Output-Feedback Global Continuous Finite-Time and Exponential Stabilization of Bounded-Input Mechanical Systems

Griselda I. Zamora-Gómez, Arturo Zavala-Río and Daniela J. López-Araujo

***Abstract*— This work proposes an output-feedback global continuous controller for the finite-time or (local) exponential stabilization of bounded-input mechanical systems. The control design is carried out within the analytical framework of local homogeneity. The proposed approach avoids velocity measurements on the feedback through the use of a dynamic dissipator expressed in a generalized non-linear form. The analytical results are supported through simulation tests. In particular, such implementations show the performance differences obtained through diverse particular dynamic dissipator structures, as well as the contrast among the finite-time and exponential convergence achievable through the proposed scheme.**

I. INTRODUCTION

Output-feedback control is often implemented involving observers. These techniques imply the exact knowledge of the system structure and parameters. Such a model dependence has been proven to be alleviated in the case of mechanical systems with unavailable velocity measurements, by involving the *dirty derivative* instead [1]. However, it is not yet clear how can such a simple dynamic dissipation technique be adapted to achieve global finite-time stabilization through a continuous control law. Such a design challenge becomes more complex under the consideration of bounded inputs. This additional realistic consideration complicates the use of traditional analytical tools that are often involved to characterize finite-time stability of continuous vector fields, such as homogeneity [2]. However, such analytical restriction can be relaxed through alternative related notions [3].

A debuting study on continuous finite-time control addressed to robotic manipulators was introduced in [4] disregarding input constraints. The proposed state-feedback controller included Proportional (P) and Derivative (D) type terms. The design was built upon the traditional framework of homogeneity.

Another work treating the finite-time control of mechanical manipulators, assuming unconstrained inputs, appeared later in [5]. The state-feedback scheme proposed therein was designed so as to compensate for the nominal dynamics of the entire system. The rest of the design is carried out employing *backstepping*. The synthesis is then completed using a *Lyapunov-redesign* type procedure to cope with system uncertainties, which *a priori* renders discontinuous the resulting controller. Alternative approximations of certain control terms were suggested so as to avoid discontinuities and singularities implied by the developed approach.

A different continuous control strategy for the finite-time stabilization of mechanical systems was more recently presented in [6] similarly disregarding input constraints. The proposed state-feedback scheme is based on the definition of a (positively invariant) manifold where the system converges to the zero (desired) state in a finite time T_1. An appropriate closed loop form guaranteeing convergence of the system variables to such a manifold in a finite time T_2 is then obtained. The controller is then synthesized through *exact dynamic compensation* so as to impose the closed-loop form found in the last step.

Lately, a state-feedback continuous controller for the global stabilization with finite-time or (local) exponential convergence of constrained-input mechanical systems was proposed in [7]. It has a generalized saturating PD-type structure involving compensation of the natural conservative terms only.

From the above-cited state-feedback approaches, only that in [4] formulates an output-feedback extension of the proposed controller. It is a finite-time-observer-based controller that achieves stabilization only locally. The considered observer involves the whole system dynamics (and parameters), and reconstructs the whole set of position and velocity variables. Although a bounded variation of such an observer-based approach, with the conventional saturation function involved in the P and D type actions, was further contemplated, no formal closed-loop analysis was presented for this case, which does not fit within the analytical framework where the proposed unconstrained schemes were developed.

Thus, in this work, we propose an output-feedback global continuous control scheme for mechanical systems with bounded inputs, giving rise to finite-time or (local) exponential stabilization. The choice on the type of convergence is fixed *via* a simple control parameter. Moreover, the formulated problem is solved avoiding the use of observers but rather ensuring motion dissipation dynamically from the exclusive feedback of the position variables. This has been made possible through a *dirty-derivative*-based nonlinear dynamic dissipator presented in a generalized form. The proposed controller keeps an SP-SD structure that does not need to compensate for the system dynamic terms other than the natural conservative force vector. A simulation section showing the efficiency of the proposed scheme is included.

Griselda I. Zamora-Gómez, Arturo Zavala-Río and Daniela J. López-Araujo are with the Department of Applied Mathematics, Instituto Potosino de Investigación Científica y Tecnológica, San Luis Potosí, SLP 78216 Mexico {griselda.zamora / azavala}@ipicyt.edu.mx, daniela.lopez.araujo@gmail.com

II. Preliminaries

Let X and y be an $m \times n$ matrix and an n-dimensional vector, respectively. We denote X_{ij} the entry of X at its i^{th} row and j^{th} column, X_i the i^{th} row of X, and y_i the i^{th} element of y. 0_n stands for the origin of $\mathbb{R}^n$. $\mathbb{R}^n_{>0}$ denotes the set of n-tuples with positive entries. $\|\cdot\|$ stands for the standard Euclidean norm for vectors and induced norm for matrices. Let $S^{n-1}_c = \{x \in \mathbb{R}^n : \|x\| = c\}$: an $(n-1)$-dimensional sphere of radius $c > 0$ on $\mathbb{R}^n$. We will consider the sign function as $\text{sign}(\varsigma) = \begin{cases} \varsigma/|\varsigma| & \text{if } \varsigma \neq 0 \\ 0 & \text{if } \varsigma = 0 \end{cases}$, and the standard saturation function as $\text{sat}(\varsigma) = \text{sign}(\varsigma)\min\{|\varsigma|, 1\}$. The contents of the following subsections were mostly included in [7]; for the sake of completeness and ease of reading, they are reproduced here.

A. Mechanical systems

The n-degree-of-freedom (DOF) fully-actuated mechanical system dynamics is given by [8, §6.1]

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \tag{1}$$

where $q, \dot{q}, \ddot{q} \in \mathbb{R}^n$ are the position (generalized coordinates), velocity, and acceleration vectors; the inertia matrix $H(q) \in \mathbb{R}^{n\times n}$ is a continuously differentiable positive definite symmetric matrix function; the Coriolis (and centrifugal effect) matrix $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$ satisfies [8, §6.1.2] [9, §2.3]

$$\dot{q}^T \left[\frac{1}{2}\dot{H}(q,\dot{q}) - C(q,\dot{q})\right]\dot{q} = 0 \tag{2}$$

$\forall (q,\dot{q}) \in \mathbb{R}^n \times \mathbb{R}^n$, where $\dot{H}$ denotes the rate of change of H, *i.e.* $\dot{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ with $\dot{H}_{ij}(q,\dot{q}) = \frac{\partial H_{ij}}{\partial q}(q)\dot{q}$, $i,j = 1,\ldots,n$, and $C(x,y)z = C(x,z)y$, $\forall x,y,z \in \mathbb{R}^n$, whence we have that

$$C(q,a\dot{q})b\dot{q} = C(q,ab\dot{q})\dot{q} = C(q,b\dot{q})a\dot{q} = C(q,\dot{q})ab\dot{q} \tag{3}$$

$\forall q,\dot{q} \in \mathbb{R}^n$, $\forall a,b \in \mathbb{R}$; $g(q) = \nabla\mathcal{U}(q)$ with $\mathcal{U} : \mathbb{R}^n \to \mathbb{R}$ being the potential energy function of the open-loop system; and $\tau \in \mathbb{R}^n$ is the external input (generalized) force vector.

We consider the bounded input case, where each input τ_i is constrained by a saturation bound $T_i > 0$. More precisely, letting u_i represent the control variable (controller output) relative to the i^{th} degree of freedom, we have that

$$\tau_i = T_i \text{sat}(u_i/T_i) \tag{4}$$

Assumption 1: The conservative (generalized) force vector $g(q)$ is bounded, or equivalently: $|g_i(q)| \leq B_{gi}$, $i = 1,\ldots,n$, $\forall q \in \mathbb{R}^n$, for some positive constants B_{gi}.

Assumption 2: $T_i > B_{gi}$, $\forall i \in \{1,\ldots,n\}$.

B. Local homogeneity, finite-time / δ-exponential stability

Fundamental in this study is the notion of *family of dilations* δ^r_ε, defined as $\delta^r_\varepsilon(x) = \left(\varepsilon^{r_1}x_1,\ldots,\varepsilon^{r_n}x_n\right)^T$, $\forall x \in \mathbb{R}^n$, $\forall \varepsilon > 0$, with $r = (r_1,\ldots,r_n)^T$, where the *dilation coefficients* $r_1,\ldots,r_n$ are positive scalars.

Definition 1: [3] A function $V : \mathbb{R}^n \to \mathbb{R}$, resp. vector field $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ (with $f_i : \mathbb{R}^n \to \mathbb{R}$), is *locally homogeneous of degree* α with respect to the family of dilations δ^r_ε —or equivalently, it is said to be *locally* r*-homogeneous of degree* α— if there is an open neighborhood of the origin $D \subset \mathbb{R}^n$ —referred to as the *domain of homogeneity*— such that, for every $x \in D$ and all $\varepsilon \in (0,1]$: $\delta^r_\varepsilon(x) \in D$ and $V(\delta^r_\varepsilon(x)) = \varepsilon^\alpha V(x)$, resp. $f_i(\delta^r_\varepsilon(x)) = \varepsilon^{\alpha+r_i} f_i(x)$, $i = 1,\ldots,n$.

Subsequently, an r-homogeneous norm [10] [11] —denoted $\|\cdot\|_r$— will conventionally be considered to refer to an r-homogeneous p-norm, *i.e.* $\|x\|_r = \left[\sum_{i=1}^n |x_i|^{p/r_i}\right]^{1/p}$, with $p > \max_i\{r_i\}$. An r*-homogeneous* $(n-1)$*-sphere* of radius $c > 0$ is the set $S^{n-1}_{r,c} = \{x \in \mathbb{R}^n : \|x\|_r = c\}$.

Consider an n-th order autonomous system

$$\dot{x} = f(x) \tag{5}$$

where the vector field f is continuous on an open neighborhood of the origin $\mathcal{D} \subset \mathbb{R}^n$ and $f(0_n) = 0_n$, and let $x(t;x_0)$ represent the system solution with initial condition $x(0;x_0) = x_0$. A fundamental concept underlying this work is that of a (globally) *finite-time stable* equilibrium, as defined in [2].

Remark 1: The origin is a globally finite-time stable equilibrium of system (5) if and only if it is globally asymptotically stable and finite-time stable.

Theorem 1: [3] Consider system (5) with $\mathcal{D} = \mathbb{R}^n$. Suppose f is a locally r-homogeneous vector field of degree α with domain of homogeneity $D \subset \mathbb{R}^n$. Then, the origin is a globally finite-time stable equilibrium of system (5) if and only if it is globally asymptotically stable and $\alpha < 0$.

The following definition is stated under the additional consideration that, for some $r \in \mathbb{R}^n_{>0}$, f in (5) is locally r-homogeneous with domain of homogeneity $D \subset \mathcal{D}$.

Definition 2: [10], [11] The equilibrium point $x = 0_n$ of (5) is δ*-exponentially stable* with respect to the homogeneous norm $\|\cdot\|_r$ if there exist a neighborhood of the origin, $\mathcal{V} \subset D$, and constants $a \geq 1$ and $b > 0$ such that $\|x(t;x_0)\|_r \leq a\|x_0\|_r e^{-bt}$, $\forall t \geq 0$, $\forall x_0 \in \mathcal{V}$.

Remark 2: Definition 2 becomes the usual definition of exponential stability when $r_i = 1$, $i = 1,\ldots,n$.

Lemma 1: [7] Assume that f in (5) is a locally r-homogeneous vector field of degree $\alpha = 0$ with domain of homogeneity $D \subset \mathcal{D}$. Then, the origin is a δ-exponentially stable equilibrium if and only if it is asymptotically stable.

Remark 3: If a vector field f is locally r-homogeneous of degree $\alpha = 0$ with dilation coefficients $r_i = r_0$, $i = 1,\ldots,n$, for some $r_0 > 0$, then f is locally r^*-homogeneous of degree $\alpha = 0$ with dilation coefficients $r^*_i = r^*_0$, $i = 1,\ldots,n$, for any $r^*_0 > 0$ [7, Remark 2.5]. Hence, if f in (5) is locally r-homogeneous of degree $\alpha = 0$ with dilation coefficients $r_i = r_0$, $i = 1,\ldots,n$, for some $r_0 > 0$, then (keeping Remark 2 in mind) the origin turns out to be exponentially stable if and only if it is δ-exponentially stable.

Consider an n-th order autonomous system of the form

$$\dot{x} = f(x) + \hat{f}(x) \tag{6}$$

where f and $\hat{f}$ are continuous vector fields such that $f(0_n) = \hat{f}(0_n) = 0_n$.

Lemma 2: [7] Assume that, for some $r \in \mathbb{R}^n_{>0}$, f in (6) is a locally r-homogeneous vector field of degree $\alpha < 0$, resp. $\alpha = 0$, with domain of homogeneity $D \subset \mathbb{R}^n$, and that 0_n is a globally asymptotically, resp. δ-exponentially, stable equilibrium of $\dot{x} = f(x)$. Then, the origin is a finite-time, resp. δ-exponentially, stable equilibrium of system (6) if

$$\lim_{\varepsilon \to 0^+} \frac{\hat{f}_i(\delta^r_\varepsilon(x))}{\varepsilon^{\alpha + r_i}} = 0$$

$i = 1, \ldots, n$, $\forall x \in S_c^{n-1}$, resp. $\forall x \in S_{r,c}^{n-1}$, for some $c > 0$ such that $S_c^{n-1} \subset D$, resp. $S_{r,c}^{n-1} \subset D$.

Remark 4: The condition required by Lemma 2 may be equivalently verified through the fulfilment of

$$\lim_{\varepsilon \to 0^+} \left\| \varepsilon^{-\alpha} \delta_\varepsilon^{-r} \big(\hat{f}(\delta^r_\varepsilon(x)) \big) \right\| = 0$$

$\forall x \in S_c^{n-1}$ (resp. $S_{r,c}^{n-1}$).

C. Scalar functions with particular properties

Definition 3: A continuous scalar function $\sigma : \mathbb{R} \to \mathbb{R}$ will be said to be:

1) *bounded* (by M) if $|\sigma(\varsigma)| \leq M$, $\forall \varsigma \in \mathbb{R}$, for some positive constant M;
2) *strictly passive* if $\varsigma\sigma(\varsigma) > 0$, $\forall \varsigma \neq 0$;
3) *strongly passive* if it is a strictly passive function satisfying $|\sigma(\varsigma)| \geq \kappa \big|a \operatorname{sat}(\varsigma/a)\big|^b = \kappa\big(\min\{|\varsigma|, a\}\big)^b$, $\forall \varsigma \in \mathbb{R}$, for some positive constants κ, a and b.

Remark 5: Equivalent characterizations of strictly passive functions are: $\varsigma\sigma(\varsigma) > 0 \iff \operatorname{sign}(\varsigma)\sigma(\varsigma) > 0 \iff \operatorname{sign}\big(\sigma(\varsigma)\big) = \operatorname{sign}(\varsigma)$, $\forall \varsigma \neq 0$.

Lemma 3: [7] Letting $\sigma : \mathbb{R} \to \mathbb{R}$, $\sigma_0 : \mathbb{R} \to \mathbb{R}$, $\sigma_1 : \mathbb{R} \to \mathbb{R}$ be strongly passive functions and k a positive constant:

1) $\int_0^\varsigma \sigma(k\nu)d\nu > 0$, $\forall \varsigma \neq 0$;
2) $\int_0^\varsigma \sigma(k\nu)d\nu \to \infty$ as $|\varsigma| \to \infty$;
3) $\sigma_0 \circ \sigma_1$ is strongly passive.

Lemma 4: [7] Let $\sigma_0 : \mathbb{R} \to \mathbb{R}$ be strictly increasing, $\sigma_1 : \mathbb{R} \to \mathbb{R}$ be strictly passive, and k be a positive constant. Then: $\varsigma_1 \big[\sigma_0\big(\sigma_1(k\varsigma_1) + \varsigma_2\big) - \sigma_0(\varsigma_2)\big] > 0$, $\forall \varsigma_1 \neq 0$, $\forall \varsigma_2 \in \mathbb{R}$, or equivalently $\operatorname{sign}\big(\sigma_0\big(\sigma_1(k\varsigma_1) + \varsigma_2\big) - \sigma_0(\varsigma_2)\big) = \operatorname{sign}(\varsigma_1)$.

As a (simple) illustrative example, let $\sigma(\varsigma) = \operatorname{sign}(\varsigma)|\varsigma|^\beta$, $\sigma_i(\varsigma) = \operatorname{sign}(\varsigma)|\varsigma|^{\beta_i}$, $i = 0, 1$, for positive constants β, β_i, $i = 0, 1$, and k be a positive constant. We have $\int_0^\varsigma \sigma(k\nu)d\nu = k^\beta |\varsigma|^{\beta+1}/(\beta + 1)$, which is clearly positive definite and radially unbounded; $\sigma_0\big(\sigma_1(\varsigma)\big) = \operatorname{sign}(\varsigma)|\varsigma|^{\beta_0\beta_1}$, whence we have that $\varsigma\sigma(\varsigma) = |\varsigma|^{\beta_0\beta_1+1} > 0$, $\forall \varsigma \neq 0$, and $\big|\sigma_0\big(\sigma_1(\varsigma)\big)\big| = |\varsigma|^{\beta_0\beta_1} \geq \kappa\big(\min\{|\varsigma|, a_0\}\big)^{\beta_0\beta_1} \geq \kappa\big(\min\{|\varsigma|, a_1\}\big)^b$ for any positive constants $\kappa \leq 1$, $a_0 \geq 1$, $a_1 \leq 1$ and $b \geq \beta_0\beta_1$. The proof of Lemma 4 is developed in [7].

III. The proposed output-feedback scheme

We define the following SP-SD type control law

$$u(q, \vartheta) = -s_1(K_1\bar{q}) - s_2(K_2\vartheta) + g(q) \tag{7}$$

where $\bar{q} = q - q_d$, for any constant (desired equilibrium configuration) $q_d \in \mathbb{R}^n$; $\vartheta \in \mathbb{R}^n$ is the output vector variable of an auxiliary subsystem defined as

$$\dot{\vartheta}_c = -As_d(\vartheta, \bar{q}) \quad , \quad \vartheta = \vartheta_c + B\bar{q} \tag{8}$$

$K_i = \operatorname{diag}[k_{i1}, \ldots, k_{in}]$, $i = 1, 2$, $A = \operatorname{diag}[a_1, \ldots, a_n]$ and $B = \operatorname{diag}[b_1, \ldots, b_n]$, with $k_{ij} > 0$, $a_j > 0$ and $b_j > 0$, $\forall j = 1, \ldots, n$; for any $x \in \mathbb{R}^n$, $s_i(x) = \big(\sigma_{i1}(x_1), \ldots, \sigma_{in}(x_n)\big)^T$, $i = 1, 2$, with, for each $j \in \{1, \ldots, n\}$, σ_{ij} being strongly passive functions, locally Lipschitz-continuous on $\mathbb{R} \setminus \{0\}$, and such that

$$B_j \triangleq \max_{\varsigma \in \mathbb{R}} \big|\sigma_{1j}(\varsigma) + \sigma_{2j}(\varsigma)\big| < T_j - B_{gj} \tag{9}$$

and s_d (in (8)) is a continuous vector function whose components satisfy

$$\operatorname{sign}\big(s_{dj}(\vartheta, \bar{q})\big) = \operatorname{sign}(\vartheta_j) \tag{10}$$

$j = 1, \ldots, n$.

Proposition 1: Consider system (1)-(4) in closed loop with the proposed control scheme in Eqs. (7)-(8). Thus, for any positive definite diagonal matrices K_1, K_2, A and B: global asymptotic stability of the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is guaranteed with $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$.

Proof: Notice that —for every $j \in \{1, \ldots, n\}$— by (9), we have that, for any $(q, \vartheta) \in \mathbb{R}^n \times \mathbb{R}^n$: $|u_j(q, \vartheta)| \leq \big|\sigma_{1j}(k_{1j}\bar{q}_j) + \sigma_{2j}(k_{2j}\vartheta_j)\big| + |g_j(q)| \leq B_j + B_{gj} < T_j$. From this and (4), one sees that $T_j > |u_j(q, \vartheta)| = |u_j| = |\tau_j|$, $\forall (q, \vartheta) \in \mathbb{R}^n \times \mathbb{R}^n$, which shows that, along the system trajectories, $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$. This shows that, under the proposed scheme, the input saturation values, T_j, are never attained. Thus, the closed-loop dynamics takes the (equivalent) form

$$H(q)\ddot{q} + C(q, \dot{q})\dot{q} = -s_1(K_1\bar{q}) - s_2(K_2\vartheta)$$
$$\dot{\vartheta} = -As_d(\vartheta, \bar{q}) + B\dot{q}$$

Defining $x_1 = \bar{q}$, $x_2 = \dot{q}$, $x_3 = \vartheta$ and $x = (x_1^T, x_2^T, x_3^T)^T$, the closed-loop dynamics adopts a $3n$-order state-space representation in the form of (6) with

$$f(x) = \begin{pmatrix} f_{(1)}(x) \\ f_{(2)}(x) \\ f_{(3)}(x) \end{pmatrix} \quad , \quad \hat{f}(x) = \begin{pmatrix} \hat{f}_{(1)}(x) \\ \hat{f}_{(2)}(x) \\ \hat{f}_{(3)}(x) \end{pmatrix} \tag{11}$$

where $f_{(1)}(x) = x_2$, $f_{(2)}(x) = -H^{-1}(q_d)\big[s_1(K_1x_1) + s_2(K_2x_3)\big]$, $f_{(3)}(x) = -As_d(x_3, x_1) + Bx_2$, $\hat{f}_{(1)}(x) = \hat{f}_{(3)}(x) = 0_n$, and

$$\begin{aligned} \hat{f}_{(2)}(x) = &-H^{-1}(x_1 + q_d)C(x_1 + q_d, x_2)x_2 \\ &- \mathcal{H}(x_1)\big[s_1(K_1x_1) + s_2(K_2x_3)\big] \end{aligned} \tag{12}$$

with $\mathcal{H}(x_1) = H^{-1}(x_1 + q_d) - H^{-1}(q_d)$. Hence, the closed-loop stability property stated through Proposition 1 is corroborated by showing that $x = 0_{3n}$ is a globally asymptotically stable equilibrium of the state equation $\dot{x} = f(x) + \hat{f}(x)$, which is proven through the following theorem. ∎

Theorem 2: Under the stated specifications, the origin is a globally asymptotically stable equilibrium of the state equation $\dot{x} = f(x)$ and the (closed-loop) system $\dot{x} = f(x) + \hat{f}(x)$, with $f(x)$ and $\hat{f}(x)$ defined through Eqs. (11).

Proof: For every $\ell \in \{0,1\}$, let us define the continuously differentiable scalar function

$$V_\ell(x) = \frac{1}{2}x_2^T H(\ell x_1 + q_d)x_2 + \mathcal{I}_1(x_1) + \mathcal{I}_2(x_3)$$

$\mathcal{I}_1(x_1) \triangleq \int_{0_n}^{x_1} s_1^T(K_1\nu)d\nu = \sum_{j=1}^n \int_0^{x_{1j}} \sigma_{1j}(k_{1j}\nu_j)d\nu_j$, $\mathcal{I}_2(x_3) \triangleq \int_{0_n}^{x_3} s_2^T(K_2\nu)B^{-1}d\nu = \sum_{j=1}^n \int_0^{x_{3j}} \frac{\sigma_{2j}(k_{2j}r_j)}{b_j}dr_j$. From the positive definiteness of the inertia matrix and Lemma 3, $V_\ell(x)$, $\ell = 0,1$, are concluded to be positive definite and radially unbounded. Further, for every $\ell \in \{0,1\}$, the derivative of V_ℓ along the trajectories of $\dot{x} = f(x) + \ell\hat{f}(x)$, is obtained as

$$\begin{aligned}
\dot{V}_\ell(x) &= x_2^T H(\ell x_1 + q_d)\dot{x}_2 + \frac{\ell}{2}x_2^T\dot{H}(x_1+q_d,x_2)x_2 \\
&\quad + s_1^T(K_1x_1)\dot{x}_1 + s_2^T(K_2x_3)B^{-1}\dot{x}_3 \\
&= x_2^T\big[-\ell C(x_1+q_d,x_2)x_2 - s_1(K_1x_1) - s_2(K_2x_3)\big] \\
&\quad + \frac{\ell}{2}x_2^T\dot{H}(x_1+q_d,x_2)x_2 + s_1^T(K_1x_1)x_2 \\
&\quad + s_2^T(K_2x_3)B^{-1}\big[-As_d(x_3,x_1) + Bx_2\big] \\
&= -s_2^T(K_2x_3)B^{-1}As_d(x_3,x_1) \\
&= -\sum_{j=1}^n \frac{a_j}{b_j}\sigma_{2j}(k_{2j}x_{3j})s_{dj}(x_3,x_1) \qquad (13)
\end{aligned}$$

where, in the case of $\ell = 1$, (2) has been applied. Note, from the strictly passive character of σ_{2j}, $j = 1,\ldots,n$ (see Remark 5), and the definition of s_d (see (10)), that $\dot{V}_\ell \leq 0$, $\forall (x_1,x_2,x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, with $Z_\ell \triangleq \{(x_1,x_2,x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : \dot{V}_\ell = 0\} = \{(x_1,x_2,x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : x_3 = 0_n\}$. Further, from the system dynamics $\dot{x} = f(x) + \ell\hat{f}(x)$ —under the consideration of the strictly passive character of σ_{1j}, $j = 1,\ldots,n$, and the positive definiteness of $H(q)$ and K_1— one sees that $x_3(t) \equiv 0_n \implies \dot{x}_3(t) \equiv 0_n \implies x_2(t) \equiv 0_n \implies \dot{x}_2(t) \equiv 0_n \implies s_1\big(K_1x_1(t)\big) \equiv 0_n \iff x_1(t) \equiv 0_n$ (which shows that $(x_1,x_2,x_3)(t) \equiv (0_n,0_n,0_n)$ is the only system solution completely remaining in Z_ℓ), and corroborates that at any $(x_1,x_2,x_3) \in Z_\ell \setminus \{(0_n,0_n,0_n)\}$, the resulting unbalanced force terms act on the closed-loop dynamics $[\dot{x} = f(x_1,x_2,0_n) + \ell\hat{f}(x_1,x_2,0_n)$ with $(x_1,x_2) \neq (0_n,0_n)]$, forcing the system trajectories to leave Z_ℓ, whence $\{(0_n,0_n,0_n)\}$ is concluded to be the only invariant set in Z_ℓ, $\ell = 0,1$. Therefore, by the invariance theory [12, §7.2], $x = 0_{3n}$ is concluded to be a globally asymptotically stable equilibrium of both the state equation $\dot{x} = f(x)$ and the (closed-loop) system $\dot{x} = f(x) + \hat{f}(x)$. ■

Remark 6: One notes from the proof of Theorem 2 that as long as the controlled-system state variables are not in equilibrium, the closed loop energy function V_1 keeps continually decreasing, showing the dissipative role of the auxiliary subsystem, through the analytical properties of s_d (and s_2, ensuring negativity of the right-hand side of (13)).

IV. Finite-time / exponential stabilization

Let $\hat{r}_i = (r_{i1},\ldots,r_{in})^T$, $i = 1,2,3$.

Proposition 2: Consider the proposed control scheme under the additional consideration that, for every $j \in \{1,\ldots,n\}$, σ_{ij}, $i = 1,2$, are locally r_i-homogeneous of degree $\alpha_i = 2r_2 - r_1 > 0$ —*i.e.* $r_{1j} = r_1$, $r_{2j} = r_2$ and $\alpha_{1j} = \alpha_1 = 2r_2 - r_1 = \alpha_2 = \alpha_{2j} > 0$ for all $j \in \{1,\ldots,n\}$— with domain of homogeneity $D_{ij} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{ij} \in (0,\infty]\}$ and s_{dj} is locally $(\hat{r}_1^T,\hat{r}_1^T)^T$-homogeneous of degree $\alpha_d = r_2$ —*i.e.* $r_{3j} = r_3 = r_1$ and $\alpha_{dj} = \alpha_d = r_2$ for all $j \in \{1,\ldots,n\}$— with (common) domain of homogeneity D_d. Thus, for any positive definite diagonal matrices K_1, K_2, A and B: $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1,\ldots,n$, $\forall t \geq 0$, and the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is:

1) globally finite-time stable if $r_2 < r_1$;
2) globally asymptotically stable with (local) exponential stability if $r_2 = r_1$.

Proof: Since the proposed control scheme is applied —with all its previously stated specifications— Proposition 1 holds and consequently $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1,\ldots,n$, $\forall t \geq 0$. Then, all that remains to be proven is that the additional considerations give rise to the specific stability properties claimed in items 1 and 2 of the statement. In this direction, let $r = (\hat{r}_1^T,\hat{r}_2^T,\hat{r}_3^T)^T$, $D_i \triangleq \{x_i \in \mathbb{R}^n : K_ix_i \in D_{i1} \times \cdots \times D_{in}\} = \{x_i \in \mathbb{R}^n : |x_{ij}| < L_{ij}/k_{ij}\,,\ j = 1,\ldots,n\}$, $i = 1,2$, $D_{ce} \triangleq D_1 \times D_2 \times \mathbb{R}^n$, $D_{de} \triangleq \{(z_1,z_2,z_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : (z_3,z_1) \in D_d\}$, $D \triangleq D_{ce} \cap D_{de}$, and consider the previously defined state (vector) variables and the consequent closed-loop state-space representation $\dot{x} = f(x) + \hat{f}(x)$, with f and $\hat{f}$ as defined through Eqs. (11). Since D defines an open neighborhood of the origin, there exists $\rho > 0$ such that $B_\rho \triangleq \{x \in \mathbb{R}^{3n} : \|x\| < \rho\} \subset D$. Moreover, for every $x \in B_\rho$ and all $\varepsilon \in (0,1]$, we have that $\delta_\varepsilon^r(x) \in B_\rho$ (since $\|\delta_\varepsilon^r(x)\| < \|x\|$, $\forall \varepsilon \in (0,1)$), and, for every $j \in \{1,\ldots,n\}$:

$$\begin{aligned}
f_{(1)j}(\delta_\varepsilon^r(x)) &= \varepsilon^{r_2}x_{2j} = \varepsilon^{(r_2-r_1)+r_1}x_{2j} \\
&= \varepsilon^{(r_2-r_1)+r_{1j}}f_{(1)j}(x)
\end{aligned}$$

$$\begin{aligned}
f_{(2)j}(\delta_\varepsilon^r(x)) &= -H_j^{-1}(q_d)\big[s_1(\varepsilon^{r_1}K_1x_1) + s_2(\varepsilon^{r_3}K_2x_3)\big] \\
&= -H_j^{-1}(q_d)\big[\varepsilon^{\alpha_1}s_1(K_1x_1) + \varepsilon^{\alpha_2}s_2(K_2x_3)\big] \\
&= -\varepsilon^{2r_2-r_1}H_j^{-1}(q_d)\big[s_1(K_1x_1) + s_2(K_2x_3)\big] \\
&= \varepsilon^{(r_2-r_1)+r_{2j}}f_{(2)j}(x)
\end{aligned}$$

$$\begin{aligned}
f_{(3)j}(\delta_\varepsilon^r(x)) &= -As_d(\varepsilon^{r_3}x_3,\varepsilon^{r_1}x_1) + \varepsilon^{r_2}Bx_2 \\
&= -A\varepsilon^{\alpha_3}s_d(x_3,x_1) + \varepsilon^{r_2}Bx_2 \\
&= \varepsilon^{r_2}\big[-As_d(x_3,x_1) + Bx_2\big] \\
&= \varepsilon^{(r_2-r_1)+r_{3j}}f_{(3)j}(x)
\end{aligned}$$

whence one concludes that f is a locally r-homogeneous vector field of degree $\alpha = r_2 - r_1$, with domain of homogeneity B_ρ. Hence, by Theorems 1 and 2, Lemma 1 and Remark 3, the origin of the state equation $\dot{x} = f(x)$ is concluded to be a globally finite-time stable equilibrium if $r_2 < r_1$, and a globally asymptotically stable equilibrium with (local) exponential stability if $r_2 = r_1$. Thus, by Theorem 2, Lemma

2, and Remarks 1 and 4, the origin of the closed-loop system $\dot{x} = f(x) + \hat{f}(x)$ is concluded to be a globally finite-time stable equilibrium provided that $r_2 < r_1$, and a globally asymptotically stable equilibrium with (local) exponential stability provided that $r_2 = r_1$, if

$$\begin{aligned}\mathcal{L}_0 &\triangleq \lim_{\varepsilon\to 0^+} \left\|\varepsilon^{-\alpha}\delta_\varepsilon^{-\hat{r}_2}\big(\hat{f}_{(2)}(\delta_\varepsilon^r(x))\big)\right\| \\ &= \lim_{\varepsilon\to 0^+} \left\|\varepsilon^{-\alpha-r_2}\hat{f}_{(2)}(\delta_\varepsilon^r(x))\right\| \\ &= \lim_{\varepsilon\to 0^+} \varepsilon^{r_1-2r_2}\left\|\hat{f}_{(2)}(\delta_\varepsilon^r(x))\right\| \\ &= 0\end{aligned} \tag{14}$$

for all $x \in S_c^{3n-1} = \{x \in \mathbb{R}^{3n} : \|x\| = c\}$ (resp. $x \in S_{r,c}^{3n-1} = \{x \in \mathbb{R}^{3n} : \|x\|_r = c\}$), for some $c > 0$ such that $S_c^{3n-1} \subset B_\rho$ (resp. $S_{r,c}^{3n-1} \subset B_\rho$). Hence, from (12) and (3), we have, for all such $x \in S_c^{3n-1}$ (resp. $x \in S_{r,c}^{3n-1}$):

$$\begin{aligned}&\left\|\hat{f}_{(2)}(\delta_\varepsilon^r(x))\right\| \\ &\quad\le \left\| -H^{-1}(\varepsilon^{r_1}x_1+q_d)C(\varepsilon^{r_1}x_1+q_d,x_2)\varepsilon^{2r_2}x_2\right\| \\ &\qquad + \left\|\mathcal{H}(\varepsilon^{r_1}x_1)\big[\varepsilon^{\alpha_1}s_1(K_1x_1)+\varepsilon^{\alpha_2}s_2(K_2x_3)\big]\right\| \\ &\quad\le \left\| -\varepsilon^{2r_2}H^{-1}(\varepsilon^{r_1}x_1+q_d)C(\varepsilon^{r_1}x_1+q_d,x_2)x_2\right\| \\ &\qquad + \left\|\mathcal{H}(\varepsilon^{r_1}x_1)\varepsilon^{2r_2-r_1}\big[s_1(K_1x_1)+s_2(K_2x_3)\big]\right\| \\ &\quad\le \varepsilon^{2r_2}\left\|H^{-1}(\varepsilon^{r_1}x_1+q_d)C(\varepsilon^{r_1}x_1+q_d,x_2)x_2\right\| \\ &\qquad +\varepsilon^{2r_2-r_1}\left\|\mathcal{H}(\varepsilon^{r_1}x_1)\big[s_1(K_1x_1)+s_2(K_2x_3)\big]\right\|\end{aligned}$$

and consequently, from (14), we get

$$\begin{aligned}\mathcal{L}_0 &\le \lim_{\varepsilon\to 0^+}\varepsilon^{r_1}\left\|H^{-1}(\varepsilon^{r_1}x_1+q_d)C(\varepsilon^{r_1}x_1+q_d,x_2)x_2\right\| \\ &\qquad + \lim_{\varepsilon\to 0^+}\left\|\mathcal{H}(\varepsilon^{r_1}x_1)\big[s_1(K_1x_1)+s_2(K_2x_3)\big]\right\| \\ &\le \left\|H^{-1}(q_d)C(q_d,x_2)x_2\right\| \lim_{\varepsilon\to 0^+}\varepsilon^{r_1} \\ &\qquad + \left\|s_1(K_1x_1)+s_2(K_2x_3)\right\| \lim_{\varepsilon\to 0^+}\left\|\mathcal{H}(\varepsilon^{r_1}x_1)\right\| \\ &\le \left\|s_1(K_1x_1)+s_2(K_2x_3)\right\|\cdot\left\|\mathcal{H}(0_n)\right\| = 0\end{aligned}$$

which completes the proof. ∎

V. PARTICULAR STRUCTURES

The vector function s_d in (8) may be defined (based on Lemma 4) for instance as

$$s_d(\vartheta,\bar{q}) = s_5\big(s_3(\vartheta)+s_4(\bar{q})\big) - s_5\big(s_4(\bar{q})\big) \tag{15}$$

where, for any $z \in \mathbb{R}^n$ and $i = 3,4,5$, $s_i(z) = \big(\sigma_{i1}(z_1),\ldots,\sigma_{in}(z_n)\big)^T$, with, for every $j = 1,\ldots,n$: $\sigma_{3j}(\cdot)$ being strictly passive, locally Lipschitz-continuous on $\mathbb{R}\setminus\{0\}$, and locally r_1-homogeneous of degree α_3 —*i.e.* $r_{3j} = r_3 = r_1$ and $\alpha_{3j} = \alpha_3$ for all $j \in \{1,\ldots,n\}$— with domain of homogeneity $D_{3j} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{3j} \in (0,\infty]\}$; $\sigma_{4j}(\cdot)$ being locally Lipschitz-continuous on $\mathbb{R}\setminus\{0\}$ and locally r_1-homogeneous of degree α_3 —*i.e.* $\alpha_{4j} = \alpha_4 = \alpha_3$ for all $j \in \{1,\ldots,n\}$— with domain of homogeneity $D_{4j} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{4j} \in (0,\infty]\}$; and $\sigma_{5j}(\cdot)$ being strictly passive, strictly increasing, locally Lipschitz-continuous on $\mathbb{R}\setminus\{0\}$ and locally α_3-homogeneous of degree r_2 —*i.e.* $r_{5j} = r_5 = \alpha_3$ and $\alpha_{5j} = \alpha_5 = r_2$ for all $j \in \{1,\ldots,n\}$— with domain of homogeneity $D_{5j} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{5j} \in (0,\infty]\}$.

Another (more direct) choice of s_d is the particular case of (15) obtained by taking $s_5(z) \equiv z$, *i.e.*

$$s_d(\vartheta,\bar{q}) = s_3(\vartheta) \tag{16}$$

with s_3 as previously defined, taking $\alpha_3 = r_2$.

A common choice for the elements of the vector functions s_i involved in the control scheme is: $\sigma_{ij}(\varsigma) = \mathrm{sign}(\varsigma)|\varsigma|^{\beta_i}$, $\forall|\varsigma| \le L_{ij} \in (0,\infty]$, $j = 1,\ldots,n$. For this choice, the stabilization goal with (15) is achieved taking $\beta_2 = \beta_1$, $\beta_4 = \beta_3$, $\beta_5\beta_3 = (1+\beta_1)/2$, and $0 < \beta_1 < 1$ or $\beta_1 = 1$ for finite-time or exponential convergence, respectively; the requirements of the proposed scheme are thus corroborated to be satisfied for any $r_1 > 0$ and $r_2 = (1+\beta_1)r_1/2$. Note that taking $\beta_5 = 1$ and $L_{5j} = \infty$, the (simple) case of (16) takes place with $\beta_3 = (1+\beta_1)/2$.

VI. SIMULATION RESULTS

The proposed scheme was tested through computer simulations considering the model of a 2-DOF mechanical manipulator corresponding to the experimental robotic arm used in [13]. For such a robot, the various terms characterizing the system dynamics in Eq. (1) are given by

$$H(q) = \begin{pmatrix} 2.351 + 0.168\cos q_2 & 0.102 + 0.084\cos q_2 \\ 0.102 + 0.084\cos q_2 & 0.102 \end{pmatrix}$$

$$C(q,\dot{q}) = \begin{pmatrix} -0.084\dot{q}_2\sin q_2 & -0.084(\dot{q}_1+\dot{q}_2)\sin q_2 \\ 0.084\dot{q}_1\sin q_2 & 0 \end{pmatrix}$$

$$g(q) = \begin{pmatrix} 38.465\sin q_1 + 1.825\sin(q_1+q_2) \\ 1.825\sin(q_1+q_2) \end{pmatrix}$$

Assumption 1 is thus fulfilled with $B_{g1} = 40.29$ Nm and $B_{g2} = 1.825$ Nm. Furthermore, the input saturation bounds are $T_1 = 150$ Nm and $T_2 = 15$ Nm for the first and second links respectively, whence one can corroborate that Assumption 2 is satisfied as well. For the sake of simplicity, units will be subsequently omitted.

For the implementation of the proposed design methodology, let us define the functions

$$\sigma_u(\varsigma;\beta,a) = \mathrm{sign}(\varsigma)\max\{|\varsigma|^\beta, a|\varsigma|\} \tag{17a}$$

$$\sigma_b(\varsigma;\beta,a,M) = \mathrm{sign}(\varsigma)\min\{|\sigma_u(\varsigma;\beta,a)|, M\} \tag{17b}$$

for constants $\beta > 0$, $a \in \{0,1\}$ and $M > 0$. Examples and alternative function definitions are presented in [7]. Based on the functions in Eqs. (17), we define —for every $j = 1,2$— those involved in the tests performed here as

$$\sigma_{ij}(\varsigma) = \begin{cases} \sigma_b(\varsigma;\beta_i,a_{ij},M_{ij}) & i = 1,2 \\ \sigma_u(\varsigma;\beta_i,a_{ij}) & i > 2 \end{cases}$$

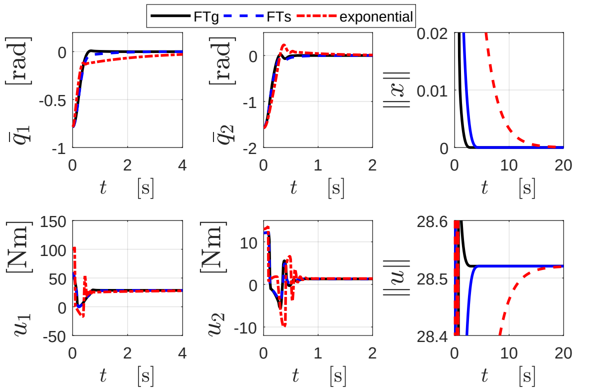

Fig. 1. Position errors $(\bar{q}_1, \bar{q}_2)$, control signals (u_1, u_2), $\|x(t)\|$, $\|u(t)\|$

Notice that through these definitions we have $B_j = M_{1j} + M_{2j}$, $j = 1, 2$ (see (9)). Hence, by fixing $M_{11} = M_{21} = 50$ and $M_{12} = M_{22} = 6.5$, the inequalities from expression (9) are satisfied. The tests were implemented taking initial conditions as $q(0) = \dot{q}(0) = \vartheta_c(0) = 0_2$ and desired position at $q_d = \left(\pi/4 \quad \pi/2\right)^T$. As a performance comparison indicator, we will obtain the ϱ-stabilization time $t_\varrho^s \triangleq \inf\{t_s \geq 0 : \|x(t)\| \leq \varrho \quad \forall t \geq t_s\}$, with $x \triangleq \left(\bar{q}^T \quad \dot{q}^T \quad \vartheta^T\right)^T$.

We implemented a test where the aim is to corroborate the performance difference among the closed-loop trajectories obtained through the proposed finite-time controller with the different cases of s_d presented in Section V, taking $\beta_1 = \beta_2 = 1/2$. For the case of (15) —referred to as FTg— we further took $\beta_3 = \beta_4 = 1$ and $\beta_5 = 3/4$, while for the case of (16) —referred to as FTs— we fixed $\beta_3 = 3/4$. The test included a further implementation with the analog exponential stabilizer, *i.e.* with $\beta_1 = \beta_2 = \beta_3 = 1$ [$= \beta_4 = \beta_5$]. All the rest, including control and auxiliary subsystem gains, remained unchanged. The implementations were run fixing $a_{ij} = 0$, for all i and j.

Fig. 1 shows results obtained taking $K_1 = K_2 = B = \text{diag}[70, 20]$ and $A = \text{diag}[30, 30]$. One sees that the stabilization objective was achieved by every implemented controller avoiding input saturation. Moreover, different performance among the use of (15) [FTg] and (16) [FTs] is appreciated as well as the contrast among the different types of trajectory convergence, in accordance to the corresponding controller nature (finite-time *vs* exponential). In particular, it is seen that, with the finite-time controllers, the position errors, and actually the (norm of the) whole state vector in the extended state space [$x = (\bar{q}^T \quad \dot{q}^T \quad \vartheta^T)^T$], converge to zero in less than 5 seconds, remaining invariant thereafter. The exponential controller, instead, generated asymptotically convergent closed-loop trajectories with $t_{0.01}^s = 7.38$ s and $t_{0.001}^s = 12.8$ s. Such convergence differences are corroborated from the graph of the control signals u_i, $i = 1, 2$, and their norm $\|u\|$ [$u = (u_1 \ u_2)^T$], where, contrarily to the finite-time stabilizers that get to their exact steady-state value in less than 5 seconds, the exponential controller is observed to perform a transient variation during a short period —where the position variables (actually the whole system states) are considerably approached to their corresponding equilibrium values— and a slow stabilization refinement thereafter. Among the finite-time controllers, the use of (15) gave rise to faster closed-loop responses: $t_{0.01} = 0.99$s and $t_{0.001}^s = 1.97$s with (15) [FTg] *vs* $t_{0.01} = 2.16$s and $t_{0.001}^s = 3.32$s with (16) [FTs].

VII. Conclusions

An output-feedback global continuous control scheme for mechanical systems with bounded inputs, guaranteeing finite-time or exponential stabilization, has been designed within the analytical context of local homogeneity. The choice among the type of trajectory convergence is left to the user through a simple control parameter. The proposed controller avoids velocity measurements on the feedback by involving a dirty-derivative-based dynamic dissipator expressed in a generalized form. Such a generalized form increases the degree of design flexibility which may be useful for closed-loop performance improvement. This has been corroborated and shown through simulation implementations, although a deeper study is still needed to draw more concrete conclusions about such a useful aspect. The contrast among the different type of trajectory convergence, namely finite-time *vs* exponential, has also been appreciated through simulation results, showing advantages of the former over the latter.

VIII. Acknowledgments

This work was financially supported by CONACYT, Mexico; grant number: CB-2014-01-239833.

References

[1] A. Loría, Observers are unnecessary for output-feedback control of Lagrangian systems, *IEEE Tr. Aut. Ctl.*, vol. 61, 2016, pp. 905–920.

[2] S.P. Bhat and D.S. Bernstein, Geometric homogeneity with applications to finite-time stability, *Mathematics of Control, Signals and Systems*, vol. 17, 2005, pp. 101–127.

[3] A. Zavala-Río and I. Fantoni, Global finite-time stability characterized through a local notion of homogeneity, *IEEE Transactions on Automatic Control*, vol. 59, 2014, pp. 471–477.

[4] Y. Hong, Y. Xu and J. Huang, Finite-time control for robot manipulators, *Systems & Control Letters*, vol. 46, 2002, pp. 243–253.

[5] D. Zhao, S. Li, Q. Zhu and S. Gao, Robust finite-time control approach for robotic manipulators, *IET Ctl. Th. & App.*, vol. 4, 2010, pp. 1–15.

[6] A.K. Sanyal and J. Bohn, Finite-time stabilisation of simple mechanical systems using continuous feedback, *International Journal of Control*, vol. 88, 2015, pp. 783–791.

[7] A. Zavala-Río and G.I. Zamora-Gómez, Local-homogeneity-based global continuous control for mechanical systems with constrained inputs: finite-time and exponential stabilization, *International Journal of Control*, vol. 90, 2017, pp. 1037–1051.

[8] B. Brogliato, R. Lozano, B. Maschke and O. Egeland, *Dissipative Systems Analysis and Control*, 2nd ed., Springer, London; 2007.

[9] R. Ortega, A. Loría, P.J. Niclasson and H. Sira-Ramírez, *Passivity-based control of Euler-Lagrange systems*, Springer, London; 1998.

[10] R.T. M'Closkey and R.M. Murray, Exponential stabilization of driftless nonlinear control systems using homogeneous feedback, *IEEE Transactions on Automatic Control*, vol. 42, 1997, pp. 614–628.

[11] M. Kawski, Homogeneous stabilizing feedback laws, *Control Theory and Advanced Technology*, vol. 6, 1990, pp. 497-516.

[12] A.N. Michel, L. Hou and D. Liu, *Stability of Dynamical Systems*, Birkhäuser, Boston; 2008.

[13] A. Zavala-Río and V.Santibáñez, Simple extensions of the PD-with-gravity-compensation control law for robot manipulators with bounded inputs, *IEEE Trans. on Control Syst. Tech.*, vol. 14, 2006, pp. 958–965.