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Adaptive tracking control of chaotic systems with applications to synchronisation

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***Abstract*— We address the problem of controlled synchronisation of a class of uncertain chaotic systems. Our approach follows techniques of adaptive tracking control and identification of dynamic systems from recent developments of control theory. In particular, we use new notions of the so-called property of persistency of excitation —known to be sufficient and necessary for parameter estimation— to construct adaptive algorithms that ensure perfect tracking/synchronisation and parameter estimation of chaotic systems with parameter uncertainty. Our theoretical findings are supported by particular examples and simulation studies on systems such as the Lorenz and Rössler oscillators and the Duffing equation.**

I. Introduction

We address adaptive tracking control of chaotic systems; in a general setting, this can be described as follows. Consider the controlled system

$$\dot{x} = f(t, x, u, \theta) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state of the system, $u \in \mathbb{R}^n$ corresponds to the control inputs, $\theta \in \mathbb{R}^m$ is a vector of unknown *constant* parameters and f satisfies the sufficient conditions for existence and uniqueness of solutions (typically, continuity in all arguments and, for each u and θ, locally Lipschitz in x, uniformly in t) and the origin is an equilibrium point, that is, $f(t, 0, 0, \theta) \equiv 0$. Consider the problem of designing a control input u such that, given any (sufficiently smooth, bounded) reference trajectory $x_d(t)$ —typically generated by an exogenous system— we have:

- the system trajectories $x(t)$ tend to the reference trajectories $x_d(t)$; and
- the parameters θ are identified, *i.e.* the estimates $\hat{\theta}(t) \to \theta$ as $t \to \infty$.

The adaptive tracking control problem of chaotic systems is partially motivated by applications in classical master-slave synchronisation, *cf.* [1], and in communications using chaotic signals, *cf.* [2], [3]. In these contexts, the reference trajectory $x_d(t)$ is assumed to be generated by a *master* system which does not necessarily have the same nature as the slave; *e.g.* a chaotic signal may be generated by a Rössler system and received, carrying significant information, by a Lorenz *slave* system. Hence, synchronisation is achieved if $x(t) \to x_d(t)$, that is if the tracking control problem is solved. Parameter adaptation is necessary since, even though the slave system's model is supposed to be known, in practise, there always exists some degree of uncertainty in the values of the dynamic parameters θ, which depend on the physical devices used to construct the slave system —*e.g.* resistances, capacitors, *etc.*, used in electrical-circuit realisations of chaotic systems.

Tracking control of chaotic systems in the presence of parametric uncertainty has been considered, for instance, in [4], [5], [6], [7], but none of these references address the problem of parametric convergence. In the context of secure communication, adaptive chaos control has been considered in [8]. In [9], parameter estimation of a chaotic dynamical system involved in a synchronisation scheme is developed; see also [10] which deals with the parameter estimation of the Lorenz system and, wrongly (abusively using LaSalle's principle), proves that parameters converge. This is commented in the interesting paper [11] where the authors correctly point out that LaSalle's principle does not hold for general non-autonomous systems but holds for periodic systems; in [11] an alternative analysis[1] is carried out to prove parameter convergence for the *particular case* of synchronisation of two Lorenz oscillators using a contradiction argument. It is also claimed, without reference nor (general) proof, that LaSalle's principle holds for chaotic systems. Another article to mention, on synchronisation of chaotic systems *via* adaptive control is [12]. However, even though tracking is achieved, parameter convergence is not included within the stability proof developed by the authors, and is consequently not guaranteed through the proposed scheme. We believe that, while the ideas in the previous references are quite interesting they lack either, of formal stability proofs or generality or both.

Two aspects of stability are crucial for general non-autonomous systems: *uniformity* and *globality*. The first guarantees certain robustness while the second ensures good performance for any initial conditions. As far as we know the problem of guaranteeing *uniform* parametric convergence, simultaneously to synchronisation, remains open. The contribution of this paper is a *general* result on adaptive tracking control for a wide variety of chaotic systems; we prove uniform global asymptotic stability for the synchronisation and parameter error dynamics. We stress that this is a much stronger property than simple convergence of parameters and synchronisation errors[2]. Our method relies on designing adaptive controllers, *i.e.* dynamic systems composed of a control law $u(t, x, x_d(t), \hat{\theta})$ and adaptation

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[1] We stress that this reference was published after the original submission of the present paper.

[2] For further discussions we invite the reader to see [13].

law $\dot{\hat{\theta}} = g(t,x)$, such that the closed-loop system

$$\begin{aligned}\dot{x} &= f(t,x,u(t,x,x_d(t),\hat{\theta}),\theta)\\ \dot{\hat{\theta}} &= g(t,x)\end{aligned}$$

can be written in the new state-space as

$$\begin{aligned}\dot{\tilde{x}} &= F(t,\tilde{x},\tilde{\theta})\\ \dot{\tilde{\theta}} &= G(t,\tilde{x})\end{aligned}$$

where we defined the tracking errors $\tilde{x} := x - x_d$ and the estimation errors $\tilde{\theta} := \hat{\theta} - \theta$.

We solve the adaptive tracking control problem by showing uniform global asymptotic stability (UGAS, *cf.* Section II and [14], [15]) for the origin of the closed-loop system, *i.e.* $\mathrm{col}[\tilde{x}, \tilde{\theta}] = 0$ where $\mathrm{col}[\cdot,\cdot]$ represents a column vector. UGAS is one of the strongest stability properties one can have for time-varying systems since it guarantees a certain degree of *robustness* with respect to "bounded disturbances"; in other words, if a system is UGAS then, small disturbances will produce small steady-state errors.

In particular, we solve the problem of adaptive synchronisation under the standing assumption that the state of the master system is known, however, no other knowledge of the master system is required —not even the type of system that generates the reference trajectories. We also assume that only the model of the slave system is known but not the exact numerical values of the dynamic parameters.

The rest of the paper is organised as follows: we first present a generalisation of a recently contributed theorem for stability of nonlinear time-varying systems (*cf.* Section II) then, in Section II we apply this theorem in the adaptive control of a general class of systems which covers chaotic oscillators, such as the Rössler and Lorenz systems as well as the Duffing equation. Simulations, in the context of synchronisation, are presented for each application to illustrate the usefulness of our findings. Finally, some concluding remarks are given in Section IV.

Notation: We start by introducing some notation and recalling some definitions. A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ ($\gamma \in \mathcal{K}$), if $\gamma(s)$ is strictly increasing and $\gamma(0) = 0$; $\gamma \in \mathcal{K}_\infty$ if, in addition, $\gamma(s) \to \infty$ as $s \to \infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(s,t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}_{\geq 0}$ and, for each $s \geq 0$, $\beta(s,\cdot)$ is strictly decreasing and $\beta(s,t) \to 0$ as $t \to \infty$. We denote by $x(t,t_\circ,x_\circ)$, the solutions of the differential equation $\dot{x} = f(t,x)$ with initial conditions $(t_\circ, x_\circ)$. For a function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ we also define $\dot{V}(t,x) := \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x)$. We denote by $\|\cdot\|$ the Euclidean norm of vectors and the induced norm of matrices. For continuous functions $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ we define $\|\phi\|_\infty := \sup_{t \geq 0} \|\phi(t)\|$. We define the closed ball $B_R := \{x \in \mathbb{R}^n : \|x\| \leq R\}$.

II. Lyapunov stability of time-varying systems

Consider the system

$$\dot{x} = F(t,x)\,, \tag{2}$$

For these systems (with F continuous and locally Lipschitz in x, uniformly in t) probably the most useful stability properties are the following since in particular, there exist converse Lyapunov theorems for them and UGAS guarantees robustness.

Definition 1 (Uniform global stability [15]) The origin of the system (2) is said to be uniformly globally stable (UGS) if there exists $\gamma \in \mathcal{K}_\infty$ such that, for each $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ each solution $x(t,t_\circ,x_\circ)$ satisfies

$$\|x(t,t_\circ,x_\circ)\| \leq \gamma(\|x_\circ\|) \qquad \forall\, t \geq t_\circ\,. \tag{3}$$

Definition 2 (Uniform global attractivity [15]) The origin of the system (2) is said to be uniformly globally attractive if for each $r, \sigma > 0$ there exists $T > 0$ such that, for each $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ each solution $x(t,t_\circ,x_\circ)$ satisfies

$$\|x_\circ\| \leq r \implies \|x(t,t_\circ,x_\circ)\| \leq \sigma \qquad \forall\, t \geq t_\circ + T\,. \tag{4}$$

Furthermore, we say that the (origin of the) system is uniformly globally asymptotically stable (UGAS) if it is UGS and uniformly globally attractive[3].

In general UGAS, is guaranteed for (2) if and only if there exists a Lyapunov function $V(t,x)$, class $\mathcal{K}_\infty$ functions α_1 and α_2 and a class $\mathcal{K}$ function α_3 such that

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|) \tag{5a}$$

$$\dot{V}(t,x) \leq -\alpha_3(\|x\|)\,. \tag{5b}$$

While UGAS necessarily implies the existence of a Lyapunov function satisfying (5), finding such a Lyapunov function is in general very hard. The contribution of this paper consists in presenting general results that guarantee UGAS, without a Lyapunov function satisfying the conditions above, for a class of systems that includes many chaotic systems. We present below the setting for these theorems.

A. Uniform δ-persistency of excitation

Our results rely on a condition of exciteness tailored for nonlinear systems, that we describe in this section. We start by recalling the definition of *persistency of excitation* introduced for linear time-varying systems and which can be found in many books on adaptive control ([16], [17], *etc.*).

Definition 3 (PE) The locally integrable function $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n \times m}$, with $n \geq m$, is said to be persistently exciting if there exist $\mu > 0$ and $T > 0$ such that

$$\int_t^{t+T} \Phi(\tau)\Phi(\tau)^\top d\tau \geq \mu I \qquad \forall\, t \in \mathbb{R}_{\geq 0}\,. \tag{6}$$

The systems that we deal with in this paper are nonlinear time-varying; therefore, we need to introduce the following

[3]Other references treating stability definitions in detail are [14], [13].

property which is a relaxed condition of persistency of excitation. The definition below was originally proposed in [18] (see also [19]).

Let $x \in \mathbb{R}^n$ be partitioned as $x := \text{col}[x_1, x_2]$ where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ and $n = n_1 + n_2$. Define the *column vector* function $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m$ to be such that $t \mapsto \phi(t, x)$ is locally integrable. Define also $\mathcal{D}_1 := \{x \in \mathbb{R}^n : x_1 \neq 0\}$.

Definition 4 The function ϕ is said to be uniformly δ-persistently exciting (Uδ-PE) with respect to x_1 if for each $x \in \mathcal{D}_1$ there exist $\delta > 0$, $T > 0$ and $\mu > 0$ such that for all $t \in \mathbb{R}_{\geq 0}$,

$$\|z - x\| \leq \delta \quad \Longrightarrow \quad \int_t^{t+T} \|\phi(\tau, z)\| d\tau \geq \mu \,. \tag{7}$$

The property of Uδ-PE defined above roughly means that for each *fixed* x such that $x_1 \neq 0$ the function $\Phi(t)^\top := \phi(t, x)$ (that is, Φ depends only on time because x is fixed) is PE and μ and T are the same for all neighbouring points of x. For functions that are continuous uniformly in the second argument we do not need to check the condition on the neighbouring points. More precisely we have the following.

Lemma 1 [19] If $\phi(t, \cdot)$ is uniformly continuous[4] then, $\phi(t, x)$ is Uδ-PE with respect to x_1 if and only if for each $x \in \mathcal{D}_1$ there exist $T > 0$ and $\mu > 0$ such that, for all $t \in \mathbb{R}$,

$$\int_t^{t+T} \|\phi(\tau, x)\| d\tau \geq \mu \,. \tag{8}$$

□

Example 1 The function $\phi(t, x) := v(x)\psi(t)$ with v positive definite, continuous and ψ PE and bounded, is Uδ-PE. □

Lemma 1 also helps to see that the following interesting fact, which is very useful in our identification problem, is true.

Fact 1 The function $\phi(t, x) := \Phi(t)^\top x$ with $\Phi(t)$, continuous and bounded, is Uδ-PE with respect to x if and only if Φ is PE.

In its turn, this fact is very useful to analyse nonlinear adaptive control systems that are linear in the unknown parameters. Such is the case for several chaotic systems, as we will see later.

B. UGAS of nonlinear systems

We introduce now, a general result for stability of nonlinear time-varying systems (2) that applies in the adaptive tracking control problems studied in this paper. Let

$$F(t, x) \quad := \quad \begin{bmatrix} A(t, x_1) + B(t, x) \\ H(t, x) \end{bmatrix} \tag{9}$$

[4]A sufficient condition for uniform continuity of a function f is that its derivative be bounded.

be all functions that are equivalently equal to zero when $x = 0$ and have the properties to ensure existence and uniqueness of solutions. For further development, let us define

$$B_\circ(t, x_2) := B(t, x)\big|_{x_1 = 0} \tag{10}$$

and notice that necessarily, $B_\circ(t, 0) \equiv 0$.

This type of systems has been studied before for instance in [20], [21] and, as pointed out in the last reference, have appeared for instance in Model Reference Adaptive Control of *linear* plants (*cf.* [20]) and in mechanical systems. We show that they also cover well-known chaotic systems such as the Duffing equation, the Lorenz and Rössler systems.

For the sequel, we split the state vector into $x := \text{col}[x_1, x_2]$ where x_1 corresponds to the trajectory tracking (position and velocity) error and x_2 the parameter estimation error. The vector $A(t, x_1)$ corresponds to the closed-loop dynamics of the system with the controller as if the dynamic parameters were known. Finally, the vectors $B(t, x)$ and $H(t, x)$ involve the regressor function, considering that the model is linear in the dynamic unknown parameters.

Thus, let us consider systems of the form given above and under the following conditions.

Assumption 1 There exists a continuously differentiable function $V_1 : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ which is positive definite, decrescent, radially unbounded, *i.e.* there exist functions α_1', $\alpha_2' \in \mathcal{K}_\infty$ such that

$$\alpha_1'(\|x\|) \leq V_1(t, x) \leq \alpha_2'(\|x\|)$$

and $\dot{V}_1(t, x) \leq 0$ for all $(t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$.

Assumption 1 is tantamount to assuming that the system is UGS, hence, for any $r > 0$ we have $\|x(t)\| \leq \gamma(r)$ where γ is defined in (3).

Assumption 2 Given any $R > 0$, there exists a continuously differentiable function $V_2 : \mathbb{R}_{\geq 0} \times B_R \to \mathbb{R}_{\geq 0}$ which is positive definite, decrescent, radially unbounded and has a negative semidefinite time-derivative on B_R. More precisely, assume that there exist functions α_1, $\alpha_2 \in \mathcal{K}_\infty$ and $U : \mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$ continuous positive definite, such that

$$\alpha_1(\|x\|) \leq V_2(t, x) \leq \alpha_2(\|x\|) \tag{11a}$$

$$\dot{V}_2(t, x) \leq -U(x_1) \tag{11b}$$

for all $(t, x) \in \mathbb{R}_{\geq 0} \times B_R$.

In the context of this paper, *i.e.* where x_1 denotes tracking errors and x_2 parameter estimation errors, Assumption 2 means that we know a strict Lyapunov function for the case when the parameters are exactly known. Such a Lyapunov function is useful, for instance, to construct another Lyapunov function involving parametric errors; then, if the derivative of this new function satisfies the bound (11b) one can show that the tracking errors converge to zero even if the parametric errors do not. The latter is due to what is called, in adaptive control theory, *certainty-equivalence principle* –*cf. e.g.* [22], [17] and it is a property of many adaptive controllers; in particular, of passivity-based adaptive controllers for mechanical systems –*cf. e.g.* [23], [24]. To some extent, a similar situation, to that of adaptive

control of mechanical systems is encountered in the context of synchronisation with parameter uncertainty *e.g.*, in [10], [11].

In many applications both Assumptions 1 and 2 may be verified with the same function V; however, in general this is not necessarily the case. Stating these assumptions separately gives the extra freedom of verifying (11b) restricting the state to closed balls as opposed to the whole state space.

The following hypothesis consists in a set of conditions on boundedness of certain functions, uniformly in time.

Assumption 3 For each $\Delta > 0$ there exist $b_M > 0$ and continuous nondecreasing functions $\rho_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $i = 1, 2$ such that $\rho_i(0) = 0$ and for all $t \in \mathbb{R}_{\geq 0}$ and all $x \in \mathbb{R}^n$

$$\max_{|x_2|\leq\Delta} \left\{ \|B_\circ(t,x_2)\|,\, \|\frac{\partial B_\circ}{\partial t}\|,\, \|\frac{\partial B_\circ}{\partial x_2}\| \right\} \leq b_M\,, \quad (12)$$

$$\max_{|x_2|\leq\Delta} \|B(t,x) - B_\circ(t,x_2)\| \leq \rho_1(\|x_1\|)\,, \quad (13)$$

$$\max_{|x_2|\leq\Delta} \{\|A(t,x_1)\|,\, \|H(t,x)\|\} \leq \rho_2(\|x_1\|)\,. \quad (14)$$

Under these assumptions, we can state the following general theorem for systems (2), (9) and that is fundamental to present our main results on adaptive control of chaotic systems.

Theorem 1 The system (2), (9) under Assumptions 1-3 is UGAS if and only if
(Assumption 4*)* the function $B_\circ(t,x_2)$ is Uδ-PE with respect to x_2.
The proof follows along the lines of the proof of Theorem 3 in [18] and is omitted here.

III. Adaptive tracking control of chaotic systems

Our control approach consists in designing a control law such that the closed-loop system has the form (2), (9) and satisfies the conditions of the Theorem 1. We study mechanical systems such as the Duffing equation as well as a class of chaotic oscillators which includes the Rössler and Lorenz systems.

A. Chaotic oscillators

Let us consider chaotic systems that have the form

$$\dot{x} + \Phi(x)x + \Psi(x)\theta = u \quad (15)$$

where $\theta \in \mathbb{R}^m$ is a vector of unknown parameters, $x \in \mathbb{R}^n$ is the system's state, $\Phi(\cdot)$ and $\Psi(\cdot)$ are locally Lipschitz continuous and $u \in \mathbb{R}^n$ is a control input. We address the problem of adaptive tracking control of system (15), with parameter convergence. In the context of synchronisation, such problem comes to synchronise the trajectories of the *controlled* chaotic slave system (15) with the *reference* trajectories $x_d(t)$ generated by a master system.

Proposition 1 Consider the system (15). Assume that there exist a continuously differentiable function $P : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, p_m, p_M and $q > 0$ such that, for all $\xi \in \mathbb{R}^n$, $p_m I \leq P(t,x) \leq p_M I$ and

$$\xi^\top \Big(\dot{\overline{P(t,x)}} + [K(x) - \Phi(x)]^\top P(t,x) + P(t,x)[K(x) - \Phi(x)] \Big) \xi \leq -q\|\xi\|^2\,. \quad (16)$$

Then, the adaptive controller

$$u = \Phi(x)x_d + K(x)\tilde{x} + \Psi(x)\hat{\theta} + \dot{x}_d \quad (17a)$$

$$\dot{\hat{\theta}} = -\gamma\Psi(x)^\top P(t,x)\tilde{x}\,, \quad (17b)$$

where each element of $K(\cdot)$ is locally Lipschitz continuous, makes the origin of the closed-loop system, uniformly globally asymptotically stable if and only if $\Psi(x_d(t))$ is persistently exciting, *i.e.* if there exist μ and $T > 0$ such that

$$\int_t^{t+T} \Psi(x_d(s))\Psi(x_d(s))^\top ds \geq \mu I \qquad \forall\, t \geq 0\,. \quad (18)$$

□

Proof. The proof follows from Theorem 1 with $x_1 = \tilde{x}$ and $x_2 = \tilde{\theta}$. Consider the function[5]

$$V(t,\tilde{x},\tilde{\theta}) := \frac{1}{2}\tilde{x}^\top P(t,x)\tilde{x} + \frac{1}{2\gamma}\|\tilde{\theta}\|^2$$

which satisfies

$$\frac{1}{2\gamma}\|\tilde{\theta}\|^2 + p_m\|\tilde{x}\|^2 \leq 2V(t,\tilde{x},\tilde{\theta}) \leq p_M\|\tilde{x}\|^2 + \frac{1}{\gamma}\|\tilde{\theta}\|^2\,.$$

Its time-derivative satisfies

$$\begin{aligned} 2\dot{V}(t,\tilde{x},\tilde{\theta}) &= \tilde{x}^\top \Big(\dot{\overline{P(t,x)}} + [K(x) - \Phi(x)]^\top P(t,x) \\ &\qquad + P(t,x)[K(x) - \Phi(x)] \Big) \tilde{x} \\ &\leq -q\|\tilde{x}\|^2\,. \end{aligned}$$

This shows that Assumptions 1 and 2 hold. To show that Assumption 3 holds, let $x_1 := \tilde{x}$, $x_2 := \tilde{\theta}$ and $A(t,x_1) := [\,K(x_1 + x_d(t)) - \Phi(x_1 + x_d(t))\,]x_1$, $B(t,x) := \Psi(x_1 + x_d(t))x_2$, and $H(t,x) := -\gamma\Psi(x_1 + x_d(t))^\top P(t, x_1 + x_d(t))x_1$. The functions $A(t,x_1)$ and $H(t,x)$ are uniformly bounded in t provided that the reference trajectories and velocities are bounded; also, $A(t,0) \equiv 0$, $H(t,x)$ depends only on x_1 and $H(t,0) \equiv 0$.

Finally, we see that $B_\circ(t,x_2)$ is Uδ-PE if and only if (18) holds (*cf.* Fact 1). ■

A.1 Example 1: the Lorenz system

The controlled Lorenz chaotic system is defined as

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) + u_1 \\ \dot{x}_2 &= rx_1 - x_2 - x_1x_3 + u_2 \\ \dot{x}_3 &= x_1x_2 - bx_3 + u_3\,. \end{aligned}$$

[5]For simplicity, with an abuse of notation, we write $P(t,x)$ instead of $P(t,\tilde{x} + x_d(t))$.

The uncontrolled ($u = 0$) Lorenz oscillator exhibits a chaotic behaviour if $\sigma = 16$, $r = 45.6$ and $b = 4$. Defining $\theta := \text{col}[\sigma, r, b]$, this system takes the form (15), *i.e.*

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & x_1\\ 0 & -x_1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}x_1 - x_2 & 0 & 0\\ 0 & -x_1 & 0\\ 0 & 0 & x_3\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix} = \begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}.$$

For this system, given the reference trajectories $x_d(t)$ — possibly generated by another Lorenz system or any other type of chaotic oscillator— the adaptive controller (17) takes the form

$$\begin{aligned} u &= \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & x_1\\ 0 & -x_1 & 0\end{bmatrix}\begin{bmatrix}x_{1d}\\ x_{2d}\\ x_{3d}\end{bmatrix} - \begin{bmatrix}k_1 & 0 & 0\\ 0 & k_2 & 0\\ 0 & 0 & k_3\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\\ \tilde{x}_3\end{bmatrix} \\ &\quad + \begin{bmatrix}x_1 - x_2 & 0 & 0\\ 0 & -x_1 & 0\\ 0 & 0 & x_3\end{bmatrix}\begin{bmatrix}\hat{\theta}_1\\ \hat{\theta}_2\\ \hat{\theta}_3\end{bmatrix} + \begin{bmatrix}\dot{x}_{d1}\\ \dot{x}_{d2}\\ \dot{x}_{d3}\end{bmatrix} \end{aligned}$$

$$\dot{\hat{\theta}} = -\gamma \begin{bmatrix}x_1 - x_2 & 0 & 0\\ 0 & -x_1 & 0\\ 0 & 0 & x_3\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\\ \tilde{x}_3\end{bmatrix}$$

where k_1, k_2 and k_3 are positive constants. Indeed, a simple calculation yields that, choosing $P(t, x) := I$, the inequality (16) becomes, simply,

$$-\xi^\top \begin{bmatrix}k_1 & 0 & 0\\ 0 & k_2 + 1 & 0\\ 0 & 0 & k_3\end{bmatrix}\xi \leq -\min\{k_1, k_2 + 1, k_3\}\|\xi\|^2.$$

We conclude that this condition is verified for any control parameters such that $k_1 > 0$, $k_2 > -1$ and $k_3 > 0$. The persistency of excitation condition may be verified numerically; yet, it is evident that this condition would not hold if $x_1(t) \equiv x_2(t)$.

We have tested in simulations the performance of the adaptive controller given above. In the simulation set-up we assume that the reference trajectories $x_d(t)$ are generated by a Rössler system for which we chose physical parameters that make the (master) system chaotic *i.e.*

$$\dot{x}_{1d}(t) = x_{2d}(t) + a x_{1d}(t) \quad (19a)$$
$$\dot{x}_{2d}(t) = -x_{1d}(t) - x_{3d}(t) \quad (19b)$$
$$\dot{x}_{3d}(t) = b + x_{3d}(t)[x_{2d}(t) - c] \quad (19c)$$

where $a = 0.15$, $b = 0.2$ and $c = 10$. The simulation set-up is as follows: for 40s the Lorenz oscillator is uncontrolled and describes a typical Lorenz chaotic behaviour — the parameters of the Lorenz system were fixed at $\sigma = 16$, $b = 45.6$ and $r = 4$. Then, the control action starts on at $t = 40s$ and the Lorenz system's trajectories synchronise, after a short transient, to the Rössler master system —the Rössler chaotic behaviour is appreciated in the plots.

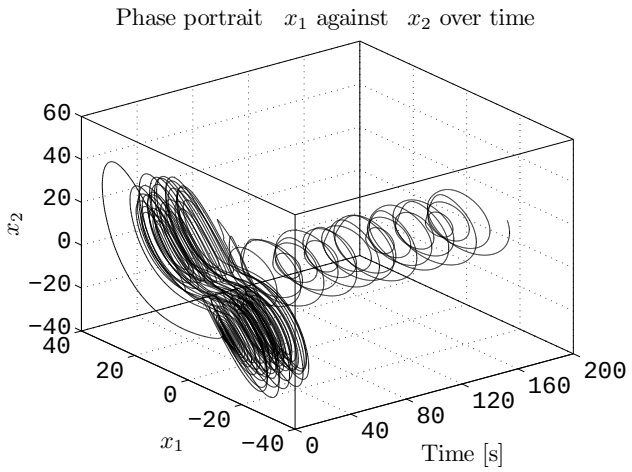

Fig. 1. Adaptive tracking control of Lorenz system: state integral curves for x_1 and x_2

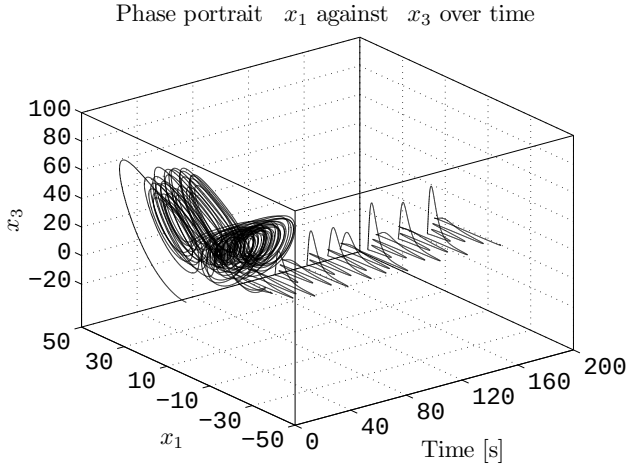

Fig. 2. Adaptive tracking control of Lorenz system: state integral curves for x_1 and x_3

The control input gains were chosen as $k_1 = k_2 = k_3 = 10$. The controller uses estimated parameters initialised at zero and that evolve according to the adaptation law with an adaptation gain $\gamma = 0.1$. Figures 1–3 show the phase portraits (integral curves) of the state responses while Figures 4–6 show the evolution, against time, of the respective estimated parameters. Note the convergence of the estimated parameters (of the Lorenz system) to their true values. We also show, in Figures 7–9, the graphs of the control inputs and in Figures 10 and 11, we represent the tracking errors between the actual Lorenz and the reference Rössler trajectories.

We stress that our theoretical conditions are, in general, both sufficient and necessary. The *particular* choice of lower-bounds ($k_1 > 0$, $k_2 > -1$ and $k_3 > 0$) for the control gains in this example remains only *sufficient*. For completeness we show, in Figure 12, the system's response for the choice $k_1 = 0$, and $k_2 = k_3 = 10$ *i.e.*, violating the conditions given. Note that only tracking in the last two variables is achieved. From this one may conclude that

UGAS is not achieved for this particular choice of the gains, which violates the numeric conditions given above, but one shall not haste to conclude that such *particular* numeric conditions are necessary.

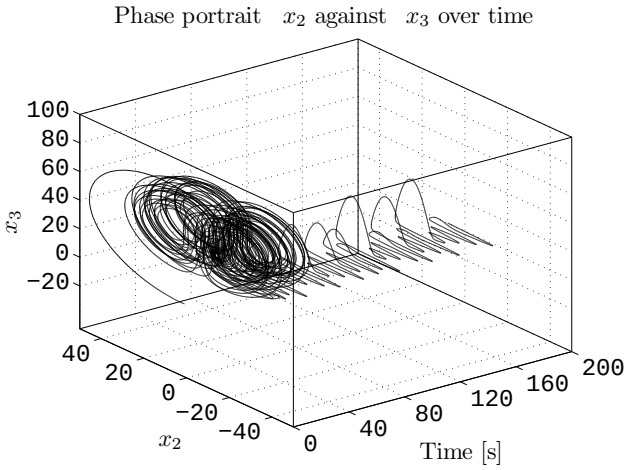

Fig. 3. Adaptive tracking control of Lorenz system: state integral curves for x_2 and x_3

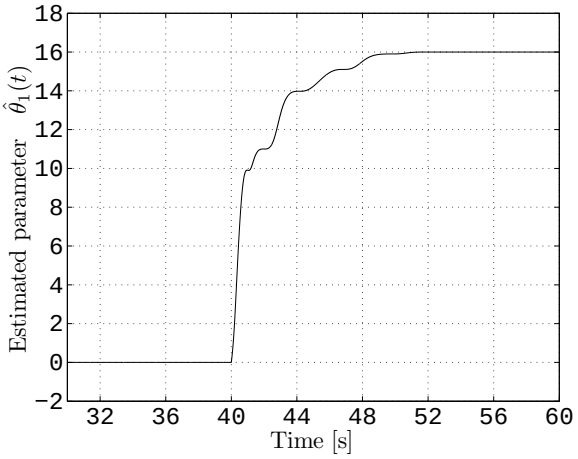

Fig. 4. Adaptive tracking control of Lorenz system: estimated parameter $\hat{\theta}_1$ converging to its true value

A.2 Example 2: the Rössler system

Let us consider now the adaptive tracking control problem for a Rössler system with input, given by

$$\begin{aligned} \dot{x}_1 &= ax_1 + x_2 + u_1 \\ \dot{x}_2 &= -x_1 - x_3 + u_2 \\ \dot{x}_3 &= b + x_3[x_2 - c] + u_3 \,. \end{aligned}$$

Notice that, defining $\theta := \text{col}[a,\, b,\, c]$, this system can also be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -x_3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -x_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & x_3 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} .$$

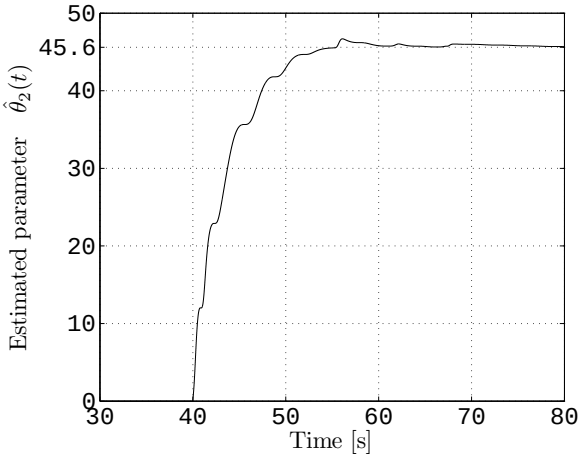

Fig. 5. Adaptive tracking control of Lorenz system: estimated parameter $\hat{\theta}_2$ converging to its true value

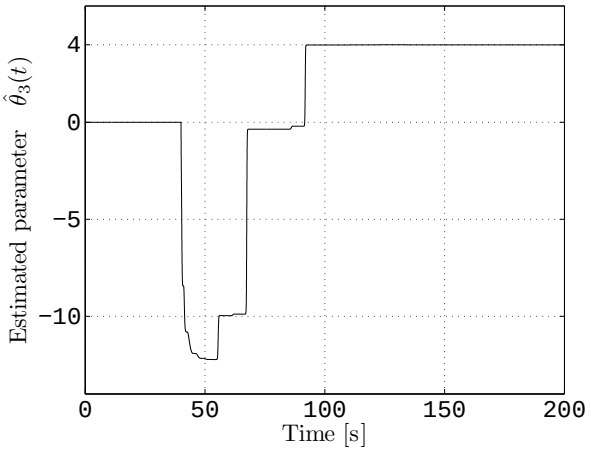

Fig. 6. Adaptive tracking control of Lorenz system: estimated parameter $\hat{\theta}_3$ converging to its true value

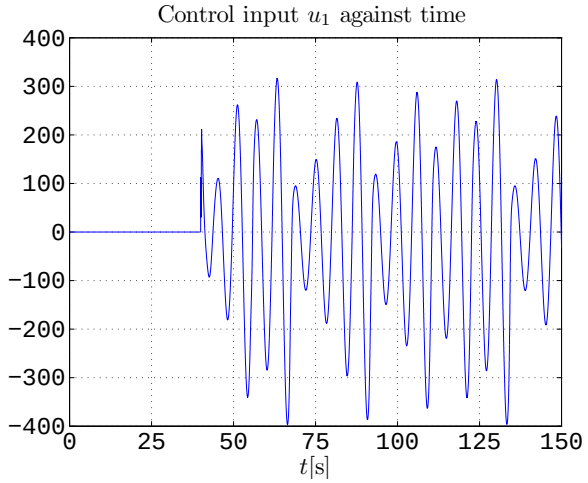

Fig. 7. Adaptive tracking control of Lorenz system: control input $u_1(t)$

For this system, given the reference trajectories $x_d(t)$ — possibly generated by another Rössler system or any type

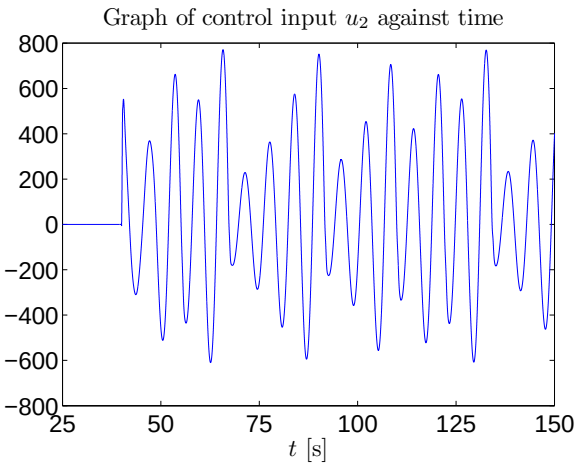

Fig. 8. Adaptive tracking control of Lorenz system: control input $u_2(t)$

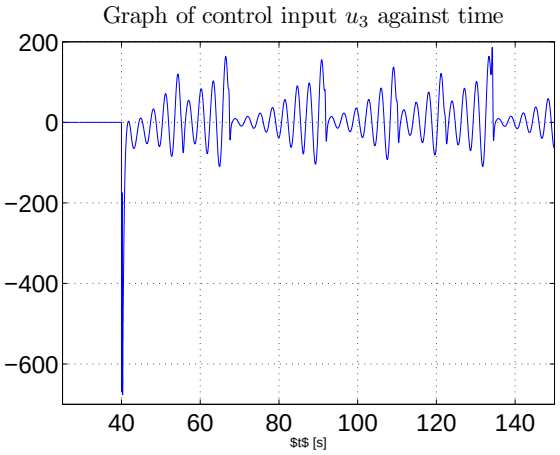

Fig. 9. Adaptive tracking control of Lorenz system: control input $u_3(t)$

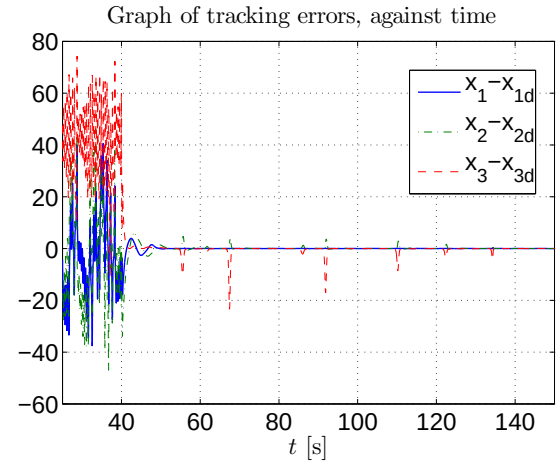

Fig. 10. Adaptive tracking control of Lorenz system: tracking errors between Lorenz and (Rössler) reference trajectories, $x(t) - x_d(t)$

of chaotic oscillator— the adaptive controller (17) takes the

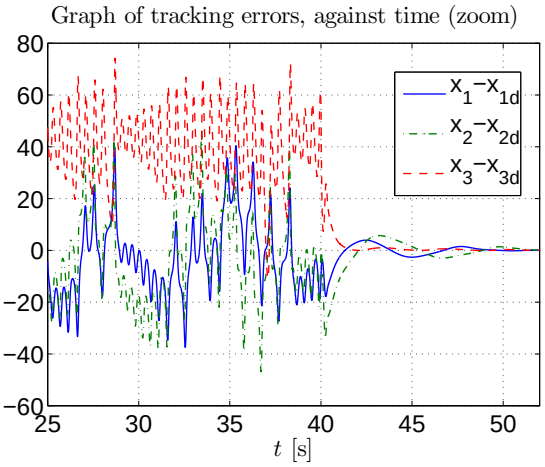

Fig. 11. Adaptive tracking control of Lorenz system: zoom on tracking errors between Lorenz and (Rössler) reference trajectories

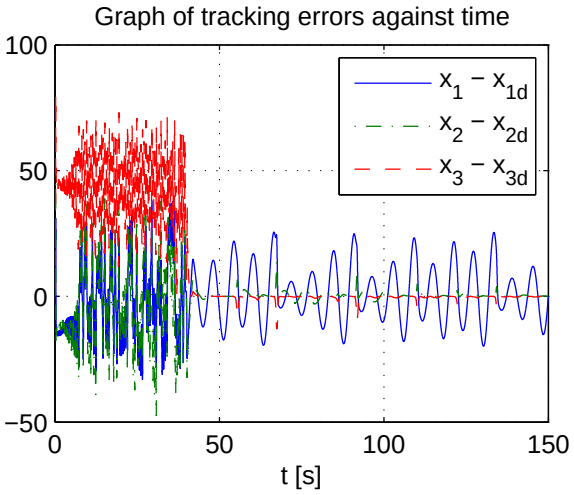

Fig. 12. Adaptive tracking control of Lorenz system: violation of stability conditions on control gains lead to tracking errors

form

$$u = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -x_3 & 0 \end{bmatrix} \begin{bmatrix} x_{1d} \\ x_{2d} \\ x_{3d} \end{bmatrix} - \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & x_3 & k_3 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} + \begin{bmatrix} -x_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & x_3 \end{bmatrix} \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \end{bmatrix} + \begin{bmatrix} \dot{x}_{d1} \\ \dot{x}_{d2} \\ \dot{x}_{d3} \end{bmatrix} \quad (20a)$$

$$\dot{\hat{\theta}} = -\gamma \begin{bmatrix} -x_1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & x_3 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} \quad (20b)$$

where k_1, k_2 and k_3 are positive constants such that $k_2 k_3 > \frac{1}{4}$. To see this, we verify the inequality (16) which, for $P(t,x) := I$, becomes

$$-\xi^\top \begin{bmatrix} 2k_1 & 0 & 0 \\ 0 & 2k_2 & 1 \\ 0 & 1 & 2k_3 \end{bmatrix} \xi \leq -q\|\xi\|^2$$

where q is a positive number if and only if $k_2k_3 > \frac{1}{4}$ and $k_i > 0$; indeed, a simple calculation shows that the minimal eigenvalue of the matrix above corresponds to

$$\min\left\{2k_1,\ k_2+k_3-\sqrt{(k_2+k_3)^2-(4k_2k_3-1)}\right\}.$$

We have tested in simulations the performance of the adaptive controller given above. In the simulation set-up we assume that the reference trajectories $x_d(t)$ are generated by a Lorenz system for which we chose physical parameters that make the (master) system chaotic *i.e.*

$$\dot{x}_{1d}(t) = \sigma(x_{2d}(t)-x_{1d}(t)) \quad (21a)$$
$$\dot{x}_{2d}(t) = rx_{1d}(t)-x_{2d}(t)-x_{1d}(t)x_{3d}(t) \quad (21b)$$
$$\dot{x}_{3d}(t) = x_{1d}(t)x_{2d}(t)-bx_{3d}(t)\,. \quad (21c)$$

with $\sigma = 16$, $r = 45.6$ and $b = 4$.

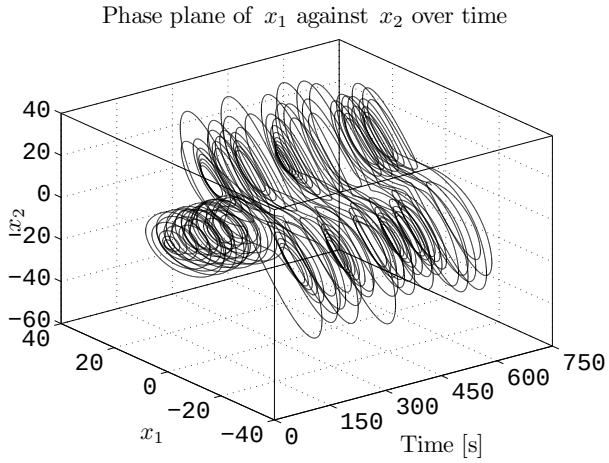

Fig. 13. Adaptive tracking control of Rössler system: state integral curves for x_1 *vs.* x_2

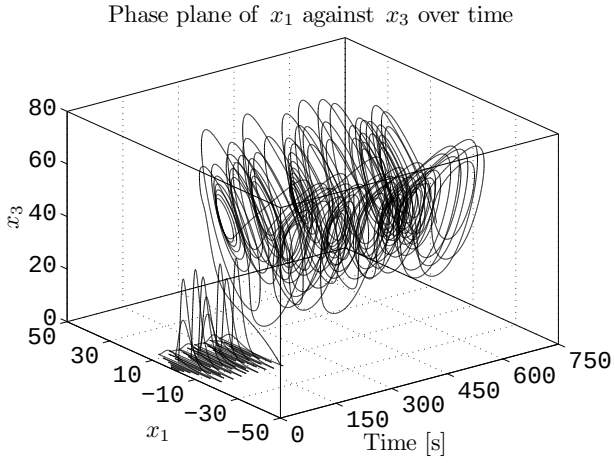

Fig. 14. Adaptive tracking control of Rössler system: state integral curves for x_1 *vs.* x_3

The simulation set-up is as follows: for 200s the Rössler oscillator is uncontrolled and describes a typical chaotic behaviour —the parameters of the Rössler system were fixed

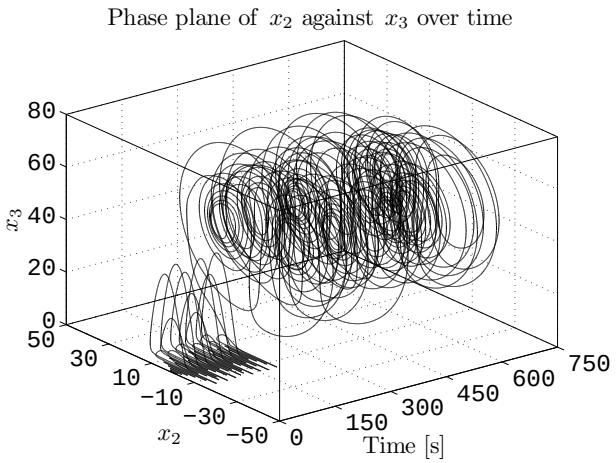

Fig. 15. Adaptive tracking control of Rössler system: state integral curves for x_2 *vs.* x_3

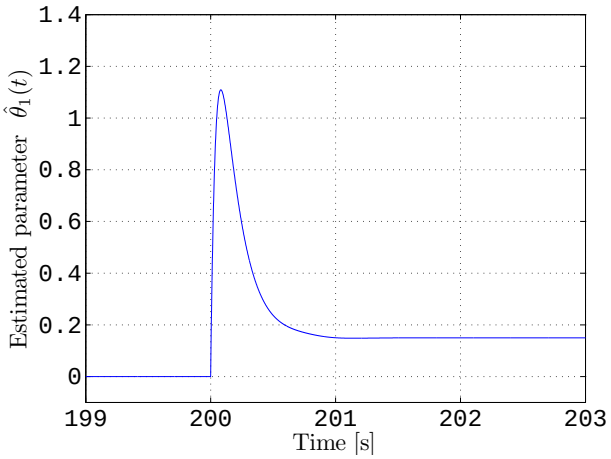

Fig. 16. Adaptive tracking control of Rössler system: estimated parameter $\hat{\theta}_1(t)$ converging to its true value

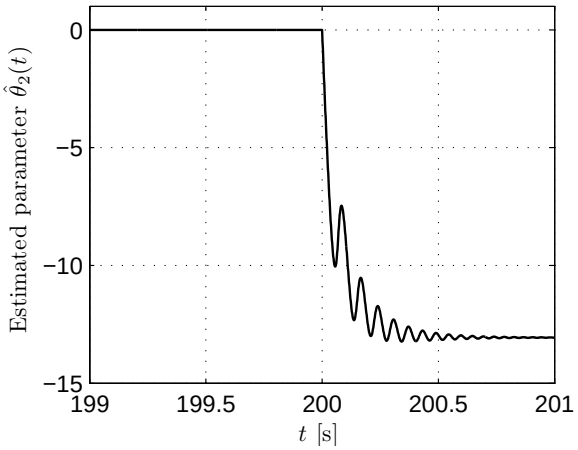

Fig. 17. Adaptive tracking control of Rössler system: estimated parameter $\hat{\theta}_2(t)$ *not* converging to its true value

at $a = 0.15$, $b = 0.2$ and $c = 10$; the control action starts on at $t = 200s$ and the Rössler system's trajectories syn-

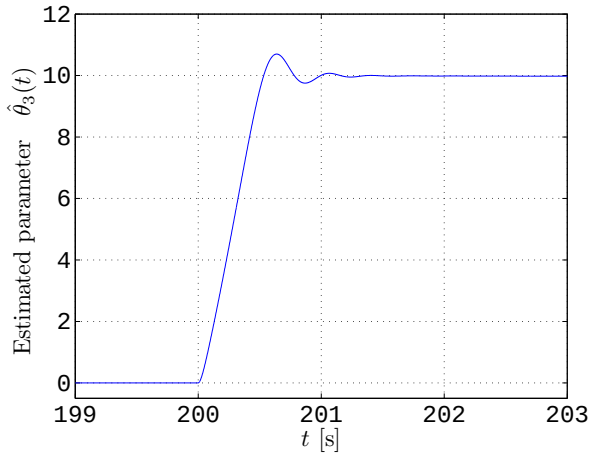

Fig. 18. Adaptive tracking control of Rössler system: estimated parameter $\hat{\theta}_3(t)$ converging to its true value

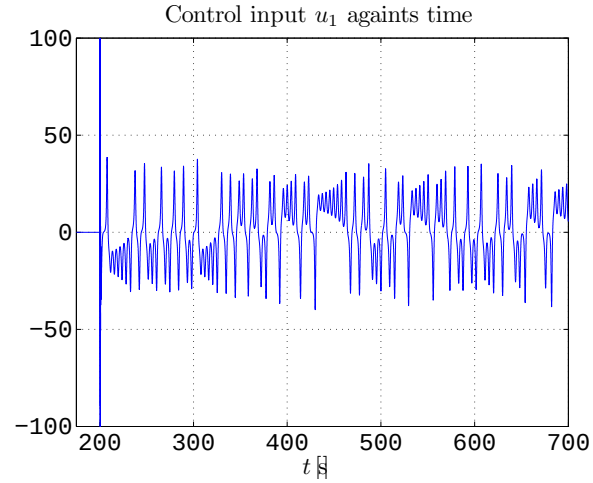

Fig. 19. Adaptive tracking control of Rössler system: control input $u_1(t)$

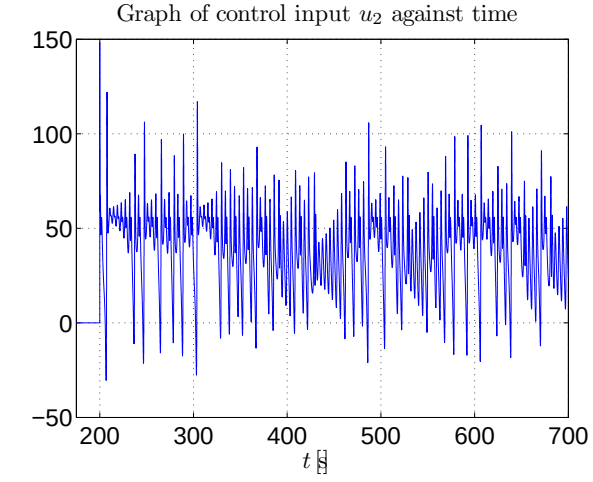

Fig. 20. Adaptive tracking control of Rössler system: control input $u_2(t)$

chronise to the Lorenz master system —the Lorenz chaotic behaviour described by the *Rössler* system is appreciated

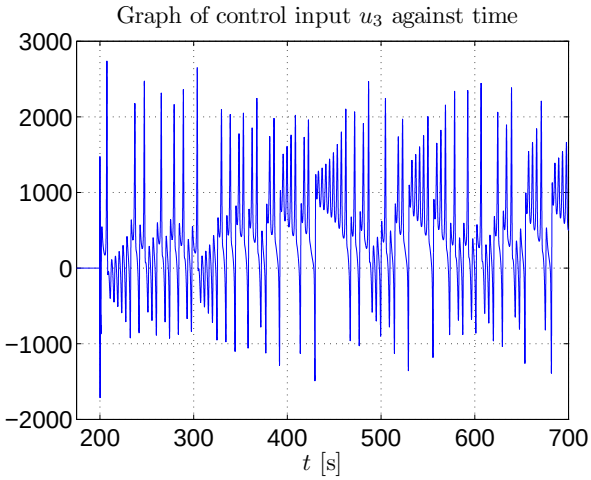

Fig. 21. Adaptive tracking control of Rössler system: control input $u_3(t)$

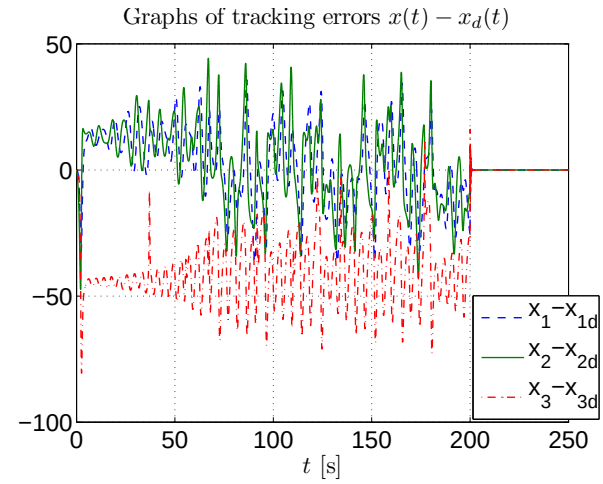

Fig. 22. Adaptive tracking control of Rössler system: tracking errors between Rössler solutions and (Lorenz) reference trajectories $x - x_d(t)$

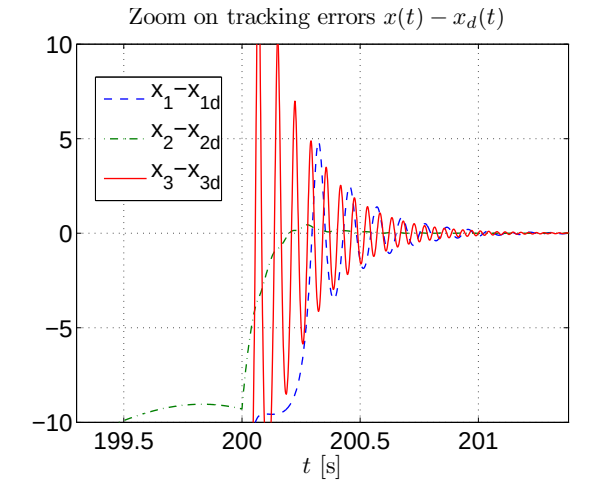

Fig. 23. Adaptive tracking control of Rössler system: zoom on tracking errors between Rössler solutions and (Lorenz) reference trajectories $x - x_d(t)$

in the plots, *cf.* Figures 13–23. For the simulation purposes, the trajectories of the Lorenz master system were slowed down by a factor of 10 since it is naturally much faster than the Rössler system.

The control input gains were chosen as $k_1 = k_2 = k_3 = 10$. The controller uses estimated parameters initialised at zero and that evolve according to the adaptation law with an adaptation gain $\gamma = 0.1$. Figures 13–15 show plots of the state integral curves corresponding to the controlled slave system and Figures 16–18 show plots of the estimated parameters, against time; notice that convergence of the estimated parameters (of the Rössler system) to their true values is achieved, except for $\hat{\theta}_2(t) = \hat{b}(t)$ which converges to a steady-state value different from $b = 0.2$. This is in accordance with the theoretical results: a closer look at the matrix $\Psi(x)$ reveals that the *necessary* condition of persistency of excitation for uniform parameter convergence cannot be met; indeed, we have

$$\int_t^{t+T} \Psi(x_d(s))\Psi(x_d(s))^\top ds = \begin{bmatrix} \int_t^{t+T} x_{d1}(s)^2 ds & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \int_t^{t+T} [x_{d3}(s)^2 + 1] ds \end{bmatrix}$$

for which matrix there is clearly no $T > 0$ to render it positive definite. Yet, the tracking errors converge to zero asymptotically; this is in view of the commonly known, in control theory, *certainty equivalence principle* (*cf.* [17]). However, in this *particular* case, one cannot conclude uniform global asymptotic stability of the origin.

Other plots are shown in Figures 19–23: specifically the control inputs on Figures 19–21 and the tracking errors on Figures 22 and 23; note from the latter that the tracking control goal is achieved almost instantaneously.

B. Duffing equation

Let us consider now the tracking control problem for the controlled (normalised) Duffing equation

$$\ddot{q} + \theta_1 q + q^3 + \theta_2 \dot{q} - \theta_3 \cos \omega t = u \tag{22}$$

where θ_i are supposed to be unknown but constant, ω is known and u is a control input. Let $\theta := \mathrm{col}[\theta_1, \theta_2, \theta_3]$ and $\hat{\theta}$ denote its estimate. Define the *regressor* function $\Psi : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \to \mathbb{R}^3$ as $\Psi(t, q, \dot{q}) := [q, \dot{q}, -\cos \omega t]$. The uncontrolled Duffing equation (with $u = 0$) exhibits chaotic behaviour if $\theta_1 = -1.1$, $\theta_2 = 0.4$, $\theta_3 = 2.1$ and $\omega = 1.8$

Proposition 2 Let $q_* : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be the reference trajectory, assumed to be twice continously differentiable, bounded and with bounded derivatives. Define the tracking velocity and position errors $\dot{\tilde{q}} := \dot{q} - \dot{q}_*$ and $\tilde{q} := q - q_*$. Consider the system (22) in closed loop with

$$\begin{aligned} u &= \ddot{q}_* + {q_*}^3 + 3q^2 q_* - 3q{q_*}^2 + \hat{\theta}_1 q + \hat{\theta}_2 \dot{q} - \hat{\theta}_3 \cos \omega t \\ &\quad -k_d \dot{\tilde{q}} - k_{p1} \tilde{q} - k_{p2} \tilde{q}^3 \qquad (23a) \\ \dot{\hat{\theta}} &= -\gamma \Psi(t, q, \dot{q})^\top [\dot{\tilde{q}} + \varepsilon \tilde{q}] \qquad (23b) \end{aligned}$$

where γ, ε, k_{p1}, k_{p2}, and k_d are positive constants. Then, the closed loop system is UGAS if and only if $\Upsilon(t) := \Psi(t, q_*(t), \dot{q}_*(t))$ is persistently exciting; in particular, the estimation errors $\tilde{\theta} := \hat{\theta} - \theta$ converge uniformly to zero for any initial conditions, if and only if the reference trajectories are such that Υ is PE. □

Proof . Define the state variables $x_1 := \mathrm{col}[x_{11}, x_{12}] = \mathrm{col}[\tilde{q}, \dot{\tilde{q}}]$ then, the closed-loop system becomes

$$\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \end{bmatrix} = \begin{bmatrix} x_{12} \\ -(k_{p2}+1)x_{11}^3 - k_d x_{12} - k_{p1} x_{11} \end{bmatrix} + \begin{bmatrix} 0 \\ \Psi(t, x_{11} + q_*(t), x_{12} + \dot{q}_*(t)) \end{bmatrix} x_2 \tag{24}$$

$$\dot{\tilde{\theta}} = -\gamma \Psi(t, x_{11} + q_*(t), x_{12} + \dot{q}_*(t))^\top [x_{12} + \varepsilon x_{11}] . \tag{25}$$

That is, the closed-loop system has the form (9). Then, stability proof follows applying Theorem 1. Assumption 2 holds with the Lyapunov function

$$V(t, x, \tilde{\theta}) := \frac{1}{2} \left[x_{12}^2 + \frac{k_{p2}+1}{2} x_{11}^4 + k_{p1} x_{11}^2 + 2\varepsilon x_{11} x_{12} + \frac{1}{\gamma} \|\tilde{\theta}\|^2 \right] .$$

By direct calculations one can show that that, if $2\varepsilon^2 \leq k_{p1}$,

$$\left[x_{12}^2 + \frac{k_{p2}+1}{2} x_{11}^4 + (k_{p1} + \varepsilon^2) x_{11}^2 + \frac{1}{\gamma} \|\tilde{\theta}\|^2 \right] \geq V(t, x, \tilde{\theta}) \geq \frac{1}{4} \left[x_{12}^2 + (k_{p2}+1) x_{11}^4 + \frac{1}{\gamma} \|\tilde{\theta}\|^2 \right]$$

hence, (11a) holds. Furthermore, the derivative of V along the trajectories of (24) yields

$$\dot{V}(t, x, \tilde{\theta}) = -\frac{k_d}{2} x_{12}^2 - \varepsilon(k_{p2}+1) x_{11}^4 - \frac{\varepsilon k_{p1}}{2} x_{11}^2 - \left[\left(\frac{k_d}{2} - \varepsilon \right) x_{12}^2 + \varepsilon k_d x_{11} x_{12} + \frac{\varepsilon k_{p1}}{2} x_{11}^2 \right] .$$

The sum of three terms in brackets on the right hand side is nonnegative if and only if

$$0 \leq \varepsilon \leq \frac{k_{p1} k_{d1}}{2k_{p1} + k_d^2}$$

hence, taking into account the condition that $k_{p1} \geq 2\varepsilon^2$, we obtain that

$$\dot{V}(t, x, \tilde{\theta}) \leq -\frac{k_d}{2} x_{12}^2 - \varepsilon(k_{p2}+1) x_{11}^4 - \frac{\varepsilon k_{p1}}{2} x_{11}^2$$

for any positive k_{p1}, k_{p2} and k_d. Thus, Assumption 2 holds. To see that Assumption 3 holds we first observe that, in this case,

$$A(t, x_1) := \begin{bmatrix} x_{12} \\ -(k_{p2}+1)x_{11}^3 - k_d x_{12} - k_{p1} x_{11} \end{bmatrix}$$

$$B := \begin{bmatrix} 0 \\ \Psi(t, x_{11} + q_*(t), x_{12} + \dot{q}_*(t)) \end{bmatrix} x_2$$

$$H(t, x)) = -\gamma \Psi(t, x_{11} + q_*(t), x_{12} + \dot{q}_*(t))^\top [x_{12} + \varepsilon x_{11}] ,$$

and hence,

$$B_\circ(t, x_2) = \begin{bmatrix} 0 \\ \Psi(t, q_*(t), \dot{q}_*(t)) \end{bmatrix} x_2 \,.$$

Note that $A(t, x_1)$ and $H(t, x)$ are uniformly bounded in t provided that the reference trajectories and velocities, q_* and $\dot{q}_*$, are bounded; also, $A(t, 0) \equiv 0$ and $H(t, x)$ depends only on x_1 and $H = 0$ if $x_1 = 0$.

Finally, notice that $\|B_\circ\| = \|\Upsilon(t)x_2\|$ therefore, in view of Fact 1, $B_\circ$ is Uδ-PE with respect to x_2 if and only if $\Upsilon(t)$ is PE. Hence, Assumption 4 holds; this ends the proof. ■

We have tested in simulations, the performance of our adaptive controller. We assume that the reference trajectory $q_*(t)$ is generated by a van der Pol oscillator with parameter values leading to a chaotic behaviour, *i.e.* the reference trajectory $x_d(t) := [q_*(t),\, \dot{q}_*(t)]^\top$ satisfies the second-order differential equation:

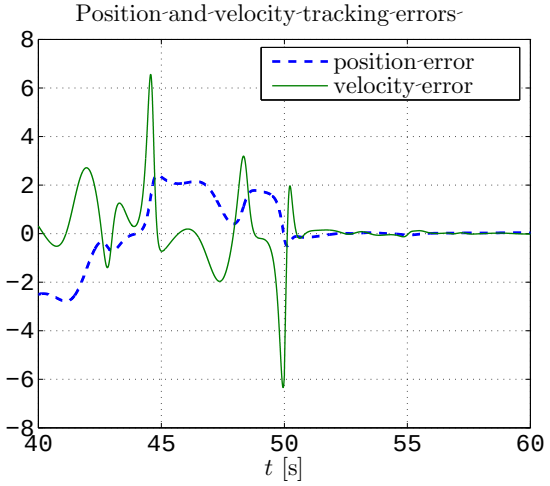

Fig. 24. Duffing equation: tracking errors

$$\ddot{q}_*(t) - \mu[1 - q_*(t)^2]\dot{q}_*(t) + q_*(t) = a\cos(\omega_v t) \qquad (26)$$

with $\mu = 3$ and $a = 5$ and $\omega_v = 1.7880$rad/s. The control gains for the controller (23) were chosen as follows: $k_{p1} = k_{p2} = k_d = \gamma = 10$. The simulation scenario is as follows: the Duffing equation is left uncontrolled, driven by the signal $\theta_3 \cos(\omega t)$ for 50s; then, the control starts and the duffing response immediately follows the reference trajectories given by the van der Pol system, *i.e.*, synchronising with the latter. We present the simulation results in Figures 24–28.

Figure 24 shows the tracking errors between the actual Duffing trajectories and the (reference) van der Pol's *i.e.*, $q - q_*(t)$ and $\dot{q} - \dot{q}_*$; the phase portrait of position against velocity is depicted on Figure 25; state integral curves are shown in Figure 26; the evolution of the three estimated parameters are shown in Figure 27 –note the convergence to their true values $\theta_1 = -1.1$, $\theta_2 = 0.4$ and $\theta_3 = 2.1$. Finally, Figure 28 shows the control inputs $u(t, q(t), \dot{q}(t))$.

IV. Conclusions

We have studied the adaptive tracking control problem of chaotic systems. We presented general results that cover different controlled chaotic systems such as the Duffing equation, the Lorenz and Rössler systems. In particular, we addressed the open problem of tracking control in the presence of parametric uncertainty and guaranteeing con-

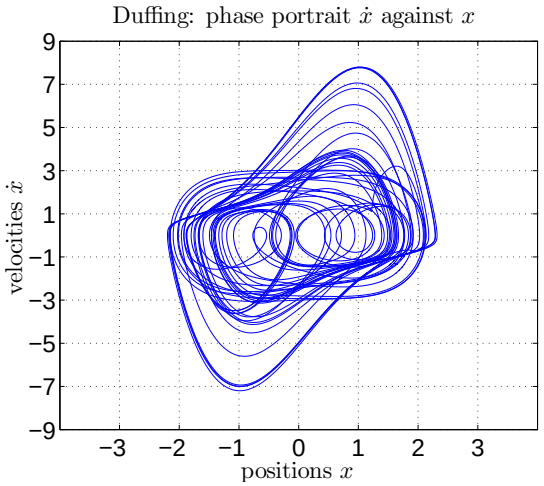

Fig. 25. Duffing equation: phase portrait of position against velocity

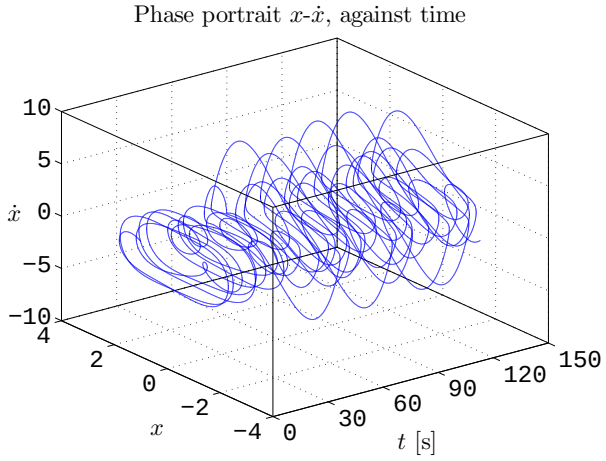

Fig. 26. Duffing equation: state integral curve

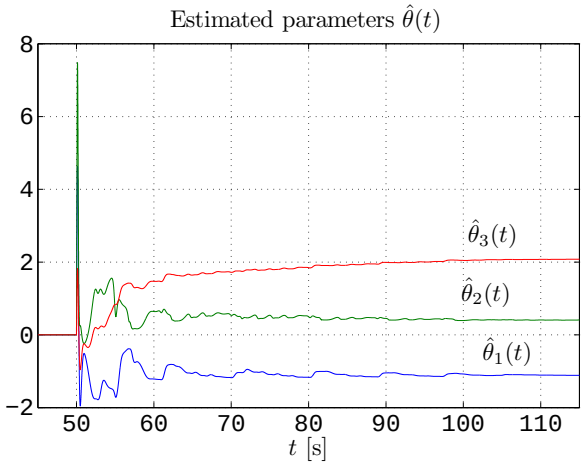

Fig. 27. Duffing equation: evolution of the estimated parameters

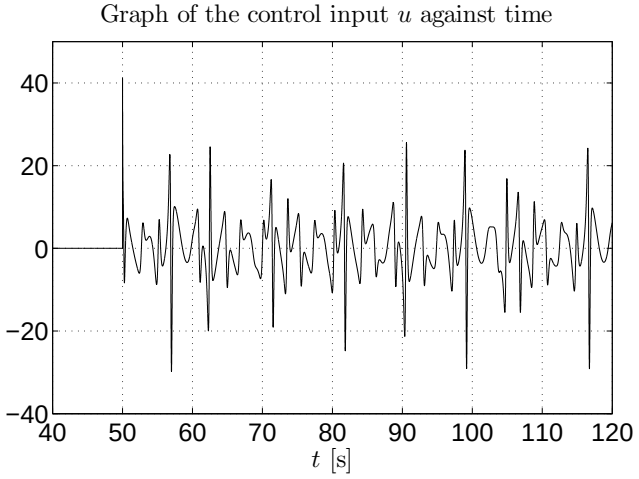

Fig. 28. Duffing equation: controlinput

vergence of the estimated parameters to their true values. A direct application of our results is in the synchronisation of chaotic systems; we have illustrated the usefulness of our findings via simulations that show that synchronisation of different systems is possible.

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