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Further results on the global continuous control for finite-time and exponential stabilization of constrained-input mechanical systems: desired conservative-force compensation and experiments

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Abstract: Saturating-Proportional-Derivative (SPD) type global continuous control for the finite-time or (local) exponential stabilization of mechanical systems with bounded inputs is achieved involving *desired* conservative-force compensation. Far from what one could expect, the proposed controller is not a simple extension of the *on-line* compensation case but it rather proves to entail a closed-loop analysis with considerably higher degree of complexity. This gives rise to more involved requirements to guarantee its successful performance and implementability. Interesting enough, the proposal even shows that actuators with higher power-supply capabilities than in the on-line compensation case are required. Other important analytical limitations are further overcome through the developed algorithm. Experimental tests on a 2-degree-of-freedom robotic arm corroborate the efficiency of the proposed scheme.

1 Introduction

A global continuous state-feedback scheme for the finite-time and exponential stabilization of mechanical systems with bounded inputs has been recently proposed and thoroughly motivated in [1]. Giving a formal solution to the corresponding formulated problem under the explicit consideration of input constraints and the explicit choice on the system trajectory convergence (among finite-time and exponential) constitute the main distinctions of such an approach with respect to continuous finite-time controllers developed for mechanical systems before its appearance: [2–4] (which were developed in an unconstrained input context; see for instance [1, §1] for a brief description of such previous works). But the distinctive features do not stop there: while the cited previous approaches mainly rely on the *dynamic inversion* technique —or exact compensation of the whole dynamics— (except for one of the two controllers presented in [2]), the scheme in [1] benefits from the inherent passive nature of mechanical systems. This is done by keeping a (Saturating) Proportional-Derivative type structure with exclusive compensation of the conservative-force (vector) term as a direct way to suitably reshape the closed-loop potential energy so as to set the desired posture as the only equilibrium position on the whole configuration space (of course, with the required stability property). Through such an *on-line* compensation of the conservative-force term —exclusively (instead of compensating the whole dynamics)—, the system model dependence of the designed scheme is considerably reduced, consequently simplifying the control structure and decreasing the inherent inconveniences of modelling inaccuracies as well as the implied computation burden. But these improvements could still be potentiated if the on-line compensation term could be replaced by the conservative-force term exclusively evaluated at the desired position. Such a *desired conservative-force compensation* idea was first introduced in an unconstrained-input conventional (infinite-time) stabilization context by [5] and, ever since its introduction in the literature, it has been much appreciated in view of its simplicity and simplification improvements. This constitutes the main motivation of this work which aims at developing a *desired-conservative-force-compensation* extension of the SPD-type (Saturating-Proportional-Derivative) finite-time/exponential stabilization scheme from [1]. Far from what one could expect, such a design task is not as simple or direct as a simple replacement of the *on-line* compensation term by the *desired* one. Such a replacement happens to keep the required (desired) closed-loop equilibrium position but not its uniqueness. Contrarily to the on-line compensation case [where the open-loop conservative forces are (ideally) cancelled out], in the desired compensation case further design requirements prove to be needed so as to guarantee that the control-induced potential energy component *dominates* the open-loop one (in order to guarantee uniqueness of the desired closed-loop equilibrium configuration). This was already pointed out in the unconstrained-input conventional case [5], where such a domination goal was shown to be achieved through a P control (vector) term with a(n absolutely) stronger growing rate than that of the open-loop conservative force term in any direction (at every point) on the configuration space; in particular, under the simple consideration of uncoupled linear P and D control actions, this was shown to be achieved by simply fixing P gains higher than the highest (induced) norm value of the Jacobian matrix of the conservative force term (assuming that such a Jacobian matrix is bounded) [6]. But the solution of the referred uniqueness issue cannot be that simple in the analytical context considered here —under the consideration of input constraints, the contemplated type of trajectory convergence (finite-time or exponential) and the generalized form of the SPD controller component— in view of the special functions involved in the SPD term to guarantee the achievement of the formulated stabilization goal. This represents an important analytical challenge to which this work succeeds to give a solution enjoying the technical benefits from desired conservative-force compensation. Interesting enough, the exhaustive analysis developed here further brings to the fore that actuators with higher power-supply capabilities than in the on-line-compensation case are required. This results from the *worst-case* type design (analytical) procedure followed to guarantee the achievement of the previously described domination feature of the controller-induced conservative force term over the open-loop system one. As a matter of fact, it is the permanence of the open-loop conservative-force term on the system dynamics which is at the origin of the design complication and higher degree of complexity of the closed-loop analysis (with respect to the on-line compensation case where such a term is absent in view of its cancellation). For instance, a further complication to be dealt with —and overcome in this work— is on the support of the controller ability to transit from finite-time to exponential stabilization through a simple control parameter. Indeed, for exponential stability purposes, the counterbalanced (in the desired-compensation sense) open-loop conservative forces happen to lack of the properties required in the *homogeneity-oriented* analytical framework within which [1] and this work are developed. Thus, such a stabilization case has to be treated differently. This work gives a suitable solution to such an additional complication of the closed-loop analysis by supporting the exponential stabilization case through a strict Lyapunov function.

During the preparation of the present work, the authors got aware of the work [7] where an energy-shaping-based state-feedback finite-time regulation design method for robot manipulators was presented. Such a method was shown to give rise to several control laws achieving the formulated control objective, generally structured through (separated) P and D type actions. These respectively result from the difference of a *closed-loop desired* and open-loop potential energy functions and an *energy dissipation* function. Since the latter is considered to be exclusively dependent on the system velocity vector variable, it is not clear how such a method could give rise to controllers with *gathered* P and D type actions [*i.e.* with both of them within a common (suitable shaping) function] like the SPD schemes of [1] and the present work. As a matter of fact, the SPD type schemes presented in [1] and this work prove to be generalized enough to include the SP-SD type structure (with separated P and D type actions) as a special case. This was already shown in the on-line compensation case developed in [1] and will be shown to be the case in the desired compensation context developed here. Both the on-line and desired compensation versions of the developed SPD scheme are experimentally corroborated here on a 2-degree-of-freedom (DOF) robot manipulator, which is a complementary distinction of this work.

2 Preliminaries

Let $X \in \mathbb{R}^{m\times n}$ and $y \in \mathbb{R}^n$. Throughout this work, X_{ij} denotes the element of X at its i^{th} row and j^{th} column, X_i represents the i^{th} row of X and y_i stands for the i^{th} element of y. With $m = n$, $X > 0$ (conventionally) denotes that X is positive definite while, for a symmetric matrix X, $\lambda_m(X)$ and $\lambda_M(X)$ respectively stand for its minimum and maximum eigenvalues. 0_n represents the origin of $\mathbb{R}^n$ and I_n the $n \times n$ identity matrix. We denote $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ for scalars, and $\mathbb{R}^n_{>0} = \{x \in \mathbb{R}^n : x_i > 0,\ i = 1,\dots,n\}$ and $\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n : x_i \geq 0,\ i = 1,\dots,n\}$ for vectors. $\|\cdot\|$ stands for the standard Euclidean norm for vectors and induced norm for matrices. An $(n-1)$-dimensional sphere of radius $c > 0$ on $\mathbb{R}^n$ is denoted S^{n-1}_c, *i.e.* $S^{n-1}_c = \{x \in \mathbb{R}^n : \|x\| = c\}$. For a continuously differentiable scalar function $f : \mathbb{R}^n \to \mathbb{R}$ and a vector function $g : \mathbb{R}^n \to \mathbb{R}^n$, we denote $D_g f$ the directional derivative of f along g, *i.e.* $D_g f(x) = \frac{\partial f}{\partial x} g(x)$. We will consider the sign function to be zero at zero, *i.e.*

$$\text{sign}(\varsigma) = \begin{cases} \frac{\varsigma}{|\varsigma|} & \text{if } \varsigma \neq 0 \\ 0 & \text{if } \varsigma = 0 \end{cases}$$

and denote $\text{sat}(\cdot)$ the standard (unitary) saturation function, *i.e.* $\text{sat}(\varsigma) = \text{sign}(\varsigma)\min\{|\varsigma|, 1\}$. The contents of the following subsections —except for some complementary properties, assumptions and considerations— were mostly included in [1, §2]; for the sake of completeness, they are reproduced here.

2.1 Mechanical systems

Consider the n-degree-of-freedom (n-DOF) fully-actuated frictionless mechanical system dynamics [8, §6.1]

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \tag{1}$$

where $q, \dot{q}, \ddot{q} \in \mathbb{R}^n$ are the position (generalized coordinates), velocity, and acceleration vectors; $H(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix; $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$ is the Coriolis and centrifugal effect matrix defined through the Christoffel symbols of the first kind; $g(q) = \nabla\mathcal{U}_{\text{ol}}(q)$, with $\mathcal{U}_{\text{ol}} : \mathbb{R}^n \to \mathbb{R}$ being the potential energy function of the open-loop system, or equivalently

$$\mathcal{U}_{\text{ol}}(q) = \mathcal{U}_{\text{ol}}(q_0) + \int_{q_0}^{q} g^T(z)dz \tag{2}$$

for any $q, q_0 \in \mathbb{R}^n$ [the integration in (2) takes into account the conservative nature of g, as pointed for instance in [9, Note 1, p. 2009]]; and $\tau \in \mathbb{R}^n$ is the external input (generalized) force vector. Some well-known properties characterizing the terms of such a dynamical model are recalled here [8, 10, 11]. Subsequently, we denote $\dot{H}$ the rate of change of H, *i.e.* $\dot{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ with $\dot{H}_{ij}(q,\dot{q}) = \frac{\partial H_{ij}}{\partial q}(q)\dot{q}$, $i,j = 1,\dots,n$.

Property 2.1. *$H(q)$ is a continuously differentiable positive definite symmetric matrix function, and actually $H(q) \geq \mu_m I_n$ —which implies $\|H(q)\| \geq \mu_m$— $\forall q \in \mathbb{R}^n$, for some $\mu_m > 0$.*

Property 2.2. *The Coriolis and centrifugal effect matrix defined through the Christoffel symbols of the first kind satisfies:*

2.2.1. $\dot{H}(q,\dot{q}) = C(q,\dot{q}) + C^T(q,\dot{q})$, $\forall q, \dot{q} \in \mathbb{R}^n$, *and consequently* $z^T\left[\frac{1}{2}\dot{H}(x,y) - C(x,y)\right]z = 0$, $\forall x,y,z \in \mathbb{R}^n$;
2.2.2. $C(x,y)z = C(x,z)y$, $\forall x,y,z \in \mathbb{R}^n$;
2.2.3. $\|C(x,y)\| \leq \psi(x)\|y\|$, $\forall x,y \in \mathbb{R}^n$, *for some* $\psi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$.

Remark 2.1. Observe from Property 2.2.2 that $C(q,a\dot{q})b\dot{q} = C(q,b\dot{q})a\dot{q} = C(q,ab\dot{q})\dot{q} = C(q,\dot{q})ab\dot{q}, \forall q,\dot{q} \in \mathbb{R}^n, \forall a,b \in \mathbb{R}$. △

In this work, we consider the (realistic) bounded input case, where the absolute value of each input τ_i is constrained to be smaller than a given saturation bound $T_i > 0$, *i.e.* $|\tau_i| \leq T_i$, $i = 1,\dots,n$. More precisely, letting u_i represent the control variable (controller output) relative to the i^{th} degree of freedom, we have that

$$\tau_i = T_i \text{sat}(u_i/T_i) \tag{3}$$

Further assumptions are stated next.

Assumption 2.1. *The inertia matrix is bounded, i.e.* $\|H(q)\| \leq \mu_M$, $\forall q \in \mathbb{R}^n$, *for some* $\mu_M \geq \mu_m > 0$.

Assumption 2.2. *$\psi(\cdot)$ in Property 2.2.3 is bounded and consequently* $\|C(x,y)\| \leq k_C\|y\|$, $\forall x,y \in \mathbb{R}^n$, *for some* $k_C \geq 0$.

Assumption 2.3. *The conservative (generalized) force vector $g(q)$ is a continuously differentiable bounded vector function with bounded Jacobian matrix $\frac{\partial g}{\partial q}$, or equivalently,*

2.3.1. every element of the conservative force vector, $g_i(q)$, $i = 1,\dots,n$, *satisfies:* $|g_i(q)| \leq B_{gi}$, $\forall q \in \mathbb{R}^n$, *for some non-negative constant* B_{gi};
2.3.2. $\frac{\partial g}{\partial x}$ *exists and is continuous and such that* $\left\|\frac{\partial g}{\partial q}(q)\right\| \leq k_g$, $\forall q \in \mathbb{R}^n$, *for some non-negative constant* k_g, *and consequently* $\|g(x) - g(y)\| \leq k_g\|x - y\|$, $\forall x,y \in \mathbb{R}^n$.

IET Research Journals, pp. 1–12

Assumption 2.4. $T_i > \eta B_{gi}$, $\forall i \in \{1, \dots, n\}$, *for some scalar* $\eta \geq 1$.

Remark 2.2. Assumptions 2.1–2.3 apply *e.g.* for robot manipulators having only revolute joints [11, §4.3]. All the stated assumptions will prove to be essential along the control design procedure —and/or closed-loop system analysis— developed in this paper. △

2.2 Local homogeneity, finite-time stability and δ-exponential stability

As in [1], this work is developed within the analytical framework of *local homogeneity*, which states a formal analytical platform permitting to handle vector fields with bounded components (and consequently, control design under the consideration of input constraints [12], which would not be formally possible within the conventional coordinate-dependent context of homogeneity [13]). Definitions and results in such an analytical context are strongly related to *family of dilations* δ_ε^r, defined as $\delta_\varepsilon^r(x) = \big(\varepsilon^{r_1}x_1, \dots, \varepsilon^{r_n}x_n\big)^T$, $\forall x \in \mathbb{R}^n$, $\forall \varepsilon > 0$, with $r = (r_1, \dots, r_n)^T$, where the *dilation coefficients* $r_1, \dots, r_n$ are positive scalars.

Definition 2.1. *[12] A function* $V : \mathbb{R}^n \to \mathbb{R}$*, resp. vector field* $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ *(with* $f_i : \mathbb{R}^n \to \mathbb{R}$*), is locally homogeneous of degree* α *with respect to the family of dilations* δ_ε^r *—or equivalently, it is said to be locally* r*-homogeneous of degree* α*— if there exists an open neighborhood of the origin* $D \subset \mathbb{R}^n$ *—referred to as the domain of homogeneity— such that, for every* $x \in D$ *and all* $\varepsilon \in (0, 1]$*:* $\delta_\varepsilon^r(x) \in D$ *and*

$$V(\delta_\varepsilon^r(x)) = \varepsilon^\alpha V(x) \tag{4}$$

resp.

$$f_i(\delta_\varepsilon^r(x)) = \varepsilon^{\alpha + r_i} f_i(x) \tag{5}$$

$i = 1, \dots, n$.

Fundamental concepts involved in the analytical context underlying this work are those of *homogeneous norm* —with respect to the family of dilations δ_ε^r, or simply r*-homogeneous norm*: a positive definite continuous function being r-homogeneous of degree 1— [1, 14, 15], denoted $\|\cdot\|_r$, and r*-homogeneous* $(n-1)$*-sphere* of radius $c > 0$: $S_{r,c}^{n-1} = \{x \in \mathbb{R}^n : \|x\|_r = c\}$.

Consider an n-th order autonomous system

$$\dot{x} = f(x) \tag{6}$$

where f is a vector field being continuous on an open neighborhood of the origin $\mathcal{D} \subset \mathbb{R}^n$ and such that $f(0_n) = 0_n$, and let $x(t; x_0)$ represent the system solution with initial condition $x(0; x_0) = x_0$.

Definition 2.2. *[13] The origin is said to be a finite-time stable equilibrium of system* (6) *if it is Lyapunov stable and there exist an open neighborhood of the origin,* $\mathcal{N} \subset \mathcal{D}$*, being positively invariant with respect to* (6)*, and a positive definite function* $T : \mathcal{N} \to \mathbb{R}_{\geq 0}$*, called the settling-time function, such that* $x(t; x_0) \neq 0_n$*,* $\forall t \in \big[0, T(x_0)\big)$*,* $\forall x_0 \in \mathcal{N} \setminus \{0_n\}$*, and* $x\big(T(x_0); x_0\big) = 0_n$*,* $\forall x_0 \in \mathcal{N}$*. The origin is said to be a globally finite-time stable equilibrium if it is finite-time stable with* $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$*.*

Remark 2.3. Note, from Definition 2.2, that the origin is a globally finite-time stable equilibrium of system (6) if and only if it is globally asymptotically stable and finite-time stable. △

Theorem 2.1. *[12] Consider system* (6) *with* $\mathcal{D} = \mathbb{R}^n$*. Suppose that* f *is a locally* r*-homogeneous vector field of degree* α *with domain of homogeneity* $D \subset \mathbb{R}^n$*. Then, the origin is a globally finite-time stable equilibrium of system* (6) *if and only if it is globally asymptotically stable and* $\alpha < 0$*.*

The next definition is stated under the additional consideration that, for some $r \in \mathbb{R}_{>0}^n$, f in (6) is locally r-homogeneous with domain of homogeneity $D \subset \mathcal{D}$.

Definition 2.3. *[14, 15] The equilibrium point* $x = 0_n$ *of* (6) *is* δ*-exponentially stable with respect to the homogeneous norm* $\|\cdot\|_r$ *if there exist a neighborhood of the origin,* $\mathcal{V} \subset D$*, and constants* $a \geq 1$ *and* $b > 0$ *such that* $\|x(t; x_0)\|_r \leq a\|x_0\|_r e^{-bt}$*,* $\forall t \geq 0$*,* $\forall x_0 \in \mathcal{V}$*.*

Remark 2.4. If f in (6) is locally r-homogeneous of degree $\alpha = 0$ with dilation coefficients $r_i = r_0$, $\forall i \in \{1, \dots, n\}$, for some $r_0 > 0$, then the origin turns out to be exponentially stable (in the usual or standard sense [16, Definition 4.5]) if and only if it is δ-exponentially stable [1, Remark 2.5]. △

Consider an n-th order autonomous system of the form

$$\dot{x} = f(x) + \hat{f}(x) \tag{7}$$

where f and $\hat{f}$ are continuous vector fields on $\mathbb{R}^n$ such that $f(0_n) = \hat{f}(0_n) = 0_n$.

Lemma 2.1. *[1, Lemma 2.2] Suppose that, for some* $r \in \mathbb{R}_{>0}^n$*,* f *in* (7) *is a locally* r*-homogeneous vector field of degree* $\alpha < 0$*, resp.* $\alpha = 0$*, with domain of homogeneity* $D \subset \mathbb{R}^n$*, and that* 0_n *is a globally asymptotically, resp.* δ*-exponentially, stable equilibrium of* $\dot{x} = f(x)$*. Then, the origin is a finite-time, resp.* δ*-exponentially, stable equilibrium of system* (7) *if*

$$\lim_{\varepsilon \to 0^+} \frac{\hat{f}_i(\delta_\varepsilon^r(x))}{\varepsilon^{\alpha + r_i}} = 0 \tag{8}$$

$i = 1, \dots, n$*,* $\forall x \in S_c^{n-1}$ *, resp.* $\forall x \in S_{r,c}^{n-1}$*, for some* $c > 0$ *such that* $S_c^{n-1} \subset D$*, resp.* $S_{r,c}^{n-1} \subset D$*.*

Remark 2.5. Notice that the condition required by Lemma 2.1 may be equivalently verified through the satisfaction of

$$\lim_{\varepsilon \to 0^+} \left\| \varepsilon^{-\alpha} \mathrm{diag}\big[\varepsilon^{-r_1}, \dots, \varepsilon^{-r_n}\big] \hat{f}(\delta_\varepsilon^r(x)) \right\| = 0 \tag{9}$$

$\forall x \in S_c^{n-1}$ (resp. $S_{r,c}^{n-1}$). In other words, (8) is fulfilled for all $i = 1, \dots, n$ and all $x \in S_c^{n-1}$ (resp. $S_{r,c}^{n-1}$) if and only if (9) is satisfied for all $x \in S_c^{n-1}$ (resp. $S_{r,c}^{n-1}$). △

2.3 Scalar functions with particular properties

Definition 2.4. *A continuous scalar function* $\sigma : \mathbb{R} \to \mathbb{R}$ *will be said to be:*

1. *positively upper-bounded —by* M^+*— if* $\sigma(\varsigma) \leq M^+$*,* $\forall \varsigma \in \mathbb{R}$*, for some positive constant* M^+*;*
2. *negatively lower-bounded —by* $-M^-$*— if* $\sigma(\varsigma) \geq -M^-$*,* $\forall \varsigma \in \mathbb{R}$*, for some positive constant* M^-*;*
3. *bounded —by* M*— if* $|\sigma(\varsigma)| \leq M$*,* $\forall \varsigma \in \mathbb{R}$*, for some positive constant* M*;*
4. *strictly passive if* $\varsigma\sigma(\varsigma) > 0$*,* $\forall \varsigma \neq 0$*;*
5. *strongly passive if it is a strictly passive function satisfying* $|\sigma(\varsigma)| \geq \kappa\big|a\,\mathrm{sat}(\varsigma/a)\big|^b = \kappa\big(\min\{|\varsigma|, a\}\big)^b$*,* $\forall \varsigma \in \mathbb{R}$*, for some positive constants* κ*,* a *and* b*.*

Remark 2.6. A non-decreasing strictly passive function σ is strongly passive [1, Remark 2.7]. △

Remark 2.7. Equivalent characterizations of strictly passive functions are: $\varsigma\sigma(\varsigma) > 0 \iff \mathrm{sign}(\varsigma)\sigma(\varsigma) > 0 \iff \mathrm{sign}(\sigma(\varsigma)) = \mathrm{sign}(\varsigma), \forall\varsigma$. △

Lemma 2.2. *[1, Lemma 2.3] Let $\sigma : \mathbb{R} \to \mathbb{R}$, $\sigma_0 : \mathbb{R} \to \mathbb{R}$ and $\sigma_1 : \mathbb{R} \to \mathbb{R}$ be strongly passive functions and k be a positive constant. Then:*

1. $\int_0^{\varsigma} \sigma(k\nu)d\nu > 0$, $\forall\varsigma \neq 0$;
2. $\int_0^{\varsigma} \sigma(k\nu)d\nu \to \infty$ *as* $|\varsigma| \to \infty$;
3. $\sigma_0 \circ \sigma_1$ *is strongly passive.*

Lemma 2.3. *[1, Lemma 2.4] Let $\sigma_0 : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function, $\sigma_2 : \mathbb{R} \to \mathbb{R}$ be strictly passive, and k be a positive constant. Then: $\varsigma_2\big[\sigma_0(\varsigma_1 + \sigma_2(k\varsigma_2)) - \sigma_0(\varsigma_1)\big] > 0$, $\forall\varsigma_2 \neq 0$, $\forall\varsigma_1 \in \mathbb{R}$.*

3 The proposed control scheme

Consider the following SPD-type controller with *desired* conservative-force compensation

$$u(q, \dot{q}) = -s_0\big(s_1(K_1\bar{q}) + s_2(K_2\dot{q})\big) + g(q_d) \tag{10}$$

where $\bar{q} = q - q_d$, for any constant —desired equilibrium position— $q_d \in \mathbb{R}^n$; $K_i \in \mathbb{R}^{n\times n}$, $i = 1, 2$, are positive definite diagonal matrices —*i.e.* $K_i = \mathrm{diag}[k_{i1}, \ldots, k_{in}]$, $k_{ij} > 0$, $i = 1, 2$, $j = 1, \ldots, n$— with K_1 involved in an additional requirement stated below (through (12)); for any $x \in \mathbb{R}^n$, $s_i(x) = \big(\sigma_{i1}(x_1), \ldots, \sigma_{in}(x_n)\big)^T$, $i = 0, 1, 2$, with —for each $j = 1, \ldots, n$— σ_{0j} being a strictly increasing strictly passive function, σ_{1j} being strongly passive and σ_{2j} being strictly passive, all three being locally Lipschitz-continuous on $\mathbb{R} \setminus \{0\}$, and such that

$$B_j \triangleq \sup_{(\varsigma_1,\varsigma_2)\in\mathbb{R}^2} \big|\sigma_{0j}\big(\sigma_{1j}(\varsigma_1) + \sigma_{2j}(\varsigma_2)\big)\big| < T_j - B_{gj} \tag{11}$$

(recall Assumption 2.3.1) [notice that if σ_{1j} and σ_{2j} are (both) chosen to be non-decreasing, then $B_j = \max\{\lim_{\varsigma\to\infty} \sigma_{0j}(\sigma_{1j}(\varsigma) + \sigma_{2j}(\varsigma))\ ,\ \lim_{\varsigma\to-\infty} -\sigma_{0j}(\sigma_{1j}(\varsigma) + \sigma_{2j}(\varsigma))\}$], all three being locally Lipschitz-continuous on $\mathbb{R} \setminus \{0\}$; and with —for each $j = 1, \ldots, n$— k_{1j}, σ_{0j} and σ_{1j} additionally required to be such that

$$\big|\sigma_{0j}\big(\sigma_{1j}(k_{1j}\varsigma)\big)\big| > \min\left\{k_g|\varsigma|\, ,\, 2B_{gj}\right\} \tag{12}$$

$\forall\varsigma \neq 0$ (recall Assumption 2.3.2).

Remark 3.1. As seen from (10), the proposed controller is based on position and velocity error correction terms $K_1\bar{q}$ and $K_2\dot{q}$, respectively, which may be seen as P and D basic actions. With the negative sign included in the right-hand side of (10), and the analytical properties of the scalar functions σ_{ij}, $i = 0, 1, 2$, $j = 1, \ldots, n$, involved in the vector functions s_i, $i = 0, 1, 2$, included in the control law (as shown in the right-hand side of (10)), the referred PD-basic-action terms carry out their correction function by generating forces/torques that oppose to the referred errors. Additionally, the referred analytical properties of the involved functions σ_{ij}, $i = 0, 1, 2$, $j = 1, \ldots, n$, are stated so as to suitably bound the referred correction actions in order to avoid input saturation along the closed-loop system trajectories (through (11)), on the one hand, and simultaneously to guarantee the required convergence (among finite-time and exponential) through their slope at zero, on the other hand; a visualization of these aspects on σ_{ij} will be graphically shown in Section 4. Thus, the error correction/opposition actions are suitably nonlinearly distorted so as to successfully achieve the just mentioned design objectives. Further, the last term in the right-hand side of (10) is the *desired* compensation term through which the conservative forces are counterbalanced at the desired position q_d. This way, q_d is ensured to be an equilibrium configuration, while its uniqueness is guaranteed through (12). The resulting (saturating) PD-type (with desired conservative-force compensation) structure of the proposed scheme has the advantage to keep the main (beneficial) features of PD-type controllers, such as the intuitive sense on the role of the P and D control gains (in K_1 and K_2), as well as the respect of the structure and essence of the controlled system by avoiding exact on-line compensation of any of its open-loop terms and keeping its passive nature with suitably shaped (potential) energy and added damping (details on this latter aspect are given later on in Remark 3.8). △

Remark 3.2. Notice that σ_{1j} and σ_{2j}, $j = 1, \ldots, n$, are not required to be non-decreasing, as long as they are strongly and strictly passive functions —respectively— that fulfill the above-stated specifications. Actually, this also applies in the (on-line) conservative-force compensation case presented in [1] (where such functions were defined to be non-decreasing strictly passive, which rendered them strongly passive, but the point is that the non-decreasing character is unnecessary, as long as they are as described above). △

Remark 3.3. Note that by (11), the proposed control law, in (10), shall be bounded. This gives rise to several possible combinations on the selection of the functions σ_{ij}, $i = 0, 1, 2$, $j = 1, \ldots, n$, (concerning items 1–3 of Definition 2.4) aiming at the satisfaction of the suitable boundedness requested by (11), as described through [1, Remark 3.1] (*i.e.* Remark 3.1 from [1] applies in the case of the controller proposed here too). △

Remark 3.4. From the formulation of the proposed scheme, one can verify that the proper satisfaction of the stated requirements entails that

$$\begin{aligned} 2B_{gj} &< \big|\sigma_{0j}\big(\sigma_{1j}(k_{1j}\varsigma)\big)\big| \\ &\leq \sup_{(\varsigma_1,\varsigma_2)\in\mathbb{R}^2} \big|\sigma_{0j}\big(\sigma_{1j}(\varsigma_1) + \sigma_{2j}(\varsigma_2)\big)\big| \\ &< T_j - B_{gj} \end{aligned}$$

$\forall|\varsigma| \geq 2B_{gj}/k_g$, whence one sees that Assumption 2.4 with $\eta = 3$ is a necessary condition for the feasibility of the simultaneous fulfilment of (11) and (12). A similar condition on the control input bounds has been required by other approaches where input constraints have been considered [17] [9, Appendix 1], generally arising from the *worst-case* procedure followed to ensure that the analytical requirements that guarantee the result are fulfilled. △

Remark 3.5. Let us note that (12) could have been alternatively stated as requiring $\big|\sigma_{0j}\big(\sigma_{1j}(k_{1j}\varsigma)\big)\big| \geq \min\left\{\hat{k}_{1j}|\varsigma|, b_j\right\}$ for some constants $\hat{k}_{1j} > k_g$ and $b_j > 2B_{gj}$. However, by stating (12), the existence of constants $\hat{k}_{1j} > k_g$ and $b_j > 2B_{gj}$ such that $\big|\sigma_{0j}\big(\sigma_{1j}(k_{1j}\varsigma)\big)\big| \geq \min\left\{\hat{k}_{1j}|\varsigma|, b_j\right\} > \min\left\{k_g|\varsigma|, 2B_{gj}\right\}$, $\forall\varsigma \neq 0$, is implied. △

Remark 3.6. Note that the (*D*) gains in K_2 are not at all restricted and are consequently free to take any positive value, while the (*P*) gains in K_1 are the only ones whose choice remains restricted in accordance to the design requirement stated through (12) (where they are involved in). △

Proposition 3.1. *Consider system* (1),(3) *in closed loop with the proposed control law* (10)*, under Assumptions* 2.1–2.3 *and* 2.4 *with $\eta = 3$, and the above stated design specifications. Thus, global asymptotic stability of the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is guaranteed with $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$.*

Proof. Observe that —for every $j = 1, \dots, n$— by (11), we have that, for any $(q, \dot{q}) \in \mathbb{R}^n \times \mathbb{R}^n$ and any $q_d \in \mathbb{R}^n$:

$$\begin{aligned} |u_j(q,\dot{q})| &= \big| -\sigma_{0j}\big(\sigma_{1j}(k_{1j}\bar{q}_j) + \sigma_{2j}(k_{2j}\dot{q}_j)\big) + g_j(q_d)\big| \\ &\leq \big|\sigma_{0j}\big(\sigma_{1j}(k_{1j}\bar{q}_j) + \sigma_{2j}(k_{2j}\dot{q}_j)\big)\big| + |g_j(q_d)| \\ &\leq B_j + B_{gj} < T_j \end{aligned}$$

From this and (3), one sees that $T_j > |u_j(q,\dot{q})| = |u_j| = |\tau_j|$, $\forall (q,\dot{q}) \in \mathbb{R}^n \times \mathbb{R}^n$, which shows that, along the system trajectories, $|\tau_j(t)| = |u_j(t)| < T_j, j = 1, \dots, n, \forall t \geq 0$. This proves that, under the proposed scheme, the input saturation values, T_j, are never reached. Hence, the closed-loop dynamics takes the form

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = -s_0\big(s_1(K_1\bar{q}) + s_2(K_2\dot{q})\big) + g(q_d)$$

By defining $x_1 = \bar{q}$ and $x_2 = \dot{q}$, the closed-loop dynamics adopts the $2n$-order state-space representation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= H^{-1}(x_1 + q_d)\Big[-s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \\ &\qquad -C(x_1+q_d, x_2)x_2 - g(x_1+q_d) + g(q_d)\Big] \end{aligned}$$

By further defining $x = (x_1^T, x_2^T)^T$, these state equations may be rewritten in the form of system (7) with

$$f(x) = \begin{pmatrix} x_2 \\ -H^{-1}(q_d)s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \end{pmatrix} \tag{13a}$$

$$\hat{f}(x) = \begin{pmatrix} 0_n \\ -H^{-1}(x_1+q_d)\left[C(x_1+q_d,x_2)x_2 + g(x_1+q_d) - g(q_d)\right] \\ -\mathcal{H}(x_1)s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \end{pmatrix} \tag{13b}$$

where

$$\mathcal{H}(x_1) = H^{-1}(x_1 + q_d) - H^{-1}(q_d) \tag{14}$$

Thus, the closed-loop stability property stated through Proposition 3.1 is corroborated by showing that $x = 0_{2n}$ is a globally asymptotically stable equilibrium of the state equation $\dot{x} = f(x) + \hat{f}(x)$, which is proven through the following theorem (whose formulation proves to be convenient for subsequent developments and proofs).

Theorem 3.1. *Under the stated specifications, the origin is a globally asymptotically stable equilibrium of $\dot{x} = f(x) + \ell\hat{f}(x)$, $\forall \ell \in \{0,1\}$, —i.e. of both the state equation $\dot{x} = f(x)$ and the (closed-loop) system $\dot{x} = f(x) + \hat{f}(x)$,— with $f(x)$ and $\hat{f}(x)$ defined through Eqs.* (13).

Proof: For every $\ell \in \{0,1\}$, let us define the continuously differentiable scalar function

$$V_\ell(x_1,x_2) = \frac{1}{2}x_2^T H(\ell x_1 + q_d)x_2 + \mathcal{U}_\ell(x_1) \tag{15}$$

where

$$\mathcal{U}_\ell(x_1) \triangleq \int_{0_n}^{x_1} s_0^T\big(s_1(K_1z)\big)dz + \ell\,\mathcal{U}(x_1) \tag{16}$$

with

$$\int_{0_n}^{x_1} s_0^T\big(s_1(K_1z)\big)dz = \sum_{j=1}^{n}\int_0^{x_{1j}} \sigma_{0j}\big(\sigma_{1j}(k_{1j}z_j)\big)dz_j \tag{17}$$

and

$$\mathcal{U}(x_1) \triangleq \mathcal{U}_{\rm ol}(x_1+q_d) - \mathcal{U}_{\rm ol}(q_d) - g^T(q_d)x_1 \tag{18a}$$

$$= \int_{0_n}^{x_1}\left[g(z+q_d) - g(q_d)\right]^T dz \tag{18b}$$

$$= \int_{0_n}^{x_1}\left[\int_{0_n}^{z}\frac{\partial g}{\partial q}(\bar{z}+q_d)d\bar{z}\right]^T dz \tag{18c}$$

Observe from Eqs. (18) and Assumption 2.3 that

$$\begin{aligned} \mathcal{U}(x_1) &\leq \int_{0_n}^{x_1}\left[\int_{0_n}^{z}\left\|\frac{\partial g}{\partial q}(\bar{z}+q_d)\right\|d\bar{z}\right]^T dz \\ &\leq \int_{0_n}^{x_1}\left[\int_{0_n}^{z}k_g d\bar{z}\right]^T dz \\ &= \int_{0_n}^{x_1}k_g z^T dz = \sum_{j=1}^{n}\int_0^{x_{1j}}k_g z_j dz_j \end{aligned} \tag{19}$$

$\forall x_1 \in \mathbb{R}^n$ (more specifically from (18c)), and simultaneously that

$$\begin{aligned} \mathcal{U}(x_1) &\leq \sum_{j=1}^{n}\int_0^{x_{1j}}\text{sign}(z_j)\big|g_j(z+q_d) - g_j(q_d)\big|dz_j \\ &\leq \sum_{j=1}^{n}\int_0^{x_{1j}}\text{sign}(z_j)2B_{gj}dz_j \end{aligned}$$

$\forall x_1 \in \mathbb{R}^n$ (more specifically from (18b)). From these inequalities, Eqs. (16) and (17), the satisfaction of (12), and Remark 3.5, we have that

$$\begin{aligned} &\mathcal{U}_\ell(x_1) \\ &\geq \sum_{j=1}^{n}\int_0^{x_{1j}}\text{sign}(z_j)\min\left\{(\hat{k}_{1j} - \ell k_g)|z_j|\,,\,(b_j - 2\ell B_{gj})\right\}dz_j \\ &\geq \sum_{j=1}^{n}\int_0^{x_{1j}}\text{sign}(z_j)\min\left\{\bar{k}_{\ell j}|z_j|\,,\,\bar{b}_{\ell j}\right\}dz_j \\ &\quad = \sum_{j=1}^{n}w_{\ell j}(x_{1j}) \triangleq S_\ell(x_1) \end{aligned} \tag{20a}$$

with

$$w_{\ell j}(x_{1j}) = \begin{cases} \frac{\bar{k}_{\ell j}}{2}x_{1j}^2 & \text{if } |x_{1j}| \leq \bar{b}_{\ell j}/\bar{k}_{\ell j} \\ \bar{b}_{\ell j}\left[|x_{1j}| - \bar{b}_{\ell j}/(2\bar{k}_{\ell j})\right] & \text{if } |x_{1j}| > \bar{b}_{\ell j}/\bar{k}_{\ell j} \end{cases} \tag{20b}$$

for some $\hat{k}_{1j} > k_g$ and $b_j > 2B_{gj}$, and any positive constants $\bar{k}_{\ell j} \leq \hat{k}_{1j} - \ell k_g$ and $\bar{b}_{\ell j} \leq b_j - 2\ell B_{gj}$.

Remark 3.7. One sees from expressions (20) that S_ℓ, $\ell = 0, 1$, are positive definite radially unbounded functions of x_1. Observe further that (involving previous arguments and Remark 2.7)

$$\begin{aligned} D_{x_1}\mathcal{U}_\ell(x_1) &= x_1^T\nabla_{x_1}\mathcal{U}_\ell(x_1) \\ &= x_1^T\Big[s_0\big(s_1(K_1x_1)\big)+\ell\big(g(x_1+q_d)-g(q_d)\big)\Big] \\ &= \sum_{j=1}^n |x_{1j}|\Big[\big|\sigma_{0j}\big(\sigma_{1j}(k_{1j}x_{1j})\big)\big| \\ &\qquad + \ell\,\mathrm{sign}(x_{1j})\big(g_j(x_1+q_d)-g_j(q_d)\big)\Big] \\ &\geq \sum_{j=1}^n |x_{1j}|\Big[\big|\sigma_{0j}\big(\sigma_{1j}(k_{1j}x_{1j})\big)\big| \\ &\qquad - \ell\big|g_j(x_1+q_d)-g_j(q_d)\big|\Big] \\ &\geq \sum_{j=1}^n |x_{1j}|\min\{(\hat{k}_{1j}-\ell k_g)|x_{1j}|\,,\,(b_j-2\ell B_{gj})\} \\ &\geq \sum_{j=1}^n |x_{1j}|\min\left\{\bar{k}_{\ell j}|x_{1j}|\,,\,\bar{b}_{\ell j}\right\}>0 \end{aligned} \quad (21)$$

$\forall x_1\neq 0_n$ [in any radial direction, $\mathcal{U}_\ell(x_1)$ is strictly increasing, and consequently $x_1=0_n$ is the unique stationary point of $\mathcal{U}_\ell(x_1)$], whence one sees that, for every $\ell=0,1$,

$$\nabla_{x_1}\mathcal{U}_\ell(x_1)=s_0\big(s_1(K_1x_1)\big)+\ell\big[g(x_1+q_d)-g(q_d)\big]=0_n \iff x_1=0_n \quad (22)$$

△

Thus, from Eqs. (15) and (20), and Property 2.1, we get that

$$V_\ell(x_1,x_2)\geq\frac{\mu_m}{2}\|x_2\|^2+S_\ell(x_1) \quad (23)$$

whence positive definiteness and radial unboundedness of V_ℓ, $\ell=0,1$, is concluded. Further, for every $\ell\in\{0,1\}$, the derivative of V_ℓ along the trajectories of $\dot{x}=f(x)+\ell\hat{f}(x)$, is obtained as

$$\begin{aligned} &\dot{V}_\ell(x_1,x_2) \\ &= x_2^TH(\ell x_1+q_d)\dot{x}_2+\frac{\ell}{2}x_2^T\dot{H}(x_1+q_d,x_2)x_2 \\ &\quad+\Big[s_0\big(s_1(K_1x_1)\big)+\ell\big[g(x_1+q_d)-g(q_d)\big]\Big]^T\dot{x}_1 \\ &= x_2^T\Big[-\ell\big[C(x_1+q_d,x_2)x_2+g(x_1+q_d)-g(q_d)\big] \\ &\quad -s_0\big(s_1(K_1x_1)+s_2(K_2x_2)\big)\Big] \\ &\quad+\frac{\ell}{2}x_2^T\dot{H}(x_1+q_d,x_2)x_2 \\ &\quad+\Big[s_0\big(s_1(K_1x_1)\big)+\ell\big[g(x_1+q_d)-g(q_d)\big]\Big]^Tx_2 \\ &= -x_2^T\Big[s_0\big(s_1(K_1x_1)+s_2(K_2x_2)\big)-s_0\big(s_1(K_1x_1)\big)\Big] \\ &= -\sum_{j=1}^n x_{2j}\Big[\sigma_{0j}\big(\sigma_{1j}(k_{1j}x_{1j})+\sigma_{2j}(k_{2j}x_{2j})\big) \\ &\qquad -\sigma_{0j}\big(\sigma_{1j}(k_{1j}x_{1j})\big)\Big] \end{aligned}$$

where, in the case of $\ell=1$, Property 2.2.1 has been applied. Note, from Lemma 2.3, that $\dot{V}_\ell(x_1,x_2)\leq 0, \forall(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n$, with $Z_\ell\triangleq\{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n:\dot{V}_\ell(x_1,x_2)=0\}=\{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n:x_2=0_n\}$. Further, from the system dynamics $\dot{x}=f(x)+\ell\hat{f}(x)$ —under the consideration of Property 2.1 and Remark 3.7 (more precisely (22))— one sees that $x_2(t)\equiv 0_n \implies \dot{x}_2(t)\equiv 0_n \implies s_0\big(s_1(K_1x_1(t))\big)+\ell\big[g(x_1(t)+q_d)-g(q_d)\big]\equiv 0_n \iff x_1(t)\equiv 0_n$ (which shows that $(x_1,x_2)(t)\equiv(0_n,0_n)$ is the only system solution completely remaining in Z_ℓ), and corroborates that at any $(x_1,x_2)\in\{(\bar{q},\dot{q})\in Z_\ell:\bar{q}\neq 0_n\}$, the resulting unbalanced force term $-s_0\big(s_1(K_1x_1)\big)+\ell\big[g(x_1+q_d)-g(q_d)\big]$ acts on the closed-loop dynamics, forcing the system trajectories to leave Z_ℓ, whence $\{(0_n,0_n)\}$ is concluded to be the only invariant set in Z_ℓ, $\ell=0,1$. Therefore, by the invariance theory [18, §7.2] —more precisely by [18, Corollary 7.2.1]— $x=0_{2n}$ is concluded to be a globally asymptotically stable equilibrium of both the state equation $\dot{x}=f(x)$ and the (closed-loop) system $\dot{x}=f(x)+\hat{f}(x)$. □

Remark 3.8. The proof of Theorem 3.1 brings to the fore how the proposed scheme shapes the closed-loop potential energy and injects damping to guarantee the stabilization goal. Indeed, one sees from the proof that, through the proposed scheme, the closed-loop potential energy is given the shape adopted from its generalized expression, $\mathcal{U}_1(\bar{q})=\int_{0_n}^{\bar{q}}s_0^T\big(s_1(K_1z)\big)dz+\mathcal{U}_{\mathrm{ol}}(q)-\mathcal{U}_{\mathrm{ol}}(q_d)-g^T(q_d)\bar{q}$, which —through the requirement stated by (12)— is guaranteed to be a positive definite radially unbounded function with global minimum at the origin, giving rise to the closed-loop conservative force $u_c(\bar{q})=\nabla_{\bar{q}}\mathcal{U}_1(\bar{q})-\nabla_q\mathcal{U}_{\mathrm{ol}}(q)=s_0\big(s_1(K_1\bar{q})\big)-g(q_d)$. Further, damping is injected through a force vector of the form $s_d(\bar{q},\dot{q})=s_0\big(s_1(K_1\bar{q})+s_2(K_2\dot{q})\big)-s_0\big(s_1(K_1\bar{q})\big)$, which —through the properties required for σ_{ij}, $i=0,1,2$, $j=1,\ldots,n$— is proven to fulfil $\dot{q}^Ts_d(\bar{q},\dot{q})>0, \forall\dot{q}\neq 0_n$, $\forall\bar{q}\in\mathbb{R}^n$. Thus, the proposed control law proves to be the addition of a dissipative force opposing to motion, $-s_d(\bar{q},\dot{q})$, and a *restituting* conservative force, $-u_c(\bar{q})$; more precisely $u(q,\dot{q})=-s_d(\bar{q},\dot{q})-u_c(\bar{q})$, giving rise to the expression in (10), which —through the additional requirement in (11)— is guaranteed to be suitably bounded (so as to avoid input saturation along the closed loop trajectories). [Such an energy-shaping-plus-damping-injection characteristic can also be observed in the on-line compensation approach of [1], where $\mathcal{U}_1(\bar{q})=\int_{0_n}^{\bar{q}}s_0^T\big(s_1(K_1z)\big)dz$, $u_c(\bar{q})=\nabla_{\bar{q}}\mathcal{U}_1(\bar{q})-\nabla_q\mathcal{U}_{\mathrm{ol}}(q)=s_0\big(s_1(K_1\bar{q})\big)-g(q)$ and $s_d(\bar{q},\dot{q})=s_0\big(s_1(K_1\bar{q})+s_2(K_2\dot{q})\big)-s_0\big(s_1(K_1\bar{q})\big)$.] △

3.1 Finite-time stabilization

Proposition 3.2. *Consider the proposed control scheme under the additional consideration that, for every $j=1,\ldots,n$, σ_{ij}, $i=1,2$, are locally r_i-homogeneous of degree $\alpha_j>0$—i.e. $r_{1j}=r_1$, $r_{2j}=r_2$ and $\alpha_{1j}=\alpha_{2j}=\alpha_j>0$ for all $j=1,\ldots,n$— with domain of homogeneity $D_{ij}=\{\varsigma\in\mathbb{R}:|\varsigma|<L_{ij}\in(0,\infty]\}$ and σ_{0j} is locally α_j-homogeneous of degree $\alpha_0=2r_2-r_1$ —i.e. $\alpha_{0j}=\alpha_0=2r_2-r_1$ for all $j=1,\ldots,n$— with domain of homogeneity $D_{0j}=\{\varsigma\in\mathbb{R}:|\varsigma|<L_{0j}\in(0,\infty]\}$, for some dilation coefficients $r_i>0$, $i=1,2$, such that $\alpha_0=2r_2-r_1>0>r_2-r_1$. Thus, global finite-time stability of the closed-loop trivial solution $\bar{q}(t)\equiv 0_n$ is guaranteed with $|\tau_j(t)|=|u_j(t)|<T_j$, $j=1,\ldots,n$, $\forall t\geq 0$.*

Proof: Since the proposed control scheme is applied —with all its previously stated specifications— Proposition 3.1 holds and consequently $|\tau_j(t)|=|u_j(t)|<T_j$, $j=1,\ldots,n$, $\forall t\geq 0$. Then, all that remains to be proven is that the additional considerations give rise to the claimed finite-time stabilization. In this direction, let $\hat{r}_i=(r_{i1},\ldots,r_{in})^T$, $i=1,2$, $r=(\hat{r}_1^T,\hat{r}_2^T)^T$, $\hat{r}_0=(\alpha_1,\ldots,\alpha_n)^T$, $\hat{\alpha}_0=(\alpha_{01},\ldots,\alpha_{0n})^T$, $D\triangleq\{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n:K_ix_i\in D_{i1}\times\cdots\times D_{in}\,,\,i=1,2\,,\,s_1(K_1x_1)+s_2(K_2x_2)\in D_{01}\times\cdots\times D_{0n}\}=\{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n:|x_{1j}|<L_{1j}/k_{1j}$, $|x_{2j}|<L_{2j}/k_{2j}\,,\,|\sigma_{1j}(k_{1j}x_{1j})+\sigma_{2j}(k_{2j}x_{2j})|<L_{0j}\,,\,j=1,\ldots,n\}$, and consider the previously defined state (vector) variables and the consequent closed-loop state-space representation $\dot{x}=f(x)+\hat{f}(x)$, with f and $\hat{f}$ as defined through Eqs. (13). Since D defines an open neighborhood of the origin, there exists

IET Research Journals, pp. 1–12

$\rho > 0$ such that $B_\rho \triangleq \{x \in \mathbb{R}^{2n} : \|x\| < \rho\} \subset D$. Moreover, for every $x \in B_\rho$ and all $\varepsilon \in (0,1]$, we have that $\delta_\varepsilon^r(x) \in B_\rho$ (since $\|\delta_\varepsilon^r(x)\| < \|x\|, \forall \varepsilon \in (0,1)$), and, for every $j \in \{1,\ldots,n\}$,

$$f_j(\delta_\varepsilon^r(x)) = \varepsilon^{r_{2j}} x_{2j} = \varepsilon^{r_2} x_{2j} = \varepsilon^{(r_2-r_1)+r_1} x_{2j} = \varepsilon^{(r_2-r_1)+r_{1j}} f_j(x)$$

and [observe, for every $x \in B_\rho$ and all $\varepsilon \in (0,1]$, that $\sigma_{ij}(k_{ij}\varepsilon^{r_{ij}}x_{ij}) = \sigma_{ij}(\varepsilon^{r_i}k_{ij}x_{ij}) = \varepsilon^{\alpha_j}\sigma_{ij}(k_{ij}x_{ij}), i=1,2, j=1,\ldots,n \iff s_i(K_i\delta_\varepsilon^{\hat{r}_i}(x_i)) = s_i(\varepsilon^{r_i}K_ix_i) = \delta_\varepsilon^{\hat{r}_0}(s_i(K_ix_i))$, $i=1,2$, and $\sigma_0(\varepsilon^{\alpha_j}\cdot) = \varepsilon^{\alpha_{0j}}\sigma_0(\cdot) = \varepsilon^{\alpha_0}\sigma_0(\cdot)$, $j=1,\ldots,n \iff s_0(\delta_\varepsilon^{\hat{r}_0}(\cdot)) = \delta_\varepsilon^{\hat{\alpha}_0}(s_0(\cdot)) = \varepsilon^{\alpha_0}s_0(\cdot)$]

$$\begin{aligned} f_{n+j}(\delta_\varepsilon^r(x)) &= -H_j^{-1}(q_d)s_0\big(s_1(K_1\delta_\varepsilon^{\hat{r}_1}(x_1)) + s_2(K_2\delta_\varepsilon^{\hat{r}_2}(x_2))\big) \\ &= -H_j^{-1}(q_d)s_0\big(s_1(\varepsilon^{r_1}K_1x_1) + s_2(\varepsilon^{r_2}K_2x_2)\big) \\ &= -H_j^{-1}(q_d)s_0\big(\delta_\varepsilon^{\hat{r}_0}(s_1(K_1x_1)) + \delta_\varepsilon^{\hat{r}_0}(s_2(K_2x_2))\big) \\ &= -H_j^{-1}(q_d)s_0\big(\delta_\varepsilon^{\hat{r}_0}(s_1(K_1x_1) + s_2(K_2x_2))\big) \\ &= -H_j^{-1}(q_d)\delta_\varepsilon^{\hat{\alpha}_0}\big(s_0(s_1(K_1x_1) + s_2(K_2x_2))\big) \\ &= -\varepsilon^{\alpha_0}H_j^{-1}(q_d)s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \\ &= \varepsilon^{(r_2-r_1)+r_{2j}} f_{n+j}(x) \end{aligned} \tag{24}$$

whence one concludes that f is a locally r-homogeneous vector field of degree $\alpha = r_2 - r_1$, with domain of homogeneity B_ρ. Hence, by Theorems 2.1 and 3.1, the origin of the state equation $\dot{x} = f(x)$ is concluded to be a globally finite-time stable equilibrium since $r_2 - r_1 < 0$. Thus, by Theorem 3.1, Lemma 2.1, and Remarks 2.3 and 2.5, the origin of the closed-loop system $\dot{x} = f(x) + \hat{f}(x)$ is concluded to be a globally finite-time stable equilibrium, provided that $r_2 - r_1 < 0$, if

$$\begin{aligned} \mathcal{L}_0 &\triangleq \lim_{\varepsilon\to0^+} \Big\|\varepsilon^{-\alpha}\text{diag}\big[\varepsilon^{-r_{11}},\ldots,\varepsilon^{-r_{1n}},\varepsilon^{-r_{21}},\ldots, \varepsilon^{-r_{2n}}\big]\hat{f}(\delta_\varepsilon^r(x))\Big\| \\ &= \lim_{\varepsilon\to0^+} \Big\|\varepsilon^{-\alpha}\text{diag}\big[\varepsilon^{-r_{21}},\ldots,\varepsilon^{-r_{2n}}\big]\big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\ldots, \hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T\Big\| \\ &= \lim_{\varepsilon\to0^+} \Big\|\varepsilon^{-\alpha-r_2}\big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\ldots,\hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T\Big\| \\ &= \lim_{\varepsilon\to0^+} \varepsilon^{r_1-2r_2}\Big\|\big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\ldots,\hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T\Big\| \\ &= 0 \end{aligned} \tag{25}$$

for all $x \in S_c^{2n-1} = \{x \in \mathbb{R}^{2n} : \|x\| = c\}$, for some $c > 0$ such that $S_c^{2n-1} \subset D$. Hence, from (13b), under the consideration of Property 2.2.2 and Remark 2.1, we have, for all such $x \in S_c^{2n-1}$:

$$\begin{aligned} &\Big\|\big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\ldots,\hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T\Big\| \\ &= \Big\| -H^{-1}(\varepsilon^{r_1}x_1 + q_d)\big[C(\varepsilon^{r_1}x_1 + q_d, \varepsilon^{r_2}x_2)\varepsilon^{r_2}x_2 \\ &\quad + g(\varepsilon^{r_1}x_1 + q_d) - g(q_d)\big] \\ &\quad - \mathcal{H}(\varepsilon^{r_1}x_1)s_0\big(s_1(\varepsilon^{r_1}K_1x_1) + s_2(\varepsilon^{r_2}K_2x_2)\big)\Big\| \\ &\le \Big\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)\varepsilon^{2r_2}x_2\Big\| \\ &\quad + \Big\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)\Big\|\Big\|g(\varepsilon^{r_1}x_1 + q_d) - g(q_d)\Big\| \\ &\quad + \Big\|\mathcal{H}(\varepsilon^{r_1}x_1)s_0\big(\delta_\varepsilon^{\hat{r}_0}\big(s_1(K_1x_1) + s_2(K_2x_2)\big)\big)\Big\| \end{aligned}$$

whence, through a procedure similar to the one developed to obtain (24), and the consideration of Assumption 2.3.2, we get

$$\begin{aligned} &\Big\|\big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)),\ldots,\hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T\Big\| \\ &\le \varepsilon^{2r_2}\Big\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2\Big\| \\ &\quad + \Big\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)\Big\|k_g\varepsilon^{r_1}\|x_1\| \\ &\quad + \varepsilon^{2r_2-r_1}\Big\|\mathcal{H}(\varepsilon^{r_1}x_1)s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big)\Big\| \end{aligned}$$

and consequently, from (25) (recalling that by design specifications: $r_1 > r_2 > 0$), we get

$$\begin{aligned} \mathcal{L}_0 &\le \lim_{\varepsilon\to0^+} \varepsilon^{r_1}\Big\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2\Big\| \\ &\quad + k_g\|x_1\| \lim_{\varepsilon\to0^+} \varepsilon^{2(r_1-r_2)}\Big\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)\Big\| \\ &\quad + \lim_{\varepsilon\to0^+} \Big\|\mathcal{H}(\varepsilon^{r_1}x_1)s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big)\Big\| \\ &\le \big\|H^{-1}(q_d)C(q_d, x_2)x_2\big\| \lim_{\varepsilon\to0^+} \varepsilon^{r_1} \\ &\quad + k_g\|x_1\|\big\|H^{-1}(q_d)\big\| \lim_{\varepsilon\to0^+} \varepsilon^{2(r_1-r_2)} \\ &\quad + \big\|s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big)\big\| \lim_{\varepsilon\to0^+} \big\|\mathcal{H}(\varepsilon^{r_1}x_1)\big\| \\ &\le \big\|s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big)\big\| \cdot \big\|\mathcal{H}(0_n)\big\| = 0 \end{aligned} \tag{26}$$

(note, from (14), that $\|\mathcal{H}(0_n)\| = \|H^{-1}(q_d) - H^{-1}(q_d)\| = 0$), which completes the proof. □

Corollary 3.1. *Consider the proposed control scheme taking σ_{ij}, $i = 0,1,2$, $j = 1,\ldots,n$, such that*

$$\sigma_{ij}(\varsigma) = \text{sign}(\varsigma)|\varsigma|^{\beta_{ij}} \quad \forall|\varsigma| \le L_{ij} \in (0,\infty) \tag{27}$$

with constants β_{ij} such that

$$\beta_{1j} > 0 \quad , \quad \beta_{2j} = \gamma\beta_{1j} \quad , \quad \beta_{0j} = \frac{2-\gamma}{\gamma\beta_{1j}} \tag{28}$$

for a constant $\gamma \in (1,2)$. Thus, global finite-time stability of the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is guaranteed with $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1,\ldots,n$, $\forall t \ge 0$.

Proof: Note that, given any $r_{ij} > 0$, for every $\varsigma \in (-L_{ij}, L_{ij})$: $\varepsilon^{r_{ij}}\varsigma \in (-L_{ij}, L_{ij})$ and $\sigma_{ij}(\varepsilon^{r_{ij}}\varsigma) = \varepsilon^{r_{ij}\beta_{ij}}\text{sign}(\varsigma)|\varsigma|^{\beta_{ij}} = \varepsilon^{r_{ij}\beta_{ij}}\sigma_{ij}(\varsigma)$, $\forall\varepsilon \in (0,1]$. Hence, under the consideration of expressions (28), for every $j = 1,\ldots,n$, we have, for any

Fig. 1: Experimental setup: 2-DOF revolute-joint mechanical manipulator

$r_{1j} = r_1 > 0$, that taking $r_{2j} = r_2 = r_1/\gamma$ and $r_{0j} = r_1\beta_{1j}$, σ_{ij}, $i = 1, 2$, are locally r_i-homogeneous of degree $\alpha_{2j} = r_2\beta_{2j} = r_1\beta_{1j} = \alpha_{1j} = \alpha_j$ with domain of homogeneity $D_{ij} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{ij}\}$, and σ_{0j} is locally α_j-homogeneous of degree $\alpha_{0j} = \alpha_0 = (2-\gamma)r_1/\gamma$ with domain of homogeneity $D_{0j} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{0j}\}$. The requirements of Proposition 3.2 are thus concluded to be satisfied with $1 < \gamma < 2 \iff r_2 < r_1 < 2r_2 \iff r_2 - r_1 < 0 < 2r_2 - r_1 = \alpha_0$. □

Remark 3.9. Since the results of this section depart from the application of the proposed control scheme, the cases of Proposition 3.2 with $r_2 \geq r_1$ and Corollary 3.1 with $\gamma \in (0, 1]$ are particular cases of Proposition 3.1 where the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is globally asymptotically (but not finite-time) stable. It is further worth pointing out that with $r_2 = r_1$ —or analogously $\gamma = 1$ in the case of Corollary 3.1— we have that $\varepsilon^{r_2 - r_1} = 1$, $\forall \varepsilon > 0$. Hence, in this case, developments analog to those giving rise to inequalities (26) lead to $\mathcal{L}_0 \leq k_g\|x_1\|\|H^{-1}(q_d)\|$, and consequently, Lemma 2.1 (under the consideration of Remark 2.4) cannot be applied to conclude (local) exponential stability (contrarily to the on-line gravity compensation case of [1]). Nevertheless, exponential stability is next proven to be achieved (locally), through an alternative (strict-Lyapunov-function-based) analytical procedure, for the special case obtained under the consideration of (27) with $\beta_{ij} = 1$, $i = 0, 1, 2$, $j = 1, \dots, n$ (which implies $\gamma = 1 \iff r_2 = r_1$). △

3.2 Exponential stabilization

Corollary 3.2. *Consider the proposed control scheme taking —for every $i = 0, 1, 2$ and $j = 1, \dots, n$— σ_{ij} as in (27) with $\beta_{ij} = 1$, i.e. such that*

$$\sigma_{ij}(\varsigma) = \varsigma \quad \forall|\varsigma| \leq L_{ij} \in (0, \infty) \tag{29}$$

Thus: $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \dots, n$, $\forall t \geq 0$, and the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is globally asymptotically stable and (locally) exponentially stable.

Proof: See the Appendix (Subsection 8.1). □

4 Experimental results

The proposed control scheme was implemented through experimental tests on a 2-DOF robot manipulator moving on a vertical plane. The experimental setup, shown in Fig. 1, is a 2-revolute-joint mechanical arm located at the *Instituto Tecnológico de la Laguna*, Mexico, previously used in [19]. The robot actuators are direct-drive brushless servomotors operated in torque mode, that is, they act as torque sources and receive an analogue voltage as a torque reference signal. The motors used in the experimental arm are DM1200-A and DM1015-B from *Parker Compumotor*, for the shoulder and elbow joints, respectively. In this configuration, the first motor is capable of delivering a maximum torque of 150 Nm and the second one delivers only 15 Nm. Joint positions are obtained from incremental encoders located on the motors, which have a resolution of 1,024,000 pulses/rev for the first motor and 655,300 for the second one (accuracy of 0.0069°for both motors), and the standard backwards difference algorithm is used to obtain the velocity signals. The setup includes a PC-host computer with an acquisition board —the *Multi-Q* I/O card form *Quanser*— to get the encoder data and generate reference voltages. The robot is programmed through *WinMechLab* [20], which is a general-purpose computer system for real time control of mechanisms that runs on a Windows platform based on C language. The control algorithm is executed at a 2.5 ms sampling period (holding constant the control signals among the samples). This has proven to be fast enough to suitably approximate the continuous control signals generated by the implemented continuous-time scheme. Thorough details of the manipulator model can be found in [19]. In particular, the gravity (conservative) force vector is expressed as

$$g(q) = \begin{pmatrix} 38.465\sin q_1 + 1.825\sin(q_1 + q_2) \\ 1.825\sin(q_1 + q_2) \end{pmatrix}$$

Assumptions 2.1–2.3 are thus satisfied, which is a direct consequence of the revolute nature of both joints of the considered manipulator, in accordance to Remark 2.2; in particular Assumption 2.3 is fulfilled with $B_{g1} = 40.29$ Nm, $B_{g2} = 1.825$ Nm and $k_g = 40.37$ Nm/rad. Furthermore, as previously mentioned, the input saturation bounds are $T_1 = 150$ Nm and $T_2 = 15$ Nm for the first and second links respectively, whence one can corroborate that Assumption 2.4 is fulfilled with $\eta = 3$ (since $T_1 = 150$ Nm > 120.87 Nm $= 3B_{g1}$ and $T_2 = 15$ Nm > 5.475 Nm $= 3B_{g2}$). For the sake of simplicity, units will be subsequently omitted.

For the application of the proposed design methodology, let us define the functions

$$\sigma_u(\varsigma; \beta, a) = \mathrm{sign}(\varsigma)\max\{|\varsigma|^{\beta}, a|\varsigma|\} \tag{30a}$$

$$\sigma_{bh}(\varsigma; \beta, a, M) = \mathrm{sign}(\varsigma)\min\{|\sigma_u(\varsigma; \beta, a)|, M\} \tag{30b}$$

$$\sigma_{bs}(\varsigma; \beta, a, M, L) = \begin{cases} \sigma_u(\varsigma; \beta, a) & \text{if } |\varsigma| \leq L \\ \mathrm{sign}(\varsigma)\sigma_{bs}^{+}(|\varsigma|; \beta, a, M, L) & \text{if } |\varsigma| > L \end{cases} \tag{30c}$$

where

$$\sigma_{bs}^{+}(\varsigma; \beta, a, M, L) = \sigma_u(L; \beta, a) + (M - \sigma_u(L; \beta, a))\tanh\left(\frac{\sigma_u(\varsigma; \beta, a) - \sigma_u(L; \beta, a)}{M - \sigma_u(L; \beta, a)}\right)$$

for constants $\beta > 0$, $a \in \{0, 1\}$, $M > 0$, and $L > 0$ such that $\sigma_u(L; \beta, a) < M$. Examples are shown in Fig. 2.

Through experimental tests that show the efficiency of the proposed approach from an actual application whence model inaccuracies constitute an unavoidable reality, we further aim at observing diverse aspects on the closed-loop responses. The first of these is to show the achievement of the finite-time stabilization in contrast to analog exponential regulation implementations. Next, finite-time stabilization tests aiming at illustrating the ability of the proposed controller to adopt different saturating structures will be shown. Finally, finite-time stabilization tests oriented to conclude on the differences or coincidences among closed-loop responses obtained through the desired and on-line compensation versions of the developed SPD type scheme —where the type of the compensation term is the only difference among the implementations— are included. All the implementations were run taking the desired configuration at $q_d = \begin{pmatrix}\pi/6 & \pi/3\end{pmatrix}^T$ [rad] and initial conditions as $q(0) = \dot{q}(0) = 0_2$.

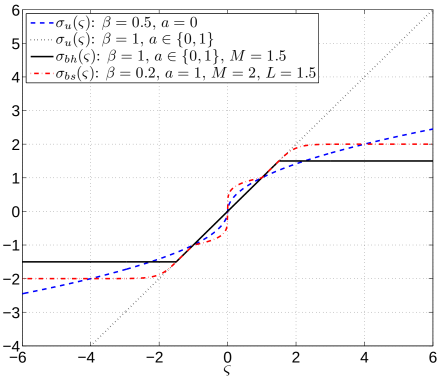

Fig. 2: Examples of $\sigma_u(\varsigma; \beta, a)$, $\sigma_{bh}(\varsigma; \beta, a, M)$ and $\sigma_{bs}(\varsigma; \beta, a, M, L)$

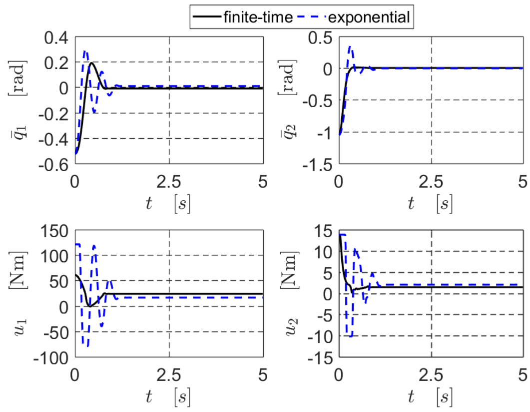

Fig. 3: Finite-time *vs* exponential stabilization

4.1 *Finite-time* vs *exponential stabilization*

Based on the functions in Eqs. (30), we define —for every $j = 1, 2$— those involved in the implementations performed in this subsection as

$$\sigma_{0j}(\varsigma) = \sigma_{bs}(\varsigma; \beta_0, a_{0j}, M_{0j}, L_{0j}) \tag{31a}$$

$$\sigma_{ij}(\varsigma) = \sigma_u(\varsigma; \beta_i, a_{ij}) \qquad i = 1, 2 \tag{31b}$$

with $a_{ij} = 0$, $i = 0, 1, 2$, $j = 1, 2$. Conditions on their parameters under which (12) is fulfilled are:

$$k_{1j} > k_g(2B_{gj})^{(1-\beta_0\beta_1)/\beta_0\beta_1} \tag{32a}$$

$$2B_{gj} \leq L_{0j}^{\beta_0} < M_{0j} \tag{32b}$$

(this is shown in the Appendix (Subsection 8.2)) [the right-most inequality in (32b) actually comes from the specifications of σ_{bs} in (30c)]. Let us note, from the involved functions, as defined through Eqs. (31), that $B_j = M_{0j}$, $j = 1, 2$ (see (11)). Hence, (11) and (32b) simultaneously require that $2B_{gj} < M_{0j} < T_j - B_{gj}$, $j = 1, 2$, which has been fulfilled by fixing $M_{01} = 100$ and $M_{02} = 12$. The rest of the control gain/parameter values were chosen taking care that inequalities (32) were always satisfied.

Figure 3 shows results obtained taking $\beta_1 = 3/5$, $\beta_2 = 18/25$ and $\beta_0 = 10/9$ for the finite-time controller with $\gamma = 6/5$, and the remaining control gain/parameters were taken, for both (finite-time and exponential) controllers, as: $K_1 = \text{diag}[500, 78]$ and $K_2 = \text{diag}[2.5, 2.5]$. We further fixed $L_{01} = 59.78$ and $L_{02} = 7.58$ for the finite-time controller, and $L_{01} = 94.17$ and $L_{02} = 9.49$ for the exponential stabilizer. One sees that control signals avoiding input saturation took place in both implementations, while the closed-loop trajectory arising through the exponential controller was observed to present a longer and more important transient. Interestingly, the finite-time stabilizer shows a more efficient ability to counteract the inertial effects through control signals with considerably less and lower variations during the transient.

4.2 *Multiple saturating structure*

We present an alternative test where the proposed desired-compensation scheme adopts two different saturating structures. It is worth pointing out that the proposed design methodology does not force to keep the same saturating structure at every one of the controlled degree of freedom but rather permits different choices among them. However, for our comparison purposes, the saturating structures are chosen different among the controllers but are kept the same among the controlled degrees of freedom for each one of the implemented stabilizer.

One of the implemented finite-time controllers adopts the same saturating structure of the precedent subsection, *i.e.* it involves the functions defined through Eqs. (31). Since this stabilizer uses, at every controlled degree of freedom, a single saturation function that includes both the P and D actions, it will be referred to as the SPD controller. The alternative finite-time controller is structured taking, for every $j = 1, 2$:

$$\sigma_{0j}(\varsigma) = \sigma_u(\varsigma; \beta_0, a_{0j}) \tag{33a}$$

$$\sigma_{ij}(\varsigma) = \sigma_{bh}(\varsigma; \beta_i, a_{ij}, M_{ij}) \qquad i = 1, 2 \tag{33b}$$

with $a_{ij} = 0$, $i = 0, 1, 2$, $j = 1, 2$. Since this stabilizer uses a saturation function for each one of the P and D actions (separately), it will be referred to as the SP-SD controller. Conditions on the parameters of the functions involved in this case —as defined through Eqs. (33)— under which (12) is fulfilled are:

$$k_{1j} > k_g(2B_{gj})^{(1-\beta_0\beta_1)/\beta_0\beta_1} \tag{34a}$$

$$M_{1j}^{\beta_0} > 2B_{gj} \tag{34b}$$

(this is shown in the Appendix (Subsection 8.3)).

For both —the SPD and SP-SD— finite-time controllers with $\gamma = 5/4$, we took $\beta_1 = 3/5$, $\beta_2 = 3/4$ and $\beta_0 = 1$. Notice that with such a unitary value of β_0, for the SP-SD algorithm we have $B_j = M_{1j} + M_{2j}$, $j = 1, 2$ (see (11)). Hence, while $M_{01} = 100$ and $M_{02} = 12$ were kept for the SPD controller (as in the precedent subsection), by taking $M_{11} = 82$, $M_{21} = 18$, and $M_{12} = M_{22} = 6$, the inequalities from expressions (11) and (34b) have been simultaneously satisfied. By further fixing $K_1 = \text{diag}[3260, 400]$ and $K_2 = \text{diag}[250, 25]$, the common inequality (32a) and (34a) has been fulfilled (for both controllers). We further fixed $L_{01} = 94.17$ and $L_{02} = 9.49$ for the SPD algorithm, under the consideration of (32b).

Figure 4 shows the results obtained from the implementations. One sees that, while both controllers achieve the finite-time stabilization objective avoiding input saturation, the closed-loop responses show different performances, with the SP-SD stabilizer giving rise to longer overshoots. Such a result corroborates the usefulness of the structural variety offered by the proposed approach in searching for performance improvement. It is worth further noticing that the SPD finite-time controller shows again —as in the previous test but this time compared to the SP-SD finite-time stabilizer— a more efficient ability to counteract the inertial effects through signals with considerably less and lower variations during the transient, concluding that such a nice feature is related not only to the finite-time nature of the controller but also to its (combined) SPD type structure.

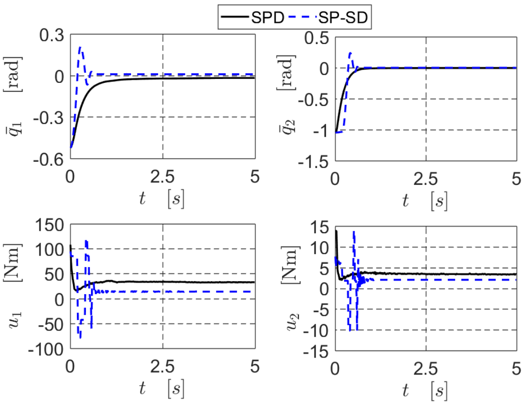

Fig. 4: SPD *vs* SP-SD finite-time controllers

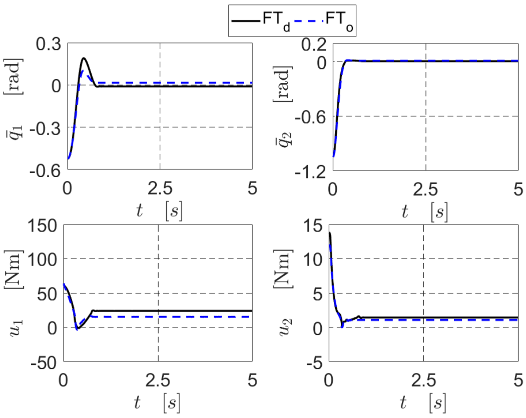

Fig. 5: Desired *vs* on-line conservative-force compensation

4.3 *Desired* vs *on-line conservative-force compensation*

The last test focuses on the comparison among finite-time control implementations involving the desired and on-line conservative-force compensation versions of the SPD-type control schemes from this work and that from [1], respectively. Of course, for every one of these cases, one can always choose control gain/parameters and/or saturating structures such that, for the same initial conditions, either of them outperforms the other. Thus, what we really focus on, in this section, is in comparing closed-loop responses when both controllers hold the same control gain/parameter values and saturating structures but differ only on the type of conservative-force (gravity) compensation. With this goal in mind, we repeated exactly the finite-time control test of Subsection 4.1 simply alternating the referred compensation term. Figure 5 shows the comparison among the tested controllers, with $\mathrm{FT_d}$ and $\mathrm{FT_o}$ denoting the finite-time controllers with desired and on-line compensation term, respectively. No considerable differences among the closed-loop performances can be appreciated. Several alternative tests performing the same comparison but with different control characteristics (*e.g.* different control gain/parameter value combinations) generally gave rise to a similar result, *i.e.* suitable closed-loop responses with no considerable differences among them. We conclude from these results that the cost on the performance for the implementation simplification earned by the desired compensation version of the controller is negligible, in spite of the open-loop conservative-force term that is left acting on the system in this case.

5 Conclusions

Global SPD-type continuous control of mechanical systems with input constraints guaranteeing finite-time or exponential stabilization has been made possible and further simplified through desired conservative-force compensation. Far from what one could have expected, this controller is not a simple extension of the on-line compensation case but it has rather proven to need more involved requirements resulting from a closed-loop analysis with considerably higher degree of complexity. Moreover, the proposed approach has overcome the proof on its transition from finite-time to exponential stabilization, which could not be solved keeping the local-homogeneity approach of the former in view of the open-loop conservative force which is kept acting on the closed loop. Experimental tests on a 2-DOF mechanical manipulator have shown the actual ability of the proposed approach to guarantee the considered types of convergence avoiding input saturation, and through different saturating configurations. In particular, the SPD finite-time stabilizer, with *external* saturation gathering unbounded internal P and D type actions, has shown the ability to counteract the system inertial transient effects through control signals with considerably less and lower variations than its analog SPD exponential and SP-SD finite-time controller versions. Furthermore, both the on-line and desired conservative-force compensation versions of the developed scheme were tested and actually compared when the only difference among them is on the type of the referred compensation term. They both gave rise to suitable results with very small differences among the corresponding closed-loop responses. Thus, the implementation simplifications earned through the desired compensation are concluded to have a negligible cost on the system performance, passing the bill rather to the closed-loop analysis. Future work will focus on robustness issues of the proposed continuous finite-time controllers.

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8 Appendix

8.1 Proof of Corollary 3.2

The global asymptotic stability follows from Proposition 3.1. Thus, all that remains to be proven is the (local) exponential stability property. In this direction, let us consider the scalar function $V_2(x_1,x_2)=V_1(x_1,x_2)+\epsilon x_1^T H(x_1+q_d)x_2$, with $V_1(x_1,x_2)$ as defined through Eq. (15) (with $\ell=1$), *i.e.*

$$V_2(x_1,x_2)=\frac{1}{2}x_2^T H(x_1+q_d)x_2+\int_{0_n}^{x_1} s_0^T\big(s_1(K_1z)\big)dz +\mathcal{U}_{\rm ol}(x_1+q_d)-\mathcal{U}_{\rm ol}(q_d)-g^T(q_d)x_1 +\epsilon x_1^T H(x_1+q_d)x_2$$

where ϵ is a positive constant such that

$$\epsilon<\min\{\epsilon_1,\epsilon_2\} \tag{35}$$

with

$$\epsilon_1=\frac{\big[\bar{k}_{1m}\mu_m\big]^{1/2}}{\mu_M}\quad,\quad \epsilon_2=\frac{\bar{k}_{1m}k_{2m}}{\bar{k}_{1m}k_C\varrho+\bar{k}_{1m}\mu_M+k_{2M}^2/4}$$

$\bar{k}_{1m}=\min_j\{\bar{k}_{1j}\}$; $k_{2m}=\min_j\{k_{2j}\}$; $k_{2M}=\max_j\{k_{2j}\}$; μ_m, μ_M and k_C as defined through Property 2.1 and Assumptions 2.1 and 2.2; and ϱ is a positive constant to be defined later on. From the proof of Theorem 3.1 (particularly, from inequality (23)), we have that $V_2(x_1,x_2)\geq\frac{\mu_m}{2}\|x_2\|^2+S_1(x_1)-\epsilon|x_1^T H(x_1)x_2|$, with $S_1(x_1)$ as defined through Eqs. (20) (with $\ell=1$). More precisely, on $\mathcal{Q}_1\times\mathbb{R}^n$, with $\mathcal{Q}_1=\{x_1\in\mathbb{R}^n:|x_{1j}|<b_{1j}/\bar{k}_{1j},\ j=1,\ldots,n\}$, we have that

$$\begin{aligned}V_2(x_1,x_2)&\geq\frac{\mu_m}{2}\|x_2\|^2+\sum_{j=1}^n\frac{\bar{k}_{1j}}{2}x_{1j}^2-\epsilon\big|x_1^T H(x_1+q_d)x_2\big|\\&\geq\frac{\mu_m}{2}\|x_2\|^2+\frac{\bar{k}_{1m}}{2}\|x_1\|^2-\epsilon\mu_M\|x_1\|\|x_2\|\\&=\frac{1}{2}\begin{pmatrix}\|x_1\|\\\|x_2\|\end{pmatrix}^T Q_1\begin{pmatrix}\|x_1\|\\\|x_2\|\end{pmatrix}\end{aligned}$$

with

$$Q_1=\begin{pmatrix}\bar{k}_{1m}&-\epsilon\mu_M\\-\epsilon\mu_M&\mu_m\end{pmatrix}$$

where Assumption 2.1 has been considered, and since (35) $\Longrightarrow$ $\epsilon<\epsilon_1$ $\Longrightarrow$ $Q_1>0$, we get

$$V_2(x)\geq c_1\|x\|^2 \tag{36}$$

$\forall x\in\mathcal{Q}_1\times\mathbb{R}^n$, with $c_1=\lambda_m(Q_1)/2>0$. On the other hand, observe that in view of (29), we have, on $\mathcal{Q}_0=\{x_1\in\mathbb{R}^n:|x_{1j}|\leq L_{1j}/k_{1j},\ |\sigma_{1j}(k_{1j}x_{1j})|=|k_{1j}x_{1j}|\leq L_{0j},\ j=1,\ldots,n\}=\{x_1\in\mathbb{R}^n:|x_{1j}|\leq\min\{L_{1j},L_{0j}\}/k_{1j},\ j=1,\ldots,n\}$, that $s_0\big(s_1(K_1x_1)\big)=K_1x_1$. From this, Assumption 2.1 and (19) we get, on $\mathcal{Q}_0\times\mathbb{R}^n$:

$$\begin{aligned}V_2(x_1,x_2)&=\frac{1}{2}x_2^T H(x_1+q_d)x_2+\frac{1}{2}x_1^T K_1x_1+\mathcal{U}_{\rm ol}(x_1+q_d)\\&\quad-\mathcal{U}_{\rm ol}(q_d)-g^T(q_d)x_1+\epsilon x_1^T H(x_1+q_d)x_2\\&\leq\frac{\mu_M}{2}\|x_2\|^2+\frac{k_{1M}}{2}\|x_1\|^2+\frac{k_g}{2}\|x_1\|^2\\&\quad+\epsilon\mu_M\|x_1\|\|x_2\|\\&=\frac{1}{2}\begin{pmatrix}\|x_1\|\\\|x_2\|\end{pmatrix}^T Q_2\begin{pmatrix}\|x_1\|\\\|x_2\|\end{pmatrix}\end{aligned}$$

with

$$Q_2=\begin{pmatrix}k_{1M}+k_g&\epsilon\mu_M\\\epsilon\mu_M&\mu_M\end{pmatrix}$$

and $k_{1M}=\max_j\{k_{1j}\}$. From simple developments, one can further verify that (35) $\Longrightarrow$ $\epsilon<\epsilon_1$ $\Longrightarrow$ $Q_2>0$, whence we get

$$V_2(x)\leq c_2\|x\|^2 \tag{37}$$

$\forall x\in\mathcal{Q}_0\times\mathbb{R}^n$, with $c_2=\lambda_M(Q_2)/2>0$. Furthermore, the derivative of V_2 along the closed-loop system trajectories is given by

$$\begin{aligned}&\dot{V}_2(x_1,x_2)\\&\quad=x_2^T H(x_1+q_d)\dot{x}_2+\frac{1}{2}x_2^T\dot{H}(x_1+q_d,x_2)x_2\\&\qquad+\big[s_0\big(s_1(K_1x_1)\big)+g(x_1+q_d)-g(q_d)\big]^T\dot{x}_1\\&\qquad+\epsilon x_1^T H(x_1+q_d)\dot{x}_2+\epsilon x_1^T\dot{H}(x_1+q_d,x_2)x_2\\&\qquad+\epsilon\dot{x}_1^T H(x_1+q_d)x_2\\&\quad=x_2^T\big[-C(x_1+q_d,x_2)x_2-g(x_1+q_d)+g(q_d)\\&\qquad-s_0\big(s_1(K_1x_1)+s_2(K_2x_2)\big)\big]+\frac{1}{2}x_2^T\dot{H}(x_1+q_d,x_2)x_2\\&\qquad+\big[s_0\big(s_1(K_1x_1)\big)+g(x_1+q_d)-g(q_d)\big]^T x_2\\&\qquad+\epsilon x_1^T\big[-C(x_1+q_d,x_2)x_2-g(x_1+q_d)\\&\qquad+g(q_d)-s_0\big(s_1(K_1x_1)+s_2(K_2x_2)\big)\big]\\&\qquad+\epsilon x_1^T\big[C(x_1+q_d,x_2)+C^T(x_1+q_d,x_2)\big]x_2\\&\qquad+\epsilon x_2^T H(x_1+q_d)x_2\\&\quad=-x_2^T\big[s_0\big(s_1(K_1x_1)+s_2(K_2x_2)\big)-s_0\big(s_1(K_1x_1)\big)\big]\\&\qquad-\epsilon x_1^T\big[s_0\big(s_1(K_1x_1)\big)+g(x_1+q_d)-g(q_d)\big]\\&\qquad-\epsilon x_1^T\big[s_0\big(s_1(K_1x_1)+s_2(K_2x_2)\big)-s_0\big(s_1(K_1x_1)\big)\big]\\&\qquad+\epsilon x_2^T C(x_1+q_d,x_2)x_1+\epsilon x_2^T H(x_1+q_d)x_2\end{aligned}$$

where Property 2.2.1 has been applied. Notice that, in view of (29), we have, on $\mathcal{S}=\{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n:|x_{1j}|\leq L_{1j}/k_{1j},\ |x_{2j}|\leq L_{2j}/k_{2j},\ |\sigma_{1j}(k_{1j}x_{1j})+\sigma_{2j}(k_{2j}x_{2j})|=|k_{1j}x_{1j}+k_{2j}x_{2j}|\leq L_{0j},\ j=1,\ldots,n\}$, that $s_0\big(s_1(K_1x_1)+s_2(K_2x_2)\big)-s_0\big(s_1(K_1x_1)\big)=K_2x_2$. From this, (21), Property 2.2.3 and

Assumptions 2.1 and 2.2, we get

$$\begin{aligned}\dot V_2(x_1,x_2) &\leq -x_2^T K_2 x_2 - \epsilon\sum_{j=1}^n \bar k_{1j} x_{1j}^2 + \epsilon\left|x_1^T K_2 x_2\right| \\ &\quad + \epsilon\left|x_2^T C(x_1+q_d,x_2)x_1\right| + \epsilon\left|x_2^T H(x_1+q_d)x_2\right| \\ &\leq -k_{2m}\|x_2\|^2 - \epsilon\bar k_{1m}\|x_1\|^2 + \epsilon k_{2M}\|x_1\|\|x_2\| \\ &\quad + \epsilon k_C \varrho\|x_2\|^2 + \epsilon\mu_M\|x_2\|^2 \\ &= -\begin{pmatrix}\|x_1\| \\ \|x_2\|\end{pmatrix}^T Q_3 \begin{pmatrix}\|x_1\| \\ \|x_2\|\end{pmatrix}\end{aligned}$$

$\forall(x_1,x_2)\in\mathcal{S}\cap\left(\mathcal{Q}_1\times\mathbb{R}^n\right)$, with

$$Q_3 = \begin{pmatrix}\epsilon\bar k_{1m} & -\epsilon k_{2M}/2 \\ -\epsilon k_{2M}/2 & k_{2m}-\epsilon k_C\varrho - \epsilon\mu_M\end{pmatrix}$$

and $\varrho = \max_{x_1\in\mathcal{Q}_1}\|x_1\| = \left[\sum_{j=1}^n\left[\bar b_{1j}/\bar k_{1j}\right]^2\right]^{1/2}$, and since (35) $\Longrightarrow$ $\epsilon < \epsilon_2$ $\Longrightarrow$ $Q_3 > 0$, we get

$$\dot V_2(x) \leq -c_3\|x\|^2 \tag{38}$$

$\forall x\in\mathcal{S}\cap\left(\mathcal{Q}_1\times\mathbb{R}^n\right)$, with $c_3=\lambda_m(Q_3)>0$. Thus, from the simultaneous satisfaction of inequalities (36)–(38) on $\mathcal{S}\cap\left[(\mathcal{Q}_0\cap\mathcal{Q}_1)\times\mathbb{R}^n\right]$, we conclude —by [16, Theorem 4.10]— that the origin $(x_1,x_2)=(0_n,0_n)$ is a (locally) exponentially stable equilibrium of the closed-loop system, whence the proof is completed.

8.2 On inequalities (32)

Noting from (28) that $\beta_0\beta_1=(2-\gamma)/\gamma$ and (in accordance to Corollaries 3.1 and 3.2) that $1\leq\gamma<2 \iff 0<(2-\gamma)/\gamma\leq 1 \iff 0<\beta_0\beta_1\leq 1$, observe that on $\{\varsigma\in\mathbb{R}:0<|\varsigma|\leq 2B_{gj}/k_g\}$ we have that

$$\begin{aligned}|\varsigma|\leq\frac{2B_{gj}}{k_g} &\iff |\varsigma|^{1-\beta_0\beta_1}\leq\left(\frac{2B_{gj}}{k_g}\right)^{1-\beta_0\beta_1} \\ &\iff k_{1j}^{\beta_0\beta_1}\left(\frac{2B_{gj}}{k_g}\right)^{\beta_0\beta_1-1}|\varsigma|\leq|k_{1j}\varsigma|^{\beta_0\beta_1}\end{aligned}$$

while from (32a) we have, for all $\varsigma\neq 0$, that:

$$\begin{aligned}\text{(32a)} &\iff k_g(2B_{gj})^{(1-\beta_0\beta_1)/(\beta_0\beta_1)}|\varsigma|^{1/(\beta_0\beta_1)} < k_{1j}|\varsigma|^{1/(\beta_0\beta_1)} \\ &\iff k_g^{\beta_0\beta_1}(2B_{gj})^{1-\beta_0\beta_1}|\varsigma| < k_{1j}^{\beta_0\beta_1}|\varsigma| \\ &\iff k_g|\varsigma| < k_{1j}^{\beta_0\beta_1}\left(\frac{2B_{gj}}{k_g}\right)^{\beta_0\beta_1-1}|\varsigma|\end{aligned}$$

From these developments we thus get, on $\{\varsigma\in\mathbb{R}:0<|\varsigma|\leq\min\{2B_{gj}/k_g\,,\,L_0^{1/\beta_1}/k_{1j}\}\}$, that: (32a) $\Longrightarrow$ $k_g|\varsigma|<|k_{1j}\varsigma|^{\beta_0\beta_1}$, and consequently, for all $\{\varsigma\in\mathbb{R}:0<|\varsigma|\leq L_0^{1/\beta_1}/k_{1j}\}$, that: (32a) $\Longrightarrow$ $\min\{k_g|\varsigma|,2B_{gj}\}<|k_{1j}\varsigma|^{\beta_0\beta_1}$, whence, under the additional consideration of (32b), we get that:

$$\begin{aligned}&\text{(32)} \Longrightarrow \min\{k_g|\varsigma|,2B_{gj}\} \\ &< \begin{cases}|k_{1j}\varsigma|^{\beta_0\beta_1} & \text{if } |\varsigma|\leq\frac{L_{0j}^{1/\beta_1}}{k_{1j}} \\ L_{0j}^{\beta_0}+(M_{0j}-L_{0j}^{\beta_0})\tanh\left(\frac{|k_{1j}\varsigma|^{\beta_0\beta_1}-L_{0j}^{\beta_0}}{M_{0j}-L_{0j}^{\beta_0}}\right) & \text{if } |\varsigma|>\frac{L_{0j}^{1/\beta_1}}{k_{1j}}\end{cases} \\ &= \begin{cases}|\sigma_{1j}(k_{1j}\varsigma)|^{\beta_0} & \text{if } |\sigma_{1j}(k_{1j}\varsigma)|\leq L_{0j} \\ \sigma_{bs}^+(|\sigma_{1j}(k_{1j}\varsigma)|;\beta_0,0,L_{0j},M_{0j}) & \text{if } |\sigma_{1j}(k_{1j}\varsigma)|>L_{0j}\end{cases} \\ &= \left|\sigma_{0j}\left(\sigma_{1j}(k_{1j}\varsigma)\right)\right|\end{aligned}$$

$\forall\varsigma\neq 0$.

8.3 On inequalities (34)

From previous arguments, used in Subsection 8.2 (since (32a) and (34a) are analog inequalities), we have, on $\{\varsigma\in\mathbb{R}:0<|\varsigma|\leq 2B_{gj}/k_g\}$, that: (34a) $\Longrightarrow$ $k_g|\varsigma|<|k_{1j}\varsigma|^{\beta_0\beta_1}$, and consequently, for all $\varsigma\neq 0$, that: (34a) $\Longrightarrow$ $\min\{k_g|\varsigma|,2B_{gj}\}<|k_{1j}\varsigma|^{\beta_0\beta_1}$, whence, under the additional consideration of (34b), we get that: (34) $\Longrightarrow$ $\min\{k_g|\varsigma|,2B_{gj}\}<\min\left\{|k_{1j}\varsigma|^{\beta_0\beta_1},M_{1j}^{\beta_0}\right\} = |\sigma_{1j}(k_{1j}\varsigma)|^{\beta_0} = \left|\sigma_{0j}\left(\sigma_{1j}(k_{1j}\varsigma)\right)\right|$, $\forall\varsigma\neq 0$.

IET Research Journals, pp. 1–12