Exponential stability of linear continuous time difference systems with multiple delays

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Abstract

Some recent results on exponential stability of linear continuous time difference systems with discrete and distributed delay terms are extended to the case of multiple delays. New sufficient conditions for the exponential stability and exponential estimates for the solutions by using Lyapunov-Krasovskii functionals are derived. Special attention is paid to the case of systems with commensurate discrete and distributed delays.

Key words: Continuous time difference systems, Exponential Stability, Lyapunov-Krasovskii functionals

1. Introduction

Recently, motivated from some limitations on the application of existing Lyapunov approaches [17, 20, 21] to the stability analysis of linear continuous time systems with discrete and distributed delay terms, a new Lyapunov-Krasovskii methodology for the exponential stability of such a class of systems has been introduced in [15] and [16].

In [16] (see also [15]) the following system is considered:

$$x(t) = Ax(t-h) + \int_{-h}^{0} G(\theta)x(t+\theta)d\theta,$$

where $A \in \mathcal{R}^{n\times n}$ is a Schur stable matrix (all eigenvalues of the matrix lie in the open unit disc of the complex plane) and $G(\theta)$ is a matrix function with piecewise continuous bounded elements defined for $\theta \in [-h, 0]$.

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This system can be found as delay approximations of the partial differential equations for describing the propagation phenomena in excitable media [2], in the stability analysis of additional dynamics introduced by some system transformations [3, 7, 8, 9], in delay-dependent stability analysis of neutral type systems [12], and in the stability analysis of some difference operators in neutral type functional differential equations [4], [11].

For such systems, sufficient delay-dependent conditions for the exponential stability and exponential estimates for the solutions by means of Lyapunov-Krasovskii functionals are derived in [16].

A natural extension of the results in [16] is to present a Lyapunov-Krasovskii approach for the exponential stability of linear continuous time difference systems of the form

$$x(t) = \sum_{j=1}^{m} A_j x(t - h_j) + \int_{-\tau}^{0} G(\theta) x(t + \theta) d\theta, \tag{1}$$

where $A_j \in \mathcal{R}^{n\times n}$, $j = 1, 2, \dots, m$, delays $0 < h_1 < h_2 < \cdots < h_m$, $0 < \tau$, and matrix function $G(\theta)$ has piecewise continuous bounded elements defined in the interval $[-\tau, 0]$.

In this contribution we present such an extension for which, to the best of our knowledge, no attempt has been made. We follow the same robust stability idea used in [16] for the case of systems with a single discrete delay term that is inspired from previous results on Lyapunov-Krasovskii functionals for integral delay systems [13], [14].

Thus, by assuming that the continuous time difference system

$$x(t) = \sum_{j=1}^{m} A_j x(t - h_j), \tag{2}$$

is *delay-independent* exponentially stable and interpreting the distributed delay term as a perturbation we will present Lyapunov-Krasovskii functionals guaranteeing the exponential stability of the whole system (1).

We start in Section 2 with some preliminaries. Basic facts about solutions and a norm-based condition for the exponential stability of (1) are given. Lyapunov-Krasovskii type conditions for the exponential stability of (1) are introduced. A Lyapunov result which allows us to derive exponential bounds for the solutions is also given. In section 3, the general Lyapunov-Krasovskii results introduced in section 2 is applied to the particular class of (1) with constant matrices and multiple distributed delay terms. Next, the special cases of commensurate discrete and commensurate discrete and distributed delays are considered. In all these cases, sufficient conditions for the exponential stability and exponential estimates for the solutions are expressed in terms of linear matrix inequalities. Several examples illustrating the results are given in section 4. Concluding remarks end the paper.

Notation: Throughout this paper, the Euclidean norm for vectors and the induced matrix norm for matrices are used, both denoted by $\|\cdot\|$. We denote by A^T the transpose of A, I and 0 stand for the identity and zero matrices, while $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalues of a symmetric matrix A, respectively. For a real symmetric matrix Q, the standard notation $Q > 0$ (respectively, $Q < 0$) is used to denote that Q is positive (respectively negative) definite.

2. Preliminaries

2.1. Solutions and stability concept

Let $r = \max\{\tau, h_m\}$. In order to define a particular solution of (1) an initial vector function $\varphi(\theta), \theta \in [-r, 0)$, should be given. We assume that $\varphi \in \mathcal{C}([-r, 0), \mathcal{R}^n)$, the space of continuous vector functions mapping $[-r, 0)$ to $\mathcal{R}^n$ equipped with the uniform convergence norm $\|\varphi\|_r = \sup_{\theta\in[-r,0)} \|\varphi(\theta)\|$.

For a given initial function $\varphi \in \mathcal{C}([-r, 0), \mathcal{R}^n)$, there exists a unique solution $x(t, \varphi)$ of (1) defined for all $t \geq 0$, see [4]. This solution presents jump discontinuities which distribution on time is very difficult (maybe impossible) to describe in the general case.

Clearly, at $t = 0$ the jump discontinuity is explicitly given by

$$\Delta x(0) = x(0) - \varphi(-0) = \sum_{j=1}^{m} A_j \varphi(-h_j) + \int_{-\tau}^{0} G(\theta)\varphi(\theta)d\theta - \varphi(-0).$$

In the particular case when the discrete and distributed delays are both commensurate, i.e., they are multiple of a basic delay, the jump discontinuities of the solutions are regularly distributed at time instants multiple of the basic delay.

Definition 1. *[4] System (1) is said to be exponentially stable if there exist $\alpha > 0$ and $\mu > 0$ such that any solution of (1) satisfies the inequality*

$$\|x(t, \varphi)\| \leq \mu e^{-\alpha t} \|\varphi\|_r, \forall t \geq 0.$$

Remark 1. *In the particular case when $G(\theta)$ is continuously differentiable on $[-\tau, 0]$, the solutions of (1) can be related with particular solutions of some neutral functional differential equations and coupled systems described by retarded functional differential equations and functional difference equations, see [16] for a detailed proof in the case of a single discrete delay. Nevertheless, as it is discussed in [16], the corresponding differential delay systems are not exponentially stable and, therefore, they cannot be directly analyzed by existing Lyapunov stability conditions as, for instance, those in [5] and [6].*

2.2. A norm condition

By using frequency domain tools, based on the characteristic function associated to (1), one can get the following result:

Lemma 1. *System (1) is exponentially stable if*

$$\sum_{j=1}^{m} \|A_j\| + \tau \left(\sup_{\theta \in [-\tau, 0]} \|G(\theta)\| \right) < 1. \tag{3}$$

Although the inequality (3) could be relatively easy to verify for some particular matrix functions $G(\theta)$, it may be pretty conservative. In the next, we will derive a Lyapunov-Krasovskii functional approach for the exponential stability of (1) that not only will provide better results than the inequality (3) but also exponential estimates for the solutions of (1).

2.3. A Lyapunov type theorem

We present here Lyapunov-Krasovskii conditions for the exponential stability of (1) that generalizes the ones presented in [16] for the case of system with a single delay.

For a given $t \geq 0$, we define the natural state $x_t(\varphi) = x(t + \theta, \varphi), \theta \in [-r, 0)$. When the initial function is irrelevant we simply write $x(t)$ and x_t instead of $x(t, \varphi)$ and $x_t(\varphi)$. Based on the discontinuities of the solutions it results that $x_t(\varphi) \in \mathcal{PC}([-r, 0), \mathcal{R}^n)$, the space of piecewise continuous bounded functions mapping the interval $[-r, 0)$ to $\mathcal{R}^n$. As a consequence, in a Lyapunov-Krasovskii functional setting, the functionals should be defined on $\mathcal{PC}([-r, 0), \mathcal{R}^n)$.

Theorem 2. *Consider system (1) and assume that the continuous time difference system (2) is delay-independent exponentially stable . System (1) is exponentially stable if there exists a functional $v : \mathcal{PC}([-r, 0), \mathcal{R}^n) \to \mathcal{R}$ such that $t \to v(x_t(\varphi))$ is differentiable and the following conditions hold:*

1. $\alpha_1 \int_{-r}^{0} \|\varphi(\theta)\|^2 d\theta \le v(\varphi) \le \alpha_2 \int_{-r}^{0} \|\varphi(\theta)\|^2 d\theta$, *for some* $0 < \alpha_1 \le \alpha_2$,
2. $\frac{d}{dt} v(x_t(\varphi)) \le -\beta \int_{-r}^{0} \|x(t, \varphi)\|^2 d\theta$, *for some* $\beta > 0$.

Proof. Given any initial function $\varphi \in \mathcal{C}([-r, 0), \mathcal{R}^n)$, it follows from the theorem conditions that for $2\sigma = \beta\alpha_2^{-1}$ the following inequality:

$$\frac{d}{dt} v(x_t(\varphi)) + 2\sigma v(x_t(\varphi)) \le 0, \forall t \ge 0,$$

holds. This inequality leads to

$$v(x_t(\varphi)) \le v(\varphi) e^{-2\sigma t}, \forall t \ge 0.$$

Thus, it follows that for $t \ge 0$

$$\alpha_1 \int_{-r}^{0} \|x(t + \theta, \varphi)\|^2 d\theta \le \alpha_2 e^{-2\sigma t} \int_{-r}^{0} \|\varphi(\theta)\|^2 d\theta. \tag{4}$$

From (1) one gets

$$\begin{aligned} \left\| x(t, \varphi) - \sum_{j=1}^{m} A_j x(t - h_j, \varphi) \right\|^2 &\le \left(\int_{-\tau}^{0} \|G(\theta) x(t + \theta, \varphi)\| d\theta \right)^2 \quad (5) \\ &\le \tau m_g^2 \int_{-r}^{0} \|x(t + \theta, \varphi)\|^2 d\theta, \end{aligned}$$

where the last inequality has been obtained by using the Cauchy-Schwarz inequality in $\mathcal{L}^2([-\tau, 0], \mathcal{R})$, the fact that $\tau \le r$, and $m_g = \sup_{\theta \in [-\tau, 0]} \|G(\theta)\|$.

Combining the inequalities (4) and (5) one obtains

$$\left\| x(t, \varphi) - \sum_{j=1}^{m} A_j x(t - h_j, \varphi) \right\| \le \left(\sqrt{\frac{\alpha_2}{\alpha_1} r \tau} \right) m_g \|\varphi\|_r e^{-\sigma t}, \forall t \ge 0.$$

This inequality implies that

$$x(t, \varphi) - \sum_{j=1}^{m} A_j x(t - h_j, \varphi) = f(t), \tag{6}$$

where $f \in \mathcal{C}\left([0,\infty),\mathcal{R}^n\right)$ satisfies

$$\|f(t)\| \leq \gamma \|\varphi\|_r e^{-\sigma t}, \forall t \geq 0, \tag{7}$$

with $\gamma = \left(\sqrt{\frac{\alpha_2}{\alpha_1}} r\tau\right) m_g$.

Now since system (2) is assumed to be delay-independent exponentially stable it then follows from Theorem 3.5 in [4] (see inequality (3.22)) that there exist constants $c(\eta) > 0$ and $\eta > 0$ such that the following inequality holds:

$$\|x(t,\varphi)\| \leq c\left[\|\varphi\|_r e^{-\alpha t} + \sup_{0\leq s\leq t}\|f(s)\|\right], \forall t \geq 0. \tag{8}$$

From (7) and (8) we get

$$\|x(t,\varphi)\| \leq c\left[1+\gamma\right]\|\varphi\|_r, t \geq 0.$$

This inequality implies that $\|x_t(\varphi)\|_r \leq c\left[1+\gamma\right]\|\varphi\|_r, \forall t \geq r$. When $t \in [0,r)$ we have

$$\begin{aligned}
\|x_t(\varphi)\| &= \max\left\{\sup_{\xi\in[t_0-r,0)}\|\varphi(\xi)\|, \sup_{\xi\in[0,t_0)}\|x(\xi,\varphi)\|, \|\Delta x(0)\|\right\} \\
&\leq \max\left\{\|\varphi\|_r, c\left[1+\gamma\right]\|\varphi\|_r, \|\Delta x(0)\|\right\}.
\end{aligned}$$

From the jump discontinuity of the solution at $t = 0$ we get

$$\|\Delta x(0)\| = \|x(0) - \varphi(-0)\| \leq \|x(0)\| + \|\varphi(-0)\| \leq \left(c\left[1+\gamma\right]+1\right)\|\varphi\|_r.$$

It follows that $\|x_t(\varphi)\|_r \leq \left(c\left[1+\gamma\right]+1\right)\|\varphi\|_r, \forall t \in [0,r)$, and therefore the following inequality holds:

$$\|x_t(\varphi)\|_r \leq \left(c\left[1+\gamma\right]+1\right)\|\varphi\|_r, \forall t \geq 0. \tag{9}$$

Now since the system (6) is time-invariant then we can apply the reasoning used in the proof of Theorem 7 in [19] and rewrite the inequality (8) for an initial instant $t_0 \geq 0$ to get

$$\|x(t,\varphi)\| \leq c\left[\|x_{t_0}\|_r e^{-\alpha(t-t_0)} + \sup_{t_0\leq s\leq t}\|f(s)\|\right]. \tag{10}$$

Setting $t_0 = \frac{t}{2}$, substituting $\|x_{t_0}\|_r$ with the right-hand side of (9) and using the inequality (7) in (10) we arrive at

$$\|x(t,\varphi)\| \leq \mu\|\varphi\|_r e^{-\alpha t}, \forall t \geq 0,$$

where

$$\mu = c\left(c\left(1+\gamma\right)+1+\gamma\right) \text{ and } 2\alpha = \min\left\{\eta, \sigma\right\},$$

which implies the exponential stability of (1). ■

Remark 2. *Analogously to the Lyapunov-Krasovskii conditions for the single discrete delay case introduced in [16], the conditions of Theorem 2 guarantee the exponential stability of (1) by means of continuous and differentiable functionals in spite of the fact that* $x_t(\varphi) \in \mathcal{PC}\left(\left[-r,0\right), \mathcal{R}^n\right)$.

Note that, in counterpart with Theorem 3 in [16] for the single delay case, Theorem 2 does not allow us to explicitly compute the exponential decay rate α and μ-factor involved in the exponential upper bound for the solutions. The problem is that we do not have a methodology for computing the positive constants $c\left(\eta\right)$ and η involved in the inequality (8).

By assuming that

$$\lambda = \sum_{j=1}^{m} \left\|A_j\right\| < 1, \tag{11}$$

which assures the delay-independent exponential stability of (2), we are able to explicitly calculate exponential estimates for the solutions of (1) and accept the additional conservatism.

Theorem 3. *Consider system (1) and assume that the inequality (11) holds. If there exists a functional* $v : \mathcal{PC}\left(\left[-r,0\right), \mathcal{R}^n\right) \to \mathcal{R}$ *satisfying the conditions 1 and 2 of Theorem 2 then any solution* $x(t,\varphi)$ *of (1) satisfies the inequality*

$$\left\|x(t,\varphi)\right\| \leq \mu \left\|\varphi\right\|_r e^{-\alpha t}, \forall t \geq 0, \tag{12}$$

with

$$\mu \geq \max\left\{1, \frac{\left(\sqrt{\frac{\alpha_2}{\alpha_1}} r\tau\right) m_g}{1-e^{(\alpha-\eta)r}}\right\} \text{ and } \alpha < \min\left\{\sigma, \eta\right\}, \tag{13}$$

where $\eta = -\frac{\ln(\lambda)}{r}$, $\sigma = 0.5\beta\alpha_2^{-1}$ *and* $m_g = \sup_{\theta\in[-\tau,0]} \left\|G(\theta)\right\|$.

Proof. Follow the same steps in Theorem 2 for arriving at the equation (6). Now since the inequality (11) holds it then follows from Lemma 3 in [10] that

$$\left\|x(t,\varphi)\right\| \leq K(\varphi)e^{-\alpha t}, \forall t \geq 0,$$

for the solutions $x(t,\varphi)$ of (6), where $\alpha < \min\{\sigma, \eta\}, \eta = -\frac{\ln(\lambda)}{r}$, and

$$K(\varphi) \geq \|\varphi\|_r \max\left\{1, \frac{\gamma}{1 - e^{(\alpha-\eta)r}}\right\}.$$

This implies that solution $x(t,\varphi)$ of (1) satisfies the inequality

$$\|x(t,\varphi)\| \leq \mu \|\varphi\|_r e^{-\alpha t}, \forall t \geq 0,$$

with μ and α given by (13) and the result follows. ■

3. Main Results

Consider the following continuous time difference system with multiple discrete and distributed delays:

$$x(t) = \sum_{j=1}^{m} A_j x(t - h_j) + \sum_{j=1}^{m} G_j \int_{-\tau_j}^{0} x(t+\theta)d\theta, \tag{14}$$

where $0 < h_1 < h_2 < \cdots < h_m$, $0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m$, $A_j, G_j \in \mathcal{R}^n, j = 1, 2, \ldots, m$. The system (14) is a particular case of (1), where $G(\theta) = \sum_{k=j}^{m} G_k, \theta \in [-\tau_j, \tau_{j-1}), j = 1, 2, \ldots, m$, with $\tau_0 = 0$ and $\tau_m = \tau$.

In order to apply Theorem 2 for the system (14) one needs to assure the delay-independent exponential stability of (2). To this aim, there are some conditions based on linear matrix inequalities. To the best of our knowledge, the first time such kind of conditions were reported in the literature is in the work of Carvalho [1]. It was shown there that the linear matrix inequalities conditions imply the necessary and sufficient ones given by Silkowski (see Theorem 6.1 in [4]) and, therefore, they are sufficient for delay-independent stability of (2).

Theorem 4. *[1] Suppose that there exist positive definite matrices $S_j, j = 1, 2, \ldots, m$ such that*

$$S - A^T \left[\sum_{j=1}^{m} S_j\right] A < 0, \tag{15}$$

where $S = diag(S_1, \ldots, S_m)$ and $A = \begin{bmatrix} A_1 & \cdots & A_m \end{bmatrix}$. Then the continuous time difference system (2) is delay-independent exponentially stable.

An equivalent condition to (15) has been naturally found in [18] by investigating the input-to-state stability of coupled delay differential and continuous time difference equations. Similarly, the condition (15) will naturally arises in our main result given in the next Theorem.

Theorem 5. *System (14) is exponentially stable if there exist positive definite matrices $P_j, Q_j, j = 1, 2, \ldots, m, R$ and S such that for $j = 1, 2, \ldots, m$,*

$$\mathcal{N}_j = \begin{bmatrix} \frac{1}{m\tau_j}\mathcal{X} & \mathcal{Y}_j \\ \mathcal{Y}_j^T & Q_j - m\tau_j G_j^T \mathcal{M} G_j \end{bmatrix} > 0, \tag{16}$$

where

$$\begin{aligned} \mathcal{X} &= \begin{bmatrix} P_1 - A_1^T\mathcal{M}A_1 & -A_1^T\mathcal{M}A_2 & \cdots & -A_1^T\mathcal{M}A_m \\ -A_2^T\mathcal{M}A_1 & P_2 - A_2^T\mathcal{M}A_2 & \cdots & -A_2^T\mathcal{M}A_m \\ \vdots & \vdots & \ddots & \vdots \\ -A_m^T\mathcal{M}A_1 & -A_m^T\mathcal{M}A_2 & \cdots & P_m - A_m^T\mathcal{M}A_m \end{bmatrix}, & (17) \\ \mathcal{Y}_j^T &= \begin{bmatrix} -G_j^T\mathcal{M}A_1 & -G_j^T\mathcal{M}A_2 & \cdots & -G_j^T\mathcal{M}A_m \end{bmatrix}, & (18) \\ \mathcal{M} &= \sum_{j=1}^m P_j + \sum_{j=1}^m \tau_j Q_j + R + rS, & (19) \end{aligned}$$

with $r = \max\{h_m, \tau_m\}$.

Proof. Consider the following functional candidate:

$$\begin{aligned} v(\varphi) &= \sum_{j=1}^m \int_{-h_j}^0 \varphi^T(\theta)P_j\varphi(\theta)d\theta + \sum_{j=1}^m \int_{-\tau_j}^0 \varphi^T(\theta)\left(\theta+\tau_j\right)Q_j\varphi(\theta)d\theta \\ &\quad + \int_{-r}^0 \varphi^T(\theta)\left[R + (\theta + r)\,S\right]\varphi(\theta)d\theta, \end{aligned} \tag{20}$$

where $P_j, Q_j, j = 1, 2, \ldots, m, R$ and S are positive definite matrices.

The functional (20) satisfies the following inequalities:

$$\alpha_1 \int_{-r}^0 \|\varphi(\theta)\|^2 d\theta \le v(\varphi) \le \alpha_2 \int_{-r}^0 \|\varphi(\theta)\|^2 d\theta,$$

with

$$\alpha_1 = \lambda_{\min}(R), \tag{21}$$

$$\alpha_2 = \sum_{j=1}^{m} \lambda_{\max}(P_j) + \sum_{j=1}^{m} \lambda_{\max}(\tau_j Q_j) + \lambda_{\max}(R + rS). \tag{22}$$

The time derivative of the functional (20) along solutions of (14) is

$$\frac{dv(x_t)}{dt} = x^T(t)\mathcal{M}x(t) - \sum_{j=1}^{m} x^T(t-h_j)P_j x(t-h_j) - x^T(t-r)Rx(t-r)$$
$$- \sum_{j=1}^{m} \int_{-\tau_j}^{0} x^T(t+\theta)Q_j x(t+\theta)d\theta - \int_{-r}^{0} x^T(t+\theta)Sx(t+\theta)d\theta,$$

where $\mathcal{M}$ is defined by (19).

Substituting the right-hand side of (14) in the term $x^T(t)\mathcal{M}x(t)$ we have

$$x^T(t)\mathcal{M}x(t) = \left(\sum_{j=1}^{m} A_j x(t-h_j)\right)^T \mathcal{M} \left(\sum_{j=1}^{m} A_j x(t-h_j)\right)$$
$$+2\left(\sum_{j=1}^{m} A_j x(t-h_j)\right)^T \mathcal{M} \left(\sum_{j=1}^{m} G_j \int_{-\tau_j}^{0} x(t+\theta)d\theta\right)$$
$$+\left(\sum_{j=1}^{m} G_j \int_{-\tau_j}^{0} x(t+\theta)d\theta\right)^T \mathcal{M} \left(\sum_{j=1}^{m} G_j \int_{-\tau_j}^{0} x(t+\theta)d\theta\right).$$

By using the Jensen inequality, the inequality

$$\left(\sum_{j=1}^{m} G_j \int_{-\tau_j}^{0} x(t+\theta)d\theta\right)^T \mathcal{M} \left(\sum_{j=1}^{m} G_j \int_{-\tau_j}^{0} x(t+\theta)d\theta\right)$$
$$\leq m\sum_{j=1}^{m} \tau_j \int_{-\tau_j}^{0} x^T(t+\theta)G_j^T \mathcal{M} G_j x(t+\theta)d\theta,$$

holds.

Then, we obtain the following upper bound for the derivative:

$$\frac{dv(x_t)}{dt} \leq \left(\sum_{j=1}^{m} A_j x(t-h_j)\right)^T \mathcal{M} \left(\sum_{j=1}^{m} A_j x(t-h_j)\right)$$

$$+2\left(\sum_{j=1}^{m} A_j x(t-h_j)\right)^T \mathcal{M} \left(\sum_{j=1}^{m} G_j \int_{-\tau_j}^{0} x(t+\theta)d\theta\right)$$

$$-\sum_{j=1}^{m} \int_{-\tau_j}^{0} x^T(t+\theta)\left[Q_j - m\tau_j G_j^T \mathcal{M} G_j\right] x(t+\theta)d\theta$$

$$-\sum_{j=1}^{m} x^T(t-h_j)P_j x(t-h_j) - x^T(t-r)Rx(t-r)$$

$$-\int_{-r}^{0} x^T(t+\theta)Sx(t+\theta)d\theta.$$

that can be rewritten as

$$\frac{dv(x_t)}{dt} \leq -\sum_{j=1}^{m} \int_{-\tau_j}^{0} \xi^T(\theta)\mathcal{N}_j \xi(\theta)d\theta - x^T(t-r)Rx(t-r)$$

$$-\int_{-r}^{0} x^T(t+\theta)Sx(t+\theta)d\theta,$$

where the matrices $\mathcal{N}_j, j = 1, 2, \ldots, m$, are defined by (16)-(19) and for $\theta \in [-\tau_m, 0]$

$$\xi^T(\theta) = \begin{bmatrix} x^T(t-h_1) & x^T(t-h_2) & \cdots & x^T(t-h_m) & x^T(t+\theta) \end{bmatrix}.$$

Since the matrix $\mathcal{X}$ defined by (17) can be rewritten as

$$\mathcal{X} = P - A^T \left[\sum_{j=1}^{m} P_j\right] A - A^T \left[\sum_{j=1}^{m} \tau_j Q_j + R + rS\right] A,$$

where $P = diag\,(P_1, \ldots, P_m)$ and $A = \begin{bmatrix} A_1 & \cdots & A_m \end{bmatrix}$, it then follows that the inequalities $\mathcal{N}_j > 0, j = 1, 2, \ldots, m$, implies that $\mathcal{X} > 0$ and by Theorem 4 the delay-independent exponential stability of (2).

Clearly, if $\mathcal{N}_j > 0, j = 1, 2, \ldots, m$, then

$$\frac{dv(x_t)}{dt} \leq -\beta \int_{-r}^{0} \|x(t+\theta)\|^2 d\theta$$

holds with

$$\beta = \lambda_{\min}(S), \tag{23}$$

and, therefore by Theorem 2, the exponential stability of (14) follows. ■

For exponential estimates of the solutions, we can assume the condition (11) and apply Theorem 3 to get the following result.

Theorem 6. *Let system (14) be given and assume that the inequality (11) holds. If there exist positive definite matrices $P_j, Q_j, j = 1, 2, \ldots, m, R$ and S such that the inequalities (16) hold, then an exponential estimate for the solutions of (14) is given by (12), where*

$$\mu \geq \max \left\{ 1, \frac{\left(\sqrt{\frac{\alpha_2}{\alpha_1}} r\tau\right) \left(\sum_{j=1}^{m} \|G_j\|\right)}{1 - e^{(\sigma-\eta)r}} \right\} \text{ and } \alpha < \min \left\{ \frac{\beta}{2\alpha_2}, \eta \right\}. \tag{24}$$

Here $r = \max\{h_m, \tau_m\}, \eta = -\frac{\ln(\lambda)}{r}, \alpha_1, \alpha_2$ and β respectively given by (21), (22) and (23).

Proof. Given positive definite matrices $P_j, Q_j, j = 1, 2, \ldots, m, R$ and S satisfying that $\mathcal{N}_j > 0, j = 1, 2, \ldots, m$, we calculate the positive constants α_1, α_2 and β determined by (21), (22) and (23). Noting that

$$m_g = \sup_{\theta \in [-\tau_m, 0]} \|G(\theta)\| \leq \sum_{j=1}^{m} \|G_j\|$$

the result follows directly from the proof of Theorem 3. ■

Remark 3. *Notice that although the stability conditions given in Theorem 5 include both discrete and distributed delays, they are actually* delay-independent *w.r.t the discrete delays and* delay-dependent *w.r.t. the distributed delays. In fact, since the discrete delays are only involved by means of rS, where $r = \max\{h_m, \tau_m\}$, then by doing the change of variable $\tilde{S} = rS$ the dependence on r can be discarded from the stability conditions to simply arrive at the above conclusion. On the other hand, as it is expected, the exponential decay rate α and the μ factor in the exponential estimates for the solutions given in Theorem 6 depend on both the discrete and distributed delays.*

In the case when the discrete delays are commensurate it is possible to write the systems as continuous time difference systems with a single discrete delay for which the main ideas introduced in [15] and [16] can be applied. We will consider the following two important cases: only commensurate discrete delays and commensurate discrete and distributed delays.

3.1. Commensurate discrete delays

Consider here the system

$$x(t) = \sum_{j=1}^{m} A_j x(t - jh) + \sum_{j=1}^{m} G_j \int_{-\tau_j}^{0} x(t+\theta)d\theta, \tag{25}$$

where $h > 0$ is the basic discrete delay and $0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m \leq h$. By defining

$$\hat{x}^T(t) = \left[\begin{array}{cccc} x^T(t) & x^T(t-h) & \cdots & x^T(t-(m-1)h) \end{array} \right], \tag{26}$$

the system (25) can be written as the following one with a single discrete delay and several distributed delays:

$$\hat{x}(t) = \hat{A}\hat{x}(t-h) + \sum_{j=1}^{m} \hat{G}_j \int_{-\tau_j}^{0} \hat{x}(t+\theta)d\theta, \tag{27}$$

where

$$\hat{A} = \begin{bmatrix} A_1 & A_2 & \cdots & A_m \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix} \tag{28}$$

and for $j = 1, 2, ..., m$,

$$\hat{G}_j = \begin{bmatrix} G_j & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{29}$$

Proposition 7. *Let system (25) be given and assume that $\hat{A}$ defined by (28) is Schur stable. System (25) is exponentially stable if there exist positive definite matrices $W_j, j = 0, 1, \ldots, m$, such that for $j = 1, 2, \ldots, m$,*

$$\hat{\mathcal{N}}_j = \begin{bmatrix} \frac{1}{m\tau_j} W_0 & -\hat{A}^T P \hat{G}_j \\ -\hat{G}_j^T P \hat{A} & W_j - m\tau_j \hat{G}_j^T P \hat{G}_j \end{bmatrix} > 0, \tag{30}$$

where $\hat{G}_j, j = 1, 2, \ldots, m$, are defined by (29) and P the unique positive solution of the Lyapunov inequality

$$\hat{A}^T P \hat{A} - P + \left(W_0 + \sum_{j=1}^{m} \tau_j W_j \right) < 0. \tag{31}$$

Moreover, for any $\varphi \in \mathcal{C}\left([-mh, 0), \mathcal{R}^n\right)$, the solution $x(t, \varphi)$ of (25) satisfies the inequality

$$\|x(t, \varphi)\| \leq m\mu \|\varphi\|_{mh} e^{-\alpha t}, \forall t \geq 0, \tag{32}$$

where $\mu > 0$ and $\alpha > 0$ are given by

$$\mu = \eta \left(1 + \gamma + \frac{\gamma}{h\varepsilon e} \right) \text{ and } \alpha = \min\left\{ \sigma, \frac{\beta}{2\alpha_2} \right\} - \varepsilon. \tag{33}$$

Here $\varepsilon \in \left(0, \min\left\{\sigma, \frac{\beta}{2\alpha_2}\right\}\right)$, $\eta > 0$ and $\sigma > 0$ are such that $\left\|\hat{A}^k\right\| \leq \eta e^{-\sigma(kh)}, k = 0, 1, 2, \ldots$, $\gamma = h\sqrt{\frac{\alpha_2}{\alpha_1}} \left(\sum_{j=1}^{m} \left\|\hat{G}_j\right\|\right)$, while α_1, α_2 and β are given respectively by

$$\alpha_1 = \lambda_{\min}\left(\hat{A}^T P \hat{A} + W_0\right), \tag{34}$$

$$\alpha_2 = \lambda_{\max}\left(\hat{A}^T P \hat{A} + W_0\right) + \sum_{j=1}^{m} \lambda_{\max}\left(\tau_j W_j\right), \tag{35}$$

$$\beta = \lambda_{\min}\left(\hat{\mathcal{N}}_m\right). \tag{36}$$

Proof. For a given initial function $\varphi \in \mathcal{C}\left([-mh, 0), \mathcal{R}^n\right)$, let $x(t, \varphi)$ be the solution of (25). Let $\hat{\varphi} \in \mathcal{C}\left([-h, 0), \mathcal{R}^{nm}\right)$ be the vector function constructed from φ according to (26) and $\hat{x}(t, \hat{\varphi})$ be the corresponding solution of the extended system (27). Since $\|x(t, \varphi)\| \leq \|\hat{x}(t, \hat{\varphi})\|$ then exponential stability of the extended system (27) implies that of the original system (25).

Consider now the following functional:

$$v(\hat{\varphi}) = \int_{-h}^{0} \hat{\varphi}^T(\theta) \left[\hat{A}^T P \hat{A} + W_0 \right] \hat{\varphi}(\theta) d\theta + \sum_{j=1}^{m} \int_{-\tau_j}^{0} \hat{\varphi}^T(\theta) \left(\theta + \tau_j \right) W_j \hat{\varphi}(\theta) d\theta,$$

where P is the unique positive solution of (31) and $W_j, j = 0, 1, \ldots, m$, are positive definite matrices. This functional satisfies the inequalities

$$\alpha_1 \int_{-h}^{0} \|\hat{\varphi}(\theta)\|^2 \, d\theta \leq v(\hat{\varphi}) \leq \alpha_2 \int_{-h}^{0} \|\hat{\varphi}(\theta)\|^2 \, d\theta,$$

with $0 < \alpha_1 \leq \alpha_2$ given by (34) and (35). By following a similar line of arguments of those used in the proof of Proposition 3 in [15], we are able to prove that if $\hat{\mathcal{N}}_j > 0, j = 1, 2, \ldots, m$, where $\hat{\mathcal{N}}_j$ are defined by (30), then

$$\frac{d}{dt} v(\hat{x}_t) \leq -\beta \int_{-h}^{0} \|\hat{x}(t + \theta)\| \, d\theta,$$

with β given by (36). Then, by Theorem 3 in [16] the exponential stability of the extended system (27) is assured. Indeed, from the proof of Theorem 3 in [16], one gets that any solution $\hat{x}(t, \hat{\varphi})$ of the extended system (27) satisfies the following exponential upper bound:

$$\|\hat{x}(t, \hat{\varphi})\| \leq \mu \|\hat{\varphi}\|_h e^{-\alpha t}, \forall t \geq 0, \tag{37}$$

with μ and α determined by (33). Now

$$\sup_{\theta \in [-h, 0)} \|\hat{\varphi}(\theta)\| \leq \sum_{j=0}^{m-1} \left(\sup_{\theta \in [-h, 0)} \|\varphi(\theta - jh)\| \right) \leq \sum_{j=0}^{m-1} \left(\sup_{\theta \in [-mh, 0)} \|\varphi(\theta)\| \right)$$

implies that $\|\hat{\varphi}\|_h \leq m \|\varphi\|_{mh}$. From this, the inequality (37) and the fact that $\|x(t, \varphi)\| \leq \|\hat{x}(t, \hat{\varphi})\|, \forall t \geq 0$, we arrive at the result that any solution $x(t, \varphi)$ of (25) satisfies the inequality (32) and thus the proof of the Proposition. ■

Remark 4. *Notice that, similar to that stated in Remark 3, the stability conditions for (25) given in Proposition 7 are* delay-independent *w.r.t. the basic discrete delay and* delay-dependent *w.r.t. the distributed delays. Of course, the exponential estimate for the solutions (32) depends on both the discrete and distributed delays.*

3.2. Commensurate discrete and distributed delays

Consider now the following system:

$$x(t) = \sum_{j=1}^{m} A_j x(t - jh) + \sum_{j=1}^{m} G_j \int_{-jh}^{0} x(t + \theta) d\theta, \tag{38}$$

where $h > 0$ is the basic delay. By using the same definition of $\hat{x}^T(t)$ given in (26) the system (38) can be written as the following one with only one discrete and distributed delay:

$$\hat{x}(t) = \hat{A}\hat{x}(t - h) + \hat{G} \int_{-h}^{0} \hat{x}(t + \theta) d\theta,$$

where $\hat{A}$ is defined by (28) and

$$\hat{G} = \begin{bmatrix} \sum_{j=1}^{m} G_j & \sum_{j=2}^{m} G_j & \cdots & G_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}. \tag{39}$$

The following result can be directly obtained from Proposition 7, see also Remark 6 in [16].

Proposition 8. *Let system (38) be given and assume that $\hat{A}$ defined by (28) is Schur stable. System (38) is exponentially stable if there exist positive definite matrices W_0 and W_1 such that*

$$\hat{\mathcal{N}} = \begin{bmatrix} \frac{1}{h} W_0 & -\hat{A}^T P \hat{G} \\ -\hat{G}^T P \hat{A} & W_1 - h\hat{G}^T P \hat{G} \end{bmatrix} > 0,$$

where $\hat{G}$ is defined by (39) and P is the unique positive solution of the Lyapunov inequality

$$\hat{A}^T P \hat{A} - P + (W_0 + hW_1) < 0. \tag{40}$$

Furthermore, an exponential estimate for the solutions of (38) is given by (32) with $\mu, \alpha, \sigma, \eta$ and ε as in Proposition 7 while $\gamma = h\sqrt{\frac{\alpha_2}{\alpha_1}} \left(\left\| \hat{G} \right\| \right)$,

$$\begin{aligned} \alpha_1 &= \lambda_{\min} \left(\hat{A}^T P \hat{A} + W_0 \right), \\ \alpha_2 &= \lambda_{\max} \left(\hat{A}^T P \hat{A} + W_0 \right) + \lambda_{\max} (hW_1), \\ \beta &= \lambda_{\min} \left(\hat{\mathcal{N}} \right). \end{aligned}$$

Remark 5. *Note here that not only the exponential estimate for the solutions but also the stability conditions given in Proposition 8 depend on the basic delay h.*

4. Illustrative Examples

Example 1. *Let consider the scalar system*

$$x(t) = ax(t-h) + g\int_{-h}^{0} x(t+\theta)d\theta, \tag{41}$$

where a, g are real numbers and $h > 0$.

In this particular case, by means of the characteristic function associated to (41), it is possible to obtain the following necessary and sufficient conditions for exponential stability of (41):

Given $h > 0$, the system (41) is exponentially stable if, and only if, the pair of coefficients (a, g) belongs to the stability region $\Omega(h)$, plotted in Fig. 1, determined by

$$\Omega(h) = \{(a, b) \mid -1 < a < 1 \text{ and } a + gh < 1\}.$$

It can be seen from Fig. 1 that the stability region $\Omega(h)$ (dark shadowed region) can be decomposed in two regions. The first one is the delay-independent *stability region determined by $-1 < a < 1$ and $g \leq 0$ and the second one is the* delay-dependent *stability region determined by $a + gh < 1$ with $-1 < a < 1$ and $g > 0$.*

It is worth noting that the subregion of the delay-independent stability region determined by $-1 < a \leq 0$ and $g \leq 0$ was derived in [2].

The corresponding stability region (light shadowed region) provided by Proposition 6 (or Remark 6 in [16]) is shown in Fig. 1. This region coincides with the stability region provided by the sufficient condition $|a| + h|g| < 1$ derived from the inequality (3). Thus, in this case, the obtained Lyapunov conditions are as conservative as the sufficient conditions based on matrix norms.

In spite of the conservatism of the Lyapunov conditions in the scalar case they allow us to derive robust stability conditions against general classes of perturbations. Thus, for instance, let us consider the following perturbed system

$$y(t) = [a + \Delta a(t)]\, y(t-h) + [g + \Delta g(t)] \int_{-h}^{0} x(t+\theta)d\theta, \tag{42}$$

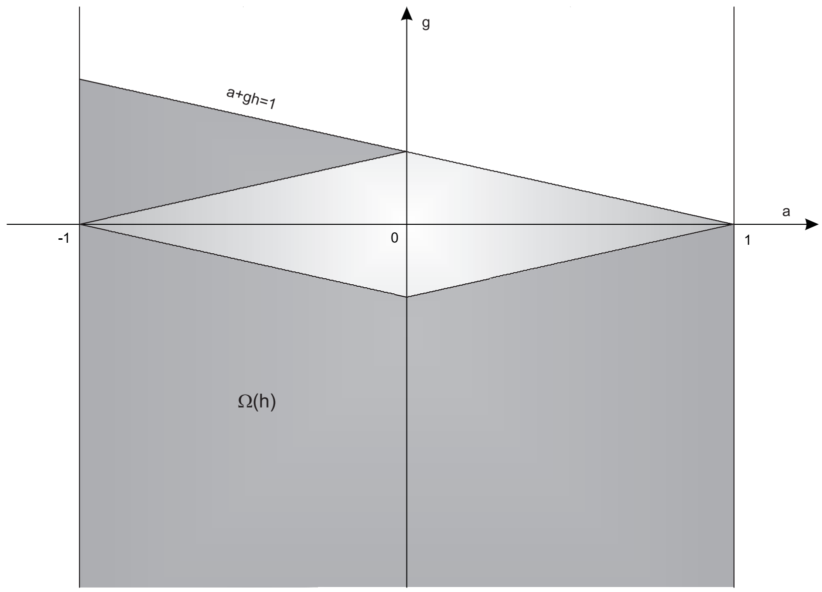

Figure 1: Stability region (dark shadowed region) of (41).

where $a = 0.5, g = 0.25,$ *and* $\Delta a(t), \Delta g(t)$ *are determined by*

$$\Delta a(t) = 0.1 \sin\left(y^2(t-h) + 5t\right) \text{ and } \Delta g(t) = 0.1\left(1 - e^{-0.25t}\right). \tag{43}$$

The time-varying nonlinear functions $\Delta a(t), \Delta g(t)$ *satisfies* $|\Delta a(t)| \leq \delta$ *and* $|\Delta g(t)| \leq \rho, \forall t \geq 0,$ *with* $\delta = \rho = 0.1.$

By applying Proposition 5 in our pervious paper [16] we have that the perturbed system (42) is exponentially stable for perturbations $\Delta a(t)$ *and* $\Delta g(t)$ *given in (43) if delay value is such that* $0 < h \leq 1.296.$

Here it is important to note that this result holds not only for the particular perturbations given in (43) but also for any time-varying nonlinear perturbations $\Delta a(t), \Delta g(t)$ *satisfying* $|\Delta a(t)| \leq 0.1$ *and* $|\Delta g(t)| \leq 0.1, \forall t \geq 0,$ *and for which the existence and uniqueness of solutions is assured.*

In Fig. 2, we present a numerical simulation of the perturbed system for an initial function $\varphi(t) = \sin(10t), t \in [-h, 0)$ *and delay value of* $h = 1.29.$ *It can be seen that the corresponding solution of the perturbed system (42) is exponentially stable as it is assured by the Lyapunov conditions.*

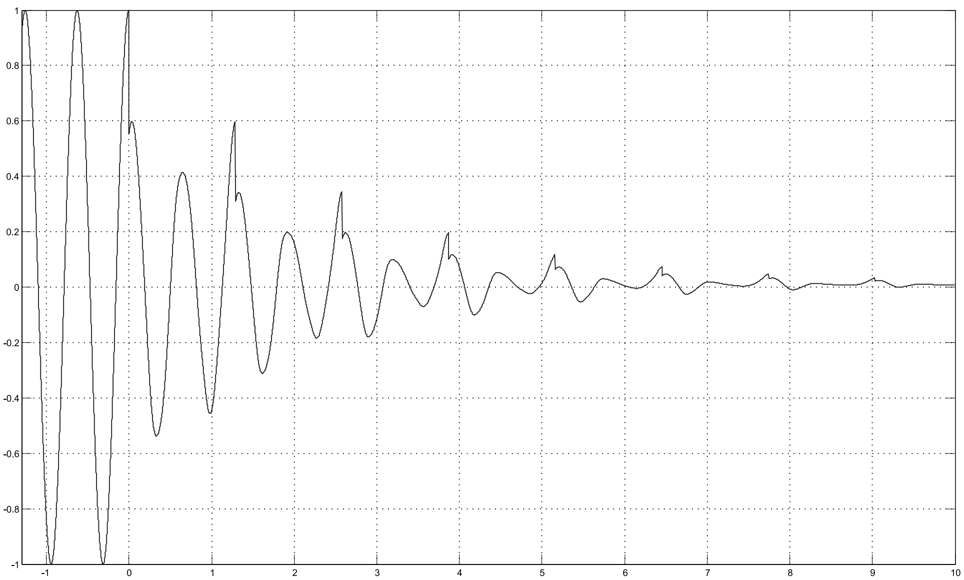

Figure 2: Numerical simulation of (42) for initial function $\varphi(t) = \sin(10t)$ and delay $h = 1.29$.

Example 2. *Consider the continuous time difference system*

$$x(t) = \sum_{j=1}^{2} A_j x(t - h_j) + \sum_{j=1}^{2} G_j \int_{-\tau_j}^{0} x(t + \theta) d\theta, \tag{44}$$

where

$$\begin{aligned} A_1 &= \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{bmatrix}, A_2 = \begin{bmatrix} 0.2 & 1 \\ -0.1 & -0.2 \end{bmatrix} \\ G_1 &= \begin{bmatrix} 0 & 0.5 \\ -0.2 & -3 \end{bmatrix}, G_2 = \begin{bmatrix} 0.1 & 0 \\ 0.5 & 0.1 \end{bmatrix}. \end{aligned}$$

In this case, the following simple sufficient condition for the exponential stability of (44) is derived from the norm condition (3):

$$\sum_{j=1}^{2} \|A_j\| + \sum_{j=1}^{2} \tau_j \|G_j\| < 1. \tag{45}$$

Since $\|A_1\| + \|A_2\| = 1.3345$ *then the inequality (45) cannot be applied to investigate the stability of (44). However, by using Theorem 5 we are able* *to provide a solution to the stability problem of (44). Let us consider that $\tau_1 = 0.25$ and search for a maximum value of $\tau_2 \geq \tau_1$ such that (44) is exponentially stable.*

We found that (44) is exponentially stable for any given $0 < h_1 < h_2$, $\tau_1 = 0.25$ and $\tau_2 \in [0.25, 0.512]$.

Unfortunately, in this case, we are not able to compute exponential estimates for solutions of (44) as it will be done in the following example.

Example 3. *Consider again system (44) but now with system matrices*

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0.2 & 0 \\ -0.1 & -0.2 \end{bmatrix}, \\ G_1 &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, G_2 = \begin{bmatrix} 0.1 & 0 \\ 1 & 0.1 \end{bmatrix}. \end{aligned}$$

In this case we have that $\|A_1\| + \|A_2\| = 0.4180$. Now if we fix $\tau_1 = 0.25$ and look for $\tau_2 \geq \tau_1$ such that (44) is exponentially stable then we have that the inequality (45) does not hold for any value of $\tau_2 > 0$ since $\|A_1\| + \|A_2\| + \tau_1 \|G_1\| = 1.0215$.

By using Theorem 5 we found that (44) is exponentially stable for any given $0 < h_1 < h_2, \tau_1 = 0.25$ and $\tau_2 \in [0.25, 0.66]$.

Now, since in this case we have that $\|A_1\| + \|A_2\| < 1$ then we can apply Theorem 6 to obtain exponential estimates for the solutions of (44).

Let us select $h_1 = 1, h_2 = \sqrt{5}, \tau_1 = 0.25$ and $\tau_2 = 0.3$. Direct calculations derived from (21), (22) and (23) yield $\alpha_1 = 1.0291, \alpha_2 = 70.7435$ and $\beta = 0.7066$. Then the corresponding $\mu > 0$ and $\alpha > 0$ in Theorem 6 satisfy $\mu \geq 40.2752$ and $\alpha < 0.0050$. By selecting $\mu = 40.3$ and $\alpha = 0.0049$ we arrive at the following exponential upper bound for the solutions of (44):

$$\|x(t, \varphi)\| \leq 40.3 \|\varphi\| e^{-0.0049t}, \forall t \geq 0,$$

which holds for any arbitrary initial function $\varphi \in \mathcal{C}\left(\left[-\sqrt{5}, 0\right), \mathcal{R}^2\right)$.

5. Conclusions

In this paper, Lyapunov-Krasovskii functionals for the exponential stability of some linear continuous time difference systems with multiple discrete and distributed delays are introduced, extending thus previous results in [15] and [16] for the case of system with a single discrete delay.

New sufficient conditions for the exponential stability expressed as linear matrix inequalities are derived. By assuming the condition (11) exponential estimates for the system solutions are also given. Further investigations are needed for relaxing the condition (11) to get exponential estimates of the solutions.

The results have been compared with the necessary and sufficient conditions that can be obtained for the simplest scalar system. It is shown that in spite of the fact that the Lyapunov conditions are as conservative as the ones derived from the norm-based inequality (3) they allow us to derive robust stability conditions against general classes of perturbations.

For the general matrix case the Lyapunov conditions provide solutions to the exponential stability problem of systems for which the norm-based inequality (3) does not allow concluding.

Future works will concern to perform robustness analysis of perturbed systems with multiple discrete and distributed delays and the application of the results to synthesis problems of continuous time difference systems.

6. Acknowledgment

This work is supported by CONACYT under grant 131587. The author wishes to thank the Associate Editor and anonymous reviewers for their careful reading and very useful comments on this paper.

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