Synchronization of chaotic systems with different order

Ricardo Femat* and Gualberto Solís-Perales†
Departamento Matemáticas Aplicadas y Sistemas Computacionales, IPICyT Apartado Postal 3-90, 78231, San Luis Potosí, Mexico

(Received 18 May 2001; revised manuscript received 24 August 2001; published 5 March 2002)

The chaotic synchronization of third-order systems and second-order driven oscillator is studied in this paper. Such a problem is related to synchronization of strictly different chaotic systems. We show that dynamical evolution of second-order driven oscillators can be synchronized with the canonical projection of a third-order chaotic system. In this sense, it is said that synchronization is achieved in reduced order. Duffing equation is chosen as slave system whereas Chua oscillator is defined as master system. The synchronization scheme has nonlinear feedback structure. The reduced-order synchronization is attained in a practical sense, i.e., the difference $e=x_3-x_1'$ is close to zero for all time $t\geqslant t_0\geqslant 0$, where t_0 denotes the time of the control activation.

DOI: 10.1103/PhysRevE.65.036226 PACS number(s): 05.45.−a, 87.10.+e

I. INTRODUCTION

Chaos synchronization is a very interesting problem that has been widely studied in recent years. Actually, there are two main directions in the research of the chaos synchronization: (i) synthesis and (ii) analysis. The problem of synchronization synthesis is to design a force for coupling two chaotic systems. The coupling force can be designed linear [1,2] or nonlinear [3,4]. The actual challenge in problem of synchronization synthesis is to achieve and to explain the synchronization between chaotic systems with different model [5,6,7]. Synchronization analysis consists in (a) classification of the synchronization phenomena [8,9] and (b) comprehension of the synchronization properties (such as robustness [10] and/or bifurcation [11]). An important challenge in synchronization analysis is to develop genuine indicators of chaotic synchronization. Such a problem arises from classical indicators failure. For example, even if Lyapunov table indicates that chaos has been controlled, small disturbances can provoke deficiency in such indicators [12].

This paper addresses the problem of the synchronization synthesis. Such a problem can be classified in the following research areas: (i) the study of potential applications for chaos synchronization and (ii) development of synchronization strategies. Concerning the first research area, chaos synchronization has application in several fields as biological systems, where the research is focussed in neurons lattices [13,14], and transmissions of secure message [15,16]. Regarding development of synchronization schemes, several strategies can be found in literature (for example, linear [2], nonlinear [4], or adaptive [17]). In particular, a synchronization strategy is developed in this paper from nonlinear feedback. Synchronization synthesis is a problem of first generation while applications of synchronization is a problem of second generation. However, there is no full knowledge about the synchronization synthesis. Thus, for instance, theoretical and experimental results show synchronous behavior in nature. For instance, in Refs. [11,14], authors have shown very interesting results about the chaotic synchronization phenomenon in neurons. Nevertheless, *on the microscopical scale the mutual interactions are not yet clearly understood*: Indeed, only phenomenon models have been used [12]. In principle, it is possible that synaptic communication can be yielded between neurons with different dynamic model. For example, synchronous activity has been observed in thalamic and hippocampal neurons networks [18]. An alternative for understanding synchronous behavior in nature is to develop synchronization strategies. In particular, a synchronization strategy is developed in this paper.

There is one interesting question in this direction: is the synchronous behavior in nature yielded from a feedback of a fed-forward coupling force? There is no definitive answer for above question. As a matter of fact, synchronization is the results of the coupling between dynamical systems. Coupling can be performed via two basic interconnections: fed-forward and feedback. Fed-forward interconnection consists of the input of a dynamical signal without return. On contrary, feedback implies that a potion of the system is returned. Both interconnections have been used for chaos control; see Refs. [19,20] for fed-forward and [2–6,8] for feedback. Indeed, a combination of both interconnections can be performed for synchronization of chaotic oscillators [21]. Since both interconnections achieve chaos synchronization, one is unable to confirm that synchronous behavior in nature is only a consequence of feedback coupling. However, synthesis of chaos synchronization via feedback allows us to expect promissory results. In this sense, some results show that synchronization between chaotic systems whose model is strictly different has been reported [5–7]. Nevertheless, until our knowledge, there is no previous results about the synchronization between chaotic systems whose order is different. Such a problem is reasonable if, for instance, we think that order of the thalamic neurons can be different than hippocampal neurons [18]. One more example is the synchronization between heart and lung. One can observe that both circulatory and respiratory systems behave in synchronous way. However, one can expect that model of the circulatory

*FAX: (+52-444) 833-5412.
Email address: rfemat@ipicyt.edu.mx
†Present address: Facultad de Inganiería, Universidad Autónoma de San Luis Potosí. Email address: gcp@cactus.iico.uaslp.mx

system is strictly different than respiratory system, which can involve different order. In addition, the synchronization of strictly-different chaotic systems is interesting by itself.

This paper addresses the synthesis of the chaos synchronization between oscillator with different model. We present a nonlinear approach for synchronizing chaotic system whose order is not equal. In particular, we present the matching of the Duffing equation attractor with a projection onto one canonical plane of the Chua circuit (double-scroll oscillator). In this sense, results show that *reduced-order synchronization* can be achieved by nonlinear systems. This means that two chaotic systems can be synchronized in spite of order of the response system is less than order of the drive system. Of course, since order of response oscillator is smaller than master system, the synchronization is only attained in reduced order. Results are focused on geometrical features of synchronization phenomenon. Chua-Duffing synchronization is allowed by a nonlinear feedback, which yields a smooth and bounded coupling force (i.e., control command). The nonlinear feedback is designed from a simple algorithm based on time derivative of system output along the master/slave vector fields. The text is organized as follows. Chua-Duffing synchronization problem is presented in Sec. II. Section III contains the proposed feedback and design details. Numerical simulations and discussion are presented in Sec. IV. Finally, text is close with some concluding remarks.

II. PROBLEM STATEMENT

Chua system is an electronic circuit with one nonlinear resistive element. The circuit equations can be written as a third-order system that is given by the following dimensionless form [22]:

$$\dot{x}_1=\gamma_1[x_1-x_2-f(x_1)],$$
$$\dot{x}_2=x_1-x_2+x_3, \qquad (1)$$
$$\dot{x}_3=-\gamma_2 x_2,$$

where $f(x_1)=\gamma_3 x_1+0.5(\gamma_4-\gamma_3)[|x_1+1|-|x_1-1|]$, γ_i's, $i=1, 2, 3, 4$, are positive constant. Let us assume that system (1) represents the drive system. Figure 1(a) shows the phase portrait of system (1) and its projection on the canonical planes. The parameters were chosen as follows: $\gamma_1=10.0$, $\gamma_2=-14.87$, $\gamma_3=-0.68$, and $\gamma_4=-1.27$. Initial condition were arbitrarily located at the point $x(0)=(0.1,-0.5,0.2)$.

Now, let us consider the Duffing equation, which is given by

$$\dot{x}_1'=x_2',$$
$$\dot{x}_2'=x_1'-x_1'^3-\delta x_2'+\tau_e(t)+u, \qquad (2)$$

where δ is a positive parameter that represents damping coefficient, $\tau_e(t)=\alpha\cos(\omega t)$ denotes driving force and u is the coupling force (controller). Figure 1(b) shows the phase portrait of Duffing equation for $u=0$ for all $t\geqslant 0$ (uncontrolled evolution). Parameters were chosen as $\delta=0.15$, $\alpha=0.3$, and $\omega=1.0$. Initial conditions were arbitrarily located at origin.

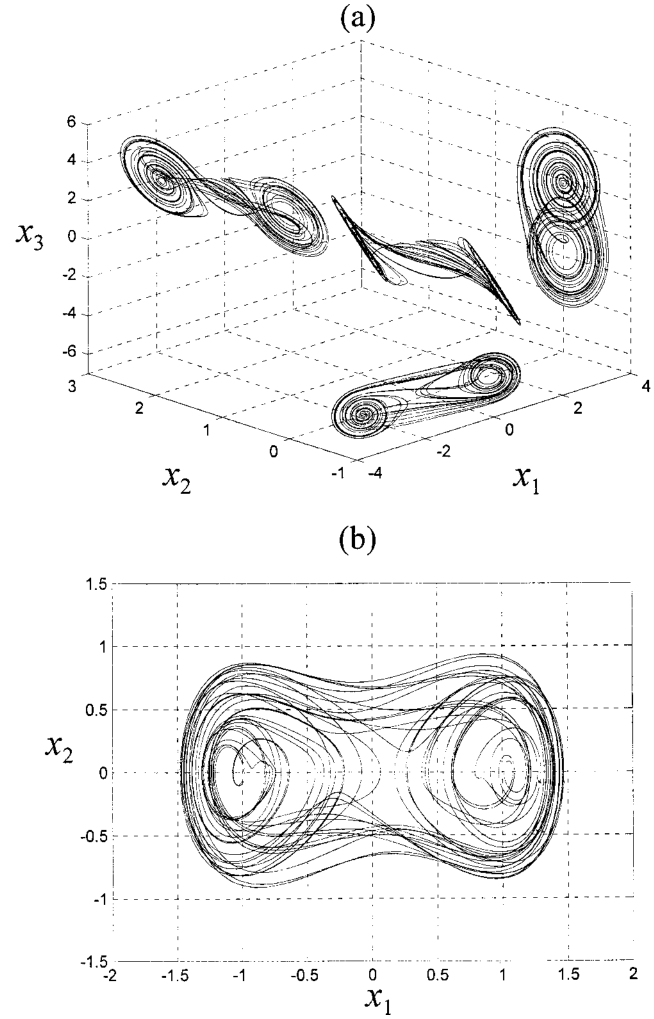

FIG. 1. Phase portrait of the Chua circuit and Duffing equation without coupling. (a) attractor of the Chua system and its projections on canonical planes. (b) attractor yielded by Duffing equation.

Obviously, there is difference between attractors of systems (1) and (2) [see phase portrait in Fig. 1(b) and projections in Fig. 1(a)], i.e., both systems (1) and (2) are not synchronized neither phase nor frequency. Moreover, there is no synchronization in any sense (see Refs. [8,9] for details concerning definition of synchronization kinds). The classical synchronization problem is somewhat distinct than problem of synchronizing different chaotic systems. In classical synchronization problem, drive and response system has similar geometrical and topological properties [23]. Thus master/slave interconnection can be sufficient to attain synchronization [24]. Latter one is understood as adjustment of master/slave dynamics due to coupling via output of both systems. Synchronization of different chaotic systems is a hard task if we think that: (i) initial conditions of master and slave systems are different and unknown, (ii) topological and geometrical properties of different chaotic systems are quite distinct and (iii) unrelated chaotic systems have strictly different time evolution.

Systems (1) and (2) have been widely used to study chaos synchronization when drive and response system have same order. However, we are interested in synchronization of Duffing attractor with a projection in a canonical plane, i.e., to lead the attractor of Eq. (2) to the geometrical properties of a projection of system (1). To study such a problem, we choose x_3 as measured state from master system [Eq. (1)] whereas x_1' is the measured observable of the slave system [Eq. (2)]. Thus the chaos control objective is *to design a feedback u such that the discrepancy error* $e=x_1'-x_3$ *tends to zero as* $t\to\infty$. Note that above goal implies that if $e\to 0$ as $t\to\infty$, then $x_1'\to x_3$ for all $t\geqslant t_0\geqslant 0$ and any initial discrepancy $e(0)=x_1'(0)-x_3(0)$. Therefore, at least, the partial synchronization [8,9] of both master and slave systems can be attained via feedback coupling. However, the control objective is to achieve the reduced order synchronization, i.e., all states of the response system should be synchronized, in some sense, with any states of the drive system.

III. PROPOSED FEEDBACK AND DESIGN DETAILS

Synchronization of chaotic systems via feedback can be addressed from the stabilization of the synchronization system error around origin (see for instance, [6]) or as the tracking of a target trajectory. First case consists in construction of the dynamical systems of the synchronization error in such way that the feedback scheme leads its trajectories to origin. The interesting problem is to attain the synchronization objective (i.e., stabilization around the origin) in spite of the synchronization error system is uncertain. Latter one comprises the leading of the slave trajectories to the master ones [e.g., to lead $x_1'(t)$ to trajectory $x_3(t)$]. That is, in second case the goal is to direct the slave system to the desired trajectory (which is provided by the master system). This problem has been recently addressed (see, for instance, [23]). However, when synchronization is not solved at all when it is addressed as a tracking problem due to tracking cannot be achieved by simple feedback [24]. On contrary, stabilization of the synchronization error around origin is promissory [2–10]. As we shall see below, in this paper the synchronization of systems with different order is addressed as the stabilization of the synchronization error at origin.

Let us assume the following. (1) Only x_3 is available for feedback from system (1); (2) x_1' and x_2' are available for feedback from slave system; (3) Vector fields of master and slave systems are smooth.

Now, let us consider the difference $e(t)=x_3(t)-x_1'(t)$. From first time derivative of the difference error one has that $\dot{e}=\dot{x}_3-\dot{x}_1'\equiv-\gamma_2x_2-x_2'$ and second time derivative of difference error one has that $\ddot{e}=-\gamma_2\dot{x}_2-\dot{x}_2'\equiv-\gamma_2(x_1-x_2+x_3)-x_1'+x_1'^3+\delta x_2'-\alpha\cos(\omega t)-u$. Now, from simple algebraic manipulations, the following equation is obtained: $\ddot{e}-\dot{e}+\gamma_2 e=(1+2\delta)x_2'+(1-2\gamma_2)x_1'-\alpha\cos(\omega t)-u$ or equivalently,

$$\dot{e}_1=e_2, \tag{3.1}$$

$$\dot{e}_2=e_1-\gamma_2e_2-\tau_e'(x',t;p)-u, \tag{3.2}$$

where $\tau_e'(x',t;p)=(1+2\delta)x_2'+(1-2\gamma_2)x_1'-\alpha\cos(\omega t)$ can be interpreted as a disturbance force acting onto the linear system (3), $p\in\mathbb{R}^4$ denotes a parameter set and $x'\in\mathbb{R}^2$ is a disturbance vector. Thus, the synchronization problem becomes to find the control command u such that system (3) is asymptotically stable at origin for any initial condition $e(0)\in\mathbb{R}^2$. Note that if control command u is able to stabilize system (3) at origin, then the synchronization error and its dynamics can be leaded to zero (i.e., u induces a steady state $e(t)=0$ for all $t\geqslant t_0\geqslant 0$, where t_0 is the time when control is activated). That is, dynamic evolution of slave system can be manipulated toward the master behavior.

A. Reduced-order synchronization under partial knowledge

We propose the following feedback:

$$u=(1+2\delta)x_2'+(1-2\gamma_2)x_2'-\alpha\cos(\omega t)+k_1e_1+k_2e_2, \tag{4}$$

where k_1, k_2 are constant parameters of the control which are computed from the following procedure. Equation (3) under controller (4) action results in the following closed-loop system: $\dot{e}=Ae$ where the matrix A is given by

$$A=\begin{bmatrix} 0 & 1 \\ 1-k_1 & -(\gamma_2+k_2) \end{bmatrix}. \tag{5}$$

Then k_1 and k_2 are chosen such that matrix has all its eigenvalues at the open left-hand complex plane (i.e., all roots of polynomial $\lambda^2+(\gamma_2+k_2)\lambda+1-k_1=0$ have negative real part), which is satisfied for any $k_2>\gamma_2$ and $0<k_1<1.0$.

Note that controller (4) yields linear behavior into system (3). That is, integration of system (3) under controller (4) yields $e(t)=e(0)\exp(At)$, where $e(0)\in\mathbb{R}^2$ is the initial condition of the synchronization error and A is a stable matrix for a given k_1 and k_2. Feedback based on Eq. (4) has two advantages: (i) it does not require full knowledge of the system (1). Indeed, feedback (4) allows coupling between system (1) and (2). Such a design is reasonable. For example, in neural systems, different neurons in one subsystem are always driven by output from neurons in higher level. Such differences can be interpreted as the time series of strictly different dynamical systems. Thus differences between signals in higher level neurons can be interpreted as the time series of strictly different dynamical systems. Hence, synchronization of dynamical systems from such time series plays an essential role in nonlinear processes. Thus, if master and slave systems have distinct properties, it is expected that direct interaction between them cannot necessarily yield synchronization. It is our belief that synchronization in nature is given by several kinds of interactions by feedback coupling. (ii) It compensates the nonlinear terms and induces a linear behavior. Indeed, this is the main desirable feature of nonlinear control.

Nevertheless, the controller (4) has the following drawbacks: (i) the stability of the system (3) under controller (4) is based on the choice of the eigenvalues of the matrix (5). This procedure is known as pole placement in control theory. Pole placement can result in poor control actions and (ii)

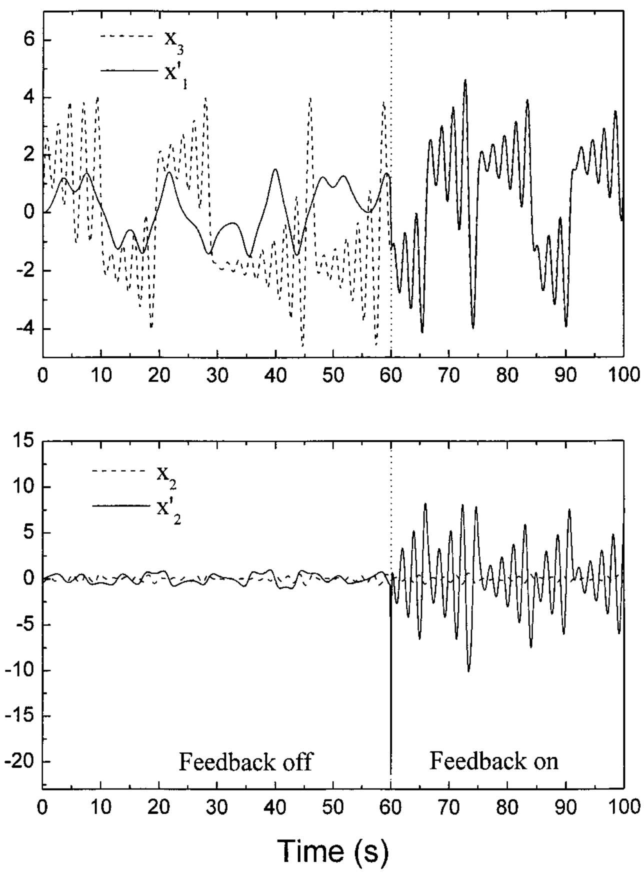

FIG. 2. The difference $e = x_3 - x_1'$ holds close to zero under feedback actions. Time evolution of $x_2(t)$ has different amplitude than $x_2'(t)$.

controller (4) requires the values of the slave parameters. The crux is obvious, what happen if the parameters δ and γ_2 are time varying or are not exactly known? Both drawbacks will be discussed in Sec. III B. However, it is pertinent to illustrate that feedback (4) yields the reduced-order synchronization. To this end, and in seek of clarity, some numerical simulations were performed. Without lost of generality, one can consider that parameters and initial conditions have same value than Fig. 1 (see above). Figure 2 shows time evolution of (x_2, x_3) and (x_1', x_2') under nonlinear feedback (4). Eigenvalues of matrix (5) were located at -30. Note that synchronization has been attained. Figure 2(a) shows that states x_3 and x_1' evolve under practical synchronous behavior. Figure 2(b) shows that x_2 and x_2' do not behave in synchronous manner. It seems that partial synchronization has been attained (see Ref. [9] for details about partial synchronization). However, as we state below, trajectories of duffing system tracks the Chua system projection, i.e., reduced-order synchronization is performed by feedback (4). It should be pointed out that the synchronization achieved by controller (4) is practical [9]; i.e., the trajectories of the synchronization error system (3) converges to a ball centered at origin. This means that error $e(t)$ does not converges exactly to zero. As a consequence, the time-evolution of the controller (4) does not go to zero. This is due to feedback (4) absorbs the structural differences between systems (1) and (2). In other words, feedback (4) "must pay the cost" of the synchronization.

B. Reduced-order synchronization with least prior knowledge

As was discussed above, the system (3) can be, if parameters are unknown, an uncertain nonlinear system. The goal is to design a feedback controller such that the synchronization error $e_1(t)$ tends to a neighborhood Ω of the origin for all $t \geqslant t_0 \geqslant 0$ and any initial conditions $[e_1(0), e_2(0)]$ in $\mathbb{R}^2$. Feedback (4) yields exponential stability of the synchronization error, $e(t)$. However, it is quite complex and requires *a priori* information about slave oscillator. Then, a modification is desirable in such manner main features of feedback (4) be held.

Let us define $\eta = \gamma_2 e_2 + \tau_e'(x', t; p)$ as a new variable, which is smooth. Of course, if $\tau_e'(x', t; p) = (1 + 2\delta)x_2' + (1 - 2\gamma_2)x_1' - \alpha\cos(\omega t)$ is uncertain, the state η is not available for feedback. We propose the following procedure for designing the feedback controller with least prior knowledge. The uncertain term η can be computed from Eq. (3.2) in the following way: $\eta = \dot{e}_2 - e_1 + u$. Now, by approaching the time derivative by means of finite differences at time $t_k \in [t_k, t_{k+1}]$, one has that $\eta(t_k) \approx \hat{\eta}(t_k) = -e_1(t_k) + u(t_{k-1}) + [e_1(t_k) - e_1(t_{k-1})]/\Delta t$, where Δt denotes the sampling rate. Such an approach provides an estimated value of the uncertain term η at time t_k from the measurements of the error $e(t_k)$ and the last control input $u(t_{k-1})$. In this way, the controller (4) becomes

$$u(t_k) = -\hat{\eta}(t_k) + k_1 e_1(t_k) + \frac{k_2}{\Delta t}[e_1(t_k) - e_1(t_{k-1})], \quad (6)$$

where k_1 and k_2 are chosen such that matrix (5) has all its eigenvalue located at open left-hand complex plane. Note that proposed controller comprises two parts: (i) the feedback (6) and (ii) the uncertainties estimator given by the finite differences approach. In addition, such a controller does not require prior knowledge neither the parameters values nor model of the master system. In principle, as $\Delta t \to 0$ as the estimated value $\hat{\eta}(t_k) \to \eta(t)$ for all $t \in [t_k, t_k + \Delta t) \geqslant t_0 \geqslant 0$, where t_0 denotes the time where controller is activated. Hence, as $\Delta t \to 0$, the controller (6) will behave as the nonlinear controller (4). This means that as $\Delta t \to 0$, nonlinear terms of the synchronization system (3) can be counteracted by the controller (6). It should be pointed out that if $\Delta t = 0$, then the controller (6) is not physically realizable. This makes sense because $\Delta t = 0$ means "no sampling rate." However, such a condition implies that noise sensitivity can be displayed by controller (6). Figure 3(a) shows the dynamics of the synchronization error under controller (6) for $\Delta t = 2.5$ Hz. The time evolution of the respective control command is shown in Fig. 3(b). Here, same values of the control parameters $k_1 = 25.0$ and $k_2 = 10.0$ were chosen for controller (4) and (6). This implies that all eigenvalues of the matrix (5) are located at -5.0. Note that, in spite of Eq. (6) requires least prior knowledge about the synchronization system, the proposed strategy is able to achieve reduced-order synchronization for relatively small sampling rates. Indeed, as

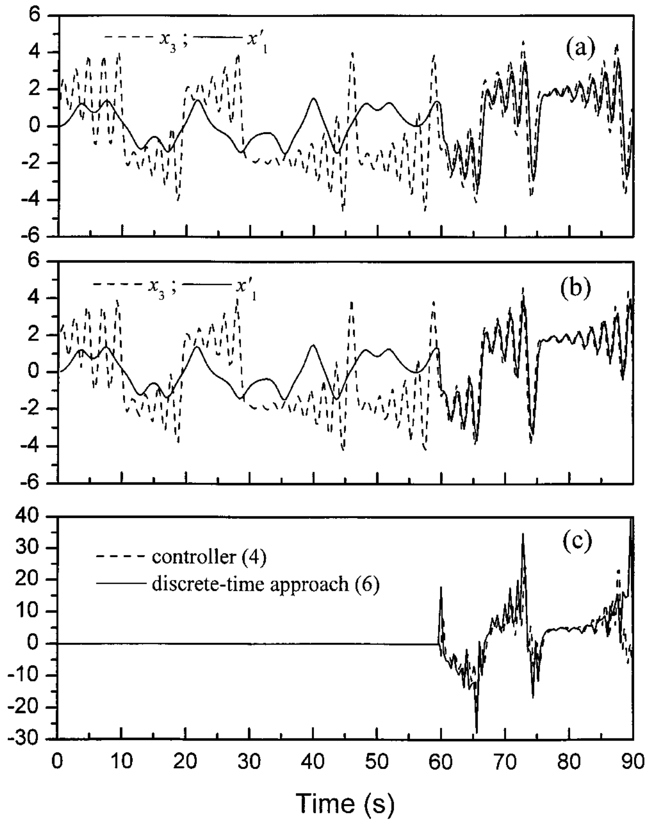

FIG. 3. Time evolution of: (a) the synchronization error for the sampling rate $\Delta t = 2.5$ Hz. As sampling rate increases as the better synchronization, (b) the coupling force given by the proposed control command (Eq. 6), (c) Comparison between continuous-time and discrete-time controllers.

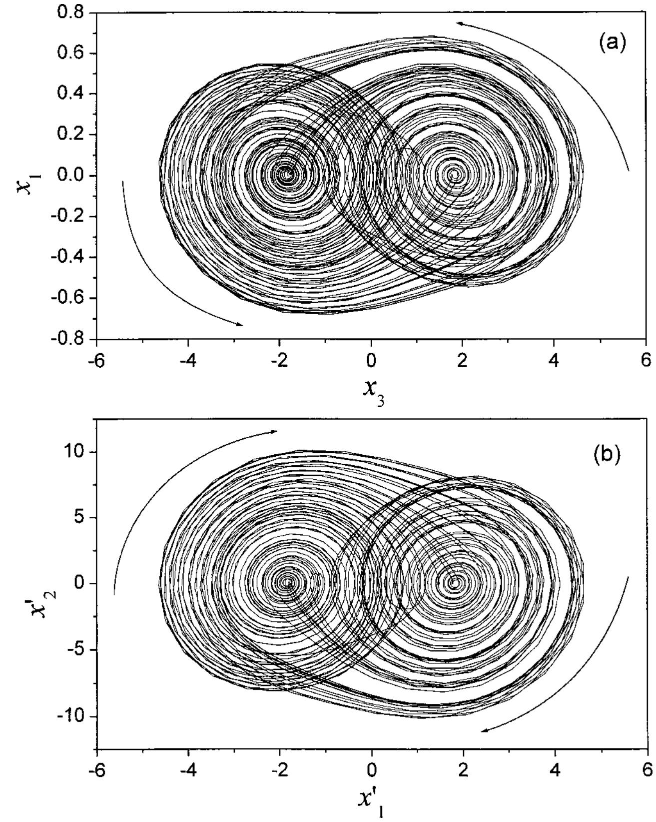

FIG. 4. Reduced-order synchronization under feedback (6). (a) Duffing equation yields an attractor which is a mirror reflection of (b) the (x_2, x_3)-plane projection of the Chua system.

smaller sampling rate as better synchronization. In addition, Figs. 3(a) and 3(b) shows that synchronization under feedback (4) and (6) are equivalent.

IV. DISCUSSION OF THE RESULTS

In this section, we briefly discuss the obtained results. Figure 4 shows the projection of Chua system [Eq. (1)] on (x_2, x_3) plane and phase portrait of the Duffing oscillator under feedback actions [Eq. (2)]. Here, eigenvalues of matrix (5) were arbitrarily located at -5 and control command was computed by feedback (6) (including the uncertainties estimator) with $\Delta t^{-1} = 2.5$ Hz. The controller was activated at time $t_0 \geqslant 0$. Two features should be noted: (a) *chirality* and (b) amplitude oscillations. The following comments are in regarding to both chirality and amplitude features:

(a) Note that Fig. 4(b) is a mirror reflection of Fig. 4(a). [Same phenomenon was observed under feedback (4)]. That is, while (x_2, x_3)-plane projection of Chua system rotates toward left, Duffing equation under feedback (6) rotates toward right. Note that Duffing attractor under actions of feedback (6) is not *superimposable* on the mirror image of Chua (x_2, x_3)-plane projection. Such property is so-called *chirality* (see Ref. [25] for introductory notion). The chirality notion has been burrowed from study of organic molecules. Thus, for example, notice that symmetry axis corresponds to state x'_1. That is, in x'_1 direction both attractor [Figs. 4(a) and 4(b)] are equal [see Figure 2(a)] where reflection is in direction of x'_2 is chiral. Now, one can expect that power spectrum peaks of x_2 and x_3 are equal than x'_1 and x'_2, respectively. Hence reduced-order synchronization of systems (1) and (3) is attained in phase and in a practical sense [8,9]. Although these results are novel, they are a consequence of how the synchronization is addressed. Thus, chirality cannot yet be claimed as a kind of synchronization; however, this feature requires detailed study. Unfortunately, such a goal is beyond of the objective in this paper. Figure 5 shows the power spectrum (PSD) of x_2, x_3, x'_1 and x'_2. PSD has been normalized by maximum amplitude peak. Such a picture shows that, at least, frequency and phase synchronization is attained by feedback (6). PSD is an important measure of synchronization. Although, PSD is not sufficient to conclude that synchronization exists; however, it is a good evidence [8].

(b) Concerning the magnitude of the attractors, we can note that, under control actions, the following steady state is obtained: $e(t) \to 0 \Rightarrow \|e(t)\| = \|x_3(t) - x'_1(t)\| \approx 0$ for all $t > t^+ \geqslant 0$, where t^+ is the time of control activation. Such a steady state implies that all time derivatives of e are zero for all $t \geqslant 0$. That is, $\dot{e} \equiv 0$ for all $t \geqslant 0$, which implies that $\|x'_2(t)\| = \gamma_2 \|x_2(t)\|$. Therefore, at steady state, an amplification factor γ_2 affects the phase portrait of controlled Duffing equation. However, as was stated above, synchronization is achieved. In order to add evidence, we have plotted x_3 vs x'_1

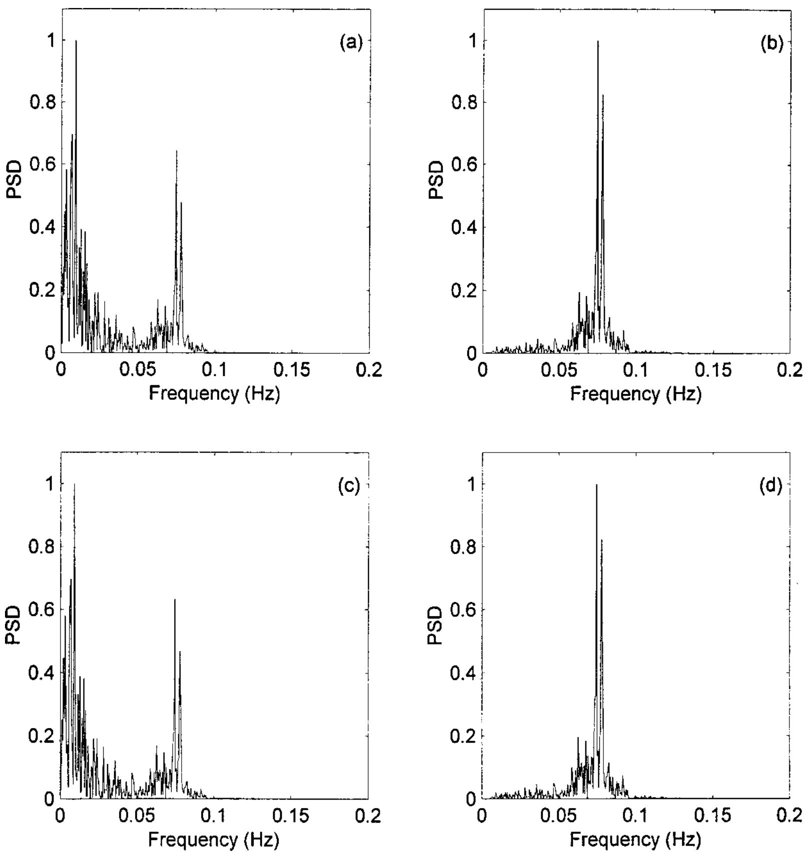

FIG. 5. Power spectrum of (a) x_3, (b) x_2, (c) x_1' (under control actions) and (d) x_2' (under control actions). Note that spectrum in Duffing equations (under control actions) are similar than Chua system (x_2,x_3)-plane projection.

and x_2 vs $-x_2'/\gamma_2$ (which can be seen as a modified phase locking diagram, see Fig. 6). Since x_2' essentially has similar dynamic properties than x_2 (see, for instance, Fig. 5). It should be pointed out that the dynamics of the synchronization error is close to zero. This means that controller (6) provides practical reduced-order synchronization [9] of the drive and response systems.

V. CONCLUDING REMARKS

This paper has one main contribution: *the synchronization of different order systems or reduced-order synchronization*. Reduced-order synchronization is the problem of synchronizing a slave system with projection of a master system. It should be noted that reduced-order synchronization is not partial synchronization. On the one hand, partial synchronization is for coupling two chaotic systems whose order is equal. A main feature of the partial synchronization is that, at least, one state of the slave system is not synchronous in some sense (see Refs. [9,21], and references therein). On the other hand, in reduced-order synchronization, all states of the slave system are synchronous, in some sense. The main feature of the reduced-order synchronization is that order of the slave system is less than master one. In this sense, the synchronization of Chua oscillator and Duffing equation is achieved in reduced order, i.e., Duffing equation can be synchronized under feedback actions with a canonical projection of the Chua system. Of course, the problem of the reduced-order synchronization has not been solved yet. Some questions have been opened: (i) Can the reduced-order synchro-

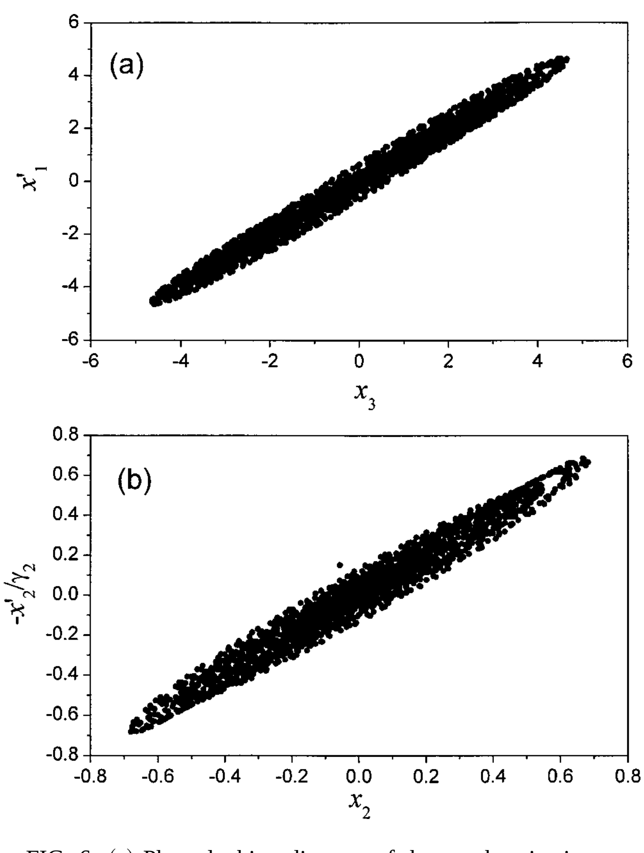

FIG. 6. (a) Phase locking diagram of the synchronization error $e_1(t)=x_1'-x_3$ variables. (b) Modified phase locking diagram for state variable in synchronization.

nization be achieved from fourth-order and third-order systems? (ii) Can the reduced-order synchronization be attained from fourth order and second-order driven systems? Results in this direction are expected. Nevertheless, we belief that this paper is a timely contribution.

Now, results show an interesting phenomenon. The reduced-order synchronization yields a mirror reflection of the drive system. This can be casual; however, one more question arises: Is the mirror reflection a synchronization phenomenon or a casualty? Unfortunately, answer is beyond the paper goal. But, if mirror reflection is a synchronization phenomenon, *chiral synchronization* can be an interesting and relevant discovery. Chiral synchronization should be characterized because the attractor of slave system is, once synchronized, a mirror reflection of the master system attractor, i.e., Duffing attractor under control actions cannot be superimposed on its mirror image. In this paper, mirror reflection of the synchronization is a consequence of how the feedback control is designed.

[1] C-W. Wu and L. O. Chua, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **42**, 430 (1995).

[2] T. Kapitaniak, M. Sekeita, and M. Ogorzalek, Int. J. Bifurcation Chaos Appl. Sci. Eng. **6**, 211 (1996).

[3] G. Grassi and S. Mascolo, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **44**, 1011 (1997).

[4] H. Neijmeijer and M. Y. Mareels, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **44**, 882 (1997).

[5] G. Xiaofeng and C. H. Lai, Chaos, Solitons Fractals **11**, 1231 (2000).

[6] R. Femat and J. Alvarez-Ramírez, Phys. Lett. A **236**, 307 (1997).

[7] J. Bragard and S. Boccalletti, Phys. Rev. E **62**, 6346 (2000).

[8] R. Brown and L. Kocarev, Chaos **10**, 344 (2000).

[9] R. Femat and G. Solís-Perales, Phys. Lett. A **262**, 50 (1999).

[10] N. F. Rulkov and Mikhail M. Sushchik, Int. J. Bifurcation Chaos Appl. Sci. Eng. **7**, 625 (1997).

[11] M. Bazhenov, R. Huerta, M. I. Rabinovich, and T. Sejnowski, Physica D **116**, 392 (1998).

[12] L. A. Aguirre and S. A. Billings, J. Nonlinear Sci. **5**, 189 (1995).

[13] P. Tass and H. Haken, Biol. Cybern. **74**, 31 (1996).

[14] R. Huerta, M. Bazhenov, and M. I. Rabinovich, Europhys. Lett. **43**, 719 (1998).

[15] K. Pyragas, Int. J. Bifurcation Chaos Appl. Sci. Eng. **8**, 1839 (1998).

[16] K. M. Short, Int. J. Bifurcation Chaos Appl. Sci. Eng. **4**, 959 (1994).

[17] F. Mossayebi, H. K. Qammar, and T. T. Hartley, Phys. Lett. A **161**, 255 (1991).

[18] D. Terman, N. Koppel, and A. Bose, Physica D **117**, 241 (1998).

[19] R. Lima and M. Pettini, Phys. Rev. A **41**, 726 (1990).

[20] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).

[21] Z. Liu, Y.-C. Lai, and F. C. Hoppensteadt, Phys. Rev. E **63**, 055201 (2000).

[22] M. di Bernardo, Int. J. Bifurcation Chaos Appl. Sci. Eng. **6**, 557 (1996).

[23] Z. Liu and C. Shigang, Phys. Rev. E **55**, 199 (1997).

[24] R. Femat, J. Capistran-Tobías, and G. Solíis-Perales, Phys. Lett. A **252**, 27 (1999).

[25] W. J. le Noble, *Highlights of Organic Chemistry* (Marcel Dekker New York, 1974), Chap. 5.