega\tau)] \left[ \frac{n_0}{\tau_0^2 c_0 \omega} - h(\omega, \tau_0) \right].$$

Since  $1 + \cos(\omega\tau) \geq 1 + \cos(\omega\tau_0)$  and  $\frac{\sin(\omega\tau)}{\tau} \geq \frac{\sin(\omega\tau_0)}{\tau_0}, \forall \omega \in \left(0, \frac{\pi}{2\tau_0}\right)$ , we have

$$\frac{k_p(\omega, n, \tau, c)}{T_i(\omega, n, \tau, c)} \geq \frac{2n_0\omega^2}{\tau_0 c_0^2} [\omega\tau_0 (1 + \cos(\omega\tau_0)) + 2 \sin(\omega\tau_0)] \left[ \frac{n_0}{\tau_0^2 c_0 \omega} - h(\omega, \tau_0) \right],$$

which implies (10). ■

**THEOREM 4.2** *Given network parameters  $(n_0, \tau_0, c_0)$  and  $(n, \tau, c)$  satisfying (9), the following property holds*

$$\Phi_{(n_0, \tau_0, c_0)} \subseteq \Phi_{(n, \tau, c)}.$$

*Proof* Obviously,  $\Phi_{(n_0, \tau_0, c_0)} = \Phi_{(n, \tau, c)}$  when  $n = n_0, \tau = \tau_0$  and  $c = c_0$ . Note that in order to get the general non trivial result, it suffices to prove that the stability regions inclusion holds for some parameters  $(n_1, \tau_1, c_1)$  satisfying  $n_1 > n_0, \tau_1 < \tau_0$  and  $c_1 < c_0$ , since we can use a continuation procedure by redefining  $n_1 = n_0, \tau_1 = \tau_0$  and  $c_1 = c_0$ . Thus, let us consider parameters  $(n_1, \tau_1, c_1)$  such that  $(T_i(\omega_0^*, n_1, \tau_1, c_1), k_p(\omega_0^*, n_1, \tau_1, c_1))$  belongs to the first quadrant of the plane  $(T_i, k_p)$  and denote by  $\omega_1^*$  the solution of (6) for such parameters.

From (10) and the properties of the boundary of the stability regions we get that, by sweeping  $\omega$  in the interval  $(0, \omega_0^*)$ , the stability region  $\Phi_{(n_0, \tau_0, c_0)}$  is partially contained in the stability region  $\Phi_{(n_1, \tau_1, c_1)}$ .

Thus, it remains to prove that by sweeping  $\omega$  in the interval  $[\omega_0^*, \omega_1^*)$ , the stability region  $\Phi_{(n_1, \tau_1, c_1)}$  still contains to the stability region  $\Phi_{(n_0, \tau_0, c_0)}$ . To prove this, consider now a pair  $(T_{i0}, k_{p0})$  belonging to the boundary of  $\Phi_{(n_0, \tau_0, c_0)}$ , where  $T_{i0} > T_i(\omega_0^*, n_1, \tau_1, c_2)$ .

From (5) it follows that there exists a unique  $\omega_0 \in (0, \omega_0^*)$  such that  $T_{i0} = T_i(\omega_0, n_0, \tau_0, c_0)$  and  $k_{p0} = k_p(\omega_0, n_0, \tau_0, c_0)$ . Since  $k_p(\omega, n_1, \tau_1, c_1)$  and  $T_i(\omega, n_1, \tau_1, c_1)$  are monotonically increasing functions as functions w.r.t. the variable  $\omega$ , when  $\omega$  goes from  $\omega_0^*$  to  $\omega_1^*$ , there exist  $\hat{\omega} \in (\omega_0^*, \omega_1^*)$  and  $\check{\omega} \in (0, \hat{\omega})$  such

that

$$\frac{k_p(\hat{\omega}, n_1, \tau_1, c_1)}{T_i(\hat{\omega}, n_1, \tau_1, c_1)} = \frac{k_p(\omega_0, n_0, \tau_0, c_0)}{T_i(\omega_0, n_0, \tau_0, c_0)} \text{ and } k_p(\check{\omega}, n_1, \tau_1, c_1) = k_p(\omega_0, n_0, \tau_0, c_0). \quad (11)$$

Since for all  $\omega \in [\check{\omega}, \hat{\omega}]$  the slope of the boundary of  $\Phi_{(n_1, \tau_1, c_1)}$  is monotonically decreasing we have

$$\frac{k_p(\check{\omega}, n_1, \tau_1, c_1)}{T_i(\check{\omega}, n_1, \tau_1, c_1)} > \frac{k_p(\hat{\omega}, n_1, \tau_1, c_1)}{T_i(\hat{\omega}, n_1, \tau_1, c_1)}.$$

From (11) and the above it follows that  $T_i(\check{\omega}, n_1, \tau_1, c_1) < T_i(\omega_0, n_0, \tau_0, c_0)$ . The monotonically increasing property of  $T_i(\omega, n_1, \tau_1, c_1)$  w.r.t.  $\omega$  implies that there exists  $\tilde{\omega} \in (\check{\omega}, \omega_1^*)$  such that  $T_i(\tilde{\omega}, n_1, \tau_1, c_1) = T_i(\omega_0, n_0, \tau_0, c_0)$ . On the other hand, the monotonically increasing property of  $k_p(\omega, n_1, \tau_1, c_1)$  w.r.t.  $\omega$  implies that

$$k_p(\tilde{\omega}, n_1, \tau_1, c_1) > k_p(\check{\omega}, n_1, \tau_1, c_1) = k_p(\omega_0, n_0, \tau_0, c_0) = k_{p0}.$$

So, we conclude that  $(T_{i0}, k_{p0}) \in \Phi_{(n_1, \tau_1, c_1)}$ . ■

As a consequence of this last property, we have the following:

**COROLLARY 4.3** *Assume that a controller (3) locally stabilizes the equilibrium point of (2) with network parameters  $(n_0, \tau_0, c_0)$ . Then it locally stabilizes the equilibrium point of (2) with network parameters  $(n, \tau, c)$  satisfying (9).*

*Remark 2* It is important to point out the links between the Corollary 4.3 and Proposition 2 from Hollot *et al.* (2002). Although some similarities exist, however, the analytical treatment and the final results here are basically different. First, we arrive at the result from the exact knowledge of the set of all stabilizing controllers and its geometric properties. This leads to improved robust stability conditions, where the controller's gains can be chosen based on the exact robust stability region, not on an estimate of it. In addition, our approach does not impose any stability condition on the controller as occurs in Hollot *et al.* (2002).

*Remark 3* Assume that the network parameters  $(n, \tau, c)$  are constants satisfying

$$n \in [n_1, n_2], \tau \in [\tau_1, \tau_2] \text{ and } c \in [c_1, c_2]. \quad (12)$$

Then from the geometry property stated in theorem 4.2 it follows that  $\Phi_{(n_1, \tau_2, c_2)} \subseteq \Phi_{(n, \tau, c)}$  for all network parameters  $(n, \tau, c)$  satisfying (12). In other words, by designing the PI controller for the largest expected values of  $\tau$  and  $c$ , and the smallest expected value of  $n$  yields a robust stabilizing controller.

Finally, in this section we would like to note that the above analysis have been developed without imposing restrictions on the controllers gains since we wanted to illustrate the real complete set of PI controllers that exponentially stabilizes (4). Of course, from the application point of view, the parameters of the AQM controllers should be positive and, therefore, one needs to restrict the controller gains to be only positive.

## 5 Fragility Analysis. A geometric approach

Now, we address the robustness to perturbations in the controller gains, i.e., the fragility of the PI AQM controller. In order to simplify our subsequent analysis we define  $k_p = k_p$  and  $k_i = \frac{k_p}{T_i}$ , as the authors of Silva *et al.* (2005), and restrict the controller gains to be positive.

Thus, we consider controller's gains  $(k_p, k_i)$  belonging to the stability region  $\Gamma_{(n, \tau, c)}$ , see Fig. 3, whose boundary in the controller's gains space  $(k_p, k_i)$  is determined by the coordinate axes  $k_p = 0$  and  $k_i = 0$ ,

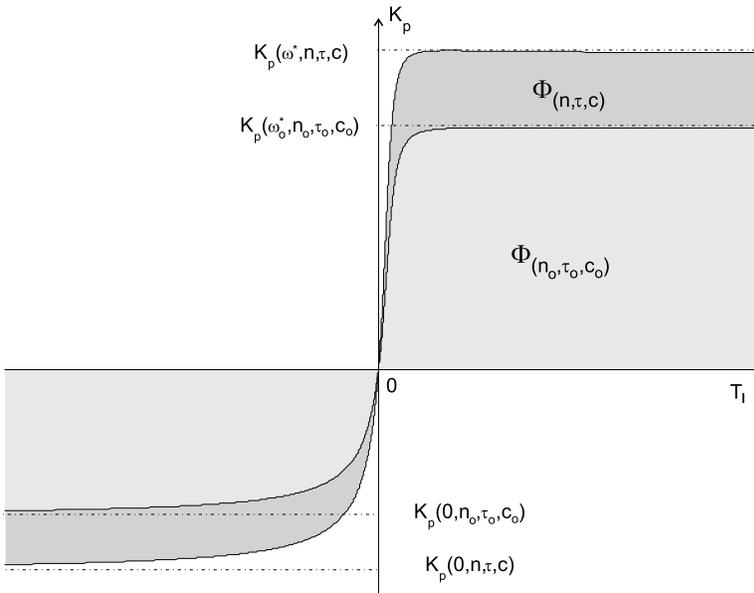


Figure 1. Stability region  $\Phi_{(n, \tau, c)}$  for  $(n, \tau, c)$  and  $(n_0, \tau_0, c_0)$  satisfying (9).

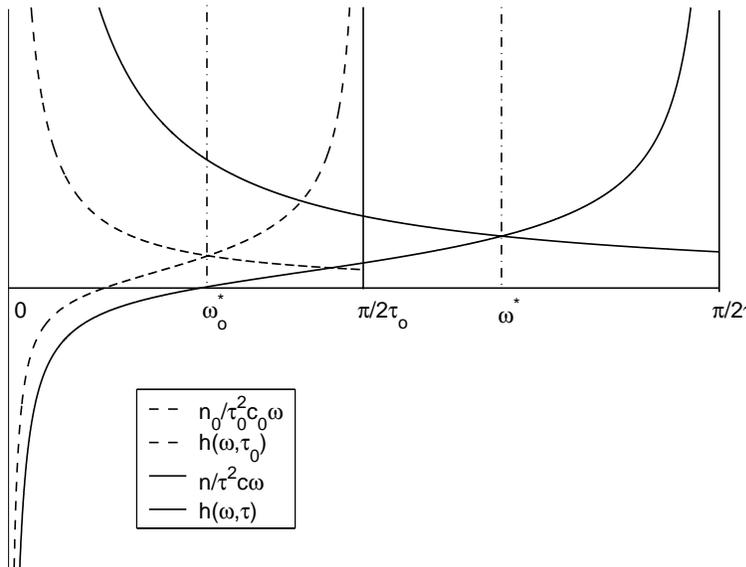


Figure 2. Numerical Solution of (6) for network parameters  $(n, \tau, c)$  and  $(n_0, \tau_0, c_0)$  satisfying (9).

and the curve defined by

$$\begin{aligned}
 k_p(\omega) &= \frac{2n}{c^2} \left[ \left( \omega^2 - \frac{2n}{\tau^3 c} \right) \cos(\omega\tau) + \frac{\omega}{\tau} \left( 1 + \frac{n}{\tau c} \right) \sin(\omega\tau) \right] \\
 k_i(\omega) &= \frac{2n\omega}{c^2} \left[ \frac{\omega}{\tau} \left( 1 + \frac{n}{\tau c} \right) \cos(\omega\tau) + \left( \frac{2n}{\tau^3 c} - \omega^2 \right) \sin(\omega\tau) + \frac{n\omega}{\tau^2 c} \right], \quad \omega \in [\bar{\omega}, \omega^*],
 \end{aligned}
 \tag{13}$$

where  $\omega^*$  is the solution of (6) and  $\bar{\omega}$  is the solution of

$$\tan(\omega\tau) = \frac{\frac{2n}{\tau^3 c} - \omega^2}{\frac{\omega}{\tau} \left( 1 + \frac{n}{\tau c} \right)}, \quad \omega \in \left( 0, \frac{\pi}{2\tau} \right).
 \tag{14}$$

The fragility problem can be formulated as follows: given nominal controller's gains  $(k_{p0}, k_{i0}) \in \Gamma_{(n, \tau, c)}$ ,

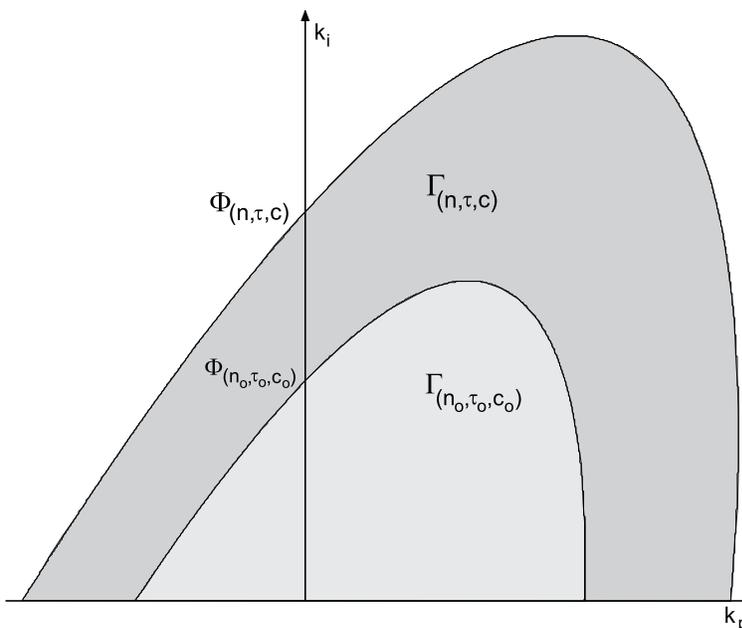


Figure 3. Stability regions  $\Phi_{(n,\tau,c)}$  and  $\Gamma_{(n,\tau,c)}$  in the  $(k_p, k_i)$  space for  $(n, \tau, c)$  and  $(n_o, \tau_o, c_o)$  satisfying (9).

find the maximum  $\rho_0 > 0$  such that for any  $k_p \geq 0$  and  $k_i > 0$  the following condition holds:

$$B_{\rho_0}(k_{p0}, k_{i0}) = \left\{ (k_p, k_i) : \sqrt{(k_p - k_{p0})^2 + (k_i - k_{i0})^2} < \rho_0 \right\} \subset \Gamma_{(n,\tau,c)}.$$

The problem is equivalent to finding the minimum distance between  $(k_{p0}, k_{i0})$  and  $\Gamma_{(n,\tau,c)}$ . Let  $C$  denotes the curve determined by varying  $\omega$  in the interval  $[\bar{\omega}, \omega^*]$  in (13). The distance from the point  $(k_{p0}, k_{i0})$  to the curve  $C$  is given by

$$d(\omega) = \sqrt{(k_p(\omega) - k_{p0})^2 + (k_i(\omega) - k_{i0})^2}, \omega \in [\bar{\omega}, \omega^*].$$

Since  $\omega \rightarrow d(\omega)$  is a continuous function of  $\omega$ , there exists always a  $\hat{\omega} \in [\bar{\omega}, \omega^*]$  such that  $d(\hat{\omega}) \leq d(\omega)$ , for all  $\omega \in [\bar{\omega}, \omega^*]$ . It is easily seen that the minimum distance from the given point  $(k_{p0}, k_{i0})$  to  $\Gamma_{(n,\tau,c)}$  is

$$\rho_0 = \min \{k_{p0}, k_{i0}, d(\hat{\omega})\}. \quad (15)$$

Formula (15) determines a procedure to compute the  $\ell_2$  parametric stability margin around the nominal point  $(k_{p0}, k_{i0})$ , which allows us to examine the fragility of a given stabilizing controller.

With this tool in our hands, we address now the problem of choosing the controller's gains  $(k_{p0}, k_{i0})$  inside the stabilizing region  $\Gamma_{(n,\tau,c)}$  at the centre of the circle  $B_{\rho_0}(k_{p0}, k_{i0})$  of maximum  $\rho_0 > 0$  such that  $B_{\rho_0}(k_{p0}, k_{i0}) \subset \Gamma_{(n,\tau,c)}$ . The radius of this circle represents the maximum  $\ell_2$  parametric stability margin in the controller's gains space, see, for instance, Keel and Bhattacharyya (1997).

In order to achieve such a goal, we select  $k_{p0}$  as the centre of the interval of allowable  $k_p$  gains and then we do a sweeping on the interval of allowable  $k_i$  gains. Thus, we choose  $k_{p0} = \frac{k_p(\omega^*)}{2}$  and compute  $\rho_0 > 0$ , according to (15), by sweeping  $k_{i0}$  in the interval  $(0, k_i(\tilde{\omega}))$ , where  $\tilde{\omega} \in [\bar{\omega}, \omega^*]$  is such that  $k_p(\tilde{\omega}) = \frac{k_p(\omega^*)}{2}$ .

This procedure determines a family of circles having different radii and centers of which we select the one with the maximum radius. Finally, we choose  $k_{i0}$  to be at the center of this circle.

### 6 Exponential estimates for the closed-loop system solutions

In this section, we compute exponential estimates for the closed-loop system solutions to evaluate the performance of PI stabilizing controllers.

From our results in Section 3, we have that, for any given stabilizing controller, the closed-loop system (4) is exponentially stable. Then, according to the definition of exponential stability for time-delay systems, there exist constants  $\mu \geq 1$  and  $\alpha > 0$  such that the following exponential upper bound holds:

$$\|\xi(t, \varphi)\| \leq \mu e^{-\alpha t} \|\varphi\|_\tau, \quad t \geq 0, \tag{16}$$

for every solution  $\xi(t, \varphi)$  of (4), see Hale and Verduyn-Lunel (1993). Here it is assumed that the initial function  $\varphi$  belongs to  $\mathcal{C}([-\tau, 0], \mathbb{R}^3)$ , the space of continuous functions mapping  $[-\tau, 0]$  to  $\mathbb{R}^3$  equipped with the uniform norm  $\|\varphi\|_\tau = \max_{\theta \in [-\tau, 0]} \|\varphi(\theta)\|$ .

As it is well known the exponential bound (16) is governed by the infinite number of zeros of the characteristic function of (4), whereas the exponential decay rate  $\alpha$  is determined by the real part of the rightmost zero, see Hale and Verduyn-Lunel (1993). Several contributions addressing the computation of an estimate for the lower bound of the decay rate can be found in the literature referred to as  $\alpha$ -stability, see, for instance, Mori *et al.* (1982) and Niculescu (1998).

Recently, an interesting method of computing simultaneously a lower bound for the decay rate  $\alpha$  and an upper bound for  $\mu$  of a given exponentially stable linear time-delay system has been developed by Kharitonov and Hinrichsen (2004). To have estimates for both constants  $\mu$  and  $\alpha$  it provides the complete characterization of the exponential behaviour of a solution, where not only an estimate for the exponential decay rate but also an upper bound for the transient response of the solutions are explicitly determined.

The method developed by Kharitonov and Hinrichsen (2004) makes use of the fact that associated to an exponentially stable system (4), there exists always a quadratic Lyapunov-Krasovskii functional  $v : \mathcal{C}([-\tau, 0], \mathbb{R}^3) \rightarrow \mathbb{R}$  of the form

$$\begin{aligned} v(\varphi) = & \varphi^T(0)U(0)\varphi(0) + 2\varphi^T(0) \int_{-\tau}^0 U(-\tau - \theta)B\varphi(\theta)d\theta \\ & + \int_{-\tau}^0 \int_{-\tau}^0 \varphi^T(\theta_1)B^T U(\theta_1 - \theta_2)B\varphi(\theta_2)d\theta_1d\theta_2 + \int_{-\tau}^0 \varphi^T(\theta) [W_1 + (\tau + \theta)W_2] \varphi(\theta)d\theta, \end{aligned} \tag{17}$$

where matrix function  $U(\cdot)$  is a solution of the second-order differential equation

$$\ddot{U}(\zeta) = \dot{U}(\zeta)A - A^T\dot{U}(\zeta) + A^TU(\zeta)A - B^TU(\zeta)B \tag{18}$$

with the boundary conditions

$$\dot{U}(0) = U(0)A + U^T(\tau)B, \tag{19}$$

$$-W = \dot{U}(0) + \dot{U}^T(0). \tag{20}$$

Here  $W = W_0 + W_1 + \tau W_2$  with  $W_j, j = 0, 1, 2$ , any positive definite matrices, see Kharitonov and Hinrichsen (2004) for details.

When (4) is exponentially stable, then given positive definite matrices  $W_j, j = 0, 1, 2$ , there exists a unique solution of (18) satisfying (19) and (20) for which a piecewise linear approximation on the interval  $[0, \tau]$  can be computed, see Kharitonov and Plischke (2006).

The following proposition is directly obtained from Theorem 4 in Kharitonov and Hinrichsen (2004), where we use  $\|\cdot\|$  to denote both the Euclidean norm for vector and the induced matrix norm for matrices,  $\lambda_{\min}(A)$  and  $\lambda_{\max}(A)$  to designate the smallest and largest eigenvalues of a symmetric matrix  $A$ , respectively.

**PROPOSITION 6.1** *Let system (4) be exponentially stable. Given any positive definite matrices  $W_j, j = 0, 1, 2$  the solution  $\xi(t, \varphi)$  with initial condition  $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{R}^3)$  satisfies the following exponential upper bound:*

$$\|\xi(t, \varphi)\| \leq \mu e^{-\alpha t} \|\varphi\|_{\tau}, \quad t \geq 0, \quad (21)$$

where

$$\mu = \sqrt{\frac{\alpha_2}{\alpha_1}} \quad \text{and} \quad \alpha = \frac{\beta}{2\kappa},$$

with

$$\begin{aligned} \beta &= \min \{ \lambda_{\min}(W_0), \lambda_{\min}(W_2) \}, \\ \kappa &= \max \{ u_0 (1 + \tau \|B\|), u_0 \|B\| (1 + \tau \|B\|) + \|W_1 + \tau W_2\| \}, \\ u_0 &= \max_{\theta \in [0, \tau]} \|U(\theta)\|, \\ \alpha_2 &\geq \kappa (1 + \tau), \end{aligned}$$

and  $\alpha_1 > 0$  such that

$$\begin{aligned} \lambda_{\min}(W_0) - \alpha_1 (2\|A\| + \|B\|) &> 0, \\ \lambda_{\min}(W_1) - \alpha_1 \|B\| &> 0. \end{aligned}$$

By combining the above result with the knowledge of the complete set of PI controllers that exponentially stabilizes (4), one can evaluate the time-domain responses of the closed-loop system solutions for a given set of PI stabilizing controllers. More explicitly, we first choose a set of positive definite matrices  $W_j, j = 0, 1, 2$ , and then compute the corresponding exponential upper bound (21) for each controller of the given set of stabilizing controller.

It is important to note that, as a Lyapunov-based approach, the exponential upper bounds determined by the Proposition 6.1 may be conservative estimates of the actual exponential upper bound for the solutions. As the estimates (21) depend on the choice of the matrices  $W_j, j = 0, 1, 2$ , these matrices can be used as free parameters to improve such estimates.

## 7 Example

Let us consider network parameters in the same setup as in Hollot *et al.* (2002), where the operation point is  $q_0 = 200$  packets,  $n = 60$  TCP flows,  $c = 3750$  packets/sec,  $\tau_p = 0.1927$  sec, so  $\tau = 0.246$  sec. Following Hollot *et al.* (2002), for a pair  $(T_i, k_p)$  designed according to the rules

$$T_i = \frac{\tau^2 c}{2n} \quad \text{and} \quad k_p = \frac{2n}{\tau^2 c^2} \beta \sqrt{1 + \beta^2}, \quad \text{where } \beta = 0.13, \quad (22)$$

the corresponding PI controller locally stabilizes the equilibrium point. On the other hand, following Üstebay and Özbay (2007), the local stabilization of the equilibrium point is achieved for a PI controller with a pair  $(T_i, k_p)$  satisfying

$$T_i = 16\tau \quad \text{and} \quad k_p = \frac{n}{2\tau^2 c^2}. \quad (23)$$

Thus, for this example, from (22) and (23) we obtain the pairs  $H = (k_{p0}, k_{i0}) = (1.8485 \times 10^{-5}, 9.7749 \times 10^{-6})$  and  $O = (k_{p0}, k_{i0}) = (3.5252 \times 10^{-5}, 8.9564 \times 10^{-6})$ , respectively.

In Fig. 4 we plot the stability region  $\Gamma_{(n,\tau,c)}$  together with the pairs  $H$  and  $O$  in the controller's gains space  $(k_p, k_i)$ . As expected, the PI controllers proposed in Hollot *et al.* (2002) and Üstebay and Özbay (2007) belong to the complete set of PI controllers that exponentially stabilizes (4).

Now let us examine the fragility of the controllers mentioned above. Using (15) we get  $\rho_0 = 9.7749 \times 10^{-6}$  for the pair  $H$  and  $\rho_0 = 8.9564 \times 10^{-6}$  for the pair  $O$ . Fig. 4 shows that the PI controllers designs proposed by Hollot *et al.* (2002) and Üstebay and Özbay (2007) are close to the stability boundary. Therefore, in this case, the resulting PI controllers are fragile controllers.

By applying the methodology proposed in section 5, we obtain that the pair  $M = (k_{p0}, k_{i0}) = (9.1044 \times 10^{-5}, 6.8 \times 10^{-5})$  provides  $\rho_0 = 6.7411 \times 10^{-5}$ , which is the maximum  $\ell_2$  parametric stability margin in the controller's gains space, see Fig. 4.

We now compute the exponential upper bound (21) for the above stabilizing controllers. Let us choose  $W_0 = 180I$ ,  $W_1 = 0.1I$  and  $W_2 = 180I$ , where  $I$  denotes the identity matrix of  $3 \times 3$  dimension. Then, a direct application of proposition 6.1 leads to the following estimates of the constants  $\mu$  and  $\alpha$ :  $\mu_H \approx 2.67 \times 10^3$  and  $\alpha_H \approx 4.25 \times 10^{-5}$  for the pair  $H$ ,  $\mu_O \approx 2.27 \times 10^3$  and  $\alpha_O \approx 5.88 \times 10^{-5}$  for the pair  $O$ , while for the pair  $M$  we get  $\mu_M \approx 1.40 \times 10^3$  and  $\alpha_M \approx 1.53 \times 10^{-4}$ .

The computed exponential estimates show that the PI controller for the pair  $M$  provides a faster closed-loop response than the PI controllers for the pairs  $H$  and  $O$ . This is corroborated by a Matlab/Simulink simulation performed on the nonlinear model (2), see Fig. 5. The faster response time of the PI controller for the pair  $M$  is clearly observed. Hence, for this example, the pair  $M$  determines a non-fragile PI controller having a good transient response. In addition, such a controller satisfies the robust stabilization properties mentioned in Corollary 4.3 and Remark 3.

In doing the simulations, we have not implemented any scheme to avoid the well-known phenomenon of windup when using an integral term in the controller, see for instance Astrom and Hagglund (1995), which certainly is a possibility in AQM control where the control signal takes values only in  $[0, 1]$ . As it can be seen in Fig. 6, for this example where the control signals have been fixed to zero when taking negative values, the probability of marking packets functions  $p(t)$  satisfy the requested property, that is to be bounded by 0 and 1.

Finally, it is important to point out that the computed constant  $\mu$  can be useful for determining an upper bound for the buffer capacity to avoid overflow, and thus lost packets and undesired retransmissions, towards an efficient queue utilization.

## 8 Conclusions

In this paper, we addressed the stabilization problem of fluid-flow delay models of TCP/AQM networks by using a PI controller as AQM strategy. The complete characterization of the set of controllers that exponentially stabilizes the linearized system is obtained in counterpart with the existing works in the literature which give only estimates of this set. We showed how the knowledge of the set of all stabilizing controller can be exploited to analyze the robustness of the controllers with respect to both network parameters uncertainty and controller coefficients perturbations. As a consequence, a simple procedure to compute the controller gains providing a non-fragile PI controller which admits uncertainty in the network parameters as well as controller coefficients perturbations is provided. We have computed exponential estimates for the closed-loop system solutions to evaluate the performance of a PI stabilizing controller. Our contribution demonstrates that we are now able to choose the PI controller gains to maintain the desired stability despite varying network conditions with a sufficient margin of tolerance around the controller design and thus to achieve better performance as compared with other typical AQM schemes. Extensions of this work to networks with heterogeneous round-trip times, multiple bottleneck links, and uncertain routing topologies deserve further study.

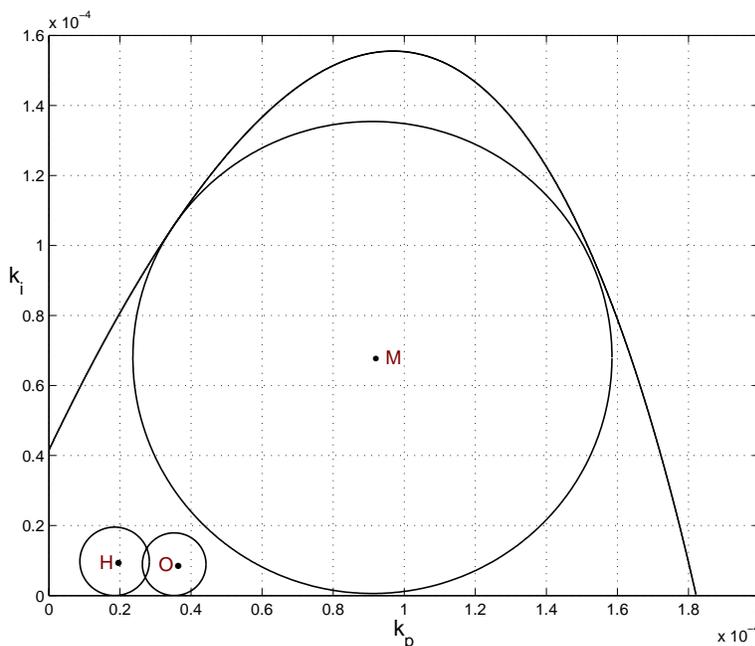


Figure 4. Fragility comparison of the PI controllers proposed in Hollot *et al.* (2002), Üstebay and Özbay (2007) (pairs  $H = (1.8485 \times 10^{-5}, 9.7749 \times 10^{-6})$  and  $O = (3.5252 \times 10^{-5}, 8.9564 \times 10^{-6})$  respectively) and the method developed in this paper (pair  $M = (9.1044 \times 10^{-5}, 6.8 \times 10^{-5})$ ).

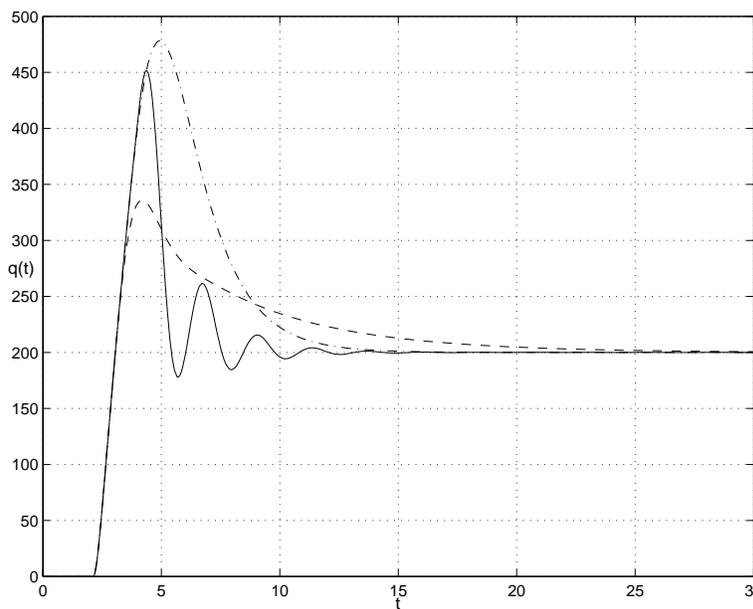


Figure 5. Closed-loop responses of  $q(t)$  for the PI controllers corresponding to the pairs  $H$  (- · -),  $O$  (- -) and  $M$  (-).

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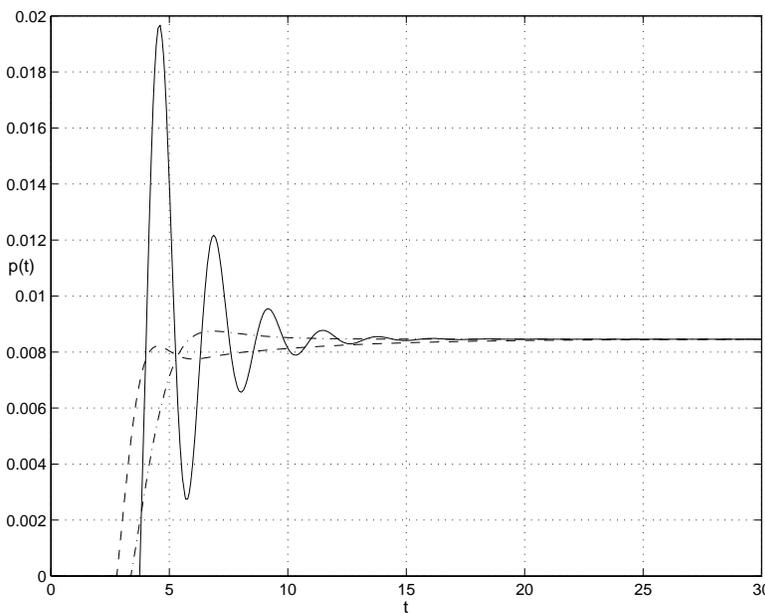


Figure 6. Control signals  $p(t)$  corresponding to the pairs  $H$  (- · -),  $O$  (- -) and  $M$  (-).

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