Robust Synchronization Of A Class Of Uncertain Complex Networks Via Discontinuous Control

J. G. Barajas-Ramírez[a]

[a]IPICyT, Instituto Potosino de Investigación Científica y Tecnológica,
División de Matemáticas Aplicadas,
Camino a la Presa San José 2055
Lomas 4a Sección CP. 78216
San Luis Potosí, S. L. P., México.

Abstract

Robust controlled synchronization is investigated for networks with uncertainties in both their node dynamics and their connections. We consider two situations: In the first case, the effect of uncertainties is assume to vanish as synchronization is achieve. On the second, the disturbances are assume non-vanishing but bounded. To achieve robust synchronization on these situations, local feedback controllers are design, which are smooth in the first case, and discontinuous in the latter. Synchronization criteria are establish for these situations, and some new observations on the design of synchronizing controllers are presented. Numerical simulations are use to illustrate our results.

Keywords: Synchronization, Dynamical networks, Robust control.

1. Introduction

In recent years synchronization of dynamical networks has become a very active area of research [1, 2, 3]. In particular, studies on the synchronization of networks with small-world and scale-free[4] and scale-free [5] topologies have significantly advanced our understanding of the synchronization phenomenon in real-world complex networks; highlighting their potential applications to the Internet, electric power distribution, social and economical groups, and even biological systems [6, 7, 8, 9]. In contrast to their wide potential applicability, the bulk of research on network synchronization has concentrated on networks with identical nodes, linearly and diffusively coupled, where full knowledge of both the dynamical description of its nodes and the structure of their interconnections is available. Under these conditions, approaches like the master stability function (MSF) [10, 11], and other methods based on linearized analysis of the network's transverse dynamics [6, 12, 13], can be used to determine the stability of the overall synchronized behavior of the network. Unfortunately, when considering more realistic situations, where complete knowledge is not available linearized approaches are not directly applicable.

Although dynamical networks may become synchronized spontaneously, in most cases is necessary to take actions to force the network into a synchronized state. This situation is referred to as *controlled synchronization*. In [14], an adaptive robust controller was proposed to achieve synchronization on uncertain networks that preserve their diffusive structure under perturbations. That is, networks where perturbations and control inputs vanish at the synchronized solution. For this type of uncertain networks in [15], synchronization is robustly achieved designing the coupling functions of the network. In [16], the problem of adaptive synchronization of uncertain networks is reconsidered describing local and global synchronization designs. Linear feedback controllers to achieve robust synchronization on uncertain networks with uniform and nonuniform inner coupling matrices are proposed in [17]. The effect of coupling delays on the synchronization of uncertain networks is considered in [18]. Following a linearized analysis under vanishing perturbations in [19] and [20] conditions for synchronization of a network with slightly different nodes were derived using the MSF approach. In the above results is require that the uncertain network remain diffusive under the effects of perturbations and controls. When considering that the topology can be perturbed, the problem becomes more complex. In [21], adaptive synchronization is considered in the context of networks under the action of slow varying time dependent network topology. In another paper [22], a similar solution is found from a MSF approach. In this paper we extend previous results by relaxing the requirements of identical nodes and vanishing coupling functions. In particular, we proposed controllers for two situations: vanishing and non-vanishing perturbations. In the first case, smooth synchronizing controllers are designed, while for the latter case, we propose discontinuous local feedback controllers to achieve robust synchronization. The proposed controller designs allow us to derive several criteria for robust synchronization of uncertain networks, which relate emergence of synchronization to: the dynamics of an isolated nominal node, topological features of the nominal network, and bounds of the uncertainties affecting the network.

The paper is organized as follows: In Section 2, the synchronization problem for uncertain networks is stated in detail. In Section 3, synchronizing controllers are designed for two distinct situations; namely, vanishing and non-vanishing perturbations. Our results are illustrated with numerical simulations presented in Section 4. Finally, the paper is concluded with some closing comments and remarks.

2. Uncertain Dynamical Network Model

The state space description of a network with uncertain couplings, where each node is a dynamical system with uncertain parameters and a local controller, is given by:

$$\dot{x}_i = \tilde{f}_i\left(x_i, \tilde{\rho}_i\right) + \tilde{g}_i(X) + u_i, \text{ for } i = 1, ..., N \tag{1}$$

where $x_i \in \mathbf{R}^n$ are the state variables of the ith node; $X = [x_1, ..., x_N] \in \mathbf{R}^{n\times N}$ is a row vector form by the state variables of all the nodes in the network; and $u_i \in \mathbf{R}^n$ is a local feedback controller to be designed.

The parameters of each dynamical node are assume to be $\tilde{\rho}_i = \rho + \Delta\hat{\rho}_i \in \mathbf{R}^p$, with $\Delta = \pm 1$, where ρ and $\hat{\rho}_i$ are the nominal and uncertain parts of the parameters of the ith node, respectively. The interactions of the ith node within the network are given by the uncertain coupling function $\tilde{g}_i(X) = g_i(X) + \Delta\hat{g}_i(X) : \mathbf{R}^{n\times N} \to \mathbf{R}^n$; where $g_i(X)$ describes the nominal coupling, and $\hat{g}_i(X)$ the uncertain part of the interactions between the ith node and the rest of the network. The uncertain function $\tilde{f}_i\left(x_i, \tilde{\rho}_i\right) = f(x_i, \rho) + \Delta\hat{f}_i(x_i, \hat{\rho}_i) : \mathbf{R}^{n\times p} \to \mathbf{R}^n$ describes the dynamics of the ith node in isolation ($\tilde{g}_i(X) = 0 \in \mathbf{R}^n$), *i.e.*, disconnected from the network; with the nonlinear Lipschitz function $f(x_i, \rho)$ describing the dynamics of the node with nominal parameters, and the unknown but bounded function $\hat{f}_i(x_i, \hat{\rho}_i)$ describing the effects of the parameter uncertainties on the ith node dynamics.

Remark 1. The uncertain dynamical network model in (1) is similar to the one used in [14, 15, 17, 18]. However, our model differs in that it considers uncertainty in both, dynamical description and coupling structure. Further, we express the uncertainties as deviations from a nominal description which directly depend on unknown parameters and perturbations. In this way different situations can be considered, including the case of perturbations which render the coupling structure not vanishing.

The nominal controlled dynamical network

$$\dot{x}_i = f(x_i, \rho) + g_i(X) + u_i, \text{ for } i = 1, ..., N \tag{2}$$

is said to asymptotically achieve synchronization, if the state solutions of every node in the network evolve at unison, in the sense that

$$\lim_{t\to\infty} \|x_i - s\| = 0, \text{ for } i = 1, ..., N \tag{3}$$

where $s \in \mathbf{R}^n$ is the *synchronized solution*.

The existence of a synchronized solution for (2) depends on the properties of the coupling function $g_i(X)$. In particular, in the case of diffusive coupling, that is, a nominal coupling function such that $g_i(X) = 0 \ \forall i$ when $x_1 = x_2 = ... = x_N$, the nominal controlled dynamical network has a synchronized solution with a dynamical evolution given by

$$\dot{s} = f(s, \rho) \tag{4}$$

where $f(\cdot)$ describes the dynamics of an isolated nominal node.

From (2) and (4) the synchronization error ($\varepsilon_i = x_i - s$) for the nominal network evolves according to:

$$\dot{\varepsilon}_i = f(x_i, \rho) - f(s, \rho) + g_i(X) + u_i, \text{ for } i = 1, ..., N \tag{5}$$

Then, the synchronization of the nominal dynamical network (2) becomes a control problem, where the objective is to design local controllers such that (5) be asymptotically stable about its zero equilibrium point.

We assume that the perturbations affecting the uncertain dynamical network (1) are small and independent of time, such that for appropriate choices of the feedback control inputs u_i the solutions of each node can be made to remain close to each other. Then, the trajectories of each individual perturbed node are well approximated by the average trajectory of the network, $\bar{s} = \frac{1}{N}\sum_{j=1}^{N} x_j$, which evolves according to:

$$\dot{\bar{s}} = \frac{1}{N}\sum_{j=1}^{N} \dot{x}_j = \frac{1}{N}\sum_{j=1}^{N} \left(\tilde{f}_j\left(x_j, \tilde{\rho}_j\right) + \tilde{g}_j(X)\right) \tag{6}$$

For the uncertain network (1) the dynamics of average trajectory becomes

$$\dot{\bar{s}} = \frac{1}{N}\sum_{j=1}^{N} f(x_j, \rho) + \frac{1}{N}\sum_{j=1}^{N} g_j(X) + \frac{1}{N}\sum_{j=1}^{N} \Delta\left(\hat{f}_j(x_j, \hat{\rho}_j) + \hat{g}_j(X)\right) \quad (7)$$

which for a diffusive nominal coupling function and vanishing perturbations when $x_i = x_j\ \forall i, j$, becomes (4). Then, we can write:

$$\dot{\bar{s}} = f(s, \rho) + \frac{1}{N}\sum_{j=1}^{N} g_j(S) + \frac{1}{N}\sum_{j=1}^{N} \Delta\left(\hat{f}_j(x_j, \hat{\rho}_j) + \hat{g}_j(X)\right) \quad (8)$$

where $S = (x_1, ..., x_N) \in \mathbf{R}^{n\times N}$ when $x_i = x_j\ \forall i, j$.

Remark 2. Under potentially non vanishing perturbations the synchronized solutions (4) is no longer possible for network (1), however, is a reasonable to presume that under small perturbations and appropriate control action the nearly identical nodes evolve on a near-synchronous state [20] which is well approximated by their average trajectory. Furthermore, if the perturbations vanish at the synchronized solution, the average trajectory becomes the dynamics of an isolated node. Then, our average trajectory can be conceived as the synchronized solution of the nominal network plus the average effects of the perturbations.

From (1) and (6) the dynamical evolution of the synchronization error $(e_i = x_i - \bar{s})$ is found to be:

$$\dot{e}_i = \tilde{f}_i(x_i, \tilde{\rho}_i) + \tilde{g}_i(X) - \dot{\bar{s}} + u_i, \text{ for } i = 1, ..., N \quad (9)$$

Then, the synchronization of the uncertain network (1) becomes a control problem, where the objective is to design local controllers u_i, such that the error dynamics (9) be robustly stable about the zero equilibrium point.

3. Robust Synchronization Design

We start our design assuming that the nominal coupling functions, $g_i(X)$, are diffusive linear combinations of the network state variables, such that:

$$g_i(X) = c\sum_{j=1}^{N} a_{ij}\Gamma x_j \quad (10)$$

where the inner coupling matrix, $\Gamma \in \mathbf{R}^{n\times n}$, is a 0-1 matrix describing the manner in which the state variables are coupled when two nodes are connected. The network topology is described by the connectivity matrix, $\mathcal{A} = \{a_{ij}\} \in \mathbf{R}^{N\times N}$, which is a matrix constructed as follows: if there is a connection between the ith and jth nodes ($j \neq i$), the entries $a_{ij} = a_{ji} = 1$; otherwise $a_{ij} = a_{ji} = 0$, with the diagonal entries given by $a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}$. The variable $c \in \mathbf{R}$ is the coupling strength between the nodes, which is taken to be uniform for the entire network.

If the network is connected such that there are no isolated clusters, the eigenvalues of $\mathcal{A}$ are real, nonpositive, and can be ordered as follows [2]:

$$0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_N \tag{11}$$

Further, the connectivity matrix can be rewritten as $\mathcal{A} = \Omega^{-1}\Lambda\Omega$, where $\Lambda = \mathrm{Diag}(\lambda_1, ..., \lambda_N)$ and $\Omega = [\omega_1, ..., \omega_N] \in \mathbf{R}^{N\times N}$, with $\omega_i = [\omega_{i1}, ..., \omega_{iN}]^\top \in \mathbf{R}^N$ the eigenvectors of $\mathcal{A}$.

Notice that with $\mathcal{A}$ constructed as described above, the nominal coupling functions vanish when synchronization is achieved, i.e., $g_i(X) = 0$, when $x_i = x_j, \forall i, j$. Then, $g_j(S) = 0$, $\forall j$.

It follows that the synchronization error dynamics in (9) become

$$\dot{e}_i = \bar{f}_i(x_i, \tilde{\rho}_i) + \bar{g}_i(X) + c\sum_{j=1}^{N} a_{ij}\Gamma e_j + u_i, \text{ for } i = 1, ..., N \tag{12}$$

where $\bar{f}_i(x_i, \tilde{\rho}_i) = f(x_i, \rho) - f(s, \rho) + \Delta\left(\hat{f}_i(x_i, \hat{\rho}_i) - \frac{1}{N}\sum_{j=1}^{N} \hat{f}_j(x_j, \hat{\rho}_j)\right) \in \mathbf{R}^n$ and $\bar{g}_i(X) = \Delta\left(\hat{g}_i(X) - \frac{1}{N}\sum_{j=1}^{N} \hat{g}_j(X)\right) \in \mathbf{R}^n$.

In what follows, the local controllers u_i in (12) are designed such that synchronization, in the sense of (2), is robustly achieved for two distinct situation: In the first case, the perturbations due to uncertainties in the network are assume to be bounded in terms of the synchronization error, as such, they vanish as the network synchronizes. In the second case, we consider these perturbations to be bounded but not vanishing.

3.1. Under Vanishing Perturbations

In this subsection we assume that the uncertain parts of the dynamical network (1) satisfy the following bounds:

$$\|\sum_{m=1}^{N} \bar{f}_m(x_m, \tilde{\rho}_m)\,\omega_{im}\| \quad \leq \quad \alpha_i\|\sum_{m=1}^{N} e_m\omega_{im}\|, \text{ and} \tag{13}$$

$$\|\sum_{m=1}^{N}\bar{g}_m\left(X\right)\omega_{im}\| \quad \leq \quad \sum_{p=1}^{N}\beta_{ip}\|\sum_{m=1}^{N}e_m\omega_{pm}\| \tag{14}$$

for $i = 1, ..., N$, where $\alpha_i \geq 0 \in \mathbf{R}$ and $\beta_{ip} \geq 0 \in \mathbf{R}$ are nonnegative constants.

In this case we have the following result:

Theorem 1: Suppose that (13) and (14) hold. If the local controllers u_i are constructed as

$$u_i = -c\; k\; \Gamma\; e_i, \;\text{ for } i = 1, ..., N \tag{15}$$

with the controller gain $k > 0$, satisfying the bound

$$k > \frac{N(\alpha + \beta)}{c} - \delta \tag{16}$$

where $\delta = \min\{|\lambda_i|\}_{\lambda_i \neq 0}$, $\alpha = \max\{\alpha_i\}$ and $\beta = N\max\{\beta_{ji}\}$. Then, the uncertain dynamical network (1) robustly synchronizes, in the sense that the zero fixed point of (12) is robustly stable.

Proof. Defining the vector variables $\mathbf{e} = [e_1, ..., e_N] \in \mathbf{R}^{n\times N}$, $\bar{\mathbf{f}} = \left[\bar{f}_1\left(\cdot\right), ..., \;\; \bar{f}_N\left(\cdot\right)\right] \in \mathbf{R}^{n\times N}$ and $\bar{\mathbf{g}} = \left[\bar{g}_1\left(\cdot\right), ..., \bar{g}_N\left(\cdot\right)\right] \in \mathbf{R}^{n\times N}$ the error dynamics (12) are rewritten as: $\dot{\mathbf{e}} = \bar{\mathbf{f}}\left(X, \rho\right) + \bar{\mathbf{g}}(X) + c\Gamma\mathbf{e}\left(\mathcal{A} - K\right)$; where $\rho = [\rho_1, ..., \rho_N]$ and $K = \text{Diag}(k, ..., k) \in \mathbf{R}^{N\times N}$. In the transformed coordinates $\eta = \mathbf{e}\Omega$, the error dynamics become $\dot{\eta} = \bar{\mathbf{f}}\left(X, \rho\right)\Omega + \bar{\mathbf{g}}\left(X\right)\Omega + c\Gamma\eta\left(\Lambda - K\right)$; where $\eta = [\eta_1, ..., \eta_N] \in \mathbf{R}^{n\times N}$, with $\eta_i = \mathbf{e}\,\omega_i \in \mathbf{R}^n$; or equivalently,

$$\dot{\eta}_i = \bar{\mathbf{f}}\left(X, \rho\right)\omega_i + \bar{\mathbf{g}}\left(X\right)\omega_i + c\left(\lambda_i - k\right)\Gamma\eta_i, \;\text{ for } i = 1, ..., N \tag{17}$$

The stability of the error dynamics (12) around its zero equilibrium point is determine using the Lyapunov candidate function: $V = \frac{1}{2}\sum_{j=1}^{N}\eta_j^\top\eta_j$, the time derivative of V along the trajectories of (17) is given by

$$\begin{aligned}\dot{V} \quad = \quad & \sum_{j=1}^{N}\eta_j^\top c\left(\lambda_j - k\right)\Gamma\eta_j + \sum_{j=1}^{N}\eta_j^\top\left(\sum_{m=1}^{N}\bar{f}_m\left(x_m, \tilde{\rho}_m\right)\omega_{jm}\right)\\ & + \sum_{j=1}^{N}\eta_j^\top\left(\sum_{m=1}^{N}\bar{g}_m(X)\omega_{jm}\right)\end{aligned} \tag{18}$$

The first term on the righthand side of (18) is bounded as:

$$\sum_{j=1}^{N} \eta_j^\top c\,(\lambda_j - k)\,\Gamma\eta_j \leq \sum_{j=1}^{N} \eta_j^\top(-\gamma)\eta_j \leq -\gamma\sum_{j=1}^{N} \|\eta_j\|^2$$

where $-\gamma I_n \geq c\,(\lambda_j - k)\,\Gamma$ for every j, with $\gamma > 0$. Since the eigenvalues of $\mathcal{A}$ are all nonpositive the bounds on γ are $-c(|\lambda_j|+k)\Gamma$ with $\lambda_j \neq 0$. Letting $\delta = \min\{|\lambda_j|\}_{\lambda_j\neq 0}$ the bound becomes $\gamma \leq c\,(\delta + k)$. From (14) we have the second term in the righthand side of (18) bounded as:

$$\sum_{j=1}^{N} \eta_j^\top \sum_{m=1}^{N} \bar{f}_m\,(x_m, \tilde{\rho}_m)\,\omega_{jm} \leq \sum_{j=1}^{N} \eta_j^\top \alpha_j \|\sum_{m=1}^{N} e_m\omega_{jm}\| \leq \alpha\sum_{j=1}^{N} \|\eta_j\|^2$$

with $\alpha = \max\{\alpha_i\}$. From (15), it follows that the third term in the righthand side of (18) is bounded as:

$$\sum_{j=1}^{N} \eta_j^\top \sum_{m=1}^{N} \bar{g}_m\,(X)\,\omega_{jm} \leq \sum_{j=1}^{N} \eta_j^\top \sum_{i=1}^{N} \beta_{ji} \|\sum_{m=1}^{N} e_m\omega_{im}\| \leq \beta\sum_{j=1}^{N}\sum_{i=1}^{N} \|\eta_j\|\|\eta_i\|$$

with $\beta = N\max\{\beta_{ji}\}$. Then, $\dot{V}$ can be rewritten as a quadratic function of $\|\eta\| = [\|\eta_1\|, ..., \|\eta_N\|]^\top \in \mathbf{R}^N$ as:

$$\dot{V} \leq -|\eta|^\top Q|\eta|$$

where Q is a $N \times N$ matrix whose elements are given by

$$q_{ij} = \begin{cases} -\beta & \text{for } i \neq j \\ -(\alpha+\beta)+\gamma & \text{for } i = j \end{cases}$$

By choosing $\gamma > N(\alpha + \beta)$, the matrix Q is positive definite ($Q > 0$) which means that $\dot{V} < 0$. Then, the error dynamics in the transform coordinates (18) are globally uniformly asymptotically stable about the zero fixed point ($\eta = 0$), which implies that the uncertain dynamical network (1) under assumptions (14) and (15) with the controller (15), achieves robust synchronization. From the above conditions on γ, the relation in (16) is readily obtained from:

$$c(\delta + k) \geq \gamma > N(\alpha + \beta) \tag{19}$$

■

In a similar way, the following result is obtained directly from the previous theorem.

Corollary 2. For the uncertain dynamical network (1) with no controllers ($u_i = 0$, for all i), assuming that the conditions on (14) and (15) hold. If the coupling strength satisfies the following criterion

$$c > \frac{N(\alpha + \beta)}{\delta} \tag{20}$$

where $\delta = \min\{|\lambda_i|\}_{\lambda_i \neq 0}$, $\alpha = \max\{\alpha_i\}$ and $\beta = N \max\{\beta_{ji}\}$. Then, the uncertain dynamical network robustly synchronizes to the solution $\bar{s}(t)$ described in (6), in the sense of (2).

Proof. From (20), when the controller is removed ($k = 0$), the criterion for robust synchronization in (19) is obtained following a similar procedure as in the proof of Theorem 1. ■

3.2. Under Non-Vanishing Perturbations

In the case where the perturbations in the network do not vanish at the synchronized state, the bounds on the uncertain parts of the network can not be expressed in terms of the synchronization error as in (14) and (15). Instead, we assume that the perturbations are bounded as follows:

$$\|\sum_{m=1}^{N} \bar{f}_m\left(x_m, \tilde{\rho}_m\right)\omega_{im}\| \leq a_i, \text{ and} \tag{21}$$

$$\|\sum_{m=1}^{N} \bar{g}_m\left(X\right)\omega_{im}\| \leq b_i \tag{22}$$

for $i = 1, ..., N$, where $a_i > 0 \in \mathbf{R}$ and $b_i > 0 \in \mathbf{R}$ are small positive constants.

To robustly achieve synchronization under these non-vanishing perturbations, we propose the use of discontinuous local controllers as described in the following result.

Theorem 3. Suppose that (21) and (22) hold. If the local controllers are constructed as:

$$u_i = -c\, k\, \Gamma\, e_i - \mu\; sgn(e\Omega)\omega_i^*, \text{ for } i = 1, ..., N \tag{23}$$

where $e = [e_1, ..., e_N] \in \mathbf{R}^{n\times N}$, the matrix $\Omega \in \mathbf{R}^{N\times N}$ is such that the connectivity matrix can be rewritten as $\mathcal{A} = \Omega^{-1}\Lambda\Omega$, with $\Lambda = \mathrm{Diag}(\lambda_1, ..., \lambda_N)$; and $\omega_i^* \in \mathbf{R}^N$ is the ith column of the matrix Ω^{-1}. $sgn(\cdot)$ represents the conventional sign function, $sgn(\epsilon) = \{1,\ \text{for } \epsilon > 0; 0,\ \text{for } \epsilon = 0; -1,\ \text{for } \epsilon < 0\}$, with $sgn(e\Omega) = sgn(\eta) = [sgn(\eta_1), ..., sgn(\eta_N)] \in \mathbf{R}^{n\times N}$, with $sgn(\eta_i) = [sgn(\eta_{i1}), ..., sgn(\eta_{in})]^\top \in \mathbf{R}^n$. Furthermore, if the smooth $k > 0$ and discontinuous $\mu > 0$ controller gains are designed such that

$$\begin{aligned} k &> \frac{\gamma}{c} - \delta, \text{ and} && (24)\\ \mu &> a + b && (25) \end{aligned}$$

where $\gamma > 0$, $a = \max\{a_i\}$, and $b = \max\{b_i\}$. Then, the uncertain dynamical network (1), robustly synchronizes to the solution $\bar{s}$ described in (6).

Proof. In terms of the vector variables described above, the error dynamics in (12) under control (23) can be rewritten as: $\dot{\mathbf{e}} = \bar{\mathbf{f}}(X, \rho) + \bar{\mathbf{g}}(X) + c\Gamma\mathbf{e}(\mathcal{A} - K) - \mu\ sgn(\eta)\Omega^{-1}$. In the transform coordinates $\eta = \mathbf{e}\Omega$, the error dynamics become $\dot{\eta} = (\bar{\mathbf{f}}(X, \rho) + \bar{\mathbf{g}}(X))\Omega - \mu\ sgn(\eta) + c\Gamma\eta(\Lambda - K)$, or equivalently:

$$\dot{\eta}_i = (\bar{\mathbf{f}}(X, \rho) + \bar{\mathbf{g}}(X))\omega_i - \mu\ sgn(\eta_i) + c(\lambda_i - k)\Gamma\eta_i,\ \text{for } i = 1, ..., N \quad (26)$$

The time derivative of the Lyapunov candidate function, $V = \frac{1}{2}\sum_{j=1}^{N} \eta_j^\top \eta_j$, along the trajectories of (26) is found to be

$$\begin{aligned} \dot{V} &= \sum_{j=1}^{N} \eta_j^\top c\left(\lambda_j - k\right)\Gamma\eta_j - \mu\sum_{j=1}^{N} \eta_j^\top sgn(\eta_i) \\ &\quad + \sum_{j=1}^{N} \eta_j^\top \left(\sum_{m=1}^{N} \bar{f}_m\left(x_m, \tilde{\rho}_m\right) + \bar{g}_m(X)\right)\omega_{jm} \\ &\leq -\gamma\sum_{j=1}^{N} \|\eta_j\|^2 + (a + b - \mu)\sum_{j=1}^{N} \|\eta_j\| \end{aligned}$$

where $a = \max\{a_i\}$, $b = \max\{b_i\}$, and as before $-\gamma \leq c(\delta + k)$ with $\delta = \min\{|\lambda_j|\}_{\lambda_j = 0}$. Then, $\dot{V} < 0$ if $\mu > a + b$ and $\gamma > 0$, from which conditions (24) and (25) follow directly. ■

4. Numerical Simulations

Case 1 *Vanishing perturbations*:
Consider a dynamical network where each node is a chaotic Chen system given by [23]:

$$\begin{aligned} \dot{x}_1 &= p_1(x_2 - x_1) \\ \dot{x}_2 &= (p_3 - p_1)x_1 - x_1x_3 + p_3x_2 \\ \dot{x}_3 &= x_1x_2 - p_2x_3 \end{aligned} \tag{27}$$

with the nominal parameters $p_1 = 35$, $p_2 = 3$, and $p_3 = 28$. Using (27) as nominal nodes, the uncertain dynamical network (1) becomes:

$$\begin{bmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{bmatrix} = \begin{bmatrix} \tilde{p}_1(x_{i2} - x_{i1}) \\ (\tilde{p}_3 - \tilde{p}_1)x_{i1} - x_{i1}x_{i3} + \tilde{p}_3x_{i2} \\ x_{i1}x_{i2} - \tilde{p}_2x_{i3} \end{bmatrix} + \tilde{g}_i(X) + u_i, \text{ for } i = 1, ..., N \tag{28}$$

where the uncertain parameters are $\tilde{p}_i = (1 + \Delta 0.1)p_i$ $(i = 1, 2, 3)$, and the uncertain component of each node is given by

$$\hat{f}_i(x_i, \hat{\rho}_i) = \begin{bmatrix} \hat{p}_1(x_{i2} - x_{i1}) \\ (\hat{p}_3 - \hat{p}_1)x_{i1} - x_{i1}x_{i3} + \hat{p}_3x_{i2} \\ x_{i1}x_{i2} - \hat{p}_2x_{i3} \end{bmatrix} \tag{29}$$

The uncertain coupling functions are $\tilde{g}_i(X) = (1 + \Delta 0.1)c\sum_{j=1}^{N} a_{ij}\Gamma x_j$, that is, the nominal coupling function is $g_i(X) = c\sum_{j=1}^{N} a_{ij}\Gamma x_j$ with $c = 1$ a nominal uniform coupling strength, and $\mathcal{A} = \{a_{ij}\}$ a 0-1 matrix satisfying the diffusive conditions such that its eigenvalue spectrum can be ordered as in (11). While the uncertain component of the coupling function for each node is:

$$\hat{g}_i(X) = \hat{c}\sum_{j=1}^{N} a_{ij}\Gamma x_j \tag{30}$$

For simplicity, the internal coupling matrix is taken to be the identity ($\Gamma = I_3$), and the connectivity matrix $\mathcal{A}$ is constructed following the scale-free network model algorithm described in [6] for $N = 50$. Under these conditions the uncertain components of the network (28), satisfies the bounds in (14) and (15), as both vanish at the synchronized state. In order to satisfy the conditions of Theorem 1, the feedback controller gain is $k = 50$, the synchronizing controller is activated at $t = 4$. As shown in Figure 1, robust synchronization is achieved shortly after the controller is activated.

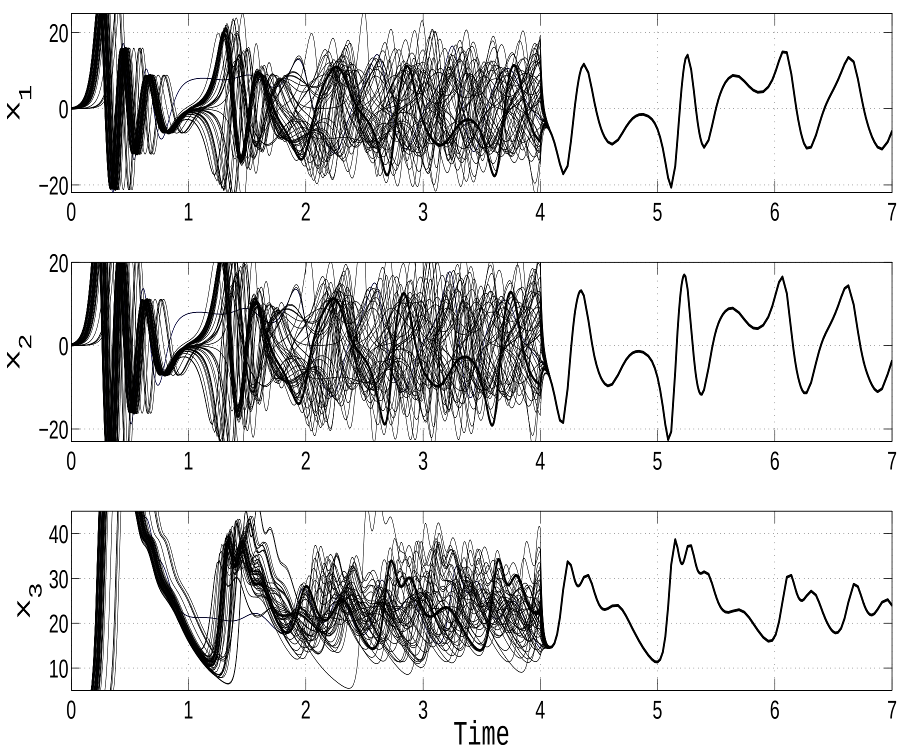

Figure 1: Synchronization on a scale-free network of fifty Chen's systems under vanishing perturbations using the controllers described in Theorem 1.

Case 2 *Non-Vanishing perturbations*: Next, we consider that each node is a chaotic Chua circuit:

$$\begin{array}{rcl} \dot{x}_1 &=& q_1(x_2 - x_1 - h(x_1)) \\ \dot{x}_2 &=& x_1 - x_2 + x_3 \\ \dot{x}_3 &=& -q_2 x_2 \end{array} \quad (31)$$

where $h(x_1) = m_o x_1 + (\frac{1}{2})(m_1 - m_o)(|x_1 + 1| - |x_1 - 1|)$ and the nominal parameters are $q_1 = 9$, $q_2 = 100/7$, $m_o = -5/7$, $m_1 = -8/7$. As such, the uncertain dynamical network (1) becomes:

$$\begin{bmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{bmatrix} = \begin{bmatrix} \tilde{q}_1(x_{i2} - x_{i1} - h(x_{i1})) + \Delta\hat{d}_1 \\ x_{i1} - x_{i2} + x_{i3} + \Delta\hat{d}_2 \\ -\tilde{q}_2 x_{i2} + \Delta\hat{d}_3 \end{bmatrix} + \tilde{g}_i(X) + u_i, \quad (32)$$

for $i = 1, ..., N$. The uncertain component of each node are given by

$$\hat{f}_i(x_i, \hat{\rho}_i) = \begin{bmatrix} \hat{q}_1(x_{i2} - x_{i1} - h(x_{i1})) + \hat{d}_1 \\ \hat{d}_2 \\ -\hat{q}_2 x_{i2} + \hat{d}_3 \end{bmatrix} \quad (33)$$

where the uncertain parameters are $\tilde{q}_i = (1 + \Delta 0.1)q_i$ $(i = 1, 2)$ and $\hat{d}_j = 0.2$ $(j = 1, 2, 3)$ are constant value perturbations affecting the uncertain nodes. The uncertain coupling functions are given by

$$\tilde{g}_i(X) = (1 + \Delta 0.1)c \sum_{j=1}^{N} a_{ij}\Gamma x_j + \Delta\hat{a}_{ii}\Gamma x_i \quad (34)$$

where $c = 1$, $\Gamma = I_3$, and $\mathcal{A}$ is a 0-1 diffusive matrix constructed with the scale-free network model algorithm of [5] for fifty nodes as in the previous case. The uncertain component of the coupling functions has the additional perturbation $\hat{a}_{ii} = 0.05$ $(\forall i)$. Under these perturbations, the uncertain parts of the uncertain network (33) do not vanish at the synchronized solution. However, since the the perturbations remain bounded, the bounds in (21) and (22) are satisfied. Then, a discontinuous local controller (23) can be designed to robustly synchronize the uncertain network to the average trajectory of the network. The synchronizing controller is designed with smooth and discontinuous gains set to $k = 10$ and $\mu = 5$, such that the conditions in (24) and (25) of Theorem 3 are satisfied. Numerical simulations are carried out, with the control action applied at $t = 40$. As shown in Figure 2, robust synchronization is achieved even in the present of these non-vanishing perturbation.

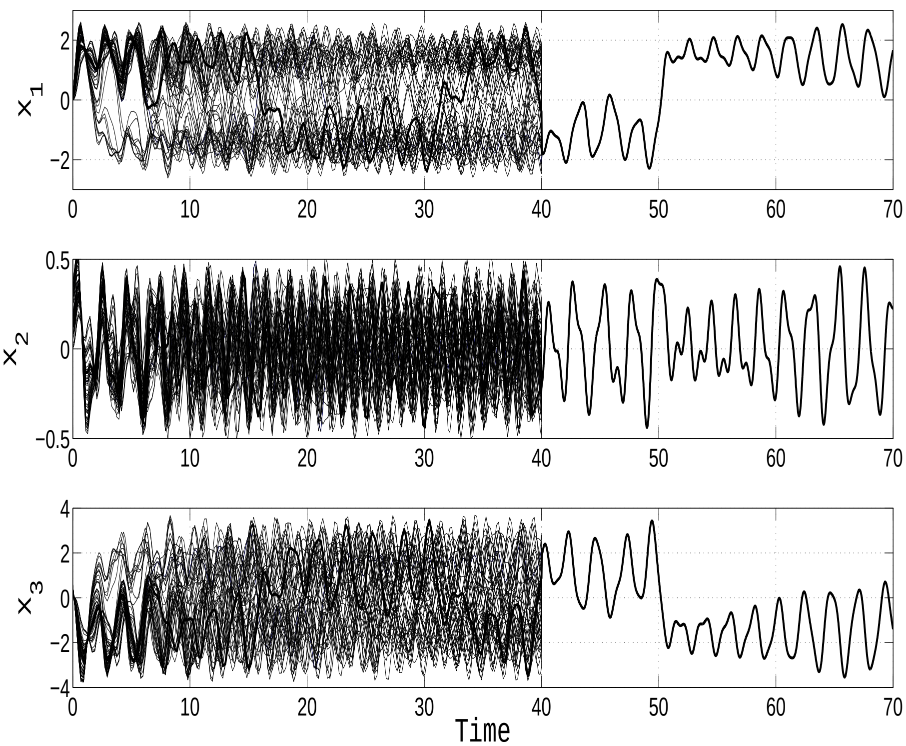

Figure 2: Synchronization on a scale-free network of fifty Chua's circuits under non-vanishing perturbations using the controllers described in Theorem 3.

5. Conclusion

This paper considers the problem of synchronization in uncertain dynamical networks with uncertainties on both their node descriptions and their interconnections. Two cases are presented, in the first uncertainties vanish at the synchronized solution, in the second perturbations are bounded but not vanishing. For the first case, its shown that local smooth feedback controllers can synchronize to the dynamical evolution of an isolated nominal node. In the second case, a discontinuous feedback controller is proposed to robustly synchronize the network even under the effect of non vanishing perturbations to the average trajectory of the network. Some new conditions are derive to ensure that the states of the uncertain dynamical network asymptotically synchronized. These conditions relate the stability of synchrony with the dynamics of the nominal node, bounds on the uncertainties, and topological features of the network. Finally, Chen system and Chua circuit are use to show the effectiveness of the proposed designs.

Acknowledgments

This research was supported by the National Council for Science and Technology of México CONACyT under Grant 106915-Y.

[1] Boccaletti, S., Latora, V., Moreno, Y., Chavez, M., Hwang, D. U. (2006). Complex Networks: Structure and Dynamics. *Phys Rep,* **424**, 175-308.

[2] Wu, C. W. (2007). *Synchronization in complex networks of nonlinear dynamical systems*, *World Scientific*, Singapure.

[3] Arenas, A., Díaz-Guilera, A., Kurths, J., Moreno, Y., Zhoug, C. (2008) Synchronization in complex networks, *Physics Reports*, 469, 93-153.

[4] Barahona, M., Pecora, L. M. (2002). Synchronization in small-world systems. *Phys Rev E,* Vol. **89**(5), 054101.

[5] Wang, X. F., Chen, G. (2002). Synchronization in scale-free dynamical networks: robust and fragility. *IEEE Trans. Circuits and Systems -I: Fundamental Theory and Applications*, 49, 54-62.

[6] Wang, X. F., Chen, G. (2003). Complex Networks: Small-world, scale-free and beyond. *IEEE Control Syst Maggazine,* 6-20.

[7] Dorogovtsev, S. N., Mendes J. F. F. (2003) *Evolution of Networks: From Biological Nets to the Internet and WWW*, Oxford University Press, New York USA.

[8] Barrat, A., Barthélemy, M., Vespignani, A. (2008) *Dynamical Processes on Complex Networks*. Cambridge University Press, Cambridge, UK.

[9] Newman, M. E. J. (2010) *Networks: An Introduction*. Oxford University Press, New York, USA.

[10] Pecora, L. M., Carroll, T. L. (1998). Master stability functions for synchronization coupled systems. *Phsical Review Letters*, 80, 2109-2112.

[11] Pecora, L., Carroll, T., Johnson, G., Mar, D., Fink, K. S. (2000). Synchronization stability in coupled oscillator arrays: solution for arbitrary configurations, Int. J. Bifurcation and Chaos, 10(2), pp. 277-290.

[12] Li, X. (2005). Sync in Complex Networks: Stability, Evolution, Control and Applications, Int. J. Computational Cognition, 3(4), 16-26.

[13] Lü, J. H., Yu, X. H., Chen, G. (2004). Chaos synchronization of general complex dynamical networks. *Physica A*, 334, 281-302.

[14] Li, Z., Chen, G. 2004. Robust adaptive synchronization of uncertain dynamical networks, Phys. Lett. A, 324, 166 - 178.

[15] Li, Z., Chen, G. 2004. Design of coupling functions for global synchronization of uncertain chaotic dynamical networks, Phys. Lett. A, 326, 333 - 339.

[16] Zhou, J., Lu, J., Lü, J. 2006. Adaptive synchronization of an uncertain complex dynamical network, IEEE Trans Automatic Control, 51(4), 652-656.

[17] Chen, M., Zhou, D. 2006. Synchronization in uncertain complex networks, Chaos, 16, 013101.

[18] Lu, J., Cao, J. 2008. Adaptive synchronization of uncertain dynamical networks with delayed coupling, Nonlinear Dyn, 53, 107-115.

[19] Restrepo, J. Ott, E., Hunt, B. 2004. Spatial patterns of desynchronization bursts in networks, Phys. Rev. E, 69, 066215.

[20] Sun, J., Bollt, E. M., Nishikawa, T. 2009. Master stabiliy functions for coupled near-identical dynamical systems, EPL, 85, 60011.

[21] Sorrentino, F., Ott, E. 2008. adaptive synchronization of dynamics on evolving complex networks, PRL, 100, 114101.

[22] Sorrentino, F., Barlev, G., Cohen, A. B., Ott, E. 2010. The stability of adaptive synchronization of chaotic systems, Chaos, 20, 013103.

[23] Chen, G., Ueta, T. 1999. Yet another chaotic attractor, Int. J. Bifurcation and Chaos, **9**(7), 1465–1466.