Robust Stability of Integral Delay Systems with Exponential Kernels

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Abstract In this chapter the stability analysis via Lyapunov-Krasovskii method is extended to perturbed integral delay systems with exponential kernels. Several sufficient robust stability conditions given in the form of linear matrix inequalities are derived.

1 Introduction

There are several stability problems in differential delay systems which involve the stability of a special class of dynamic systems which are described by integral delay equations. Problems such as the stability of additional dynamics introduced by some system transformations [2], [5–7] and the stability of some difference operators in neutral functional differential equations [3] are examples where the stability of integral delay equations play an essential role.

Another source of problems where integral delay equations can be found is in the design of feedback schemes involving delay compensation as the finite spectrum assignment [9], stabilization problems [10], [20], and optimal control [19], [14], [12] of differential delay systems.

In these feedback schemes the compensators necessarily include an infinite-dimensional dynamic governed by an integral delay equation and it has been shown that the practical implementation of the compensators demand their internal stability, i.e., the stability of the corresponding integral delay equation, see [1, 8, 15] for details.

Recently in [11], Lyapunov-Krasovskii theorems for the exponential stability of integral delay systems have been introduced. It has been shown there that a new type of Lyapunov functionals is required in order to properly address the dynamics of such class of systems. A constructive converse Lyapunov theorem was also pre-

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sented and general expressions of quadratic functionals with a given time derivative were provided. The proposed functionals were used for calculations of robustness bounds and exponential estimates for the solutions of exponentially stable integral delay systems.

However, there are still some technical problems associated with the positivity check of such functionals limiting their practical application to the stability analysis of integral delay system. Motivated from these limitations some reduced type functionals were constructed in [13] to obtain stability conditions formulated directly in terms of the coefficients of integral delay systems. Following these works the recent paper [16] applies the Lyapunov-Krasovskii functional approach to some classes of integral delay systems with analytic kernels and delay-dependent conditions for unperturbed systems were provided.

In this chapter, we continue in the direction of [13], [16] and extend to perturbed integral delay systems with exponential kernels the Lyapunov-Krasovskii methodology. Some preliminary results in this direction has been reported in [17].

The chapter is organized as follows: Section 2 presents the problem formulation. Some preliminaries are introduced in section 3. The main results are given in section 4. Illustrative examples are given in section 5, and some concluding remarks end the contribution.

Notation: Throughout this chapter, the Euclidean norm for vectors and the induced matrix norm for matrices are used, both denoted by $\|\cdot\|$. We denote by A^T the transpose of A, I_p stands for the $p \times p$ identity matrix, while $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalues of a symmetric matrix A, respectively. For a real symmetric matrix Q, the standard notation $Q > 0$ (respectively, $Q < 0$) is used to denote that Q is positive (respectively negative) definite.

2 Problem Formulation

We consider the following class of integral delay systems:

$$x(t) = \int_{-h}^{0} Ce^{A\theta} Bx(t+\theta)d\theta, \quad \forall t \geq 0, \tag{1}$$

where $C \in \mathbb{R}^{m\times n}, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and $h > 0$.

As it is mentioned in the introduction section, integral delay systems with exponential kernels of the form in (1) play a fundamental role in the internal stability problem of feedback schemes involving delay compensation.

One of such problems that has received a sustained attention during the last years, see for instance [1,8,15], and that was proposed as an interesting open problem in the survey paper [18], is the internal stability problem of the finite spectrum assignment scheme for input delay systems.

This problem concerns with input delay systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t-h), \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ represents the control input. The control law

$$u(t) = C\left(e^{Ah}x(t) + \int_{-h}^{0} e^{-A\theta}Bu(t+\theta)\,d\theta\right) \tag{3}$$

assigns a finite spectrum to the delay free closed-loop system (2)-(3) which, under the absence of perturbations, coincides with the spectrum of the matrix $A+BC$ [9].

It was demonstrated in [1] that if the integral is approximated by a finite sum (by using any type of integration rule) then the closed-loop system may become unstable if the ideal controller is not internally stable.

Indeed, it results that the stability of the integral delay system (1) and that of the ideal closed-loop system, i.e., $A+BC$ is a Hurwitz matrix, is a necessary and sufficient condition for a numerically safe implementation of the controller [8].

In this context, if one aims at applying the control law (3) to real systems of the form (2) then one needs to assure the robust stability of the internal dynamics of the controller (3). In other words, in practical applications, we affront the exponential stability problem of integral delay systems of the form

$$y(t) = \int_{-h}^{0} (C + \Delta C)\, e^{(A+\Delta A)\theta} (B + \Delta B)\, y(t+\theta)d\theta, \tag{4}$$

where ΔA, ΔB and ΔC are unknown constant matrices satisfying

$$\|\Delta A\| \leq \rho_A, \|\Delta B\| \leq \rho_B \text{ and } \|\Delta C\| \leq \rho_C. \tag{5}$$

To consider uncertainties on the matrices A and B obey to the fact that these matrices come from the input delay system (2) and they depend on physical parameters which may be subject to uncertainties and perturbations.

On the other hand, the motivation of considering uncertainties on the matrix C is due to the fact that, in practice, it could be necessary to adjust the nominal designed feedback gain in order to achieve a desired closed-loop performance and also consider possible round errors and finite length word implementation on a digital computer of the controller. This issue of sensitivity on the gain matrix C is referred to as fragility analysis in the control literature, see for instance [4].

Our goal is to derive conditions guaranteeing the exponential stability of (4) for all perturbations ΔA, ΔB and ΔC satisfying (5) by means of the Lyapunov-Krasovskii functional methodology.

3 Preliminaries

In order to define a particular solution of (1) an initial vector function $\varphi(\theta), \theta \in [-h, 0)$ should be given. We assume that $\varphi \in \mathcal{PC}([-h, 0), \mathbb{R}^m)$, the space of piecewise continuous bounded functions mapping the interval $[-h,0)$ to $\mathbb{R}^m$, equipped with the norm of uniform convergence $\|\varphi\|_h = \sup_{\theta\in[-h,0)} \|\varphi(\theta)\|$.

Given any initial function $\varphi \in \mathcal{PC}\left([-h,0), \mathbb{R}^m\right)$, there exists a unique solution $x(t,\varphi)$ of (1) which is defined for all $t \in [-h,\infty)$. This solution is continuous for all $t > 0$ and at $t = 0$ presents a jump discontinuity given by

$$\Delta x(0,\varphi) \triangleq x(0,\varphi) - x(-0,\varphi) = \int_{-h}^{0} Ce^{A\theta}B\varphi(\theta)d\theta - \varphi(-0).$$

Definition 1. [3] System (1) is said to be exponentially stable if there exist $\alpha > 0$ and $\mu > 0$ such that every solution of (1) satisfies the inequality

$$\|x(t,\varphi)\| \le \mu e^{-\alpha t} \|\varphi\|_h, \quad \forall t \ge 0.$$

In order to present the Lyapunov-Krasovskii conditions for the exponential stability of (1) given in [11] we need to introduce a little of terminology.

As usual, we define the natural state of (1) by

$$x_t(\theta,\varphi) \triangleq x(t+\theta,\varphi), \theta \in [-h,0).$$

Due to the jump discontinuity of the solutions at $t = 0$, it follows that $x_t(\theta,\varphi) \in \mathcal{PC}\left([-h,0), \mathbb{R}^m\right)$ for $t \in [0,h)$, while $x_t(\theta,\varphi) \in \mathcal{C}\left([-h,0), \mathbb{R}^m\right)$ for $t \ge h$. As a consequence, in a Lyapunov-Krasovskii setting, the functionals should be defined on the infinite-dimensional space $\mathcal{PC}\left([-h,0), \mathbb{R}^m\right)$.

For simplicity of the notation, one writes $x_t(\varphi)$ instead of $x_t(\theta,\varphi), \theta \in [-h,0)$. Also when the initial function is irrelevant from the context, we simply write $x(t)$ and x_t instead of $x(t,\varphi)$ and $x_t(\varphi)$.

Theorem 1. *[11] System (1) is exponentially stable if there exists a continuous functional $v : \mathcal{PC}\left([-h,0), \mathbb{R}^m\right) \to \mathbb{R}$ such that $t \to v(x_t(\varphi))$ is differentiable and the following conditions hold:*

1. $\alpha_1 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta \le v(\varphi) \le \alpha_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta$, for some constants $0 < \alpha_1 \le \alpha_2$,
2. $\frac{d}{dt} v(x_t(\varphi)) \le -\beta \int_{-h}^{0} \|x(t+\theta,\varphi)\|^2 d\theta$, for a constant $\beta > 0$.

4 Main Results

Constructing Lyapunov functionals for the perturbed system (4) is rather difficult due to the multiplicative way that the perturbations are involved in the exponential kernel. Therefore, we will consider an alternative perturbed system which is equivalent to (4) from the stability point of view and it has a more suitable form for the analysis by means of Lyapunov functionals.

Given matrices A, B and C, let us consider the integral delay system

$$z(t) = \int_{-h}^{0} BCe^{A\theta} z(t+\theta)d\theta, \tag{6}$$

Lemma 1. *The spectrums of (1) and (6) are equal.*

Proof. The characteristic function associated to (1) is

$$f(s) = \det\left(I_m - CM(s)B\right),$$

where $M(s) = \int_{-h}^{0} e^{(sI+A)\theta}d\theta$. By the properties of the determinant we have that

$$\det\left(I_m - CM(s)B\right) = \det\left(I_n - BCM(s)\right) = g(s)$$

Since $g(s)$ is the characteristic function associated to (6) it then follows that the spectrums of (1) and (6) are equal. □

The above Lemma implies that in despite of the fact that the systems (1) and (6) evolve in different functional spaces, $x_t(\varphi) \in \mathcal{PC}\left([-h,0), \mathbb{R}^m\right)$ while $z_t(\tilde{\varphi}) \in \mathcal{PC}\left([-h,0), \mathbb{R}^n\right)$, they are equivalent from the stability point of view.

Thus, based on these observations, instead of considering the perturbed system (4) we consider the following one:

$$z(t) = \int_{-h}^{0} \left(B + \Delta B\right)\left(C + \Delta C\right) e^{(A+\Delta A)\theta} z(t+\theta)d\theta, \tag{7}$$

where $\Delta A, \Delta B$ and ΔC are unknown constant matrices satisfying (5).

Proposition 1. *The perturbed system described by (7) and (5) is exponentially stable if there exist positive definite matrices P, Q, X, Y, and positive scalars $\gamma_1, \gamma_2, \gamma_3$ such that*

$$\mathcal{N}_{n_1}(P,Q,X,Y) - \gamma_1 \mathcal{N}_{p_1}\left(\rho_A\right) - \gamma_2 \mathcal{N}_{p_2}\left(\rho_B, \rho_C\right) - \gamma_3 \mathcal{N}_{p_3}\left(\rho_B\right) > 0, \tag{8}$$

$$\mathcal{N}_{n_2}(P,Q,X,Y) - \gamma_1 \mathcal{N}_{p_1}\left(\rho_A\right) - \gamma_2 \mathcal{N}_{p_2}\left(\rho_B, \rho_C\right) - \gamma_3 \mathcal{N}_{p_3}\left(\rho_B\right) > 0, \tag{9}$$

$$\begin{pmatrix} X & \mathcal{M} \\ \mathcal{M} & \gamma_2 I_n - \mathcal{M} \end{pmatrix} > 0, \tag{10}$$

$$\begin{pmatrix} Y & \mathcal{M} + X \\ \mathcal{M} + X & \gamma_3 I_n - \mathcal{M} - X \end{pmatrix} > 0, \tag{11}$$

$$\gamma_1 I_n - \mathcal{M} > 0, \tag{12}$$

where $\mathcal{M} = P + hQ$ and

$$\mathcal{N}_{n_1}(P,Q,X,Y) = Q + A^T\mathcal{M} + \mathcal{M}A - hC^TB^T\left[\mathcal{M} + X + Y\right]BC, \quad (13)$$
$$\mathcal{N}_{n_2}(P,Q,X,Y) = Q + A^TP + PA - hC^TB^T\left[\mathcal{M} + X + Y\right]BC, \quad (14)$$
$$\mathcal{N}_{p_1}\left(\rho_A\right) = 2\rho_A I_n, \quad (15)$$
$$\mathcal{N}_{p_2}\left(\rho_B,\rho_C\right) = h\rho_C^2\left(\left\|B\right\|^2 + 2\rho_B\left\|B\right\| + \rho_B^2\right)I_n, \quad (16)$$
$$\mathcal{N}_{p_3}\left(\rho_B\right) = h\rho_B^2\left\|C\right\|^2 I_n. \quad (17)$$

Proof. For any arbitrary $\varphi \in \mathcal{PC}\left(\left[-h,0\right),\mathbb{R}^n\right)$, let us consider the following functional:

$$v(\varphi) = \int_{-h}^{0}\varphi^T(\theta)\left(e^{(A+\varDelta A)\theta}\right)^T\left[P + (\theta+h)\,Q\right]e^{(A+\varDelta A)\theta}\varphi(\theta)d\theta, \quad (18)$$

where P and Q are $n \times n$ positive definite matrices. From (18) it follows that

$$v(\varphi) \le \lambda_{\max}\left(P + hQ\right)\int_{-h}^{0}\lambda_{\max}\left(\left(e^{(A+\varDelta A)\theta}\right)^T e^{(A+\varDelta A)\theta}\right)\left\|\varphi(\theta)\right\|^2 d\theta,$$

and

$$v(\varphi) \ge \lambda_{\min}\left(P\right)\int_{-h}^{0}\lambda_{\min}\left(\left(e^{(A+\varDelta A)\theta}\right)^T e^{(A+\varDelta A)\theta}\right)\left\|\varphi(\theta)\right\|^2 d\theta.$$

Since $e^{(A+\varDelta A)\theta}$ is nonsingular for all $\theta \in [-h,0]$ and any matrices A and $\varDelta A$, we have

$$\lambda_{\max}\left(\left(e^{(A+\varDelta A)\theta}\right)^T e^{(A+\varDelta A)\theta}\right) \ge \lambda_{\min}\left(\left(e^{(A+\varDelta A)\theta}\right)^T e^{(A+\varDelta A)\theta}\right) > 0$$

and, therefore, the functional (18) satisfies the inequalities

$$\alpha_1\int_{-h}^{0}\left\|\varphi(\theta)\right\|^2 d\theta \le v(\varphi) \le \alpha_2\int_{-h}^{0}\left\|\varphi(\theta)\right\|^2 d\theta,$$

with $0 < \alpha_1 \le \alpha_2$ given by

$$\alpha_1 = \lambda_{\min}\left(P\right)\min_{\theta\in[-h,0]}\left\{\lambda_{\min}\left(\left(e^{(A+\varDelta A)\theta}\right)^T e^{(A+\varDelta A)\theta}\right)\right\},$$
$$\alpha_2 = \lambda_{\max}\left(P + hQ\right)\max_{\theta\in[-h,0]}\left\{\lambda_{\max}\left(\left(e^{(A+\varDelta A)\theta}\right)^T e^{(A+\varDelta A)\theta}\right)\right\}.$$

The time derivative of the functional (18) along the solutions of (7) is

$$\frac{dv(z_t)}{dt} = -\int_{-h}^{0} \xi^T(\theta) \left\{ Q + A^T \mathcal{M}(\theta) + \mathcal{M}(\theta) A \right\} \xi(\theta) d\theta$$
$$+ \left(\int_{-h}^{0} (B + \Delta B) (C + \Delta C) \xi(\theta) d\theta \right)^T \mathcal{M}(0) \times$$
$$\times \left(\int_{-h}^{0} (B + \Delta B) (C + \Delta C) \xi(\theta) d\theta \right)$$
$$-z^T(t-h) \left(e^{-(A+\Delta A)h} \right)^T P \left(e^{-(A+\Delta A)h} \right) z(t-h)$$
$$- \int_{-h}^{0} \xi^T(\theta) \left\{ (\Delta A)^T \mathcal{M}(\theta) + \mathcal{M}(\theta) (\Delta A) \right\} \xi(\theta) d\theta.$$

where $\mathcal{M}(\theta) = P + (\theta + h) Q, \theta \in [-h, 0]$. Here, in order to simplify the notation, we have defined

$$\xi(\theta) \triangleq e^{(A+\Delta A)\theta} z(t+\theta), \theta \in [-h, 0].$$

Now we will derive an upper estimation of the terms involving perturbations in the derivative of the functional. Let us start with the perturbed integral term

$$IP_1 \triangleq -\int_{-h}^{0} \xi^T(\theta) \left\{ (\Delta A)^T \mathcal{M}(\theta) + \mathcal{M}(\theta) (\Delta A) \right\} \xi(\theta) d\theta.$$

We have

$$-\xi^T(\theta) \left\{ (\Delta A)^T \mathcal{M}(\theta) + \mathcal{M}(\theta) (\Delta A) \right\} \xi(\theta) \leq 2 \left\| (\Delta A) \xi(\theta) \right\| \left\| \mathcal{M}(\theta) \xi(\theta) \right\|. \tag{19}$$

Let $\gamma_1 > 0$ such that

$$\mathcal{M}(\theta) < \gamma_1 I_n, \forall \theta \in [-h, 0]. \tag{20}$$

Then the following inequality holds:

$$\left\| \mathcal{M}(\theta) \xi(\theta) \right\| \leq \gamma_1 \left\| \xi(\theta) \right\|.$$

Using the above inequality and the upper bound for the matrix ΔA in (19) we get the following estimation:

$$IP_1 \leq 2 \rho_A \gamma_1 \int_{-h}^{0} \left\| \xi(\theta) \right\|^2 d\theta. \tag{21}$$

We now consider the perturbed integral term

$$IP_2 \triangleq \left(\int_{-h}^{0} (B + \Delta B) (C + \Delta C) \xi(\theta) d\theta \right)^T \mathcal{M}(0) \times$$
$$\times \left(\int_{-h}^{0} (B + \Delta B) (C + \Delta C) \xi(\theta) d\theta \right).$$

By using the Jensen integral inequality, the inequality

$$IP_2 \leq h \int_{-h}^{0} \xi^T(\theta) (C + \Delta C)^T (B + \Delta B)^T \mathcal{M}(0) (B + \Delta B) (C + \Delta C) \xi(\theta) d\theta$$

holds. Let

$$\chi(\theta) \triangleq \xi^T(\theta) (C + \Delta C)^T (B + \Delta B)^T \mathcal{M}(0) (B + \Delta B) (C + \Delta C) \xi(\theta).$$

We have

$$\begin{aligned} \chi(\theta) = {} & \xi^T(\theta) C^T (B + \Delta B)^T \mathcal{M}(0) (B + \Delta B) C \xi(\theta) \\ & + 2\xi^T(\theta) C^T (B + \Delta B)^T \mathcal{M}(0) (B + \Delta B) (\Delta C) \xi(\theta) \\ & + \xi^T(\theta) (\Delta C)^T (B + \Delta B)^T \mathcal{M}(0) (B + \Delta B) (\Delta C) \xi(\theta). \end{aligned}$$

Observing that for any positive definite matrix X the inequality

$$\begin{aligned} & 2\xi^T(\theta) C^T (B + \Delta B)^T \mathcal{M}(0) (B + \Delta B) (\Delta C) \xi(\theta) \\ & \leq \xi^T(\theta) C^T (B + \Delta B)^T X (B + \Delta B) C \xi(\theta) \\ & + \xi^T(\theta) (\Delta C)^T (B + \Delta B)^T \mathcal{M}(0) X^{-1} \mathcal{M}(0) (B + \Delta B) (\Delta C) \xi(\theta) \end{aligned}$$

holds, we have

$$\begin{aligned} & \chi(\theta) \leq \xi^T(\theta) C^T B^T [\mathcal{M}(0) + X] B C \xi(\theta) \\ & + 2\xi^T(\theta) C^T B^T [\mathcal{M}(0) + X] (\Delta B) C \xi(\theta) \\ & + \xi^T(\theta) (\Delta C)^T (B + \Delta B)^T \left[\mathcal{M}(0) + \mathcal{M}(0) X^{-1} \mathcal{M}(0)\right] (B + \Delta B) (\Delta C) \xi(\theta) \\ & + \xi^T(\theta) C^T (\Delta B)^T [\mathcal{M}(0) + X] (\Delta B) C \xi(\theta). \end{aligned} \tag{22}$$

Using the inequality

$$\begin{aligned} 2\xi^T(\theta) C^T B^T [\mathcal{M}(0) + X] (\Delta B) C \xi(\theta) \leq {} & \xi^T(\theta) C^T B^T Y B C \xi(\theta) \\ & + \xi^T(\theta) C^T (\Delta B)^T (\mathcal{M}(0) + X) Y^{-1} (\mathcal{M}(0) + X) (\Delta B) C \xi(\theta), \end{aligned}$$

where Y is any positive definite matrix, in (22) we get the following estimation for $\chi(\theta)$:

$$\begin{aligned} & \chi(\theta) \leq \xi^T(\theta) C^T B^T [\mathcal{M}(0) + X + Y] B C \xi(\theta) \\ & + \xi^T(\theta) C^T (\Delta B)^T \left[\mathcal{M}(0) + X + (\mathcal{M}(0) + X) Y^{-1} (\mathcal{M}(0) + X)\right] (\Delta B) C \xi(\theta) \\ & + \xi^T(\theta) (\Delta C)^T (B + \Delta B)^T \left[\mathcal{M}(0) + \mathcal{M}(0) X^{-1} \mathcal{M}(0)\right] (B + \Delta B) (\Delta C) \xi(\theta). \end{aligned} \tag{23}$$

Let $\gamma_2 > 0$ and $\gamma_3 > 0$ such that

$$\mathcal{M}(0) + \mathcal{M}(0)X^{-1}\mathcal{M}(0) < \gamma_2 I_n, \tag{24}$$

$$\mathcal{M}(0) + X + (\mathcal{M}(0) + X)\, Y^{-1} \,(\mathcal{M}(0) + X) < \gamma_3 I_n. \tag{25}$$

Then, the inequalities

$$\begin{aligned}
&\xi^T(\theta) C^T (\Delta B)^T \left[\mathcal{M}(0) + X + (\mathcal{M}(0) + X)\, Y^{-1} (\mathcal{M}(0) + X)\right] (\Delta B)\, C\xi(\theta) \\
&\leq \gamma_3 \left\|(\Delta B)\, C\xi\right\|^2 \leq \gamma_3 \rho_B^2 \left\|C\right\|^2 \left\|\xi\right\|^2
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
&\xi^T(\theta)\,(\Delta C)^T (B + \Delta B)^T \left[\mathcal{M}(0) + \mathcal{M}(0)X^{-1}\mathcal{M}(0)\right] (B + \Delta B)\,(\Delta C)\,\xi(\theta) \\
&\leq \gamma_2 \xi^T(\theta)\,(\Delta C)^T (B + \Delta B)^T (B + \Delta B)\,(\Delta C)\,\xi(\theta) \\
&\leq \gamma_2 \left\|B\,(\Delta C)\,\xi(\theta)\right\|^2 + 2\gamma_2 \left\|B\,(\Delta C)\,\xi(\theta)\right\| \left\|(\Delta B)\,(\Delta C)\,\xi(\theta)\right\| \\
&+ \gamma_2 \left\|(\Delta B)\,(\Delta C)\,\xi(\theta)\right\|^2 \\
&\leq \gamma_2 \rho_C^2 \left\|B\right\|^2 \left\|\xi(\theta)\right\|^2 + 2\gamma_2 \rho_C^2 \rho_B \left\|B\right\| \left\|\xi(\theta)\right\|^2 + \gamma_2 \rho_B^2 \rho_C^2 \left\|\xi(\theta)\right\|^2
\end{aligned} \tag{27}$$

hold. Taking into account the inequalities (26) and (27) into (23) we obtain the following estimation for $\chi(\theta)$:

$$\begin{aligned}
&\chi(\theta) \leq \xi^T(\theta) C^T B^T \left[\mathcal{M}(0) + X + Y\right] BC\xi(\theta) \\
&+ \gamma_2 \rho_C^2 \left(\left\|B\right\|^2 + 2\rho_B \left\|B\right\| + \rho_B^2\right) \left\|\xi(\theta)\right\|^2 + \gamma_3 \rho_B^2 \left\|C\right\|^2 \left\|\xi(\theta)\right\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&IP_2 \leq h \int_{-h}^{0} \xi^T(\theta) \left\{C^T B^T \left[\mathcal{M}(0) + X + Y\right] BC \right.\\
&\left. + \gamma_3 \rho_B^2 \left\|C\right\|^2 I_n + \gamma_2 \rho_C^2 \left(\left\|B\right\|^2 + 2\rho_B \left\|B\right\| + \rho_B^2\right)\right\} \xi(\theta) d\theta.
\end{aligned}$$

From this inequality and (21) we arrive at the following upper bound for the derivative of the functional:

$$\begin{aligned}
&\frac{dv(z_t)}{dt} \leq -\int_{-h}^{0} \xi^T(\theta) \left\{Q + A^T\mathcal{M}(\theta) + \mathcal{M}(\theta)A\right\} \xi(\theta) d\theta \\
&+ h \int_{-h}^{0} \xi^T(\theta) \left\{C^T B^T \left[\mathcal{M}(0) + X + Y\right] BC + \gamma_3 \rho_B^2 \left\|C\right\|^2 I_n \right.\\
&\left. + \gamma_2 \rho_C^2 \left(\left\|B\right\|^2 + 2\rho_B \left\|B\right\| + \rho_B^2\right) I_n\right\} \xi(\theta) d\theta + 2\rho_A \gamma_1 \int_{-h}^{0} \left\|\xi(\theta)\right\|^2 d\theta
\end{aligned}$$

that can be rewritten as

$$\frac{dv(z_t)}{dt} \leq -\int_{-h}^{0} \xi^T(\theta)\varGamma(\theta)\xi(\theta)d\theta,$$

where $\varGamma(\theta) \in \mathcal{R}^{n\times n}$ for $\theta \in [-h, 0]$ is given by

$$\begin{aligned}\varGamma(\theta) = Q + A^T\mathcal{M}(\theta) + \mathcal{M}(\theta)A - hC^T B^T \left[M + X + Y\right] BC - \gamma_1 \mathcal{N}_{p_1}\left(\rho_A\right) \\ -\gamma_2\mathcal{N}_{p_2}\left(\rho_B, \rho_C\right) - \gamma_3\mathcal{N}_{p_2}\left(\rho_B\right),\end{aligned}$$

with $\mathcal{N}_{p_1}\left(\rho_A\right), \mathcal{N}_{p2}\left(\rho_B, \rho_C\right)$ and $\mathcal{N}_{p2}\left(\rho_B\right)$ defined by (15), (16) and (17), respectively.

Clearly, if $\varGamma(\theta) > 0, \forall\theta \in [-h, 0]$, then there exists

$$\beta = \min_{\theta\in[-h,0]} \left\{\lambda_{\min}\left(e^{(A+\Delta A)\theta}\right)^T \varGamma(\theta)e^{(A+\Delta A)\theta}\right\} > 0$$

such that

$$\frac{dv(z_t)}{dt} < -\beta \int_{-h}^{0} \|z(t+\theta)\|^2 d\theta,$$

and the exponential stability of the perturbed system is assured.

Now since

$$\left(\frac{\theta + h}{h}\right)\varGamma(0) + \left(-\frac{\theta}{h}\right)\varGamma(-h) = \varGamma(\theta), \forall\theta \in [-h, 0],$$

it follows that $\varGamma(\theta) > 0, \forall\theta \in [-h, 0]$, if, and only if, $\varGamma(0) > 0$ and $\varGamma(-h) > 0$.

By evaluating $\varGamma(\theta)$ for $\theta = 0$ and $\theta = -h$ we have

$$\begin{aligned}\varGamma(0) &= \mathcal{N}_{n_1}(P, Q, X, Y) - \gamma_1\mathcal{N}_{p_1}\left(\rho_A\right) - \gamma_2\mathcal{N}_{p_1}\left(\rho_B, \rho_C\right) - \gamma_3\mathcal{N}_{p_2}\left(\rho_B\right), \\ \varGamma(-h) &= \mathcal{N}_{n_2}(P, Q, X, Y) - \gamma_1\mathcal{N}_{p_1}\left(\rho_A\right) - \gamma_2\mathcal{N}_{p_1}\left(\rho_B, \rho_C\right) - \gamma_3\mathcal{N}_{p_2}\left(\rho_B\right),\end{aligned}$$

where $\mathcal{N}_{n_1}(P, Q, X, Y)$ and $\mathcal{N}_{n_2}(P, Q, X, Y)$ are defined by (13) and (14) respectively.

$\varGamma(0) > 0$ and $\varGamma(-h) > 0$ lead to the inequalities (8) and (9).

Observing that, by Schur complement, the inequalities (24) and (25) are respectively equivalent to (10) and (11), and that the inequality (12), i.e.,

$$\mathcal{M}(0) = \mathcal{M} = P + hQ < \gamma_1 I_n,$$

implies (20) the proof ends. □

Remark 1. Note that the functional (18) involves an exponential matrix depending not only on the nominal matrix A but also on the perturbation matrix ΔA. This special characteristic of the functional allows us to derive robust stability conditions for any arbitrary perturbation matrix ΔA with the only assumption of being bounded in norm.

4.1 Nominal Case

In the nominal case, when (7) does not have uncertainty

$$z(t) = \int_{-h}^{0} BCe^{A\theta} z(t+\theta) d\theta, \tag{28}$$

we have the following result:

Corollary 1. *The integral delay system (28) is exponentially stable if there exist positive definite matrices P and Q such that*

$$Q + A^T \mathcal{M} + \mathcal{M}A - hC^T B^T \mathcal{M}BC > 0, \tag{29}$$
$$Q + A^T P + PA - hC^T B^T \mathcal{M}BC > 0, \tag{30}$$

where $\mathcal{M} = P + hQ$.

Proof. Since we have that $\rho_A = \rho_B = \rho_C = 0$ then it follows from (15), (16) and (17) that the matrices $\mathcal{N}_{p_1}(\rho_A), \mathcal{N}_{p_2}(\rho_B, \rho_C)$ and $\mathcal{N}_{p_3}(\rho_B)$ are equal to zero and, therefore, the inequalities (8) and (9) hold for any arbitrary positive constants γ_1, γ_2 and γ_3.

Selecting γ_1 sufficiently large the restriction imposed on matrix $\mathcal{M}$ by the inequality (12), i.e.,

$$\gamma_1 I_n > \mathcal{M},$$

can be removed. Now, by Schur complement, the inequality (10) is equivalent to

$$\gamma_2 I_n > \mathcal{M} + \mathcal{M}X^{-1}\mathcal{M}.$$

Since γ_2 can be arbitrarily chosen then the above inequality holds for $X = \varepsilon_1 I$ with $\varepsilon_1 > 0$ sufficiently small and $\gamma_2 > 0$ sufficiently large.

Similarly, the inequality (11) is equivalent to

$$\gamma_3 I_n > (\mathcal{M} + X) + (\mathcal{M} + X) Y^{-1} (\mathcal{M} + X).$$

Again, since γ_3 can be arbitrarily chosen then the above inequality holds for $Y = \varepsilon_2 I$ with $\varepsilon_2 > 0$ sufficiently small and $\gamma_3 > 0$ sufficiently large.

By doing $\varepsilon_1 \to +0$ and $\varepsilon_2 \to +0$, the restrictions given by the inequalities (10) and (11) can be removed while the matrices $\mathcal{N}_{n_1}(P, Q, X, Y)$ and $\mathcal{N}_{n_2}(P, Q, X, Y)$ respectively become (29) and (30). □

The Proposition 1 provides robust stability conditions when there exist perturbations on all system matrices while Corollary 1 does in the nominal case when there are not perturbations on the system matrices. Of course, in certain applications, one could have the situation when not all system matrices but only some of them are subject to perturbations and it is hence convenient to have the explicit robust stability conditions for such cases.

In the following we establish robust stability conditions for all possible combinations of perturbations on system matrices. In all the cases the results are directly derived from Theorem 1 and the proof of Corollary 1 by considering as zero the corresponding combination of perturbations.

4.2 Case $\rho_A = 0$

Corollary 2. *The perturbed system described by (7) and (5), where $\rho_A = 0$, is exponentially stable if there exist positive definite matrices P, Q, X, Y and positive constants γ_1, γ_2 such that*

$$\begin{aligned} \mathcal{N}_{n_1}(P,Q,X,Y) - \gamma_1 \mathcal{N}_{p_2}(\rho_B,\rho_C) - \gamma_2 \mathcal{N}_{p_3}(\rho_B) &> 0, \\ \mathcal{N}_{n_2}(P,Q,X,Y) - \gamma_1 \mathcal{N}_{p_2}(\rho_B,\rho_C) - \gamma_2 \mathcal{N}_{p_3}(\rho_B) &> 0, \\ \begin{pmatrix} X & \mathcal{M} \\ \mathcal{M} & \gamma_1 I_n - \mathcal{M} \end{pmatrix} &> 0, \\ \begin{pmatrix} Y & \mathcal{M}+X \\ \mathcal{M}+X & \gamma_2 I_n - \mathcal{M} - X \end{pmatrix} &> 0, \end{aligned}$$

where $\mathcal{M} = P + hQ$ and $\mathcal{N}_{n_1}(P,Q,X,Y)$, $\mathcal{N}_{n_2}(P,Q,X,Y)$, $\mathcal{N}_{p_2}(\rho_B,\rho_C)$, $\mathcal{N}_{p_3}(\rho_B)$ are respectively given by (13), (14), (16) and (17).

4.3 Case $\rho_A = \rho_B = 0$

Corollary 3. *The perturbed system described by (7) and (5), where $\rho_A = \rho_B = 0$, is exponentially stable if there exist positive definite matrices P, Q, X and a positive constant γ such that*

$$\begin{aligned} Q + A^T\mathcal{M} + \mathcal{M}A - hC^TB^T(\mathcal{M}+X)BC - \gamma h \rho_C^2 \|B\|^2 I_n &> 0, \\ Q + A^TP + PA - hC^TB^T(\mathcal{M}+X)BC - \gamma h \rho_C^2 \|B\|^2 I_n &> 0, \\ \begin{pmatrix} X & \mathcal{M} \\ \mathcal{M} & \gamma I_n - \mathcal{M} \end{pmatrix} &> 0. \end{aligned}$$

4.4 Case $\rho_A = \rho_C = 0$

Corollary 4. *The perturbed system described by (7) and (5), where $\rho_A = \rho_C = 0$, is exponentially stable if there exist positive definite matrices P, Q, Y and a positive constant γ such that*

$$\begin{aligned} Q + A^T\mathcal{M} + \mathcal{M}A - hC^TB^T(\mathcal{M}+Y)BC - \gamma h\rho_B^2 \|C\|^2 I_n &> 0, \\ Q + A^TP + PA - hC^TB^T(\mathcal{M}+Y)BC - \gamma h\rho_B^2 \|C\|^2 I_n &> 0, \\ \begin{pmatrix} Y & \mathcal{M} \\ \mathcal{M} & \gamma I_n - \mathcal{M} \end{pmatrix} &> 0. \end{aligned}$$

4.5 Case $\rho_B = \rho_C = 0$

Corollary 5. *The perturbed system described by (7) and (5), where $\rho_B = \rho_C = 0$, is exponentially stable if there exist positive definite matrices P, Q and a positive constant γ such that*

$$\begin{aligned} Q + A^T\mathcal{M} + \mathcal{M}A - hC^TB^T\mathcal{M}BC - 2\gamma\rho_A I_n &> 0, \\ Q + A^TP + PA - hC^TB^T\mathcal{M}BC - 2\gamma\rho_A I_n &> 0, \\ \gamma I_n - \mathcal{M} &> 0. \end{aligned}$$

4.6 Case $\rho_B = 0$

Corollary 6. *The perturbed system described by (7) and (5), where $\rho_B = 0$, is exponentially stable if there exist positive definite matrices P, Q, Y and positive constants γ_1 and γ_2 such that*

$$\begin{aligned} Q + A^T\mathcal{M} + \mathcal{M}A - hC^TB^T(\mathcal{M}+X)BC - 2\gamma_1\rho_A I_n - \gamma_2 h\rho_C^2 \|B\|^2 I_n &> 0, \\ Q + A^TP + PA - hC^TB^T(\mathcal{M}+X)BC - 2\gamma_1\rho_A I_n - \gamma_2 h\rho_C^2 \|B\|^2 I_n &> 0, \\ \begin{pmatrix} X & \mathcal{M} \\ \mathcal{M} & \gamma_2 I_n - \mathcal{M} \end{pmatrix} &> 0, \\ \gamma_1 I_n - \mathcal{M} &> 0. \end{aligned}$$

4.7 Case $\rho_C = 0$

Corollary 7. *The perturbed system described by (7) and (5), where $\rho_C = 0$, is exponentially stable if there exist positive definite matrices P, Q, Y and positive constants γ_1 and γ_2 such that*

Table 1 Maximum ρ_A for different delay values

h	0.2	0.5	1.1
ρ_A	2.325	0.704	0.104

$$Q + A^T\mathcal{M} + \mathcal{M}A - hC^TB^T(\mathcal{M}+Y)BC - 2\gamma_1\rho_A I_n - h\gamma_2\rho_B^2\|C\|^2 I_n > 0,$$
$$Q + A^TP + PA - hC^TB^T(\mathcal{M}+Y)BC - 2\gamma_1\rho_A I_n - h\gamma_2\rho_B^2\|C\|^2 I_n > 0,$$
$$\begin{pmatrix} Y & \mathcal{M} \\ \mathcal{M} & \gamma_2 I_n - \mathcal{M} \end{pmatrix} > 0,$$
$$\gamma_1 I_n - \mathcal{M} > 0.$$

5 Examples

In this section, we will present two numerical examples illustrating the main results. The first one is only for academic purposes while the second one comes from certain mathematical models found in mechanical systems.

Example 1. Consider the following perturbed integral delay system:

$$z(t) = \int_{-h}^{0} e^{(A+\varDelta A)\theta} z(t+\theta)\,d\theta, \tag{31}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$$

and $\varDelta A$ is an unknown matrix satisfying $\|\varDelta A\| \leq \rho_A$.

Clearly, system (31) is a particular case of the perturbed system (7) when $m = n$, $B = C = I_n$ and $\rho_B = \rho_C = 0$. The corresponding nominal system has been studied in [16].

By using Corollary 1 we found that the corresponding nominal system of (31) is exponentially stable for all constant delay values $0 \leq h \leq 1.999$, a result which coincides with that reported in [16].

In Table 1 we present the maximum bound for ρ_A computed for different delay values by means of Corollary 5. It can be seen that the maximum bound for ρ_A decreases when the delay value increases.

Example 2. Let us consider the following nominal integral delay system:

$$z(t) = \int_{-h}^{0} Ce^{A\theta}Bz(t+\theta)\,d\theta, \tag{32}$$

where

Table 2 Maximum ρ_C for different delay values

h	2	5	10
ρ_C	0.1439	0.0219	0.00441

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} c_1 & c_2 \end{pmatrix}.$$

For these matrices A, B and C, the corresponding system (2) represents a double integrator with a delay in the input which it is very commonly found in mechanical systems. The integral delay system (32) describes the internal dynamics of the controller (3).

Let $c_1 = -0.0005$ and $c_2 = -0.0267$ be the nominal gains. For this vector gain the ideal controller (3) assigns the eigenvalues $\lambda_{1,2} = -0.0134 \pm 0.0179i$ to the ideal closed-loop system.

By using Corollary 1 we found that the system (32) is exponentially stable for all constant delay values $0 \leq h \leq 24.375$.

Now let us consider the perturbed integral delay system

$$y(t) = \int_{-h}^{0} (C + \Delta C) e^{A\theta} B y(t + \theta) d\theta, \tag{33}$$

where ΔC is an unknown vector satisfying $\|\Delta C\| \leq \rho_C$.

For the perturbed system (33), we compute the maximum bound ρ_C for different delay values by using Corollary 3. The results are presented in Table 2.

For instance for $h = 5$, the above results imply that for any constant vector $C = \begin{pmatrix} c_1 & c_2 \end{pmatrix}$ belonging to the ball with center at $C = \begin{pmatrix} -0.0005 & -0.0267 \end{pmatrix}$ and radius $\rho_C = 0.0219$, the corresponding nominal integral delay system (32) is exponentially stable.

Let us now to complicate the robust stability problem by considering perturbations not only on the matrix C but also on the matrix B.

Namely, we consider the perturbed integral system

$$y(t) = \int_{-h}^{0} (C + \Delta C) e^{A\theta} (B + \Delta B) y(t + \theta) d\theta, \tag{34}$$

where $\Delta C, \Delta B$ are unknown vectors satisfying $\|\Delta C\| \leq \rho_C$ and $\|\Delta B\| \leq \rho_B$.

For the perturbed system (34) the Corollary 2 can be used. To solve this problem we fix a delay $h > 0$ and an upper bound $\rho_C > 0$, for which we know from the previous analysis that the perturbed system is stable when $\rho_B = 0$, and then search for an upper bound $\rho_B > 0$ such that the inequalities in Corollary 2 are feasible.

For instance, for $h = 5$ and $\rho_C = 0.002$ we found that the exponential stability of the perturbed system (34) is assured for $\rho_B \leq 0.5696$.

Finally, let us consider the more complicated problem when we have perturbations on all system matrices A, B and C that yields at the perturbed integral delay system

$$y(t) = \int_{-h}^{0} (C + \Delta C)\, e^{(A+\Delta A)\theta}\, (B + \Delta B)\, y(t+\theta)\, d\theta, \tag{35}$$

where $\Delta C, \Delta B$ are unknown vector satisfying $\|\Delta C\| \leq \rho_C$, $\|\Delta B\| \leq \rho_B$ and ΔA is an unknown matrix satisfying $\|\Delta A\| \leq \rho_A$.

We address the problem as above and fix a delay $h > 0$, upper bounds $\rho_C, \rho_B > 0$, for which we have stability of the perturbed system when $\rho_A = 0$, and then search for an upper bound $\rho_A > 0$ such that the inequalities in Proposition 1 are feasible.

Thus, for $h = 5$, $\rho_C = 0.002$ and $\rho_B = 0.05$ we found that the perturbed system (35) is exponentially stable for $\rho_A \leq 0.0015$.

6 Conclusions

The robust exponential stability of integral delay systems with kernels of exponential type subject to norm bounded uncertainties is investigated. New delay-dependent robust stability conditions expressed in terms of linear matrix inequalities are derived by using the Lyapunov-Krasovskii functional approach.

The robust stability results found important application in several stability problems of differential delay systems as well as in the internal stability of feedback schemes involving delay compensation.

Acknowledgements This work was partially supported by CONACYT grant 151587.

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