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RESEARCH ARTICLE

New Results on Robust Exponential Stability of Integral Delay Systems

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The robust exponential stability of integral delay systems with exponential kernels is investigated. Sufficient delay-dependent robust conditions expressed in terms of linear matrix inequalities and matrix norms are derived by using the Lyapunov-Krasovskii functional approach. The results are combined with a new result on quadratic stabilizability of the state-feedback synthesis problem in order to derive a new linear matrix inequality methodology of designing a robust non-fragile controller for the finite spectrum assignment of input delay systems that guarantees simultaneously a numerically safe implementation and also the robustness to uncertainty in the system matrices and to perturbation in the feedback gain.

Keywords: Integral delay systems; robust exponential stability; Lyapunov-Krasovskii functionals

1. Introduction

Several problems in time-delay systems involve the stability of a special class of systems which infinite-dimensional dynamics are described by integral delay equations. The problems where such class of systems is involved can be divided in two groups:

1) Stability problems as the stability of additional dynamics introduced by some system transformations used for obtaining delay-dependent conditions of differential delay systems Gu and Niculescu (2000, 2001), Kharitonov and Melchor-Aguilar (2000, 2002, 2003), and the stability of difference operators in neutral functional differential equations Hale and Verduyn-Lunel (1993).
2) Feedback schemes involving delay compensation as the finite spectrum assignment Manitius and Olbrot (1979), stabilization problems Mayne (1968), Watanabe and Ito (1981), and optimal control Tadmord (2000), Mirkin (2006), Meinsma and Zwart (2000) of systems with time-delay. In these problems the compensators necessarily include an infinite-dimensional dynamic governed by an integral delay system. The practical implementation of the compensators demand their internal stability, i.e., the stability of the integral delay system, see Engelborghs *et al.* (2001), Michiels *et al.* (2004), Mondié and Michiels (2003) and Richard (2003).

Recently in Melchor-Aguilar *et al.* (2010), Lyapunov-Krasovskii theorems for the exponential stability of integral delay systems have been introduced. It was shown there that a new type of Lyapunov functionals is required in order to properly address the dynamics of such class of systems. Based on the general expressions of Lyapunov functionals introduced in Melchor-Aguilar

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et al. (2010), some particular functionals were constructed for the case of constant kernels in Melchor-Aguilar (2010) and for a class of analytic kernels in Mondié and Melchor-Aguilar (2012) to obtain stability conditions formulated directly in terms of the coefficients of integral delay systems. The forthcoming paper Ochoa *et al.* (2014) gives a methodology for computing the exact critical delays of the class of integral delay systems with analytic kernels via an auxiliary delay free system. These three works Melchor-Aguilar (2010),Mondié and Melchor-Aguilar (2012) and Ochoa *et al.* (2014) deal only with the unperturbed case of integral delay systems.

In the current paper, we address the robust exponential stability of integral delay systems. We consider integral delay systems with kernels of exponential type subject to norm bounded uncertainties. For these perturbed systems, we derive new conditions for the robust exponential stability expressed in terms of linear matrix inequalities and norm of matrices by using the Lyapunov-Krasovskii functional approach. Some preliminary results in this direction has been reported in Morales-Sánchez and Melchor-Aguilar (2013). We then apply the obtained results to the internal stability problem of infinite-dimensional controllers used for the finite spectrum assignment of input delay systems in the original spirit of Manitius and Olbrot (1979). We present a methodology to design robust non-fragile controllers guaranteeing a numerically safe implementation and, at the same time, robustness to uncertainty in the system matrices and to perturbation in the controller coefficient. The proof follows from new results on quadratic stabilizability of the state-feedback synthesis problem combined with the obtained ones on robust exponential stability of the integral delay systems. To the best of our knowledge, such kind of results has not been presented before in the reported literature about this interesting problem.

The paper is organized as follows: Section 2 presents the precise problem formulation. Some preliminaries are introduced in section 3. Section 4 presents the main results on robust exponential stability of integral delay systems. The results on the design of robust non-fragile controllers for the finite spectrum assignment of input delay systems are given in section 5, and some concluding remarks end the paper.

Notation: Throughout this paper, the Euclidean norm for vectors and the induced matrix norm for matrices are used, both denoted by $\|\cdot\|$. We denote by A^T the transpose of A, I_p and 0_p stand respectively for the $p \times p$ identity and zero matrices, while $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalues of a symmetric matrix A, respectively. For a real symmetric matrix Q, the standard notation $Q > 0$ (respectively, $Q < 0$) is used to denote that Q is positive (respectively negative) definite.

2. Problem Formulation

We consider the following class of integral delay systems:

$$x(t) = \int_{-h}^{0} Ce^{A\theta} Bx(t+\theta)d\theta, \quad \forall t \geq 0, \tag{1}$$

where $C \in \mathbb{R}^{m\times n}, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and $h > 0$.

The class of kernels considered in (1) is not only limited to exponential type. Evidently, the particular case of a constant kernel $G \in \mathbb{R}^{p\times p}$ is included by choosing $A = 0_n$ and $G = CB$ with $m = p$. In fact, the class of analytic kernels considered in Mondié and Melchor-Aguilar (2012) of the form $G^T\mathcal{B}(\theta)$, where $G, \mathcal{B}(\theta) \in \mathbb{R}^{p\times q}$ and the matrix function $\mathcal{B}(\theta)$ satisfies the differential matrix equation

$$\dot{\mathcal{B}}(\theta) = M\mathcal{B}(\theta), \tag{2}$$

for some constant matrix $M \in \mathbb{R}^{p\times p}$, is equivalent to the class of exponential kernels in (1).

To see this, we firstly observe that any matrix function $\mathcal{B}(\theta)$ satisfying (2) is of the form $\mathcal{B}(\theta) = e^{M\theta}\mathcal{B}(0)$. Thus, for $C = G^T, A = M$ and $B = \mathcal{B}(0)$, we have that $G^T\mathcal{B}(\theta) = Ce^{A\theta}B$. Conversely, for an exponential kernel $Ce^{A\theta}B$, by defining $\mathcal{B}(\theta) = e^{A\theta}B$ and $G^T = C$ we have that $\mathcal{B}(\theta)$ satisfies (2) with $M = A$. Of course, here for matrix dimensions compatibility $p = n$ and $q = m$.

For $\varphi \in \mathcal{PC}([-h, 0), \mathbb{R}^m)$, the space of piecewise continuous bounded functions mapping the interval $[-h, 0)$ to $\mathbb{R}^m$, equipped with the norm of uniform convergence $\|\varphi\|_h = \sup_{\theta\in[-h,0)} \|\varphi(\theta)\|$, let $x(t, \varphi)$ be the corresponding solution of (1).

Definition 2.1: Hale and Verduyn-Lunel (1993) System (1) is said to be exponentially stable if there exist $\alpha > 0$ and $\mu > 0$ such that every solution of (1) satisfies the inequality

$$\|x(t, \varphi)\| \leq \mu e^{-\alpha t} \|\varphi\|_h, \quad \forall t \geq 0.$$

Our goal is to derive conditions for the robust exponential stability of (1) by using the Lyapunov-Krasovskii functional approach. More precisely, we address the exponential stability problem of integral delay systems of the form

$$y(t) = \int_{-h}^{0} (C + \Delta C)\, e^{(A+\Delta A)\theta} (B + \Delta B)\, y(t + \theta) d\theta, \tag{3}$$

where $\Delta A, \Delta B$ and ΔC are unknown constant matrices satisfying

$$\|\Delta A\| \leq \rho_A, \|\Delta B\| \leq \rho_B \text{ and } \|\Delta C\| \leq \rho_C. \tag{4}$$

3. Preliminaries

In order to present the Lyapunov-Krasovskii conditions for the exponential stability of (1) given in Melchor-Aguilar *et al.* (2010) we need to introduce a little of terminology.

As usual, we define the natural state of (1) by $x_t(\theta, \varphi) \triangleq x(t + \theta, \varphi), \theta \in [-h, 0)$. Due to the jump discontinuity of the solutions at $t = 0$, see Melchor-Aguilar (2010) for details, it follows that $x_t(\theta, \varphi) \in \mathcal{PC}([-h, 0), \mathbb{R}^m)$ for $t \in [0, h)$, while $x_t(\theta, \varphi) \in \mathcal{C}([-h, 0), \mathbb{R}^m)$ for $t \geq h$. As a consequence, in a Lyapunov-Krasovskii setting, the functionals should be defined on the infinite-dimensional space $\mathcal{PC}([-h, 0), \mathbb{R}^m)$. For simplicity of the notation, one writes $x_t(\varphi)$ instead of $x_t(\theta, \varphi), \theta \in [-h, 0)$. Also when the initial function is irrelevant from the context, we simply write $x(t)$ and x_t instead of $x(t, \varphi)$ and $x_t(\varphi)$.

The following fundamental result gives Lyapunov-Krasovskii conditions for the exponential stability of (1):

Theorem 3.1: *Melchor-Aguilar et al. (2010) System (1) is exponentially stable if there exists a continuous functional $v : \mathcal{PC}([-h, 0), \mathbb{R}^m) \to \mathbb{R}$ such that $t \to v(x_t(\varphi))$ is differentiable and the following conditions hold:*

(1) $\alpha_1 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta \leq v(\varphi) \leq \alpha_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta$, *for some constants* $0 < \alpha_1 \leq \alpha_2$,

(2) $\frac{d}{dt} v(x_t(\varphi)) \leq -\beta \int_{-h}^{0} \|x(t + \theta, \varphi)\|^2 d\theta$, *for a constant* $\beta > 0$.

Based on this result the following sufficient stability condition is obtained in Melchor-Aguilar (2010) by constructing a particular Lyapunov functional satisfying the theorem conditions.

Lemma 3.2: *Melchor-Aguilar (2010) System (1) is exponentially stable if*

$$h\left(\max_{\theta\in[-h,0]}\left\|Ce^{A\theta}B\right\|\right)<1. \tag{5}$$

From (5) one can easily get robust stability conditions for the perturbed system (3).

Lemma 3.3: *The perturbed system described by (3) and (4) is exponentially stable if*

(1) When $\rho_A = 0$ the following inequality holds:

$$h\left(\|C\|+\rho_c\right)\left(\|B\|+\rho_B\right)\left(\max_{\theta\in[-h,0]}\left\|e^{A\theta}\right\|\right)<1. \tag{6}$$

(2) When $\rho_A \neq 0$ the following inequality holds:

$$h\left(\|C\|+\rho_c\right)\left(\|B\|+\rho_B\right)e^{(\|A\|+\rho_A)h}<1. \tag{7}$$

Note that when $\rho_A = 0$ in (7) the inequality does not reduce to (6). Indeed, in the case when $\rho_A = 0$, the inequality (7) is more conservative than (6).

In the following section we will derive additional robust stability condition by constructing special Lyapunov functionals satisfying the conditions of Theorem 3.1 that will complement the above two conditions (6) and (7) based on matrix norms.

4. Main Results

Constructing appropriate Lyapunov functionals for the perturbed system (3) is rather difficult due to the multiplicative way that the perturbations are involved in the exponential kernel. Thus, we will consider an alternative perturbed system which is equivalent to (3) from the stability point of view and it has a more suitable form for the robust stability analysis by means of Lyapunov functionals.

Given matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$, let us consider the integral delay system

$$\tilde{x}(t)=\int_{-h}^{0}BCe^{A\theta}\tilde{x}(t+\theta)d\theta. \tag{8}$$

Remark 1: System (1) is exponentially stable if and only if system (8) is exponentially stable. Indeed, by comparing the characteristic functions associated to (1) and (8) one can easily conclude that their spectrums are equal.

The above remark implies that in despite of the fact that the systems (1) and (8) evolve in different functional spaces, $x_t(\varphi) \in \mathcal{PC}\left([-h,0),\mathbb{R}^m\right)$ while $\tilde{x}_t(\tilde{\varphi}) \in \mathcal{PC}\left([-h,0),\mathbb{R}^n\right)$, they are equivalent from the stability point of view.

Thus, based on these observations, instead of considering the perturbed system (3) we consider the following one:

$$z(t)=\int_{-h}^{0}\left(B+\Delta B\right)\left(C+\Delta C\right)e^{(A+\Delta A)\theta}z(t+\theta)d\theta, \tag{9}$$

where $\Delta A, \Delta B$ and ΔC are unknown constant matrices satisfying (4).

Proposition 4.1: *The perturbed system described by (9) and (4) is exponentially stable if there exist positive definite matrices $P, Q, X, Y,$ and positive scalars $\gamma_1, \gamma_2, \gamma_3$ such that*

$$\mathcal{N}_{n_1}(P,Q,X,Y) - \gamma_1 \mathcal{N}_{p_1}(\rho_A) - \gamma_2 \mathcal{N}_{p_2}(\rho_B, \rho_C) - \gamma_3 \mathcal{N}_{p_3}(\rho_B) > 0, \tag{10}$$

$$\mathcal{N}_{n_2}(P,Q,X,Y) - \gamma_1 \mathcal{N}_{p_1}(\rho_A) - \gamma_2 \mathcal{N}_{p_2}(\rho_B, \rho_C) - \gamma_3 \mathcal{N}_{p_3}(\rho_B) > 0, \tag{11}$$

$$\begin{pmatrix} X & \mathcal{M} \\ \mathcal{M} & \gamma_2 I_n - \mathcal{M} \end{pmatrix} > 0, \tag{12}$$

$$\begin{pmatrix} Y & \mathcal{M} + X \\ \mathcal{M} + X & \gamma_3 I_n - \mathcal{M} - X \end{pmatrix} > 0, \tag{13}$$

$$\gamma_1 I_n - \mathcal{M} > 0, \tag{14}$$

where $\mathcal{M} = P + hQ$ and

$$\mathcal{N}_{n_1}(P,Q,X,Y) = Q + A^T \mathcal{M} + \mathcal{M} A - hC^T B^T \left[\mathcal{M} + X + Y\right] BC, \tag{15}$$

$$\mathcal{N}_{n_2}(P,Q,X,Y) = Q + A^T P + PA - hC^T B^T \left[\mathcal{M} + X + Y\right] BC, \tag{16}$$

$$\mathcal{N}_{p_1}(\rho_A) = 2\rho_A I_n, \tag{17}$$

$$\mathcal{N}_{p_2}(\rho_B, \rho_C) = h\rho_C^2 \left(\|B\| + \rho_B\right)^2 I_n, \tag{18}$$

$$\mathcal{N}_{p_3}(\rho_B) = h\rho_B^2 C^T C. \tag{19}$$

Proof: For any arbitrary $\varphi \in \mathcal{PC}\left([-h, 0), \mathbb{R}^n\right)$, let us consider the following functional:

$$v(\varphi) = \int_{-h}^{0} \varphi^T(\theta) \left(e^{(A+\Delta A)\theta}\right)^T \left[P + (\theta + h) Q\right] e^{(A+\Delta A)\theta} \varphi(\theta) d\theta, \tag{20}$$

where P and Q are $n \times n$ positive definite matrices. From (20) it follows that

$$v(\varphi) \leq \lambda_{\max}(P + hQ) \int_{-h}^{0} \lambda_{\max}\left(\left(e^{(A+\Delta A)\theta}\right)^T e^{(A+\Delta A)\theta}\right) \|\varphi(\theta)\|^2 d\theta,$$

and

$$v(\varphi) \geq \lambda_{\min}(P) \int_{-h}^{0} \lambda_{\min}\left(\left(e^{(A+\Delta A)\theta}\right)^T e^{(A+\Delta A)\theta}\right) \|\varphi(\theta)\|^2 d\theta.$$

Since $e^{(A+\Delta A)\theta}$ is nonsingular for all $\theta \in [-h, 0]$ and any matrices A and ΔA, we have

$$\lambda_{\max}\left(\left(e^{(A+\Delta A)\theta}\right)^T e^{(A+\Delta A)\theta}\right) \geq \lambda_{\min}\left(\left(e^{(A+\Delta A)\theta}\right)^T e^{(A+\Delta A)\theta}\right) > 0.$$

Thus, the functional (20) satisfies the inequalities

$$\alpha_1 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta \leq v(\varphi) \leq \alpha_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta,$$

with $0 < \alpha_1 \leq \alpha_2$ given by

$$\alpha_1 = \lambda_{\min}(P) \min_{\theta \in [-h,0]} \left\{ \lambda_{\min} \left(\left(e^{(A+\Delta A)\theta} \right)^T e^{(A+\Delta A)\theta} \right) \right\},$$

$$\alpha_2 = \lambda_{\max}(P + hQ) \max_{\theta \in [-h,0]} \left\{ \lambda_{\max} \left(\left(e^{(A+\Delta A)\theta} \right)^T e^{(A+\Delta A)\theta} \right) \right\}.$$

The time derivative of the functional (20) along the solutions of the perturbed system (9) is

$$\frac{dv(z_t)}{dt} = \left(\int_{-h}^{0} (B + \Delta B)(C + \Delta C) \xi(\theta) d\theta \right)^T \mathcal{M}(0) \left(\int_{-h}^{0} (B + \Delta B)(C + \Delta C) \xi(\theta) d\theta \right)$$
$$- \int_{-h}^{0} \xi^T(\theta) \left\{ Q + A^T \mathcal{M}(\theta) + \mathcal{M}(\theta) A \right\} \xi(\theta) d\theta - z^T(t-h) \left(e^{-(A+\Delta A)h} \right)^T P \left(e^{-(A+\Delta A)h} \right) z(t-h)$$
$$- \int_{-h}^{0} \xi^T(\theta) \left\{ (\Delta A)^T \mathcal{M}(\theta) + \mathcal{M}(\theta)(\Delta A) \right\} \xi(\theta) d\theta,$$

where $\mathcal{M}(\theta) = P + (\theta + h) Q, \theta \in [-h, 0]$. Here, in order to simplify the notation, we have defined $\xi(\theta) \triangleq e^{(A+\Delta A)\theta} z(t + \theta), \theta \in [-h, 0]$. Now we will derive an upper estimation of the terms involving perturbations in the derivative of the functional. Let us start with the perturbed integral term

$$IP_1 \triangleq - \int_{-h}^{0} \xi^T(\theta) \left\{ (\Delta A)^T \mathcal{M}(\theta) + \mathcal{M}(\theta)(\Delta A) \right\} \xi(\theta) d\theta.$$

We have

$$-\xi^T(\theta) \left\{ (\Delta A)^T \mathcal{M}(\theta) + \mathcal{M}(\theta)(\Delta A) \right\} \xi(\theta) \leq 2 \left\| (\Delta A) \xi(\theta) \right\| \left\| \mathcal{M}(\theta) \xi(\theta) \right\|. \tag{21}$$

Let $\gamma_1 > 0$ such that

$$\mathcal{M}(\theta) < \gamma_1 I_n, \forall \theta \in [-h, 0]. \tag{22}$$

Then the inequality $\|\mathcal{M}(\theta)\xi(\theta)\| \leq \gamma_1 \|\xi(\theta)\|$ holds. Using this inequality and the upper bound for the matrix ΔA in (21) we get the following estimation:

$$IP_1 \leq 2\rho_A \gamma_1 \int_{-h}^{0} \|\xi(\theta)\|^2 d\theta. \tag{23}$$

We now consider the perturbed integral term

$$IP_2 \triangleq \left(\int_{-h}^{0} (B + \Delta B)(C + \Delta C) \xi(\theta) d\theta \right)^T \mathcal{M}(0) \left(\int_{-h}^{0} (B + \Delta B)(C + \Delta C) \xi(\theta) d\theta \right).$$

By using the Jensen integral inequality, the inequality

$$IP_2 \leq h \int_{-h}^{0} \xi^T(\theta) (C + \Delta C)^T (B + \Delta B)^T \mathcal{M}(0) (B + \Delta B)(C + \Delta C) \xi(\theta) d\theta$$

holds. Let

$$\chi(\theta) \triangleq \xi^T(\theta)\left(C+\Delta C\right)^T\left(B+\Delta B\right)^T\mathcal{M}(0)\left(B+\Delta B\right)\left(C+\Delta C\right)\xi(\theta).$$

We have

$$\begin{aligned}\chi(\theta) = {} & \xi^T(\theta)C^T\left(B+\Delta B\right)^T\mathcal{M}(0)\left(B+\Delta B\right)C\xi(\theta)\\ & +2\xi^T(\theta)C^T\left(B+\Delta B\right)^T\mathcal{M}(0)\left(B+\Delta B\right)\left(\Delta C\right)\xi(\theta)\\ & +\xi^T(\theta)\left(\Delta C\right)^T\left(B+\Delta B\right)^T\mathcal{M}(0)\left(B+\Delta B\right)\left(\Delta C\right)\xi(\theta).\end{aligned}$$

Observing that for any positive definite matrix X the inequality

$$\begin{aligned}& 2\xi^T(\theta)C^T\left(B+\Delta B\right)^T\mathcal{M}(0)\left(B+\Delta B\right)\left(\Delta C\right)\xi(\theta)\\ \leq {} & \xi^T(\theta)C^T\left(B+\Delta B\right)^T X\left(B+\Delta B\right)C\xi(\theta)\\ & +\xi^T(\theta)\left(\Delta C\right)^T\left(B+\Delta B\right)^T\mathcal{M}(0)X^{-1}\mathcal{M}(0)\left(B+\Delta B\right)\left(\Delta C\right)\xi(\theta)\end{aligned}$$

holds, we have

$$\begin{aligned}\chi(\theta) \leq {} & \xi^T(\theta)C^TB^T\left[\mathcal{M}(0)+X\right]BC\xi(\theta)+2\xi^T(\theta)C^TB^T\left[\mathcal{M}(0)+X\right]\left(\Delta B\right)C\xi(\theta)\\ & +\xi^T(\theta)C^T\left(\Delta B\right)^T\left[\mathcal{M}(0)+X\right]\left(\Delta B\right)C\xi(\theta)\\ & +\xi^T(\theta)\left(\Delta C\right)^T\left(B+\Delta B\right)^T\left[\mathcal{M}(0)+\mathcal{M}(0)X^{-1}\mathcal{M}(0)\right]\left(B+\Delta B\right)\left(\Delta C\right)\xi(\theta). \quad (24)\end{aligned}$$

Using the inequality

$$\begin{aligned}& 2\xi^T(\theta)C^TB^T\left[\mathcal{M}(0)+X\right]\left(\Delta B\right)C\xi(\theta)\\ \leq {} & \xi^T(\theta)C^TB^TYBC\xi(\theta)+\xi^T(\theta)C^T\left(\Delta B\right)^T\left(\mathcal{M}(0)+X\right)Y^{-1}\left(\mathcal{M}(0)+X\right)\left(\Delta B\right)C\xi(\theta),\end{aligned}$$

where Y is any positive definite matrix, in (24) we get the following estimation for $\chi(\theta)$:

$$\begin{aligned}\chi(\theta) \leq {} & \xi^T(\theta)C^TB^T\left[\mathcal{M}(0)+X+Y\right]BC\xi(\theta)\\ & +\xi^T(\theta)C^T\left(\Delta B\right)^T\left[\mathcal{M}(0)+X+\left(\mathcal{M}(0)+X\right)Y^{-1}\left(\mathcal{M}(0)+X\right)\right]\left(\Delta B\right)C\xi(\theta)\\ & +\xi^T(\theta)\left(\Delta C\right)^T\left(B+\Delta B\right)^T\left[\mathcal{M}(0)+\mathcal{M}(0)X^{-1}\mathcal{M}(0)\right]\left(B+\Delta B\right)\left(\Delta C\right)\xi(\theta). \quad (25)\end{aligned}$$

Let $\gamma_2 > 0$ and $\gamma_3 > 0$ such that

$$\mathcal{M}(0)+\mathcal{M}(0)X^{-1}\mathcal{M}(0) < \gamma_2 I_n, \quad (26)$$

$$\mathcal{M}(0)+X+\left(\mathcal{M}(0)+X\right)Y^{-1}\left(\mathcal{M}(0)+X\right) < \gamma_3 I_n. \quad (27)$$

Then, the inequalities

$$\begin{aligned}& \xi^T(\theta)C^T\left(\Delta B\right)^T\left[\mathcal{M}(0)+X+\left(\mathcal{M}(0)+X\right)Y^{-1}\left(\mathcal{M}(0)+X\right)\right]\left(\Delta B\right)C\xi(\theta)\\ & \leq \gamma_3\left\|\left(\Delta B\right)C\xi(\theta)\right\|^2 \leq \gamma_3\rho_B^2\left\|C\xi(\theta)\right\|^2 \quad (28)\end{aligned}$$

and

$$\xi^T(\theta)(\Delta C)^T(B+\Delta B)^T\left[\mathcal{M}(0)+\mathcal{M}(0)X^{-1}\mathcal{M}(0)\right](B+\Delta B)(\Delta C)\xi(\theta)$$
$$\leq \gamma_2\|(B+\Delta B)(\Delta C)\xi(\theta)\|^2$$
$$\leq \gamma_2\rho_C^2(\|B\|+\rho_B)^2\|\xi(\theta)\|^2 \tag{29}$$

hold. Taking into account the inequalities (28) and (29) in (25) we obtain the following estimation for $\chi(\theta)$:

$$\chi(\theta) \leq \xi^T(\theta)C^TB^T[\mathcal{M}(0)+X+Y]BC\xi(\theta)+\gamma_2\rho_C^2(\|B\|+\rho_B)^2\|\xi(\theta)\|^2+\gamma_3\rho_B^2\|C\xi(\theta)\|^2,$$

which implies that

$$IP_2 \leq h\int_{-h}^{0}\xi^T(\theta)\left\{C^TB^T[\mathcal{M}(0)+X+Y]BC+\gamma_3\rho_B^2C^TC+\gamma_2\rho_C^2(\|B\|+\rho_B)^2I_n\right\}\xi(\theta)d\theta.$$

From this inequality and (23) we arrive at the following upper bound for the derivative of the functional:

$$\frac{dv(z_t)}{dt} \leq -\int_{-h}^{0}\xi^T(\theta)\Gamma(\theta)\xi(\theta)d\theta,$$

where $\Gamma(\theta)\in\mathcal{R}^{n\times n}$ for $\theta\in[-h,0]$ is given by

$$\Gamma(\theta)=Q+A^T\mathcal{M}(\theta)+\mathcal{M}(\theta)A-hC^TB^T[M+X+Y]BC$$
$$-\gamma_1\mathcal{N}_{p_1}(\rho_A)-\gamma_2\mathcal{N}_{p_2}(\rho_B,\rho_C)-\gamma_3\mathcal{N}_{p_2}(\rho_B),$$

with $\mathcal{N}_{p_1}(\rho_A),\mathcal{N}_{p_2}(\rho_B,\rho_C)$ and $\mathcal{N}_{p_3}(\rho_B)$ defined by (17), (18) and (19), respectively.

Clearly, if $\Gamma(\theta)>0,\forall\theta\in[-h,0]$, then there exists

$$\beta=\min_{\theta\in[-h,0]}\left\{\lambda_{\min}\left(e^{(A+\Delta A)\theta}\right)^T\Gamma(\theta)e^{(A+\Delta A)\theta}\right\}>0$$

such that

$$\frac{dv(z_t)}{dt} < -\beta\int_{-h}^{0}\|z(t+\theta)\|^2d\theta,$$

and the exponential stability of the perturbed system (9) is assured by Theorem 3.1.

Now since

$$\left(\frac{\theta+h}{h}\right)\Gamma(0)+\left(-\frac{\theta}{h}\right)\Gamma(-h)=\Gamma(\theta),\forall\theta\in[-h,0],$$

it follows that $\Gamma(\theta)>0,\forall\theta\in[-h,0]$, if, and only if, $\Gamma(0)>0$ and $\Gamma(-h)>0$, see Mondié and Melchor-Aguilar (2012).

By evaluating $\Gamma(\theta)$ for $\theta = 0$ and $\theta = -h$ we have

$$\Gamma(0) = \mathcal{N}_{n_1}(P, Q, X, Y) - \gamma_1 \mathcal{N}_{p_1}(\rho_A) - \gamma_2 \mathcal{N}_{p_1}(\rho_B, \rho_C) - \gamma_3 \mathcal{N}_{p_2}(\rho_B),$$
$$\Gamma(-h) = \mathcal{N}_{n_2}(P, Q, X, Y) - \gamma_1 \mathcal{N}_{p_1}(\rho_A) - \gamma_2 \mathcal{N}_{p_1}(\rho_B, \rho_C) - \gamma_3 \mathcal{N}_{p_2}(\rho_B),$$

where $\mathcal{N}_{n_1}(P, Q, X, Y)$ and $\mathcal{N}_{n_2}(P, Q, X, Y)$ are respectively defined by (15) and (16).

$\Gamma(0) > 0$ and $\Gamma(-h) > 0$ lead to the inequalities (10) and (11).

Observing that, by Schur complement, the inequalities (26) and (27) are respectively equivalent to (12) and (13), and that the inequality (14), i.e.,

$$\mathcal{M}(0) = \mathcal{M} = P + hQ < \gamma_1 I_n,$$

implies (22), the proof ends. ■

Remark 2: If there exist positive definite matrices P, Q, X, Y and positive scalars $\gamma_1, \gamma_2, \gamma_3$ such that the inequalities (10)-(14) hold then the perturbations $\Delta A, \Delta B, \Delta C$ as well as the delay h may be time-varying and/or state depending. The only assumption needed for the perturbations functions $\Delta A(t), \Delta B(t), \Delta C(t)$ is that they should be continuous and bounded in norm while for the delay function $h(t)$ the continuity and the bounding $0 < h(t) \leq h$ are required.

4.1. *Nominal case*

In the nominal case, when (9) does not have uncertainty

$$z(t) = \int_{-h}^{0} BCe^{A\theta} z(t+\theta) d\theta, \tag{30}$$

we have the following result:

Corollary 4.2: *The integral delay system (30) is exponentially stable if there exist positive definite matrices P and Q such that*

$$Q + A^T \mathcal{M} + \mathcal{M} A - hC^T B^T \mathcal{M} BC > 0, \tag{31}$$
$$Q + A^T P + PA - hC^T B^T \mathcal{M} BC > 0, \tag{32}$$

where $\mathcal{M} = P + hQ$.

Proof: Since we have that $\rho_A = \rho_B = \rho_C = 0$ it follows from (17), (18) and (19) that the matrices $\mathcal{N}_{p_1}(\rho_A)$, $\mathcal{N}_{p_2}(\rho_B, \rho_C)$ and $\mathcal{N}_{p_3}(\rho_B)$ are equal to zero and, therefore, the inequalities (10) and (11) hold for any arbitrary positive constants γ_1, γ_2 and γ_3.

Selecting γ_1 sufficiently large the restriction imposed on matrix $\mathcal{M}$ by the inequality (14), i.e., $\gamma_1 I_n > \mathcal{M}$, can be removed. Now, by Schur complement, the inequality (12) is equivalent to

$$\gamma_2 I_n > \mathcal{M} + \mathcal{M} X^{-1} \mathcal{M}.$$

Since γ_2 can be arbitrarily chosen the above inequality holds for $X = \varepsilon_1 I$ with $\varepsilon_1 > 0$ sufficiently small and $\gamma_2 > 0$ sufficiently large. Similarly, the inequality (13) is equivalent to

$$\gamma_3 I_n > (\mathcal{M} + X) + (\mathcal{M} + X) Y^{-1} (\mathcal{M} + X).$$

Table 1. Maximum delay value for different perturbation bounds ρ_A in Example 4.3

ρ_A	0.5	1	2	4
Norm condition (7)	0.292	0.274	0.245	0.205
Proposition (4.1)	0.616	0.402	0.228	0.120

Again, since γ_3 can be arbitrarily chosen the above inequality holds for $Y = \varepsilon_2 I$ with $\varepsilon_2 > 0$ sufficiently small and $\gamma_3 > 0$ sufficiently large.

By doing $\varepsilon_1 \to +0$ and $\varepsilon_2 \to +0$, the restrictions given by the inequalities (12) and (13) can be removed while the matrices $\mathcal{N}_{n_1}(P,Q,X,Y)$ and $\mathcal{N}_{n_2}(P,Q,X,Y)$ respectively become (31) and (32). ■

Remark 3: From the Remark **1** and the equivalence between the exponential kernel $Ce^{A\theta}B$ and those of the form $G^T\mathcal{B}(\theta)$, where $\mathcal{B}(\theta)$ satisfies the matrix differential equation (2), it follows that Corollary 4.2 summarizes the results reported in Mondié and Melchor-Aguilar (2012). In fact, by substituting $C = G^T$, $A = M$ and $B = \mathcal{B}(0)$ in (31) and (32) we arrive at the inequalities (12) and (13), for $\beta = 0$, in Theorem 7 of Mondié and Melchor-Aguilar (2012).

Remark 4: Corollary 4.2 shows that the assumption (3) imposed to $\mathcal{B}(\theta)$ in Mondié and Melchor-Aguilar (2012) in order to assure that the proposed functionals satisfy the condition (1) of Theorem 3.1, namely, that there exists $\gamma > 0$ such that $\gamma < \lambda_{\min}\left\{\mathcal{B}^T(\theta)\mathcal{B}(\theta)\right\}, \forall\theta \in [-h, 0]$, is indeed not required for the exponential stability of the integral delay systems.

The Proposition 4.1 provides robust stability conditions when there exist perturbations on all system matrices A, B and C. Of course, in certain applications, one could have the situation when not all system matrices but only some of them are subject to perturbations. In such cases one can derive robust stability conditions from Proposition 4.1 and the proof of Corollary 4.2 by considering as zero the corresponding combination of perturbations, see Morales-Sánchez and Melchor-Aguilar (2013) for some examples illustrating the above.

Example 4.3 Let us consider the following perturbed integral delay system:

$$y(t) = \int_{-h}^{0} e^{(A+\Delta A)\theta} y(t+\theta)\, d\theta, \tag{33}$$

where $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ and ΔA an unknown matrix satisfying $\|\Delta A\| \le \rho_A$.

The nominal system (33), corresponding to $\rho_A = 0$, was investigated in Mondié and Melchor-Aguilar (2012). The maximum delay value for exponential stability obtained by using the norm condition (5) is $h_{\max} = 0.988$, by means of Corollary 4.2 (Corollary 9 in Mondié and Melchor-Aguilar (2012)) one gets $h_{\max} = 1.999$, while that the critical delay obtained by using the methodology proposed in Ochoa *et al.* (2014) is $h^* \approx 38.529$.

In Table 1 we present the maximum delay value for exponential stability of the perturbed system (33) for different perturbations bounds ρ_A computed by means of the norm condition (7) and the linear matrix inequalities in Proposition 4.1.

It can be seen that for small perturbation bounds the Proposition 4.1 provides better results than the norm condition (7), but for large perturbation bounds the situation is the inverse one, i.e., the norm condition (7) gives a better result than the Proposition 4.1.

One could think that the aforementioned conclusion is due to a certain amount of conservatism in our robust stability results since Melchor-Aguilar (2010) and Mondié and Melchor-Aguilar (2012) give numerical examples illustrating a superiority of the linear matrix inequalities conditions on the norm-based conditions for nominal systems. Nevertheless, the following example

Table 2. Maximum delay value for different perturbation bounds $\rho = \rho_A = \rho_B = \rho_C$ in Example 4.4

ρ	0.01	0.1	1	2
Norm condition (7)	0.835	0.700	0.2160	0.099
Proposition (4.1)	1.212	0.822	0.194	0.080

shows that indeed one can have such a kind of conclusion even in the nominal case.

Consider the nominal integral delay system

$$z(t) = \int_{-h}^{0} Ce^{A\theta} z(t+\theta)\, d\theta, \tag{34}$$

where A is the same matrix in (33) and $C = \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix}$.

The maximum delay value for exponential stability of (34) obtained from the norm condition (5) is $h_{\max} = 0.1850$ that it is significantly better than $h_{\max} = 0.1420$ obtained from the linear matrix inequalities conditions in Corollary 4.2. For comparison, the critical delay computed from the approach in Ochoa *et al.* (2014) is $h^* \approx 41.537$.

Example 4.4 Now let us consider the perturbed scalar integral delay system

$$y(t) = \int_{-h}^{0} (C + \Delta C)\, e^{(A+\Delta A)\theta} (B + \Delta B)\, y(t+\theta)\, d\theta, \tag{35}$$

where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0.5 \end{bmatrix}$, and $\Delta A, \Delta B, \Delta C$ are unknown satisfying (4).

The nominal system (35) corresponding to $\Delta A = \Delta B = \Delta C = 0$ and $C = B = I_n$ was studied in Mondié and Melchor-Aguilar (2012), where was shown that the linear matrix inequalities conditions of Corollary 4.2 (Corollary 9 in Mondié and Melchor-Aguilar (2012)) gives $h_{\max} = 0.999$ which coincides with the exact critical delay value and improves $h_{\max} = 0.707$ obtained from the norm condition (5).

For the nominal system (35), when $\Delta A = \Delta B = \Delta C = 0$, we can compute the critical delay value for exponential stability by means of frequency domain techniques. Simple calculations based on the characteristic function

$$f(s) = 1 - 0.5 \left(\frac{1 - e^{-hs}}{s} \right),$$

leads to the critical delay value $h^* = 2$. From the norm condition (5) we get the maximum delay $h_{\max} = 2$, that coincides with the critical delay one, while that by using Corollary 4.2 we obtain $h_{\max} = 1.414$.

In Table 2 we present the maximum delay value for exponential stability of the perturbed system (35) for different perturbations bounds ρ, where $\rho = \rho_A = \rho_B = \rho_C$, computed by means of the norm condition (7) and Proposition 4.1. Similarly to that reported in Table 2 for the example 4.3, for small perturbation bounds the Proposition 4.1 provides better results than the norm condition (7), but for large perturbation bounds we have the inverse situation.

These numerical examples show that, in general, we cannot conclude about the superiority of the linear matrix inequalities conditions given in Proposition 4.1 or Corollary 4.2 (Corollary 9 in Mondié and Melchor-Aguilar (2012)) on the norm conditions (5), (6), (7) given in Lemmas 3.3 and 3.2 respectively.

Finally to end this section we would like to recall that in despite of the conservatism of the norm and linear matrix inequality conditions they allow delays and perturbations to be functions of time and/or the state in counterpart with the results for deriving the critical parameters based frequency domain techniques.

5. Robust non-fragile controllers for FSA of input delay systems

In this section we address the robust non-fragile controller problem for the finite spectrum assignment (FSA) of input delay systems extending thus the results for the nominal case considered in Mondié and Melchor-Aguilar (2012).

As it is mentioned in the introduction section, integral delay systems with exponential kernels of the form in (1) play a fundamental role in the internal stability problem of feedback schemes involving delay compensation. We here address the internal stability problem of one of such feedback schemes used for the FSA of input delay systems.

This problem concerns with the following feedback control problem:

$$\dot{x}(t) = Ax(t) + Bu(t-h), \tag{36}$$

$$u(t) = C\left(e^{Ah}x(t) + \int_{-h}^{0} e^{-A\theta}Bu(t+\theta)\,d\theta\right), \tag{37}$$

where $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ represents the control input.

Since the seminal work Manitius and Olbrot (1979) was published a long time ago it was established that if all system parameters, controller gain and delay value are exactly known then the spectrum of the closed-loop system (36)-(37) is finite and coincides with that of the matrix $A+BC$, but when the system parameters and/or the delay value are not known precisely, and/or the controller coefficients are not exactly equal to those computed from the design method, then the finite closed-loop spectrum is not preserved as far as the closed-loop system becomes again a delay system with an infinite spectrum. However, as it was also proved in Manitius and Olbrot (1979), the stability of the resulting closed-loop delay system is preserved under small perturbations bounds.

It is also well-known that there is another robustness problem concerning the numerical implementation of the controller (37) Michiels *et al.* (2004), Mondié and Michiels (2003). It has been demonstrated that a necessary and sufficient condition for a numerically safe implementation of the control law (37) is the Hurwitz stability of $A+BC$ and the stability of the internal dynamics of the controller described by the integral delay system

$$z(t) = \int_{-h}^{0} Ce^{-A\theta}Bz(t+\theta)\,d\theta.$$

The aforementioned robustness problems have been extensively studied in the literature and are still investigated. For instance, the robustness to uncertainties in the delay value has been studied in Michiels and Niculescu (2003) and Mondié *et al.* (2001) while the robustness w.r.t. the numerical implementation has been investigated in Michiels *et al.* (2004) and Mondié and Michiels (2003), see also the survey paper Richard (2003) and the references therein.

However, to the best of our knowledge, one does not find a contribution affronting the most difficult problem of designing a controller (37) that not only admit system parameters uncertainty and tolerate controller perturbations, but also has a numerically safe implementation. Our robust stability results on integral delay systems allow us present here a solution of such a complicated robust synthesis problem.

The robust synthesis problem can be precisely formulated as follows: to provide a methodology for designing the matrix gain C of the controller (37) such that the following is satisfied:

1) $(A+\Delta A)+(B+\Delta B)(C+\Delta C)$ is Hurwitz, and
2) The perturbed integral delay system

$$y(t)=\int_{-h}^{0}(C+\Delta C)\,e^{-(A+\delta A)\theta}\,(B+\delta B)\,y\,(t+\theta)\,d\theta \tag{38}$$

is exponentially stable, for all perturbations $\Delta A, \Delta B, \Delta C$ satisfying (4) and $\delta A, \delta B$ satisfying

$$\|\delta A\| \leq \kappa_A \text{ and } \|\delta B\| \leq \kappa_B. \tag{39}$$

The first issue can be formulated under the existence of a positive definite matrix R such that

$$\begin{aligned}[A+BC]^T R + R\,[A+BC] + [\Delta A + B\Delta C + \Delta BC + \Delta B\Delta C]^T R \\ +R\,[\Delta A + B\Delta C + \Delta BC + \Delta B\Delta C] < 0\end{aligned} \tag{40}$$

After pre- and post multiply the inequality (40) by the matrix $S=R^{-1}$ and then setting $L=CS$ we get

$$\begin{aligned}SA^T + AS + L^T\left[B^T+(\Delta B)^T\right] + [B+(\Delta B)]\,L + \left[S\,(\Delta A)^T + (\Delta A)\,S\right] \\ +\left[S\,(\Delta C)^T B^T + B\,(\Delta C)\,S\right] + \left[S\,(\Delta C)^T(\Delta B)^T + (\Delta B)\,(\Delta C)\,S\right] < 0.\end{aligned} \tag{41}$$

Lemma 5.1: *If there exist a positive definite matrix S and positive scalars σ and λ such that*

$$SA^T + AS - \sigma BB^T + \sigma\rho_B\,\|B\|\,I_n + \lambda\eta(\rho_A,\rho_B,\rho_C)I_n < 0, \tag{42}$$
$$S-\lambda I_n < 0, \tag{43}$$

where

$$\eta(\rho_A,\rho_B,\rho_C) = 2\left(\rho_A + \rho_C\,\|B\| + \rho_C\rho_B\right), \tag{44}$$

then the inequality (41) holds with $L=-\frac{\sigma}{2}B^T$.

Proof: We firstly observe that the inequalities

$$-\frac{\sigma}{2}\left[B\,(\Delta B)^T + (\Delta B)\,B^T\right] \leq \sigma\rho_B\,\|B\|\,I_n,$$

and, for $S<\lambda I_n$,

$$\begin{aligned}S\,(\Delta A)^T + (\Delta A)\,S &\leq 2\lambda\rho_A I_n,\\ S\,(\Delta C)^T B^T + B\,(\Delta C)\,S &\leq 2\lambda\,\|B\|\,\rho_C I_n,\\ S\,(\Delta C)^T(\Delta B)^T + (\Delta B)\,(\Delta C)\,S &\leq 2\lambda\rho_C\rho_B I_n,\end{aligned}$$

hold. It then follows that

$$
\begin{aligned}
& SA^T + AS - \sigma BB^T - \frac{\sigma}{2}\left[B(\Delta B)^T + (\Delta B)B^T\right] + S(\Delta A)^T + (\Delta A)S \\
& + S(\Delta C)^T B^T + B(\Delta C)S + S(\Delta C)^T(\Delta B)^T + (\Delta B)(\Delta C)S \\
\leq\; & SA^T + AS - \sigma BB^T + \sigma\rho_B \|B\| I_n + \lambda\eta(\rho_A, \rho_B, \rho_C)I_n. \qquad (45)
\end{aligned}
$$

Simple calculations show that

$$
-\sigma BB^T - \frac{\sigma}{2}\left[B(\Delta B)^T + (\Delta B)B^T\right] = -\frac{\sigma}{2}B\left[B^T + (\Delta B)^T\right] - \frac{\sigma}{2}\left[B + (\Delta B)\right]B^T.
$$

This equality allows us to rewrite the left-hand side of (45) as

$$
\begin{aligned}
& SA^T + AS + L^T\left(B^T + (\Delta B)^T\right) + (B + (\Delta B))L \qquad (46) \\
& + S(\Delta A)^T + (\Delta A)S + S(\Delta C)^T B^T + B(\Delta C)S + S(\Delta C)^T(\Delta B)^T + (\Delta B)(\Delta C)S,
\end{aligned}
$$

where $L = -\frac{\sigma}{2}B^T$. From (46) and the inequality (45) the result follows. ■

Remark 1 : In the nominal case, when $\rho_A, \rho_B, \rho_C = 0$, the inequalities (42) and (43) reduces to $SA^T + AS - \sigma BB^T < 0$, while (41) takes the form $SA^T + AS + L^T B^T + BL < 0$, and it is well-known that these two inequalities are equivalent by virtue of the elimination procedure and the Finsler's lemma, see Boyd *et al.* (1994).

By noting that (42) is homogeneous in S, σ and λ we can fix $\sigma = 2$ and thus reduce the number of variables by one. From Lemma 5.1 we directly obtain the following result of interest in its own right:

Proposition 5.2: *Consider the state-feedback synthesis problem*

$$
\dot{x} = (A + \Delta A)x + (B + \Delta B)u, \qquad (47)
$$

$$
u = (C + \Delta C)x, \qquad (48)
$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times n}$ are known matrices and $\Delta A, \Delta B, \Delta C$ are unknown perturbations matrices satisfying (4). If there exist a positive definite matrix S and a positive scalar λ such that

$$
SA^T + AS - 2BB^T + 2\rho_B \|B\| I_n + \lambda\eta(\rho_A, \rho_B, \rho_C)I_n < 0,
$$

$$
S < \lambda I_n,
$$

where $\eta(\rho_A, \rho_B, \rho_C)$ is given by (44), then the feedback gain

$$
C = -B^T S^{-1}, \qquad (49)
$$

assures that the perturbed closed-loop system (47)-(48) is quadratically stable for all perturbations $\Delta A, \Delta B$ and ΔC satisfying (4).

Remark 2 : The Proposition 5.2 provides a method of designing a controller that is robust not only to system parameters uncertainties but also to controller perturbations which is referred to as a robust non-fragile controller Keel and Bhattacharyya (1997). The result is essentially different to other existing approaches as, for instance, Famularo *et al.* (2000), Haddad and

Corrado (2000), Yang *et al.* (2000), Yang and Wang (2001), since it considers simultaneously perturbations in all system matrices and controller gain and under the absence of perturbations the design becomes the standard well-known conditions for stabilizability in the nominal case reported in Boyd *et al.* (1994).

Now we need to address the second issue of our robust synthesis problem, i.e., to assure the exponential stability of the perturbed integral delay system (38). To this end, we substitute (49) in the corresponding inequalities (10) and (11) of Proposition 4.1 for the robust exponential stability of the perturbed system (38) for getting

$$\begin{aligned}
&Q - A^T\mathcal{M} - \mathcal{M}A - hS^{-1}BB^T\left[\mathcal{M} + X + Y\right]BB^TS^{-1} \\
&\quad -\gamma_1\mathcal{N}_{p_1}\left(\kappa_A\right) - \gamma_2\mathcal{N}_{p_2}\left(\kappa_B,\rho_C\right) - \gamma_3 h\kappa_B^2 S^{-1}BB^TS^{-1} > 0, \\
&\quad Q - A^TP - PA - hS^{-1}BB^T\left[\mathcal{M} + X + Y\right]BB^TS^{-1} \\
&\quad -\gamma_1\mathcal{N}_{p_1}\left(\kappa_A\right) - \gamma_2\mathcal{N}_{p_2}\left(\kappa_B,\rho_C\right) - \gamma_3 h\kappa_B^2 S^{-1}BB^TS^{-1} > 0,
\end{aligned}$$

where $\mathcal{N}_{p_1}\left(\kappa_A\right), \mathcal{N}_{p_2}\left(\kappa_B,\rho_C\right)$ are respectively given by (17) and (18).

By introducing a new matrix variable Z and a positive scalar λ_2 such that $Z > S^{-1}$ and $\left[\mathcal{M} + X + Y\right]^{-1} > \lambda_2 I_n$ the above two inequalities can be replaced by

$$\begin{aligned}
&Q - A^T\mathcal{M} - \mathcal{M}A - h\lambda_2^{-1}ZBB^TBB^TZ - \gamma_1\mathcal{N}_{p_1}\left(\kappa_A\right) - \gamma_2\mathcal{N}_{p_2}\left(\kappa_B,\rho_C\right) - \gamma_3 h\kappa_B^2 ZBB^TZ > 0, \\
&Q - A^TP - PA - h\lambda_2^{-1}ZBB^TBB^TZ - \gamma_1\mathcal{N}_{p_1}\left(\kappa_A\right) - \gamma_2\mathcal{N}_{p_2}\left(\kappa_B,\rho_C\right) - \gamma_3 h\kappa_B^2 ZBB^TZ > 0, \\
&\begin{pmatrix} Z & I_n \\ I_n & S \end{pmatrix} > 0, \begin{pmatrix} \mathcal{M} + X + Y & I_n \\ I_n & \lambda_2 I_n \end{pmatrix} < 0.
\end{aligned}$$

From these inequalities and Proposition 5.2 we can directly establish the following result that provides a solution to the robust non-fragile synthesis problem of controllers for the FSA of input delay systems:

Proposition 5.3: *Let there exist positive definite matrices P, Q, X, Y, S, Z, and positive scalars $\gamma_1, \gamma_2, \gamma_3, \lambda_1, \lambda_2$, such that*

$$\begin{aligned}
&\begin{pmatrix} X & \mathcal{M} \\ \mathcal{M} & \gamma_2 I_n - \mathcal{M} \end{pmatrix} > 0, \qquad \begin{pmatrix} Y & \mathcal{M} + X \\ \mathcal{M} + X & \gamma_3 I_n - \mathcal{M} - X \end{pmatrix} > 0, \\
&\begin{pmatrix} Z & I_n \\ I_n & S \end{pmatrix} > 0, \qquad \gamma_1 I_n - \mathcal{M} > 0, \\
&S - \lambda_1 I_n < 0, \qquad \lambda_2\left(\mathcal{M} + X + Y\right) - I_n < 0, \\
&SA^T + AS - 2BB^T + 2\rho_B\left\|B\right\| I_n + \lambda_1\eta(\rho_A,\rho_B,\rho_C)I_n < 0,
\end{aligned}$$

where $\mathcal{M} = P + hQ, \eta(\rho_A,\rho_B,\rho_C)$ is defined by (44) and

Table 3. Maximum delay value, feedback gain and ideal closed-loop eigenvalues for different perturbation bounds ρ_A,ρ_B and ρ_C

$(\rho_A,\ \rho_B,\ \rho_C)$	(0.1, 0.1, 0)	(0.1, 0.1, 0.05)	(0.1, 0.1, 0.1)
h_{max}	0.549	0.324	0.231
C	(-0.490, -1.261)	(-0.855, -2.2094)	(-1.599, -3.233)
$s_{1,2}$	$-0.630 \pm 0.303i$	$-1.708, -0.500$	$-2.624, -0.609$

(1) When $\kappa_B \neq 0$, the following inequalities hold:

$$\begin{pmatrix} Q - A^T\mathcal{M} - \mathcal{M}A - \gamma_1\mathcal{N}_{p_1}(\kappa_A) - \gamma_2\mathcal{N}_{p_2}(\kappa_B,\rho_C) & 0_n & ZB \\ 0_n & \frac{\lambda_2}{h}I_n + \frac{1}{\gamma_3 h\kappa_B^2}BB^T & -\frac{1}{\gamma_3 h\kappa_B^2}B \\ B^TZ & -\frac{1}{\gamma_3 h\kappa_B^2}B^T & \frac{1}{\gamma_3 h\kappa_B^2}I_m \end{pmatrix} > 0,$$

$$\begin{pmatrix} Q - A^TP - PA - \gamma_1\mathcal{N}_{p_1}(\kappa_A) - \gamma_2\mathcal{N}_{p_2}(\kappa_B,\rho_C) & 0_n & ZB \\ 0_n & \frac{\lambda_2}{h}I_n + \frac{1}{\gamma_3 h\kappa_B^2}BB^T & -\frac{1}{\gamma_3 h\kappa_B^2}B \\ B^TZ & -\frac{1}{\gamma_3 h\kappa_B^2}B^T & \frac{1}{\gamma_3 h\kappa_B^2}I_m \end{pmatrix} > 0,$$

where $\mathcal{N}_{p_1}(\kappa_A)$, $\mathcal{N}_{p_2}(\kappa_B,\rho_C)$, are respectively given by (17) and (18).

(2) When $\kappa_B = 0$, the following inequalities hold:

$$\begin{pmatrix} Q - A^T\mathcal{M} - \mathcal{M}A - \gamma_1\mathcal{N}_{p_1}(\kappa_A) - \gamma_2\mathcal{N}_{p_2}^0(\rho_C) & ZBB^T \\ BB^TZ & \frac{\lambda_2}{h}I_n \end{pmatrix} > 0,$$

$$\begin{pmatrix} Q - A^T\mathcal{M} - \mathcal{M}A - \gamma_1\mathcal{N}_{p_1}(\kappa_A) - \gamma_2\mathcal{N}_{p_2}^0(\rho_C) & ZBB^T \\ BB^TZ & \frac{\lambda_2}{h}I_n \end{pmatrix} > 0,$$

where $\mathcal{N}_{p_1}(\kappa_A)$ is defined by (17) and $\mathcal{N}_{p_2}^0(\rho_C) = h\rho_C^2\|B\|^2 I_n$.

Then a feedback gain guaranteeing a robust non-fragile controller (37), that assures the exponential stability of the closed-loop system (36)-(37) for all perturbations $\Delta A, \Delta B, \Delta C$ satisfying (4), $\delta A, \delta B$ satisfying (39) and, at the same time, a numerically safe implementation, is given by $C = -B^TS^{-1}$.

Example 5.4 Consider the input delay system (36) with system matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and perturbations $\Delta A, \Delta B$ satisfying $\|\Delta A\| \leq \rho_A$ and $\|\Delta B\| \leq \rho_B$.

Let us consider the controller (37) having perturbations $\delta A, \delta B, \rho_C$, satisfying $\|\delta A\| \leq \kappa_A, \|\delta B\| \leq \kappa_B, \|\Delta C\| \leq \rho_c$, on the controller parameters A, B, C respectively. We address the problem of searching for delay h and feedback matrix gain C such that a robust non-fragile controller (37) having a numerically safe implementation can be achieved.

For simplicity of the calculations we consider that the uncertainties on the system matrices and perturbations on the controller parameters corresponding to matrices A and B have the same upper bound, i.e., $\rho_A = \kappa_A, \rho_B = \kappa_B$.

The maximum delay value $h_{\max}$ and the corresponding gain C, computed by solving the inequalities in Proposition 5.3 for different perturbation bounds ρ_A, ρ_B and ρ_C, as well as the ideal closed-loop eigenvalues s_{12} are displayed in Table 3.

6. Conclusions

New conditions for the robust exponential stability of integral delay systems with an exponential kernel are derived by using the Lyapunov-Krasovskii functional approach. Sufficient delay-dependent robust conditions expressed as linear matrix inequalities and norm of matrices are given. The linear matrix inequalities conditions extend to the perturbed case the previous ones reported in Mondié and Melchor-Aguilar (2012) for the unperturbed case. In doing the extension, a restrictive condition imposed on the class of analytic kernels considered in Mondié and Melchor-Aguilar (2012) has been removed.

It is shown by some numerical examples that in some cases the linear matrix inequality conditions are better than the conditions based on matrix norms but that there are also integral delay systems for which the norm conditions provide better results than the linear matrix inequalities ones. As a consequence, we can conclude that both the norm and matrix inequality conditions are in fact complementary.

The combination of the results with a new result on quadratic stabilizability of state-feedback controllers allows us give a new linear matrix inequality methodology of designing a robust non-fragile controller for the finite spectrum assignment of input delay systems that assures simultaneously a numerical safe implementation and robustness to perturbations in system parameters and controller coefficients.

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