A Lie-based approach to the general framework of chaotic synchronization

Ricardo Femat[a] and Ljupco Kocarev[b]

[a]Matemáticas Aplicadas y Sistemas Computacionales, IPICYT,

Apdo Postal 3-90, 78231, Tangamanga, San Luis Potosí.

[b]Institute for Nonlinear Science, UCSD.

March 18, 2003

ABSTRACT

Diverese phenomena have been reported on the synchornization of chaotic systems. Therefore, the generalized framework of the chaotic synchronization is an actual scientific debate. Here, a Lie-based geometrical approach is presented to remark some geometrical properties of the nonlinear (chaotic) systems toward their synchronization. That is, we address the general problem of finding the conditions for the existence of the synchronization function $y = \lambda(x)$. The contribution is focused on the 2 and 3 dimensional (unidirectionally coupled) systems. Illustrative examples are provided along the text.

I. Introduction.

Synchronous behavior can be found in diverse dynamical systems (as, to mention some, biological [1] or communication [2]) and its study is theoretically interesting and technologically important. The greek root of the word "synchronization" means "to share the common time"; that is, by de¯nition, synchronous behavior signify that two or more systems occur at same time. Such a de¯nition can be directly used on regular dynamical systems. Nevertheless, there are diverse synchronization phenomena on the chaotic systems [3]. Indeed, synchronization has been induced in strictly di®erent oscillators [4],[5] and systems with di®erent order [6]. Such a fact increases the complexity on the meaning of the chaotic synchronization. In the nonlinear science, the de¯nition of the synchronous behavior means that the trajectories of two or more dynamical systems evolve, in some sense, close along their trajectories. As a consequence of the diversity of the synchronization phenomena, the actual discussion point is the uni¯cation toward a general de¯nition of the synchronous behavior in chaotic systems [7], [8]. The idea is that the di®erent kinds of synchronization can be captured by a formalism [7] by searching the existence of a di®ermorphism between attractors of the coupled systems [9] whose properties involve a time-invariant synchronization manifold (some authors called it "the synchronization function" [8]).

The e®orts have been focused on the analysis of the time-invariant manifolds related with synchronization [8]-[10]. Two basic approaches have been exploited. On the one hand, chaotic synchronization has been interpreted as the prediction of the chaotic system, i.e., observability approach. In this sense, the reconstruction of the drive system attractor from the response system is interpreted as an observer [10]. An interesting point about observability of the synchronization systems is that differential geometry allows to find an invariant space under vector fields where the attractor can be reconstructed (see Chapter 1 in [11]). Synchronization of chaotic systems, on the other hand, has been also studied from the measurable variables (system output) [5]-[9]. In such a case, the synchronization is understood as a control (stabilization) problem. In other words, to compute the controller such that the difference between trajectories of the slave system $x_S(t)$ remains close to the trajectories of the master system $x_M(t)$. That is, the point is to find the invariant space such that the origin of the synchronization error system $\|x_M(t) - x_S(t)\| = 0$ can be stabilized. Both observability and controlability of nonlinear systems are included in the geometrical control theory [11],[13].

Here, the goal is to provide some remarks on the diffeomorphism between chaotic attractors by exploiting the Lie algebra of the chaotic systems [12] to compute the synchronization function. In this manner, in this paper we discuss both observation and stabilization approaches. That is, we are interested in the Lie-based geometric properties of the class of dynamical system given by $\dot{x} = \tau_M(x_M) - \tau_S(x_S) - g(x)u$, where $x \in R^n$ is, by definition, $x := x_M - x_S$ and stands for the state vector of the synchronization error system, $\tau_M : R^n \to R^n$, $\tau_S : R^n \to R^n$ and $g : R^n \to R^n$ are smooth vector fields. The product $g(x)u$ is related to the synchronization command. The scalar function $u = u(x)$ can be computed from the construction of accessibility spaces [13]. Since we are interested in the output $y = \lambda(x)$ of the above-mentioned dynamical system, the synchronization is fomuled in terms of geometrical properties of the system along the vector fields $\tau(x; x_M) = \tau_M(x_M) - \tau_S(x_S)$ and $g(x)$. Thus the question is: what are the geometrical properties of any synchronization error system such that the scalar function $u = u(x)$ guarantees the existence of the synchronization function $y = \lambda(x)$? This is addressed via Lie algebra of the vector field related to the synchronization error system.

The text is organized as follows. Preliminaries on the Lie-based geometry of the synchronization systems are presented in Sect. II. Third section has main results. The text is closed with some conclusions. Illustrative examples are presented along the text.

II. Lie-based geometry of nonlinear systems

The idea behind the Lie-based geometry is to find the coordinates decompositions of the synchronization error system. Thus, an invariant space (called distribution) can be constructed and analyzed. Such a space is spanned by the vector fields of the synchronization error system and the Lie Brackets between them. In seek of completeness, this section contains a brief description of the decompositions. Then, the geometrical properties, from complete (generalized) synchronization of two non-identical chaotic systems -same model with different initial conditions and parameter values - is shown as illustrative example.

A. Lie brackets and distribution notion

The starting point is in the properties of the Lie Brackets. A Lie bracket is a kind of differential operation on a dynamical system along its vector fields. Such an operation will be denoted by $[\tau_1; \tau_2]$ for the vector fields $\tau_1; \tau_2 : R^n \rightarrow R^n$ and is defined by

$$[\tau_1; \tau_2](x) \triangleq \frac{\partial \tau_2}{\partial x}\tau_1(x) - \frac{\partial \tau_1}{\partial x}\tau_2(x) \tag{1}$$

at each x in the subset $U \subset R^n$, where

$$\frac{\partial \tau_i}{\partial x} = \begin{pmatrix} \frac{\partial \tau_{i;1}}{\partial x_1} & \frac{\partial \tau_{i;1}}{\partial x_2} & \cdots & \frac{\partial \tau_{i;1}}{\partial x_n} \\ \frac{\partial \tau_{i;2}}{\partial x_1} & \frac{\partial \tau_{i;2}}{\partial x_2} & \cdots & \frac{\partial \tau_{i;2}}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \tau_{i;n}}{\partial x_1} & \frac{\partial \tau_{i;n}}{\partial x_2} & \cdots & \frac{\partial \tau_{i;n}}{\partial x_n} \end{pmatrix} \tag{2}$$

stands for the Jacobian matrices of the mappings τ_i, $i = 1;2$. The differential operation (1) of a vector field τ_2 with the same vector, namely τ_1, is given by $[\tau_1[\ldots\tau_1[\tau_1;\tau_2]]\ldots]$. For the sake of simplicity in notation, the recursive $k-$th Lie bracket of the vector field τ_2 along the vector field τ_1 valuated at the point $x \in R^n$ is denoted by the expression $ad^k_{\tau_1}\tau_2(x) = [\tau_1; ad^{k-1}_{\tau_1}\tau_2](x)$ for any $k \geq 1$ and $ad^0_{\tau_1}\tau_2(x) = \tau_2(x)$ [11]. The Lie bracket between the vector fields τ_1 and τ_2 has the following properties:

- is bilinear over R, that is, if $\theta_1;\theta_2;\tau_1$and τ_2 are vector fields, $\pi_1$and π_2 are real numbers, then (i) $[\pi_1\tau_1 + \pi_2\tau_2;\theta_1] = \pi_1[\tau_1;\theta_1] + \pi_2[\tau_2;\theta_1]$ and (ii) $[\tau_1;\pi_1\theta_1 + \pi_2\theta_2] = \pi_1[\tau_1;\theta_1] + \pi_2[\tau_2;\theta_2]$
- satisfies $[\tau_1;\tau_2] = -[\tau_2;\tau_1]$.

Now, suppose that on a open set U there are d vector fields τ_i such that, for any given point x in U, the vector $\tau_i(x)$ span a vector space. Let us denote the assignment a vector space to each point x of U as $\Delta(x) = \text{span}\,\{\tau_1(x)\ldots\tau_d(x)\}$; Δ is called distribution. In seek of illustrate such a notion, let us define F as a matrix having n columns whose entries are smooth functions of any variable, namely x. The columns of the matrix F can be interpreted as vector fields. Hence, such a matrix identifies the distribution spanned by its columns and can be valued by $\Delta(x) = \mathrm{Im}(F(x))$, where $\mathrm{Im}(F(x))$ denotes the image of the matrix F at any x in U. Of course, the dimension of the distribution can be associated to the rank of the matrix F at the point $x \in U$. Finally, let x^0 be any open set U, it is said that x^0 is a regular point of a distribution Δ if there exists a neighborhood U^0 of x^0 such that the distribution Δ is nonsingular at any x in U^0. Thus, if Δ_1 and Δ_2 are distributions, then:

- the sum $\Delta_1 + \Delta_2$ is defined by taking the pointwise the sum of the subspaces $\Delta_1(x)$ and $\Delta_2(x)$, namely $(\Delta_1 + \Delta_2)(x) = \Delta_1(x) + \Delta_2(x)$,
- the intersection $\Delta_1 \cap \Delta_2$ is defined as $(\Delta_1 \cap \Delta_2)(x) = \Delta_1(x) \cap \Delta_2(x)$.

B. Involutive distributions and flows

Definition 1 Let τ_1 and τ_2 two vector fields belonging to the distribution Δ. It is said that the distribution Δ is involutive if the Lie bracket $[\tau_1; \tau_2]$ is a vector field belonging to the distribution Δ. $\diamond$

Example 2 Let $\pi_i, \mu_i;\ i = 1; 2; 3$ be real positive constants and consider two vector fields τ_1 and τ_2 in R^3 given by $\tau_1(x) = (\pi_1(x_2 - x_1); \pi_2 x_1 - x_2 - x_1 x_3; -\pi_3 x_3 + x_1 x_2)^T$ and $\tau_2(x) = (\mu_1(x_2 - x_1); \mu_2 x_1 - x_2 - x_1 x_3; -\mu_3 x_3 + x_1 x_2)^T$. Note that $\tau_1(x)$ and $\tau_2(x)$ define the vector fields of two Lorenz equations with parameters π_i and μ_i, respectively. In addition, $\dim(\mathrm{span}\{\tau_1(x); \tau_2(x)\}) = 2$ at any point $x_1 \neq x_2$ and $x_3 \neq \pi_2$ far away the origin and for all parameters values $\pi_i; \mu_i \neq 0$. Thus, since the Lie bracket $[\tau_1; \tau_2](x) =$

$$\begin{pmatrix} (\mu_1\pi_2 - \pi_1\mu_2)x_1 + (\pi_1 - \mu_1)x_2 + (\pi_1 - \mu_1)x_1x_3 \\ (\mu_1\pi_2 + \mu_2 - \pi_1\mu_2 - \pi_2)x_1 + (\pi_1\mu_2 - \mu_1\pi_2)x_2 + (\pi_1 - \mu_1 + \pi_3 - \mu_3)x_1x_3 + (\mu_1 - \pi_1)x_3x_2 \\ (\mu_1 - \pi_1)x_2x_1 + (\pi_2 - \mu_2)x_1^2 + (\pi_1 - \mu_1)x_2^2 + (\pi_3 - \mu_3)x_1x_2 \end{pmatrix}$$

belongs to the distribution $\Delta(x)$ for any parameters values $\pi_i = \mu_i \neq 0$, hence, at any point far away the origin with $x_1 \neq x_2$ and $x_3 \neq \pi_2$, the distribution $\Delta(x)$ is involutive. Finally, note that the condition $\mu_i = \pi_i; i = 1; 2; 3$ means that both Lorenz equations are identical. □

Definition 3 A distribution Δ, defined on an open set U, is no singular if there exists an integer ρ such that $\dim(\Delta(x)) = \rho$ for all x in U. Otherwise, the distribution is called singular. △

Definition 4 A point x^0 of any U is called regular point of a distribution $\Delta(x)$ if there exists a neighborhood U^0 of x^0 with the property that Δ is no singular on U^0. △

Example 5 Let us consider the vector fields $\tau_1(x) = (f_1(x); f_2(x))^T$ and $\tau_2(x) = (0; 1)^T$, where $f_1(x) = x_2$ and $f_2(x)$ are analytical functions of $x \in R^2$. Note that, for this example, the vector field $\tau(x)$ represents the second-order oscillators (e.g., if the function is given by $f_2(x) = -\pi_1 x_2 + x_1 - x_1^3$, the vector field $\tau_1(x)$ represents the vector field of the Duffing equation). Now, the distribution $\Delta(x) = \text{span}\{\tau_1(x); \tau_2(x)\}$ has dimension 2 on the set $U = \{x \in R^2 : x_2 \neq 0\}$ and dimension 1 on the set $\Phi = \{x \in R^2 : x_2 = 0\}$. Thus, the distribution $\Delta = \text{span}\{\tau_1; \tau_2\}$ is singular at the origin; however, it is regular elsewhere. The point $x = 0$ is called no regular point or point of singularity. Note that the distribution Δ is involutive on U; however, Δ is not involutive at R^2. □

An involutive distribution is directly related to the flow of dynamical systems by the

Frobenius Theorem.

Theorem 6 (Frobenius Theorem, [11])A no singular distribution Δ is integrable if and only if it is involutive. ■

Example 7 Consider the following distribution defined on $\mathbb{R}^2$, $\Delta = \operatorname{span}\{\tau_1, \tau_2\}$, where

$$\tau_1(x) = \begin{pmatrix} x_1^2 \\ -1 \end{pmatrix}; \quad \tau_2(x) = \begin{pmatrix} x_2 + x_1(1 - x_2^2 - x_1^2) \\ -x_1 + x_2(1 - x_2^2 - x_1^2) \end{pmatrix}$$

therefore $\dim(\Delta(x)) = 2$ for any x in $U = \left\{ x \in \mathbb{R}^2 : x_2 = x_1; 0 < |x_1| < 2\sqrt{\sqrt{5} - 1} \right\}$. Note that the distribution $\Delta(x)$ is involutive on U. The flow of $\tau_1(x)$ and $\tau_2(x)$ is related to the solution of $\dot{x} = \tau_i(x); i = 1, 2$. Thus,by defining $\xi_0 = (\xi_{1,0}, \xi_{2,0}) \in \mathbb{R}^2$ as the conditions at $t_0 = 0$ (initial conditions). About $\tau_1(x)$, the equation $\dot{x} = \tau_1(x)$ is solved by

$$x_1 = \xi_{1,0}(1 - \xi_{1,0} t)^{-1}$$

$$x_2 = -t + \xi_{2,0}$$

from where, for any $\xi_0 \in \mathbb{R}^2$, we have the flow along τ_1 as follows

$$\Phi^{\tau_1}_{z_1}(\xi_0) = \begin{pmatrix} \xi_{1,0}(1 - \xi_{1,0} z_1)^{-1} \\ -z_1 + \xi_{2,0} \end{pmatrix} \qquad (3)$$

whereas, about $\tau_2(x)$, the equation $\dot{x} = \tau_2(x)$ is solved by

$$x_1 = \cos(t)(1 + \xi_{1,0} \exp(-2t))^{-1/2}$$

$$x_2 = -\sin(t)(1 + \xi_{2,0} \exp(-2t))^{-1/2}$$

from where, for any $\xi_0 \in \mathbb{R}^2$, we have the flow along τ_2 as

$$\Phi^{\tau_2}_{z_2}(\xi_0) = \begin{pmatrix} \cos(z_2)(1 + \xi_{1;0}\exp(-2z_2))^{-1/2} \\ \\ -\sin(z_2)(1 + \xi_{2;0}\exp(-2z_2))^{-1/2} \end{pmatrix} \tag{4}$$

Now, let us define $\Psi(z_1; z_2) \mapsto \Phi^{\tau_1}_{z_1} \circ \Phi^{\tau_2}_{z_2}(x)$, where "$\circ$" denotes the composition with respect to the argument x. The mapping $\Psi(z_1; z_2)$ corresponds to the integration of both τ_1 and τ_2 vector fields. Thus, one has that the mapping becomes

$$\Psi(z_1; z_2) = \begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos(z_2)(1 - \xi_{1;0}\exp(-2z_2))^{-1/2}(1 - \xi_{1;0}z_1)^{-1} \\ \\ -\sin(z_2)(1 - \xi_{1;0}\exp(-2z_2))^{-1/2} - z_1 \end{pmatrix} \tag{5}$$

Note that the map $\Psi(z_1; z_2)$ is not defined on all R^2. In fact, if initial condition $\xi_{1;0}$ is such that $0 < |\xi_{1;0}| < 2\sqrt{\sqrt{5} - 1}$, then there exists any $z_1$such that $1 - \xi_{1;0}z_1 = 0$ even if the distribution $\Delta(x) = \mathrm{span}\,\{\tau_1(x); \tau_2(x)\}$ is involutive on $U = \left\{ x \in R^2 : x_2 = x_1; 0 < |x_1| < 2\sqrt{\sqrt{5} - 1} \right\}$. That is, the trajectory $x_1(t; \xi_{1;0})$ is not defined for all initial conditions $\xi_{1;0}$ from the composition of the flows along the vector fields $\tau_1(x)$ and $\tau_2(x)$. However, provided $|1 - \xi_{1;0}z_1| > 0$ and $|1 - \xi_{1;0}\exp(-2z_2)| > 0$, the trajectory $x(t; \xi_0) = (x_1(t; \xi_{1;0}); x_2(t; \xi_{2;0}))$ is defined and, as a consequence, there exists the inverse $z = \Psi^{-1}(x)$ of the mapping $\Psi(z_1; z_2)$ at any x in the neighborhood U^0 of $x^0 \in U$. Thus, the function $z_2(x_1; x_2)$ is defined and, solves the partial differential equation $\frac{\partial z_2}{\partial x}\tau_1(x) = 0$. $\square$

III. Computing the synchronization function

Example 7 shows the importance of the Frobenius Theorem for the decompositions along vector fields. However, such an example doe snot illustrate the usage of the Frobenius Tehorem in computing the synchronization function. In this section we show how the Lie-based geometry of dynamical systems can be exploited in the chaos synchronization context. In seek of simplicity and completeness, the synchronization of nonchaotic systems is illustrated. Then, the synchronization function on chaotic system is shown.

In what follows, let us denote $\tau_1(x_M); \tau_2(x_S)$ and $g(x_S)$ as vector fields such that the synchronization error systems can be obtained from the dynamical systems $\dot{x}_M = \tau_1(x_M)$ and $\dot{x}_S = \tau_2(x_S) + g(x_S)u$, where $x_M \in R^n; x_S \in R^m$ and $u \in R$ stands for the control input. Thus, by defining $x \triangleq x_M - x_S$ as the synchronization error, the synchronization system can be written by $\dot{x} = \tau_1(x_M) - \tau_2(x_S) - g(x_S)u$. Note that $\tau_1(x_M)$ corresponds to the vector fields of the master sysetms whereas $\tau_2(x_S)$ regards the slave system. Then, the aim is to compute the synchronization function $y = \lambda(x)$ from the vector fields $\Delta\tau : R^n \to R^n$ (where $\Delta\tau \triangleq \tau_1 - \tau_2$) and $g : R^n \to R^n$ by exploiting the Lie-bracket algebra. To this end, we use some properties of the Lie derivative, which is related to the inner product $\langle d\lambda(x); \Delta\tau(x)\rangle$ between the vector $d\lambda(x) = \left(\frac{\partial}{\partial x_1}; \frac{\partial}{\partial x_2}; \ldots; \frac{\partial}{\partial x_n}\right)$ and a given vector field $\Delta\tau(x)$ (see Appendix for some details on the covector field $d\lambda(x)$).

Now, let $\Delta\tau : R^n \to R^n; g : R^n \to R^n$ be two smooth vector fields. Let us denote $\lambda(x)$ as the real-valued function of $x \in R^n$ such that its exact derivative is given by $d\lambda(x) = \left[\frac{\partial\lambda(x)}{\partial x_1}; \ldots; \frac{\partial\lambda(x)}{\partial x_n}\right]$. The output $y = \lambda(x)$ of the synchronization error system $\dot{x} = \Delta\tau(x) + g(x)u$ is a synchronization function if there is a real-valued function $\lambda(x)$ such that: (i) inner product $\langle d\lambda(x); ad^{i}_{\Delta\tau} g(x)\rangle = 0$ for $i = 1; 2; \ldots; \rho - 2$ and (ii) $\left\langle d\lambda(x); ad^{\rho-1}_{\Delta\tau} g(x)\right\rangle \neq 0$, for any integer $\rho \leq n$. Note that the integer ρ corresponds to the dimension of the involutive distribution (relative degree).

It should be noted that conditions (i) and (ii) imply the existence of a functional relationship, via the real-valued fucntion $\lambda(x)$, between states of the synchronization error system (i.e., generalized synchronization). Such conditions are relevant because they signify that the flow of the synchronization-error system can be affected by the scalar function $u = u(x)$ throughout the vector field $g(x)$ (i.e., since $\left\langle d\lambda(x); \mathrm{ad}^k_{\Delta\tau} g(x) \right\rangle$ denotes the inner product, it relates the orthogonal (tangent) space spaned by the covector field $d\lambda(x)$ and those resulting of the $k-$th Lie bracket $\mathrm{ad}^k_{\Delta\tau} g(x)$. To demonstrate above claim is beyond the goal of this paper; however a sketch of the proof has been discussed by Solís-Perales in [14] (also see [15]).

A. Synchronization function in nonchaotic systems

Let $\alpha \in \mathbb{R}$ and $p \in \mathbb{R}^2$ be real positive constants and consider two vector fields τ_1 and τ_2 in $\mathbb{R}^2$ given by

$$\tau_1(x) = \begin{pmatrix} x_2 \\ -\alpha x_1 \end{pmatrix}; \tau_2(x) = \begin{pmatrix} x_2 + x_1(1 - x_2^2 - x_1^2) \\ -x_1 + x_2(1 - x_2^2 - x_1^2) \end{pmatrix}$$

Note that the vector field $\tau_1(x)$ is related to the harmonic oscillator $\dot{x}_M = \tau_1(x_M)$ (whose fundamental frequency is given by α and amplitude is given by initial conditions) whereas as $\tau_2(x)$ is a vector field (which can be related to a dynamical system $\dot{x}_S = \tau_2(x_S)$ whose attractor is a periodic orbit with fundamental frequency equal to 1; see Example 7). Thus, the discrepancy $\Delta\tau(x) \triangleq \tau_1(x) - \tau_2(x)$ can be interpreted as the vector field related to the synchronization between two second-order oscillators. Now, let us consider the vector field $g(x) = (g_1(x)\ g_2(x))^T$, where $g_1(x) = 0$ and $g_2(x) = 1$ are given constants. Thus, a synchronization force given by $u = u(x) \in \mathbb{R}$ can be computed such that the synchronization error system takes the affine form $\dot{x} = \Delta\tau(x) - g(x)u$ [5]. In this manner, the problem can be worded as follows: is there any synchronization function $\lambda(x)$ such that the flow along the vector field $\tau_1(x)$ and $\tau_2(x)$ is synchronous? In other words, we shall compute the output $y = \lambda(x)$ such that the synchronization command $g(x)u$ asymptotically steers the trajectories $x(t) \triangleq x_M(t) - x_S(t)$ around the origin. That is, the flow along the vector field $\Delta\tau(x)$ converges to zero and, to this end, the atngent (accesability) space is spanned at the origin. Thus, we need to find the distribution $\Delta = \text{span}\{g; lb_{\Delta\tau}g\}$ such that it is involutive, at least, at the origin. By taking the duiscrepancy vector field, we have that

$$\Delta\tau(x) \triangleq \tau_1(x) - \tau_2(x) = \begin{pmatrix} -x_1(1 - x_2^2 - x_1^2) \\ (1-\alpha)x_1 - x_2(1 - x_2^2 - x_1^2) \end{pmatrix} \quad (6)$$

from where

$$lb_{\Delta\tau}g(x) = [\Delta\tau; g](x) = -\begin{pmatrix} (x_1^2 + x_2^2 + 3)g_1 + 2g_2x_1x_2 \\ (1 - \alpha + 2x_1x_2)g_1 - (x_1^2 + 3x_2^2 - 1)g_2 \end{pmatrix}$$

Thus, by substituting the constants $g_1 = 0$ and $g_2 = 1$, the distribution becomes

$$\Delta(x) = \text{span}\{g; lb_{\Delta\tau}g\} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \begin{pmatrix} -2x_1x_2g_2 \\ (x_1^2 + 3x_2^2 - 1)g_2 \end{pmatrix} \right\} \quad (7)$$

has dimension 1 at eighborhood U of the origin of the synchronization system. Hence, $\Delta = \text{span}\{g; lb_{\Delta\tau}g\}$ is involutive on $U \subset R^2$ and, as a consequence of $\Delta = \text{span}\{g; lb_{\Delta\tau}g\}$ is involutive (see Frobenuis Theorem), the system $\dot{x} = \Delta\tau(x) - g(x)u$ is integrable. That is, the solution $x(t) = \Phi_t^{\Delta\tau}(x)$ exists for any point x in $U \subset R^2$. Note that the integer $\rho = 1$.

Now, in order to obtain the synchronization function, we proceed to find the real-valued function $\lambda(x)$ which satisfies conditions (i) and (ii). Thus, we have that the synchronization function $y = \lambda(x)$ should satisfy the partial differential equations

$$\left\langle d\lambda(x), \mathrm{ad}^0_{\Delta\tau} g(x) \right\rangle = \begin{pmatrix} \frac{\partial \lambda(x)}{\partial x_1} & \frac{\partial \lambda(x)}{\partial x_2} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = 0 \tag{8}$$

Conditions (8) is equivalent to $\frac{\partial \lambda(x)}{\partial x_1} g_1 + \frac{\partial \lambda(x)}{\partial x_2} g_2 = 0$. In this manner, since by definition $g = (0\ 1)^T$, the real-valued function $\lambda(x) = a x_1$ satisfies condition (8) if $a \neq 0$ and it can be consequently used as synchronization function. Note that, if synchronization is achieved, the synchronization error $x \triangleq x_M - x_S = 0$ then the synchronization function implies that $x_{1;M} \equiv x_{1;S}$; i.e., the phase locking corresponds to a straight line with slope equal to a.

B. Finding the synchronization function in chaotic systems

Now, let us consider the following vector fields

$$\tau_1(x) = \begin{bmatrix} \sigma_1(x_2 - x_1) \\ \sigma_2 x_1 - x_2 - x_3 \\ -\sigma_3 x_3 + x_1 x_2 \end{bmatrix}; \tau_2(x) = \begin{bmatrix} \mu_1(x_2 - x_1) \\ \mu_2 x_1 - x_2 - x_3 \\ -\mu_3 x_3 + x_1 x_2 \end{bmatrix} \text{ and } g = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} \tag{9}$$

Note that the synchronization associated to the vector fields $\tau_1(x)$ and $\tau_2(x)$ can be interpreted as the synchronization of two Lorenz systems with different parameters (i.e., non-identical chaotic systems). Thus, the master system can be written as $\dot{x}_M = \tau_1(x_M)$ and the response system becomes $\dot{x}_S = \tau_2(x_S) + gu$. Therefore, the vector field of the synchronization error is given by

$$\Delta\tau(x) \triangleq \tau_1(x_M) - \tau_2(x_S) = \begin{bmatrix} \sigma_1(x_2 - x_1) - \mu_1(x_2 - x_1) \\ \sigma_2 x_1 - \mu_2 x_1 \\ (\mu_3 - \sigma_3) x_3 \end{bmatrix}. \tag{10}$$

The maximum dimension of the spanned space is 3. Thus, by compting the distribution

$$\Delta(x) = \text{span}\left\{ g; \; \hat{L}_{\Delta\tau} g \;\; \hat{L}^2_{\Delta\tau} g \right\}.$$

By taking the recursive inner product, we have that

$$\Delta(x) = \text{span}\left\{ \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}; \begin{bmatrix} (\pi_1 + \mu_1) g_1 - (\pi_1 - \mu_1) g_2 \\ (\mu_2 - \pi_2) g_2 \\ (\pi_3 - \mu_3) g_3 \end{bmatrix}; \begin{bmatrix} (\pi_1 - \mu_1)^2 g_1 - (\pi_1^2 - \mu_1^2) g_2 \\ (\mu_2 - \pi_2)^2 g_2 \\ (\pi_3 - \mu_3)^2 g_3 \end{bmatrix} \right\}$$

since third vector is linearly dependent of the second one, we have that the integer $\rho = 2$ for any difference $\pi_i - \mu_i \neq 0$ and all constant value $g_i \neq 0$, $i = 1;2;3$. Thus conditions (i) and (ii) becomes

$$\left\langle d\lambda; \hat{L}^k_{\Delta\tau} g \right\rangle \left\{ \begin{array}{l} = 0 \text{ for any } k = 0 \\ \neq 0 \text{ for } k = 1 \end{array} \right\} \tag{11}$$

In this manner, we have that the vector fields (9), which are related to the Lorenz equation, renders a maximum value of the integer $\rho = 2$ such that the dimension of the distribution holds at any point x in R^3 for any parameters values satisfying $\pi_i \neq \mu_i; i = 1;2;3$ and constrants $g_i \neq 0$. In this manner, the function $\lambda(x)$ should satisfy the partial differential equations

$$\begin{aligned} \langle d\lambda; g \rangle &= \tfrac{\partial}{\partial x_1} g_1 + \tfrac{\partial}{\partial x_2} g_2 + \tfrac{\partial}{\partial x_3} g_3 = 0 \\ \langle d\lambda; \hat{L}_{\Delta\tau} g \rangle &= ((\pi_1 - \mu_1) g_1 - (\pi_1 + \mu_1) g_2) \tfrac{\partial}{\partial x_1} + (-\pi_2 + \mu_2) g_1 \tfrac{\partial}{\partial x_2} + (\pi_3 - \mu_3) g_3 \tfrac{\partial}{\partial x_3} \neq 0 \end{aligned} \tag{12}$$

Note that $\lambda(x) = g_2 x_1 - g_1 x_2$ is solution of the equations (12), and, as consequence, it corresponds to the synchronization function. Since generalized synchronization is related to the existence of a functional relationship between the states of two systems. In this sense the

Lie-based geometry can be exploited to compute the synchronization function ¸(x) toward a general framework of the synchronization theory, and provides the properties of teh chaotic systems such that they can be synchornized (i.e., synchronizability of chaotic systems).

IV. Concludign remarks

The Lie-based geometry has been used in this contribution toward the general framework of the chaotic synchronization. The main idea is to exploit the Lie-algebra of vector ¯elds realetd to the synchronization-error system for analyzing synchronizability (that is, the property of two dynamical systems such that the synchrony can be induced between them). The novelty in this contribution consists of the geometry inetrpretation on the synchronizability property. Two examples have been presented. The former illustrates the details for computing synchronization funtion from Lie-based geometry. The later shows the obtainment of the synchronization function on the chaos theory context.

Some questions are still open in the synchornization problem of the chaotic systems. Is there any function relationship between states for the synchronization of strictly-di®erent chaotic systems (i.e., generalized synchronization)? Do generalized synchronization comprise partial-state synchronization or reduced-synchronization ?. We believe that Lie-based geometry approach can provide important information in this direction.

References

[1] Holstein-Rathlou, N.H., Yip, K.P., Sosnovtseva, O.V. and Mosekilde, E., "Synchronization phenomena in nephron-nephron interaction", Chaos, vol. 11, no. 2 (2001) 417-426.

[2] Kocarev L. and Parlitz U., "General approach for chaotic synchronization with applications to communications", Phys. Rev. Letts., vol. 74, no. 25 (1995) 5028-5031.

[3] Femat R. and Solis-Perales, G. "On the chaos synchronization phenomena", Phys. Letts. A, vol. 262 (1999) 50-60.

[4] Xiaofeng, G. and Lai C.H., "On the synchronization of di®erent chaotic oscillations", Chaos Solitons and Fractals, vol. 11 (2000) 1231-1235

[5] Femat R. and J. Alvarez-Ramirez, " Synchronization of two strictly di®erent chaotic oscillators", Phys. Letts. A, vol. 236 (1997), 307-313.

[6] Femat R. and Solis-Perales, G. "Synchronization of chaotic systems with di®erent order", Phys. Rev. E,vol. 65, no. 03, (2002) 036226-1.

[7] Brown, R and Kocarev, L., "An unifying de¯nition of synchronization for dynamical systems", Chaos, vol. 10, no. 2, (2000) 344-349

[8] Boccaletti S., Pecora L.M. and Pelaez A., "Unifying framework for synchronization of coupled dynamical systems", Phys. Rev. E, vol. 63, no. 06 (2001) 066219-1.

[9] Josic K., "Synchronization of chaotic systems and invariant manifolds", Nonlinearity, vol 13 (2000) 1321-1336.

[10] Nijmeijer, H. and Mareels, M. Y., "An observer looks synchronizaton", IEEE Trans. Circuits and Syst. I., vol. 44 (1997) 307-313.

[11] Isidori, A., Nonlinear control systems, 3th Ed., Springer-Verlag, 1995, London.

[12] Kocarev, L., Parlitz, U and Hu, B., "Lie derivatives in dynamical systems", Chaos, Solitons and Fractals, vol. 9 (1998) 1359-1366.

[13] Nijmeijer H. and van der Schaft A., Nonlinear control dynamical systems, Springer-Verlag, 1990.

[14] Solís-Perales G., "Synchronization of nonlinear systems", Ph. D. Thesis, Universidad Autónoma de San Luis Potosí, México, 2002 (in spanish).

[15] Solís-Perales G, Ayala V., Kliemann W. and Femat R., "Complete synchronizability of chaotic systems: a geometric approach", Chaos, To appear

[16] Femat R., "An extension to chaos control via Lie derivatives: Fully linearizable systems", vol. 12, No. 4, (2002) 1027-1033

Appendix: Some properties of the exact differential $d\lambda$

In this appendix the properties of the rela-vaklued funciton λ and its gradient are briefly discusssed. A detailed analysis can be found in geometrical control theory literature. Here, we have included the required properties to complete discussion in the contribution.

Definition 8 (A.1) Let λ be a smooth real-valued function defined on an open set U of R^n. The covector field $d\lambda$ defined as the $1 \times n$ row vector, whose $i-$th element is the partial derivative of λ with respect to $x_i; i = 1; 2; \ldots; n$, is called exact differential or simplely gradient of λ and its value at any point $x \in U \subset R^n$ becomes $d\lambda(x) = \left(\frac{\partial\lambda}{\partial x_1} \frac{\partial\lambda}{\partial x_2} \ldots \frac{\partial\lambda}{\partial x_n}\right) = \frac{\partial\lambda}{\partial x}$. $\diamond$

The following operation can be defined on the exact differential $d\lambda(x)$: The inner product between the covector field $d\lambda(x)$ and any vector field $\tau_1(x)$; both defined on the open set U of R^n, is defined as $\langle d\lambda(x), \tau_1(x)\rangle = \sum_{i=1} \frac{\partial\lambda}{\partial x_i}\tau_1(x) = \frac{\partial\lambda}{\partial x}\tau_1(x)$. By definition, above operation corresponds to the Lie derivative $L_\tau\lambda(x)$ of the real-valued function $\lambda(x)$ along the vector field $\tau_1(x)$ (see [12],[16]). Thus, the Lie derivative of the real-valued function has the following properties:

- if λ_1 and λ_2 are a real-valued functions and τ_1 a vector field defined on an open set U of R^n, then $\langle d\lambda_2(x), \lambda_1(x)\tau_1(x)\rangle = \langle d\lambda_2(x), \tau_1(x)\rangle\lambda_1(x) = \left(\frac{\partial\lambda_2}{\partial x}\tau_1(x)\right)\lambda_1(x)$.

- if λ_1 and λ_2 are a real-valued functions and τ_1, τ_2 are vector fields defined on an open set U of R^n, then $\langle \lambda_2(x)\tau_2(x), \lambda_1(x)\tau_1(x)\rangle = \lambda_2(x)\lambda_1(x)[\tau_2, \tau_1](x) + \langle d\lambda_1(x), \tau_2(x)\rangle\lambda_2(x)\tau_1(x) - \langle d\lambda_2(x), \tau_1(x)\rangle\lambda_1(x)\tau_2(x)$, where $[\tau_2, \tau_1](x)$ stands for the Lie bracket between the vector fields $\tau_1(x)$ and $\tau_2(x)$.

- if $\tau(x)$ is a vector field defined on a open set U of R^n and $\lambda(x)$ is a real-valued function, then $L_\tau d\lambda = dL_\tau\lambda$, where $L_\tau\lambda$ denotes the Lie derivative of the function $\lambda(x)$ along the vector field $\tau(x)$.

Lemma 9 ([11]) Let $\lambda(x)$ be a real-valued function and $\tau_1(x)$ a vector field, all defined on the open set U of R^n. For any point x^0 belonging to $U \subset R^n$ and any integer (relative degree) $\rho \leq n$, the rows vectors $d\lambda(x^0); dL_\tau\lambda(x^0); \ldots dL_\tau^{\rho-1}\lambda(x)$ are linearly independent.