# Cluster synchronization in networks of structured communities

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**Abstract**

We investigate the synchronization problem for a network of subnetwork with community structure. We consider a model with different levels of interconnection: At the first, strongly coupled adjacent nodes are modeled as a compact unit (CU) where connections are uniformed and undirected. A second level of connection occurs as CUs with similar dynamic behaviors from communities by weighted and undirected interactions. Finally a third level of connection, where directed and weighted connections between these communities form the entire network of subnetworks. Using the Lyapunov approach we show that cluster synchronization is possible for our network model where even though all nodes are interconnected each community achieves complete synchronization within, while the behavior between communities is not synchronized. Our main results are illustrated using numerical simulations of well-known benchmark systems.

*Keywords:* Network of networks, Cluster synchronization, Community structures.

## 1. Introduction

Networks of subnetworks are everywhere in the real world, many natural and artificial systems, such as the Internet and biological networks, are clear examples of this type of systems. In many instances, proper operation in complex systems depend on collaborative efforts by distinct groups connected to each other [1–3]. A network of subnetworks is composed by a large set of interconnected nodes, in which one can identify subnetworks (clusters, communities or groups) that share a common specific dynamical or topological classifying characteristic.

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To model a complex system as a dynamical network, in most cases one starts with suppositions like identical dynamical nodes coupled with undirected and unweighted connections in statical structures. Moreover, to investigate its collective behavior a basic assumption is the diffusive connections condition. That is, when all nodes have the same value the average effect of all neighbors on the dynamics of a given node is zero. The diffusive requirement arises naturally when considering systems with a large number of elements, in that case the effect of all neighbors on the behavior of a given node can be approximated in the so-called "mean-field" sense, then the effect all nodes is average over the entire system [4]. For example, in a network of neurons, where the connections are modeled as proportional electrical synapses, one way to average the effect of all the neurons in a neighborhood is to require that the connection matrix have zero sum by rows. Another example are Kirchhoff's circuit laws where the sum of voltages of all the elements in a circuit averages to zero, that is, their connections are diffusive.

The overall dynamics of a network are the result of the interplay between its edge structure and its node dynamics, this interaction gives rise to very interesting patterns of collective behavior, such as synchronization [5]. For a pair of coupled systems, as well as for networks of dynamical systems, when investigating the stability of synchronized behavior, a common approach is to linearize the error dynamics around the synchronized solution, in this way one obtains local results [6]. In contrast, global synchronization is establish if the validity of the results hold for the entire state space of the coupled systems [7]. The synchronization problem can be solved from two different perspectives: On the one hand, the connections between the systems can be designed to make the synchronized solution stable. On the other hand, controllers can be designed to impose a synchronized behavior, this can be achieved using control methodologies like: activation feedback control [8], linear separation [9], or sliding mode control [10], among many other approaches [11].

Recently, the basic patters of synchronization that occur in the master-slave configuration such as identical [12], generalized [13] or module-phase synchronization have be extended to the case of networks. For example, in Zhang *et al* [14] the module-phase synchronization of neural network with time delay is considered. In this sense, the outer synchronization of networks can also be seen as an extension of a basic synchronization pattern usually associated with the master-slave configuration, where an entire network of slave systems synchronizes to a system that is outside the network, basically functioning as an external master for the entire network [15].

The study of networks of subnetworks has received increasing attention from various disciplines in recent years. These investigations can be divided into two main lines. The first investigates the emergence of their structural or spectral properties, where the main objective is to characterize the "community structure" of the complex system, *i.e.*, identify groups of nodes that are densely related to each other [3, 16]. In this same line of research one can include investigations of different structural properties such as transitivity, degree distribution, the existence of motifs, and the spectral properties of their Laplacian matrices

[17–19]. The second line of research investigates their dynamical properties, that is, each subnetwork is form by node with identical or similar dynamical properties [20]. In this case the main concern is to describe the nature of their collective motion. For example, the emergence of different patterns of synchronized behavior [21]. In this context, cluster synchronization refers to the case where a network can be divided into several groups, inside of them the nodes synchronize with each other; while between nodes of different groups there is no synchronization. In [22] the emergence of cluster synchronization has been investigated in terms of the structure and number of identical oscillators on lattices of two and three dimensions. The different forms of cluster synchronization in terms of its structure was completely characterized in [23] for networks of identical nodes. The cluster synchronization problem for nonidentical nodes was investigated for a dynamical network with two clusters of nonidentical nodes in [24]. While [25] considered the case of nonlinearly coupled subnetworks for nonidentical nodes. The cluster synchronization problem of Boolean networks was solved from the control perspective in [26]. Also using the controller design approach a solution for the cluster synchronization problem on networks of identical nodes with time delays was proposed in [27] by designing adaptive controllers. In this paper, for a particular class of complex networks we investigate the synchronization problem using Lyapunov stability theory [28], in this way we derive sufficient condition to guarantee cluster synchronization in a network of structured communities. An advantage of using Lyapunov analysis is that nonlinearities, uncertainties and other considerations like those generated by time delays can be included in the analysis, which other tools like the graph stability approach [29], or the Master Stability Function (MSF) [6] are unable to take into consideration. Additionally, by using the Lyapunov theory approach we derive analytical instead of numerical arguments to establish the stability of the synchronized solution.

Considering the structural complexity of real world networks the assumption that all node in the network are coupled in the same way is difficult to justify. For that reason we proposed a model that remarks the existence of different types of couplings for different levels of organization. Our model of a network of subnetworks has three different levels of organization with distinct coupling characteristics at each level. At the first one, the connection is tight, adjacent nodes are coupled bidirectionally and uniformly, the groups at this level are called "compact units" (CUs). Within each CU we assume that all nodes are identical and with exactly the same coupling structure. Due to this assumption, the stability of its collective dynamics can easily be determine using previous results like the $\lambda_2$ criterion [28, 30]. At the second level of organization, the CUs that share dynamical characteristics are coupled together into communities. Each subnetwork at this level, is formed by the weighted connection of CUs. Under the assumption that each CU in the community is identically synchronized, its behavior is that of an isolated node, then the collective behavior of the community can be analyzed as a weighted network as in [7]. Finally, at the third level of organization, the entire network is formed as a directed and weighted connection of different communities. At this level of organization, we investigate the emergence of cluster synchronization. That is, complete identical synchronization within each community, while the dynamics between communities remains unsynchronized.

The remainder of the paper is organized as follows: In Section 2, our model of a network of networks is presented. In Section 3, the stability of the cluster synchronization is investigated and our main results are presented. While in Section 4, numerical simulations are used to illustrate the cluster synchronization in networks of chaotic oscillators. Finally, in Section 5 conclusions and closing remarks are given.

## 2. A model for a network of networks

We consider a network with $N$ nodes grouped in $M$ communities each with $N_k$ subnetworks called CUs. To make the network description clearer, we start at the internal most level. We assume that each CU in the network consists $r$ identical $n$-dimensional systems linearly and diffusively coupled. Thus, the state equation of a CU within a community is given by

$$\dot{\psi}_i(t) = f(\psi_i(t)) + g \sum_{j=1}^{r} a_{ij} \Gamma \psi_j(t), \text{ for } i = 1, 2, \ldots, r \tag{1}$$

where $\psi_i(t) = [\psi_{i1}(t), \psi_{i2}(t), \cdots, \psi_{in}(t)]^\top \in \mathbb{R}^n$ is the state vector of $i$-th node; $f : \mathbb{R}^n \to \mathbb{R}^n$ is at least locally Lipschitz and describes the dynamics of an isolated node. The constant $g > 0$ denotes the uniform coupling strength of the CU; $\Gamma \in \mathbb{R}^{n \times n}$ is a zero-one diagonal matrix describing the internal coupling between nodes in the CU; while its external coupling configuration of the CU is given by the matrix $\mathscr{A} = \{a_{ij}\} \in \mathbb{R}^{r \times r}$, which is constructed as follows: if there exist an edge between nodes $i$ and $j$ (with $i \neq j$), then $a_{ij} = a_{ji} = 1$, otherwise $a_{ij} = a_{ji} = 0$. Additionally, to satisfy the diffusive coupling condition, the diagonal elements of $\mathscr{A}$ are given by

$$a_{ii} = - \sum_{j=1, j\neq i}^{r} a_{ij} = - \sum_{j=1, j\neq i}^{r} a_{ji}, \text{ for } i = 1, 2, \ldots, r \tag{2}$$

We assume that the CU is connected, therefore $\mathscr{A}$ is symmetric, irreducible, and has zero as an eigenvalue of multiplicity one with all other eigenvalues strictly negative, which can be ordered as follows [28]:

$$0 = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \tag{3}$$

Defining $x(t) = [\psi_1(t)^\top, \psi_2(t)^\top, \ldots, \psi_r(t)^\top]^\top \in \mathbb{R}^{nr}$ as the state variable for the CU. In shorthand notation (1) becomes

$$\dot{x}(t) = F(x(t), g, \Gamma, \mathscr{A}) \tag{4}$$

where $F : \mathbb{R}^{nr} \to \mathbb{R}^{nr}$ describes the dynamics within the CU.

For simplicity, in the remainder we will impose the following assumptions:

**A1.** All CUs have the same connection structure and the same number of nodes,

**A2.** Within the CU all nodes have the same dynamical descriptions and only differ in their initial conditions.

At the second level of organization, the CUs are grouped into a community in the following manner: if two CU have the same dynamical description, then they belong to the same community, otherwise they must be in different communities.

The connection structure of the $k$-th community is taken to be a weighted network of $N_k$ CUs. Accordingly the state equation of the $k$-th community within the network is given by

$$\dot{x}_i^{[k]}(t) = F^{[k]}(x_i^{[k]}(t), g^{[k]}, \Gamma^{[k]}, \mathscr{A}^{[k]}) + \sum_{j=1}^{N_k} c_{ij}^{[k]} \hat{\Gamma}_{ij}^{[k]} x_j^{[k]}(t), \tag{5}$$

for $i = 1, 2, \ldots, N_k$ where $x_i^{[k]}(t) \in \mathbb{R}^{nr}$ is the state variable of the $i$-th CU in the $k$-th community. The connection between the states of the $i$-th and $j$-th CUs within the $k$-th community are described by the zero-one diagonal matrix

$$\hat{\Gamma}_{ij}^{[k]} = \operatorname{diag}(\gamma_1^\top, \gamma_2^\top, \cdots, \gamma_r^\top) \in \mathbb{R}^{nr \times nr} \tag{6}$$

where a nonzero entry of the vectors $\gamma_p = \left[\gamma_{q+(p-1)}\right] \in \mathbb{R}^n$ ($q = 1, 2, ..., n$, $p = 1, 2, \cdots, r$) indicates that the $q$-th state of the $p$-th node of the i-th CU is coupled to the $q$-th state of the $p$-th node in the $j$-th CU, both within the $k$-th community.

In the same sense as above, we assume that the states that are connected between any two CUs are the same for the entire community, *i.e.*, $\hat{\Gamma}_{ij}^{[k]} = \hat{\Gamma}, \forall i, j$.

The matrix $\mathscr{C}^{[k]} = \left[c_{ij}^{[k]}\right] \in \mathbb{R}^{N_k \times N_k}$ describes which CUs are connected in the following manner: if there exist a connection between the $i$-th and the $j$-th CUs with $i \neq j$, then $c_{ij}^{[k]} = c_{ji}^{[k]} > 0$, otherwise $c_{ij}^{[k]} = c_{ji}^{[k]} = 0$. The diagonal elements of the matrix $\mathscr{C}^{[k]}$ are

$$c_{ii}^{[k]} = -\sum_{j=1, j\neq i}^{N_k} c_{ij}^{[k]} = -\sum_{j=1, j\neq i}^{N_k} c_{ji}^{[k]}, \text{ for } i = 1, 2, \cdots, N_k \tag{7}$$

As such, $\mathscr{C}^{[k]}$ is zero row sum by rows and columns, i.e., the coupling between the CUs in the community is diffusive, therefore the eigenvalues of $\mathscr{C}^{[k]}$ can be ordered as in (3) [28].

An example of our proposed structure is shown in Figure 1. In this case the network has one community of $N_1 = 10$ CUs, each one represented by a gray octagon with dotted lines representing their weighted connections. In turn, each CU has $r = 13$ nodes, represented here by dots with solid lines representing the uniform connection between the nodes.

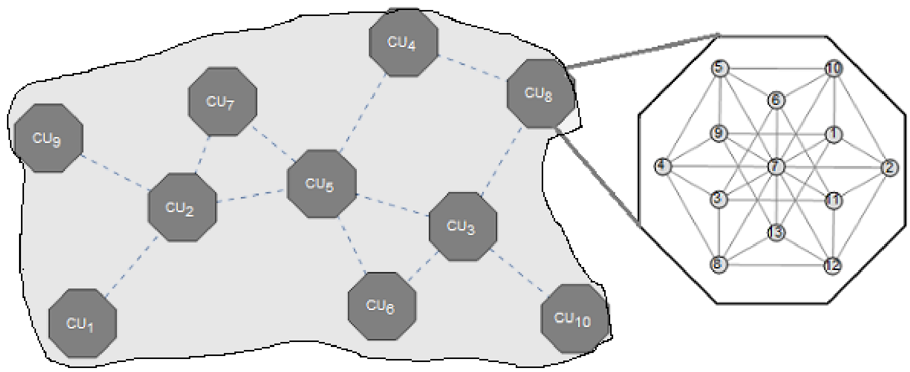



Figure 1: A schematic illustration of network with one community of ten CUs with thirteen nodes each ($M = 1$, $N_1 = 10$ and $r = 13$).

Let $\chi^{[k]}(t) = [x_1^{[k]}(t)^\top, x_2^{[k]}(t)^\top, \ldots, x_{N_k}^{[k]}(t)^\top]^\top \in \mathbb{R}^{nrN_k}$ be the state variable for the $k$-th community of the network. Then, (5) can be expressed as

$$\dot{\chi}^{[k]}(t) = \mathcal{F}^{[k]}(\chi^{(k)}(t), g^{[k]}, \Gamma^{[k]}, \mathscr{A}^{[k]}, \hat{\Gamma}, \mathscr{C}^{[k]}) \tag{8}$$

where $\mathcal{F}^{[k]} : \mathbb{R}^{nrN_k} \to \mathbb{R}^{rnN_k}$ describes the dynamics of the $k$-th community. For simplicity, in what follows we will use the following shorthand notation, $\mathcal{F}^{[k]}(\chi^{[k]}(t))$, for the dynamics of the $k$-th community.

At the third level, the entire network of networks is constructed as a directed and weighted connection of communities. Then, we have the following state equation

$$\dot{\chi}^{[i]}(t) = \mathcal{F}^{[i]}(\chi^{[i]}(t)) + \sum_{j=1}^{M} D_{ij}\chi^{[j]}(t), \text{ for } i = 1, 2, \cdots, M. \tag{9}$$

where $D_{ij} \in \mathbb{R}^{nrN_i \times nrN_j}$ is the coupling matrix between the $i$-th and the $j$-th communities within the network. In vector form, the dynamics of the entire network can be written as:

$$\dot{\chi}(t) = \mathcal{F}(\chi(t)) + D\chi(t) \tag{10}$$

where the state variable of the entire network is $\chi(t) = [\chi^{[1]}(t)^\top, \ldots, \chi^{[M]}(t)^\top]^\top \in \mathbb{R}^{nN}$, and $\mathcal{F}(\chi(t)) = \left[\mathcal{F}^{[1]}(\chi^{[1]}(t))^\top, \ldots, \mathcal{F}^{[M]}(\chi^{[M]}(t))^\top\right]^\top : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ describes the dynamics of the isolated communities. While, the interconnections between the communities is described by $D = [D_{ij}] \in \mathbb{R}^{nN \times nN}$.

Notice that the $N$ nodes of the network are divided into $M$ communities each with $N_k$ CUs, which in turn have $r$ nodes. In other words, the number of nodes is

$$N = r \sum_{k=1}^{M} N_k, \text{ with } N_k \geq 1, \forall k \tag{11}$$

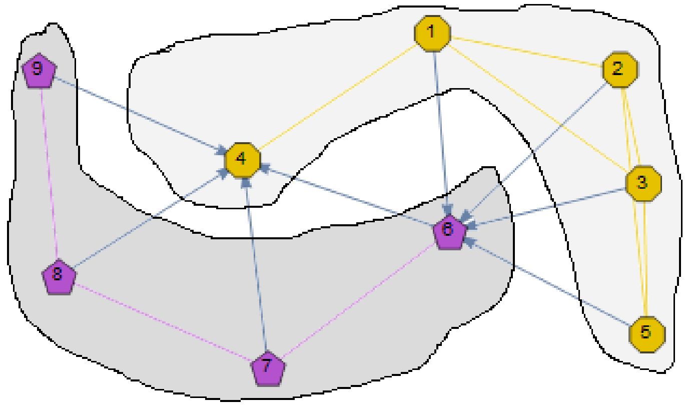



Figure 2: A schematic illustration of a network of networks with two communities ($M = 2$, $N_1 = 4$, and $N_2 = 5$).

Additional, since each node is a $n$-dimensional dynamical system, the coupling matrix between the $i$-th and $j$-th communities are of dimension $nrN_i \times nrN_j$, which in general, are rectangular. While, the diagonal blocks of $D$ ($D_{kk} \in \mathbb{R}^{nrN_k \times nrN_k}$, $k = 1, \ldots, M$) are square. As a result, the coupling matrix $D$ is square but not symmetric.

As mention above, all CUs have the same dimension, then the coupling between different communities can be rewritten as

$$D_{ij} = \mathcal{D}_{ij} \otimes \hat{\Gamma} \tag{12}$$

where $\otimes$ is the Kronecker product, the entries of $\mathcal{D}_{ij} = \left[d_{kl}^{[ij]}\right] \in \mathbb{R}^{N_i \times N_j}$, are such, that $d_{kl}^{[ij]} > 0$ indicates that the $k$-th CU of the $i$-th community is coupled to the $l$-th CU of the $j$-th community, otherwise $d_{kl}^{[ij]} = 0$. Note that in our model, we assume that if two CUs are connected, then they are coupled in the same way and by the same states.

An example of this network of networks structure is shown in Figure 2. There are two communities: one of five CUs, represented by octagons, and the other with four CUs, which are represented by pentagons. The solid lines describe the connection between CUs in the same community, and the arrows represent the directed and weighted connections between different communities.

In the following Section, the conditions for cluster synchronization in our network of networks model (10) are derived.

## 3. Stability analysis of cluster synchronization

A network of uniformly, linearly and diffusively coupled identical dynamical systems with a state equation description given by (1) is said to achieved complete (asymptotic) synchronization to the solution

$$\psi_1(t) = \psi_2(t) = \cdots = \psi_r(t) = s(t) \tag{13}$$

If for any initial condition in the neighborhood of the synchronization solution $s(t)$, one has that

$$\lim_{t\to\infty} \|\psi_i(t) - s(t)\| = 0, \text{ for } i = 1, 2, \cdots, r. \tag{14}$$

where $s(t) \in \mathbb{R}^n$ satisfies the dynamics of an isolated node $\dot{s}(t) = f(s(t))$.

There are different methods to establish the stability of $s(t)$ as a solution of (1). In [28] the error dynamics are linearized around $s(t)$ and diagonalized in terms of the eigenvalues of $\mathscr{A} \in \mathbb{R}^{r\times r}$, resulting on the $\lambda_2$ criterion for stability of the synchronized solution.

An isolated CU of the form (1) achieves complete synchronization if the coupling strength $g$, satisfies the following condition

$$g \geq \left|\frac{\alpha}{\lambda_2}\right| \tag{15}$$

where $\lambda_2$ is the largest nonzero eigenvalue of $\mathscr{A}$, and $\alpha < 0$ is constant lower bound that satisfies the Lyapunov inequality

$$\left[Df(s(t)) + \alpha\Gamma\right]^\top P + P\left[Df(s(t)) + \alpha\Gamma\right] \leq -Q \tag{16}$$

where $Df(s(t))$ is the Jacobian of the node dynamics evaluated at $s(t)$, $P = P^\top > 0$ and $Q = Q^\top > 0$ are positive define matrices [28].

In the remainder of the manuscript we assume the following:

**A3.** The uniform coupling strength $g$ in all CUs is large enough to satisfy the condition (15).

In the same sense, the isolated $k$-th community (5) is said to achieved complete (asymptotic) synchronization to the solution

$$x_1^{[k]}(t) = x_2^{[k]}(t) = \cdots = x_{N_k}^{[k]}(t) = S^{[k]}(t) \tag{17}$$

If for any initial condition in the neighborhood of $S^{[k]}(t)$

$$\lim_{t\to\infty} \|x_i^{[k]}(t) - S^{[k]}(t)\| = 0, \text{ for } i = 1, 2, \cdots, N_k. \tag{18}$$

where $S^{[k]}(t) = \left[s(t)^\top, \ldots, s(t)^\top\right]^\top \in \mathbb{R}^{nr}$ is the synchronized solution of the $k$-th community and satisfies the dynamics of an isolated CU of the $k$-th community $\dot{S}^{[k]}(t) = F^{[k]}(S^{[k]}(t), g^{[k]}, \Gamma^{[k]}, \mathscr{A}^{[k]})$.

The stability of the solution $S^{[k]}(t)$ for the weighted community of CUs can be establish following a similar procedure as in [7]. That is, from the weighted coupling matrix $\mathscr{C}^{[k]}$ we obtain the unweighted matrix $\mathscr{B}^{[k]} = \left\{b_{ij}^{[k]}\right\} \in \mathbb{R}^{N_k\times N_k}$ in the following manner:

$$b_{ij}^{[k]} = b_{ji}^{[k]} = \begin{cases} 1, & c_{ij}^{[k]} \neq 0 \\ 0, & c_{ij}^{[k]} = 0 \end{cases}, \quad \text{and } b_{ii}^{[k]} = -\sum_{j=1, i\neq j}^{N_k} b_{ij}^{[k]} \tag{19}$$

for $i, j = 1, 2, \cdots, N_k$.

The dynamics associated with the unweighted coupling matrix $\mathscr{B}^{[k]}$ correspond to the state equation

$$\dot{x}_i^{[k]}(t) = F^{[k]}(x_i^{[k]}(t), g^{[k]}, \Gamma^{[k]}, \mathscr{A}^{[k]}) + \tilde{c} \sum_{j=1}^{N_k} b_{ij}^{[k]} \hat{\Gamma}_{ij}^{[k]} x_j^{[k]}(t) \tag{20}$$

with $\tilde{c} > 0$ the corresponding uniformed coupling strength. Therefore, the unweighted community (20) achieves complete synchronization if it is coupled with a $\tilde{c}$ satisfies the condition

$$\tilde{c}\lambda_2(\mathscr{B}^{[k]}) \leq \alpha \tag{21}$$

where $\lambda_2(\mathscr{B}^{[k]})$ is the largest nonzero eigenvalues of $\mathscr{B}^{[k]}$ and $\alpha < 0$ is a constant lower bound that satisfies the Lyapunov inequality

$$[DF^{[k]}(S^{[k]}) + \alpha\hat{\Gamma})]^\top P + P[DF^{[k]}(S^{[k]}) + \alpha\hat{\Gamma})] \leq -Q \tag{22}$$

where $DF^{[k]}(S^{[k]})$ is the Jacobian of the community dynamics evaluated at $S^{[k]}$.

Condition (21) can be expressed in terms of the entries of the weighted matrix $\mathscr{C}^{[k]}$ as follows:

**Theorem 1.** *The community of CUs (5) achieves the complete synchronization, if the elements of the matrix $\mathscr{C}^{[k]}$ satisfy*

$$c_{ij}^{[k]} > \left| \frac{\alpha}{\lambda_2(\mathscr{B}^{[k]})} \right|, \ (i \neq j) \ \ i, j = 1, 2, \cdots, N_k \tag{23}$$

**Proof:** Consider the error matrix $\mathscr{Q}^{[k]} = \tilde{c}\mathscr{B}^{[k]} - \mathscr{C}^{[k]}$, whose entries are

$$q_{ij}^{[k]} = \tilde{c}b_{ij}^{[k]} - c_{ij}^{[k]}, \ \ (i \neq j), \text{ and} \tag{24}$$

$$q_{ii}^{[k]} = \sum_{\substack{l=1 \\ l \neq i}}^{N_k} \left(c_{il}^{[k]} - \tilde{c}b_{il}^{[k]}\right) \tag{25}$$

for $i, j = 1, 2, \cdots, N_k$. Under the restriction $c_{ij}^{[k]} > \tilde{c}$ the diagonal entries of $\mathscr{Q}^{[k]}$ are positive, $q_{ii}^{[k]} > 0$ for $i = 1, 2, \cdots, N_k$, and taking the absolute value of the off-diagonal elements to be $\left|q_{ij}^{[k]}\right| = \left|\tilde{c} - c_{ij}^{[k]}\right|$. Then,

$$\left|q_{ii}^{[k]}\right| \geq \sum_{\substack{j=1 \\ i \neq j}}^{N_k} \left|q_{ij}^{[k]}\right| \tag{26}$$

Therefore, $\mathscr{Q}^{[k]} \geq 0$ is a positive semidefinite matrix, if

$$c_{ij}^{[k]} > \tilde{c}, \ \ (i \neq j) \ \ i, j = 1, 2, \cdots, N_k \tag{27}$$

It follows that $\tilde{c}\mathscr{B}^{[k]} \geq \mathscr{C}^{[k]}$, which can be expressed in terms of their eigenvalues as $\lambda_j(\tilde{c}\mathscr{B}^{[k]}) \geq \lambda_j(\mathscr{C}^{[k]})$ for all $j = 1, 2, \cdots, N_k$. Since the matrices are negative semidefinite the largest nonzero eigenvalues of $\mathscr{B}^{[k]}$ is $\lambda_2(\mathscr{B}^{[k]})$ and one has

$$\lambda_2(\mathscr{C}^{[k]}) \leq \lambda_2(\tilde{c}\mathscr{B}^{[k]}) = \tilde{c}\lambda_2(\mathscr{B}^{[k]}) \tag{28}$$

Using the previous synchronization result and (21), the condition (23) is obtained. ■

In the same sense as before, we have the following assumption:

**A4.** The entries of the community coupling matrix $c_{ij}^{[k]}$ in all communities satisfy the condition (23).

At the interconnected communities level, the objective is to achieve cluster synchronization, in the sense that any CUs within the same community are identical synchronized, *i.e.*,

$$\lim_{t\to\infty} \|x_i^{[k]}(t) - S^{[k]}\| = 0, \quad \text{for} \quad i = 1, 2, \cdots, N_k \tag{29}$$

while CUs form different communities remain unsynchronized, that is,

$$\lim_{t\to\infty} \|S^{[k]}(t) - S^{[l]}(t)\| \neq 0 \quad \text{for} k, l = 1, 2, \cdots, M \quad \text{with } k \neq l. \tag{30}$$

where $S^{[k]}(t)$ and $S^{[l]}$ are the synchronous solutions of $k$-th and $l$-th community, respectively.

In order to have cluster synchronization for our network of networks model (10), the following assumptions are required

**A5. The same-input condition.** The matrices $\mathcal{D}_{ij} \in \mathbb{R}^{N_i \times N_j}$ that represent the connection between $i$-th and $j$-th communities (with $i \neq j$) is said to satisfy same-input condition, if their elements satisfy

$$d_{kl}^{[ij]} = d_{ml}^{[ij]}, \quad \text{with } i \neq j \text{ and } k \neq m \tag{31}$$

for $k, m = 1, 2, \cdots, N_i$ and $l = 1, 2, \cdots, N_l$.

**A6. The diagonal condition.** The matrices $\mathcal{D}_{ii} \in \mathbb{R}^{N_i \times N_i}$ are diagonal matrices that show the input strengths of each community, which satisfy the following

$$d_{kk}^{[ii]} = -\sum_{\substack{j=1 \\ i\neq j}}^{M} \sum_{l=1}^{N_i} d_{kl}^{[ij]} \tag{32}$$

for $i = 1, 2, \cdots, M$.

According to (31) and (32), it is easy to show that the sum by rows of $\mathcal{D}$ is zero, so we have $S^{[k]}(t)$ is a synchronized solution for the $k$-th community even under external coupling.

Now, we can give the following result:

**Theorem 2.** *Consider a network of networks (10), where each community without external coupling achieve the complete synchronization, and the synchronous states of different community are different. If each non-diagonal block $\mathcal{D}_{ij}$ with $i \neq j$ satisfies the eqs.* (31) *and diagonal blocks $D_{ii} \in \mathbb{R}^{N_i \times N_i}$ satisfy* (32) *and the following*

$$\mathcal{D}_{ii} \leq d_k \mathcal{I}_{N_i} \tag{33}$$

*where $d_k$ is a negative constant such that*

$$\omega_{pq}^{[k]} + d_k \leq 0 \tag{34}$$

*where $\omega_{pq}^{[k]}$ is a lower bound constant to ensure the complete synchronization of $k$-th community without external coupling. Then, the network of network achieves the cluster synchronization.*

**Proof:** At first, consider the synchronization error for any pair of CUs within the same community we defined as

$$e_{pq}^{[k]}(t) = x_q^{[k]}(t) - x_p^{[k]}(t) \tag{35}$$

for $p \neq q$ and $p, q = 1, 2, \cdots, N_k$ and $k = 1, 2, \cdots, M$.

The time derivative of this synchronization error is given by

$$\begin{aligned} \dot{e}_{pq}^{[k]}(t) =& F^{[k]}(x_q^{[k]}(t)) - F^{[k]}(x_p^{[k]}(t)) \\ &+ d_{qq}^{[kk]}\hat{\Gamma} x_q^{[k]}(t) - d_{pp}^{[kk]}\hat{\Gamma} x_p^{[k]}(t) \\ &+ \sum_{\substack{l=1 \\ k \neq l}}^{M} \sum_{\substack{j=1 \\ p \neq j}}^{N_j} (d_{qj}^{[kl]} - d_{pj}^{[kl]})\hat{\Gamma} x_j^{[l]} \end{aligned} \tag{36}$$

Using the eqs. (31) and (32), we obtain

$$\dot{e}_{pq}^{[k]}(t) = F^{[k]}(x_q^{[k]}(t)) - F^{[k]}(x_p^{[k]}(t)) + d_{pq}^{[k]}\hat{\Gamma} e_{pq}^{[k]}(t) \tag{37}$$

Since the coupling satisfies the same-input condition we have $d_{pq}^{[k]} = d_{qq}^{[kk]} = d_{pp}^{[kk]}$.

Consider the candidate Lyapunov function

$$\mathcal{V}_k(t) = e_{pq}^{[k]\top} e_{pq}^{[k]}, \tag{38}$$

the time derivative of the equation (38) along the trajectory of (37) is given by

$$\dot{\mathcal{V}}_k(t) = e_{pq}^{[k]\top(t)} \left[F^{[k]}(x_q^{[k]}(t)) - F^{[k]}(x_p^{[k]}(t))\right] + e_{pq}^{[k]\top(t)} \left[d_{pq}^{[k]}\hat{\Gamma} e_{pq}^{[k]}\right] \tag{39}$$

To proof that $\mathcal{V}_k(t)$ is negative definitive. Notice that with assumptions **A.3** and **A.4**, the community has strong enough coupling strength to achieve internally complete synchronization. Therefore, there exist a constant $\omega_{pq}$ such that

$$\dot{\mathcal{V}}_k(t) \leq e_{pq}^{[k]}(t)^\top [\omega_{pq}\mathcal{I}_{nr} + d_{pq}^{[k]}\hat{\Gamma}] e_{pq}^{[k]}(t) \tag{40}$$

Thus, if $\omega_{pq}\mathcal{I}_{nr} + d_{pq}^{[k]}\hat{\Gamma} \leq 0$, which satisfies when $\omega_{pq} + d_{pq}^{[k]} \leq 0$ and we obtain $\dot{\mathcal{V}}_k(t) \leq 0$. Then, $e_{pq}^{[k]}$ for $p, q = 1, 2, \cdots, N_k$ and $k = 1, 2, \cdots, M$ are stable around zero. Consequently, the $k$-th community achieve the complete synchronization even under inter-community coupling.

Now, we consider the error between communities, then according to the assumptions **A.3** and **A.4** along with the previous proof, each community achieve the complete synchronization even under external coupling, *i.e.*, on the synchronization manifold one can say the trajectories of each community network collapse to those a single solution $S^{[k]}$. Also, according to the definition of community, these solutions are different for each community $S^{[k]} \neq S^{[l]}$ for $k \neq l$. Then, we define the error synchronization between communities as $E^{[kl]} = S^{[k]} - S^{[l]}$, and the time derivative of this error is

$$\dot{E}^{[kl]} = \dot{S}^{[k]} - \dot{S}^{[l]} = F^{[k]}(S^{[k]}) - F^{[l]}(S^{[l]}) \tag{41}$$

and as $F^{[k]}(S^{[k]}) \neq F^{[l]}(S^{[l]})$ holds for all $k \neq l$. We have that error between communities is not stable at zero. Therefore, the network of networks achieves cluster synchronization. ■

Note that each coupling matrix determines if the organization levels are synchronized or not, $\mathscr{A}^{[k]}$ determines if there is complete synchronization at each CU, while $\mathscr{C}^{(k)}$ determines the conditions to achieve complete synchronization at the $k$-th community. Finally, if $\mathcal{D}$ satisfies conditions in **A.5**, **A.6** from **Theorem 2** the entire network achieves cluster synchronization. Since our results are based on Lyapunov's theory they are inherently conservative. However, the flexibility of this approach allows for the relatively easy extension our results to other types of network models, like the ones that consider nonlinearities or even time delays in their connections [31].

## 4. Numerical simulation

In this section, we present numerical examples to illustrate the effectiveness of our theoretical results.

We consider a network of ninety nodes coupled into eighteen CUs with five nodes each. Here the communities are determine in terms of node dynamical description, therefore there are three communities of $N_1 = 6$ Lorenz [28], $N_2 = 8$ Chen [4], and $N_3 = 4$ Lu systems [21], respectively as shown in Figure 3. Here we take the node dynamics to be in their chaotic regime and be given by the following equations:

- Lorenz system:

$$\begin{bmatrix} \dot{\psi}_{i1} \\ \dot{\psi}_{i2} \\ \dot{\psi}_{i3} \end{bmatrix} = \begin{bmatrix} 10(\psi_{i2} - \psi_{i1}) \\ 28\psi_{i1} - \psi_{i1}\psi_{i3} - \psi_{i2} \\ \psi_{i1}\psi_{i2} - \frac{8}{3}\psi_{i3} \end{bmatrix} \tag{42}$$

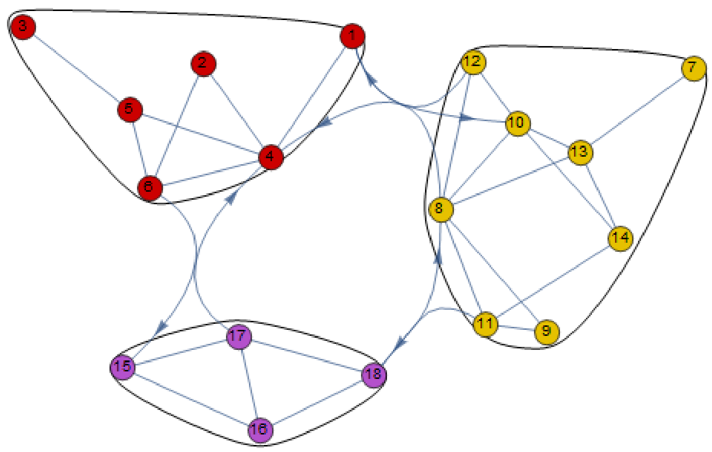



Figure 3: A network with three communities.

- Chen system:

$$\begin{bmatrix}\dot{\psi}_{i1}\\ \dot{\psi}_{i2}\\ \dot{\psi}_{i3}\end{bmatrix}=\begin{bmatrix}35(\psi_{i2}-\psi_{i1})\\ -7\psi_{i1}-\psi_{i1}\psi_{i3}-28\psi_{i2}\\ \psi_{i1}\psi_{i2}-3\psi_{i3}\end{bmatrix} \tag{43}$$

- Lu systems:

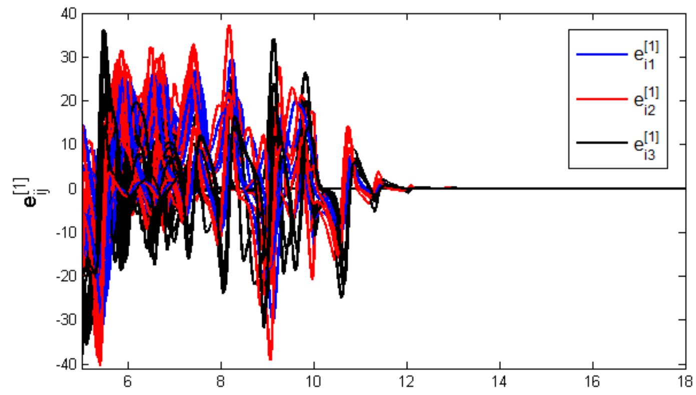
$$\begin{bmatrix}\dot{\psi}_{i1}\\ \dot{\psi}_{i2}\\ \dot{\psi}_{i3}\end{bmatrix}=\begin{bmatrix}36(\psi_{i2}-\psi_{i1})\\ 20\psi_{i2}-\psi_{i1}\psi_{i3}\\ \psi_{i1}\psi_{i2}-3\psi_{i3}\end{bmatrix} \tag{44}$$

The structure of each CUs is taken to be a 5-regular graphs and to satisfy **A.3** have the minimum coupling strength of each CUs within the communities. In this case, for CUs in the first community is $g_1 = 1$, for CUs in the second community is $g_2 = 0.84$ and the for the CUs in the third community is $g_3 = 0.368$. Moreover using **Theorem 1** we obtain the critical values for the entries of $\mathscr{C}^{[k]}$ with $k = 1, 2, 3$ as follows:

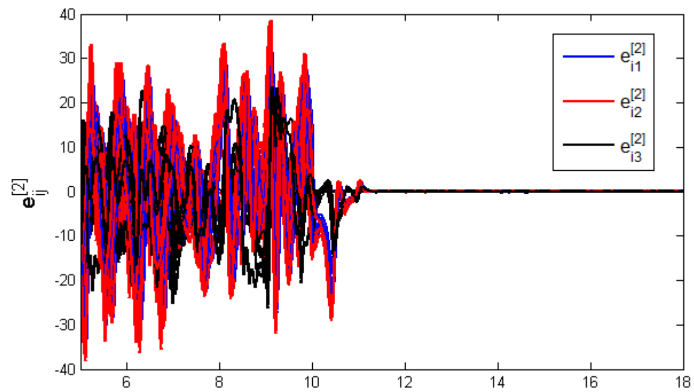
$$\begin{aligned} c_{ij}^{[1]} &> 2.67 \quad \text{for} \quad i \neq j, \text{with } i, j = 1, \ldots, 6 \\ c_{ij}^{[2]} &> 2.58 \quad \text{for} \quad i \neq j, \text{with } i, j = 7, \ldots, 14 \\ c_{ij}^{[3]} &> 0.46 \quad \text{for} \quad i \neq j, \text{with } i, j = 15, \ldots, 18 \end{aligned} \tag{45}$$

Finally, the inter communities coupling matrices $\mathcal{D}_{ij}$ are

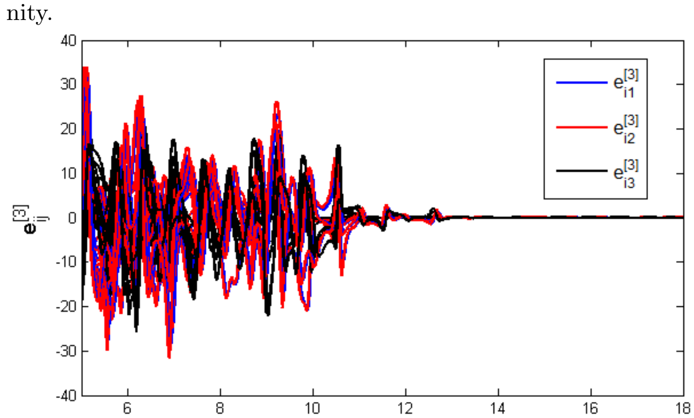
$$\mathcal{D}_{12}=\begin{bmatrix}0&0&0.43&0&0&0.4&0.3&0.2\\0&0&0.43&0&0&0.4&0.3&0.2\\0&0&0.43&0&0&0.4&0.3&0.2\\0&0&0.43&0&0&0.4&0.3&0.2\\0&0&0.43&0&0&0.4&0.3&0.2\\0&0&0.43&0&0&0.4&0.3&0.2\end{bmatrix}\ \mathcal{D}_{13}=\begin{bmatrix}0&0&0.25&0\\0&0&0.25&0\\0&0&0.25&0\\0&0&0.25&0\\0&0&0.25&0\\0&0&0.25&0\end{bmatrix}$$




(a) Evolution of the error synchronization in the first community.




(b) Evolution of the error synchronization in the second community.




(c) Evolution of the error synchronization in the third community.

Figure 4: Synchronization errors within each community

$$
\mathcal{D}_{21} = \begin{bmatrix} 0.18 & 0.28 & 0.16 & 0 & 0.14 & 0 \\ 0.18 & 0.28 & 0.16 & 0 & 0.14 & 0 \\ 0.18 & 0.28 & 0.16 & 0 & 0.14 & 0 \\ 0.18 & 0.28 & 0.16 & 0 & 0.14 & 0 \\ 0.18 & 0.28 & 0.16 & 0 & 0.14 & 0 \\ 0.18 & 0.28 & 0.16 & 0 & 0.14 & 0 \\ 0.18 & 0.28 & 0.16 & 0 & 0.14 & 0 \\ 0.18 & 0.28 & 0.16 & 0 & 0.14 & 0 \end{bmatrix} \mathcal{D}_{23} = \begin{bmatrix} 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}
$$

$$
\mathcal{D}_{31} = \begin{bmatrix} 0 & 0 & 0 & 0.25 & 0.25 & 0 \\ 0 & 0 & 0 & 0.25 & 0.25 & 0 \\ 0 & 0 & 0 & 0.25 & 0.25 & 0 \\ 0 & 0 & 0 & 0.25 & 0.25 & 0 \end{bmatrix} \mathcal{D}_{32} = \begin{bmatrix} 0 & 0.27 & 0 & 0.3 & 0 & 0.28 & 0.19 & 0.3 \\ 0 & 0.27 & 0 & 0.3 & 0 & 0.28 & 0.19 & 0.3 \\ 0 & 0.27 & 0 & 0.3 & 0 & 0.28 & 0.19 & 0.3 \\ 0 & 0.27 & 0 & 0.3 & 0 & 0.28 & 0.19 & 0.3 \end{bmatrix}
$$

Then, to satisfy **Theorem 2**, we have $\mathcal{D}_{11} \leq 1.58\mathcal{I}_{N_1}$, $\mathcal{D}_{22} \leq 0.96\mathcal{I}_{N_2}$ and $\mathcal{D}_{33} \leq 1.85\mathcal{I}_{N_3}$.

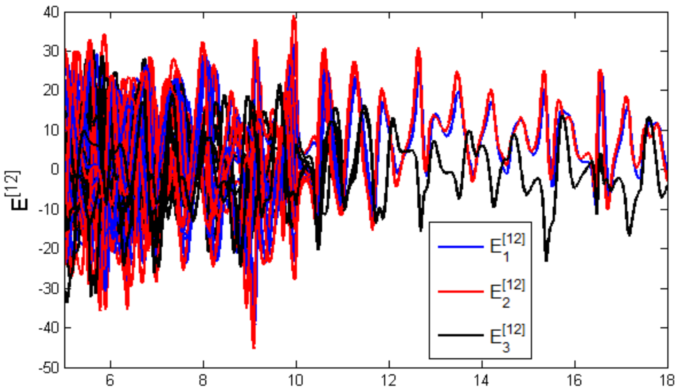
The results of the numerical simulation are presented in the following Figures. In Figure 4, the dynamics at each community are shown, the ninety nodes are uncoupled until the ten second mark in the simulation, then each community achieves internal complete synchronization as the error dynamics within each community go to zero. In Figure 5, we show the error synchronization between the $i$-th and $j$-th communities $E_{ij}$. Since the network of networks achieves cluster synchronization the communities are not identically synchronized, as such the errors $e_{ij}^{(k)}$ converge to zero while the errors between communities $E_{12}, E_{13}$ and $E_{23}$ do not.

## 5. Conclusions

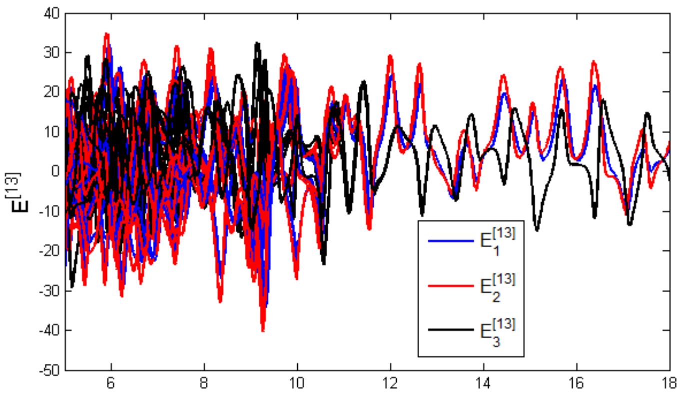
Our proposed model of a network of networks is inspired by biology, is based on the organization of cells groups, where structure is determine by dynamical description of the components as is the case in the pancreatic islet and potentially in the arrangement of neuronal cells. In this complex system, the study of alternative forms of coordination are importance. Therefore, the present investigation of cluster synchronization is a step in understanding the complex patterns of synchrony observed in real-world systems. Our proposal consists on modeling complex systems as networks with three different levels of organization, we consider that nodes form small structures with strong local couplings, these compact groups connect with different weights to form communities, which interact with each other through directed and weighted connections to form our network of subnetworks. Using this hierarchical structure is particularly important on networks where the number of nodes is high and their dynamics are non identical, because it allows us to analyze different forms of collective behavior with well-establish tools. In the results show, that cluster synchronization is achieved by placing relatively simple restrictions on the coupling structure, that is, between communities of dynamically different nodes, the coupling needs to satisfy the same-input and diagonal conditions, while within each community, the coupling must be strong enough to satisfy local synchronization conditions like the $\lambda_2$ criterion.

## Acknowledgments

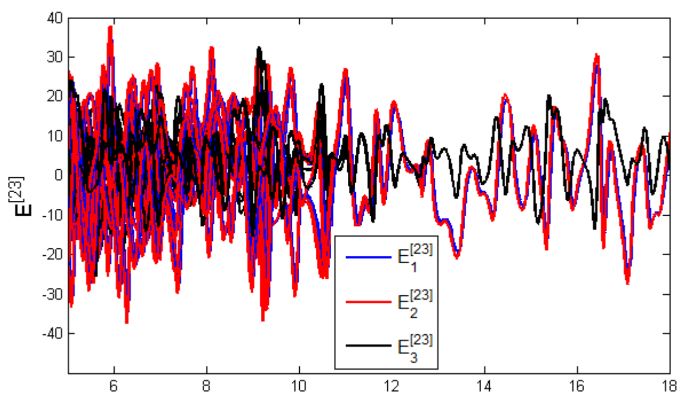
This work was partly supported by the National Consul for Science and Technology of México, CONACYT.




(a) Synchronization errors between first and second communities




(b) Synchronization errors between first and third communities




(c) Synchronization errors between second and third communities

Figure 5: Synchronization errors between communities

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