Opamp-based Synthesis of a Fractional Order Switched System

E. Zambrano-Serrano*, J.M. Muñoz-Pacheco† and E. Campos-Cantón*
*Department of Applied Mathematics, Instituto Potosino de Investigación Científica y Tecnológica,
San Luis Potosí, SLP., MEXICO 78216
Email: ernesto.zambrano@ipicyt.edu.mx
†Faculty of Electronics Sciences, Benemérita Universidad Autónoma de Puebla,
Puebla, Pue., MEXICO 72570.

***Abstract*—The analysis, design and circuit synthesis of a fractional order switched system is presented in this paper. That system is capable of showing chaotic oscillations with a fractional order less than three, i.e., 2.4. The dynamical system is called fractional order unstable dissipative system (FOUDS); because it consists of a switching law to display strange attractors. Its dynamical behavior is explored and a circuit synthesis system is realized considering operational amplifiers. SPICE simulations agree with the numerical results.**

I. Introduction

The concept of fractional calculus was proposed by Leibniz more than 300 year ago. It begins to attract much interest and importance in various fields of engineering and physics, where the number of applications has increased rapidly. A fractional model allows one describe a real object more adequately and accurately than corresponding integer model [1], [2] due to the unlimited memory and hereditary properties of a fractional order operator. Moreover the fractional order parameter improves the system performance by adding one degree of freedom. The transcendence of the fractional calculus has been demonstrated to be effective in different contexts; for example in image processing a fractional differential algorithm could preserve the information of weak texture, while enhancing the edge of image [3]. In control theory a fractional order controller has been reported in some applications such as backlash vibration suppression control of torsional systems [4]. In viscoelastic materials, the fractional order damping element gives a superior model because it is modeled as a force proportional to the fractional order derivative of the displacement [5]. In electronic circuits the fractance device is referred to as a constant phase element (CPE), where the CPE has shown important applications in the field of bioimpedance, which measure the passive electrical properties of biological materials [6]; and so on [7], [8]. More recently, many researchers have shown growing interest in fractional order dynamical systems with chaotic behavior. Up to now, a large number of fractional order systems have been proposed, such as the fractional order Rössler system [9], fractional order Chua system [10], Fractional order Chen system [11], among others [12]. Specifically, engineering applications are considering fractional order chaotic systems, such as a digital cryptography approach, and an image encryption method [13], [14] due the fractional derivatives have complex geometrical interpretation because the power spectrum of fractional order chaotic systems fluctuates complexly increasing the chaotic behavior in frequency domain.

This paper presents the design of a fractional order unstable dissipative system that generates 2-scroll chaotic attractor. We propose a electronic circuit for a value of fractional order $\alpha = 0.8$. The fractional operator has been approximated considering the frequency domain approximation, therefore the fractance device is considered to emulate such approximation, from practical point of view. We apply operational amplifiers to design the fractional integrator and nonlinear function. Finally, we compare the SPICE simulations with the results of the numerical simulation to illustrate the performance of the proposed circuit synthesis.

II. Preliminaries

Different definitions of fractional order integration and differentiation have emerged during the development of fractional order theory. Some definitions are the Grünwald-Letnikov definition, the Cauchy integral formula, the Riemann-Liouville definition and the Caputo definition [1].

Let $L^1 = L^1[a, b]$, $0 \le a < b < \infty$, be a class of Lebesgue integrable function on [a,b].

The Riemman-Liouville definition of a fractional integral for the function $f(t) \in L^1$, of order $\alpha > 0$, and $t > 0$, is given by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \qquad (1)$$

where $\Gamma(\cdot)$ is Euler's Gamma function. The Caputo definition of fractional derivative of order α, $0 < \alpha < 1$ for continuous $f(t)$ is given by

$$D^{\alpha} f(t) = I^{1-\alpha} \frac{df(t)}{dt} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(t)}{(t-\tau)^{\alpha}} dt. \qquad (2)$$

The main advantage of using the Caputo definition is due to the fact of the initial conditions for the fractional order differential equations, because are the same form as those integer order differential equations, and there are clear interpretations of the initial conditions for integer orders, moreover, it has the benefit of possessing a value of zero when it is applied to a constant.

The Laplace transform of the fractional integration operator is

$$I^{\alpha}(s) = L\left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right\} = \frac{1}{s^{\alpha}}, \tag{3}$$

the implicit fractional differentiation is defined as the dual operation of the fractional integration. If $y(t) = I^{\alpha}(x(t))$ or $Y(s) = \frac{1}{s^{\alpha}}X(s)$, then $x(t)$ is the αth order fractional derivative of $y(t)$ defined as

$$x(t) = D^{\alpha}(y(t)) \quad \text{or} \quad X(s) = s^{\alpha}Y(s). \tag{4}$$

A linear time-invariant fractional order system can be written in the following state space form

$$\frac{d^{\alpha}x}{dt^{\alpha}} = Ax + Bu, \tag{5}$$

where $x \in R^n$, $u \in R^m$, $A \in R^{n\times n}$, $B \in R^{n\times m}$, with fractional order $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_n]$. If α_i's are rationals numbers in $(0, 1]$, a sufficient condition for stability of system (5) can be given as follows: If (5) is a commensurate order system, and its stability can be checked by [15], then (5) is asymptotically stable if

$$|\arg(\lambda)| > \frac{\pi\alpha}{2}, \tag{6}$$

is satisfied for all eigenvalues (λ) of matrix A [4].

III. Frequency domain approach

In order to implement the fractional order unstable dissipative system throughout this work is considered the Charef method. The method consists of finding a rational approximation for the fractional operator based on Bode diagrams [16], the idea is obtain a linear approximation that has a similar frequency response as $1/s^{\alpha}$, by finding zeros and poles of a transfer function. The order and accuracy of transfer function depends on the desired bandwidth and discrepancy between the actual and the approximate magnitude. It is important to consider that the operator $1/s^{\alpha}$ has a Bode diagram typical by $-20\alpha\ dB/decade$. So we use the approach to develop an approximation of fractional order in Laplace domain [17]. The fractional integrator of order α can be represented by the transfer function as shown in (3), furthermore in [16] the slope with $-20\alpha\ dB/decade$ is approximated by a set of zig zag lines joined together with alternate slopes of 0 $dB/decade$ and -20 $dB/decade$. According to [16] the fractional operator $1/s^{\alpha}$ can be approximated if a frequency range ω_{max}, a corner frequency p_T and the discrepancy $y\ dB$ between the actual and approximate line are specified as

$$H(s) = \frac{1}{s^{\alpha}} \approx \frac{1}{\left(1+\frac{s}{p_T}\right)} \approx \frac{\prod_{i=0}^{N-1}\left(1+\frac{s}{z_i}\right)}{\prod_{i=0}^{N-1}\left(1+\frac{s}{p_i}\right)}. \tag{7}$$

Then the poles and zeros of the function (7) can be obtained as follows

first pole, $p_0 = p_T 10^{[y/20\alpha]}$
first zero, $z_0 = p_0 10^{[y/10(1-\alpha)]}$
second pole, $p_1 = z_0 10^{[y/10\alpha]}$
second zero, $z_1 = p_1 10^{[y/10(1-\alpha)]}$
⋮
N**th zero,** $P_{N-1}10^{[y/10(1-\alpha)]}$
$(N+1)$**th pole,** $p_N = z_{N-1}10^{[y/10\alpha]}$.

Where N is given by

$$N = \text{Integer}\left[\frac{\log\left(\frac{\omega_{max}}{p_0}\right)}{\log(ab)}\right] + 1, \tag{8}$$

the frequency corner p_T is determined in $-3\alpha\ dB$, p_0 is determined by the specified error, and p_N is determined by N, and a, b are given by

$$\begin{aligned} a &= 10^{[y/10(1-\alpha)]}, \\ b &= 10^{[y/10\alpha]}, \end{aligned} \tag{9}$$

where a, b are defined as the ratio of a zero to a previous pole and the ratio of a pole to a previous zero, respectively. Therefore, the approximation to $1/s^{0.8}$ with discrepancy error of $y = 2dB$, $p_T = 10^{-2}\text{rad}/s$, $\omega_{\max} = 10^2\text{rad}/s$, $N = 4$ is given by

$$\frac{1}{s^{0.8}} \approx \frac{5.235s^3 + 1453s^2 + 5306s + 254.9}{s^4 + 658.1s^3 + 5700s^2 + 658.2s + 1}, \tag{10}$$

more approximations can be found in [10], [18].

IV. Fractional order chaotic system

The fractional order unstable disspative system is described by [19]

$$\begin{aligned} D^{\alpha}x &= y, \\ D^{\alpha}y &= z, \\ D^{\alpha}z &= -ax - by - cz + f(x), \end{aligned} \tag{11}$$

with

$$f(x) = \begin{cases} \gamma, & \text{if} \quad x \geq 0.35, \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

where α is the fractional order satisfying $0 < \alpha < 1$. When $(a, b, c, \gamma) = (3.75, 0.7, 0.7, 2.5)$, the equilibrium points and their corresponding eigenvalues are

$$\begin{aligned} E_1 &= (0, 0, 0), & -1.6513, 0.4757 \pm 1.4299i \\ E_2 &= (0.66, 0, 0), & -1.6513, 0.4757 \pm 1.4299i. \end{aligned}$$

Considering the equation (6) we can determine a minimal commensurate order to keep the chaotic behavior in the system. In this case the order it is $\alpha > 0.7956$, then we selected $\alpha = 0.8$. The numerical simulation of (11) with $\alpha = 0.8$ is shown in Fig. 1, and the spectrum of Lyapunov exponents are $LE_1 = 0.1216$, $LE_2 = 0$ and $LE_3 = -2.5481$. The spectrum of Lyapunov exponents is determined from a time serie, taken into account the algorithm proposed in [20].

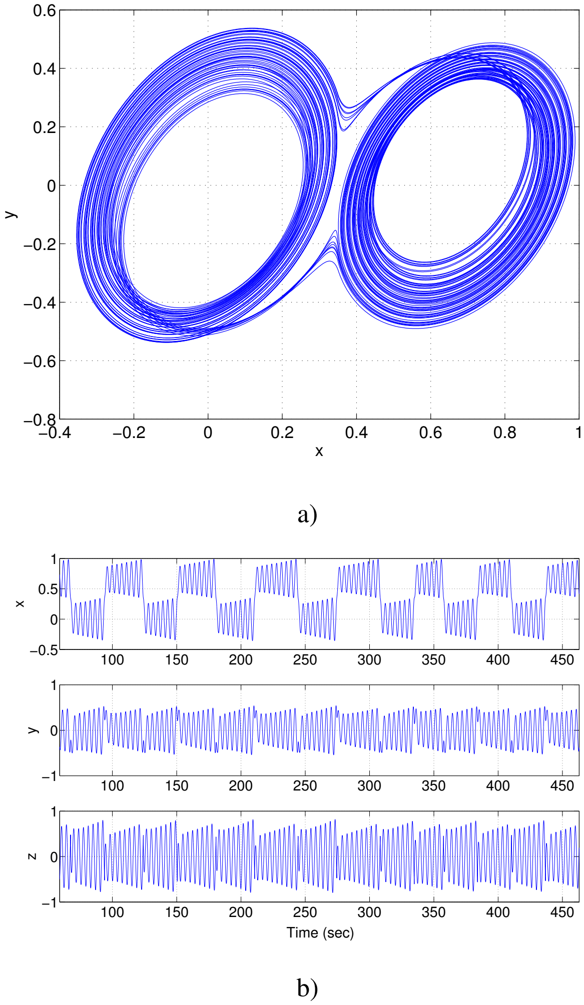


Fig. 1. a) Projections of the attractors onto the xy-plane. b) Chaotic signals of the fractional order chaotic system against to time.

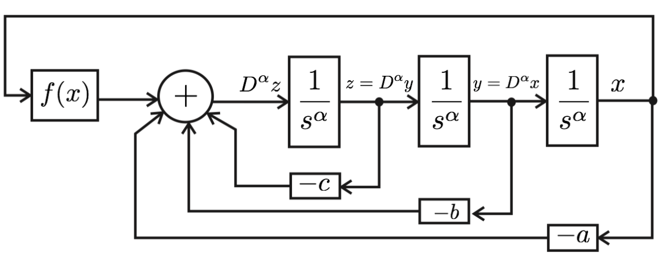


Fig. 2. The diagrammatic sketch of (11).

V. Circuit realization

This section shows how to link the electronic circuit to generate the desired dynamical behaviors based on the fractional order unstable dissipative system. The basic design idea is outlined as follows. Fig. 2 displays the diagrammatic sketch of system (11), which depicts the basic strategy of analog computing. As a straightforward approach, three fractional order integrators are cascaded, the summing is employed to form a feedback loop, and a piecewise linear is then utilized

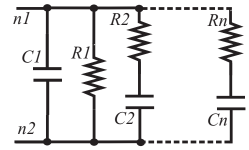


Fig. 3. Fractance device of $1/s^\alpha$.

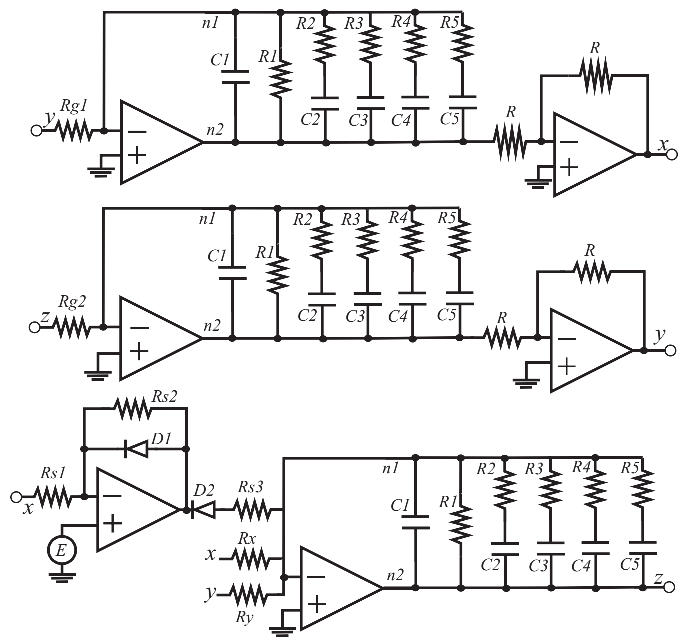


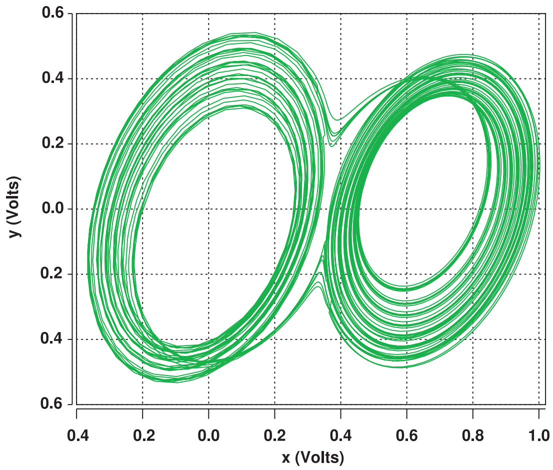
Fig. 4. Circuit diagram to realize the fractional order unstable dissipative system (11) for $\alpha = 0.8$.

to realized the function $f(x)$. From circuit theory, a circuit may has noninteger order properties. This kind of circuits are called fractance [1]. The fractance device is an electrical element which exhibits fractional order impedance properties. The fractance can be realized as chain, tree or a net grid type networks [21]. The guidelines to design a fractance were developed by authors of [1], [18], [21] for any order as shown in Fig. 3. We select a fractance that approximate fractional order $\alpha = 0.8$, that means for the design is considered an arrangement of five capacitors and five resistors from Fig. 3. Using circuit theory, we can obtain the transfer function $H(s)$ between $n1$ and $n2$ as follows

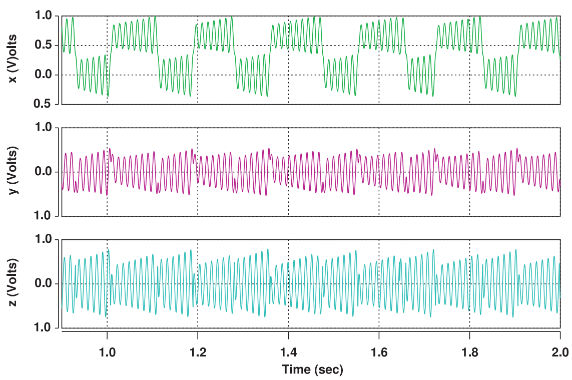
$$H(s) = R1//\frac{1}{sC1}//(\Delta)//(\Theta)//(\Lambda)//(\Psi), \qquad (13)$$

where $\Delta = R2 + \frac{1}{sC2}$, $\Theta = R3 + \frac{1}{sC3}$, $\Lambda = R4 + \frac{1}{sC4}$ and $\Psi = R5 + \frac{1}{sC5}$. Where C_0 is the unit parameter. Let $G(s) = H(s)C_0 = \frac{1}{s^{0.8}}$ and $C_0 = 1\mu F$ and matching (10) with (13) the values of resistances and capacitances as follows: $R1 = 39.8M\Omega$, $R2 = 9.839M\Omega$, $R3 = 0.9330M\Omega$, $R4 = 0.09319M\Omega$, $R5 = 0.009555M\Omega$, $C1 = 0.1884\mu F$, $C2 = 0.7619\mu F$, $C3 = 0.4520\mu F$, $C4 = 0.2545\mu F$ and $C5 = 0.1396\mu F$.

The electronic implementation of the nonlinear function (12) it is realized by considering the piecewise linear approach [22] because this method is used to model the behavior of non-



a)



b)

Fig. 5. a) Circuit simulation of the chaotic attractor with fractional order $\alpha = 0.8$. b) Time response of the states variables of the circuit

linear functions. The piecewise linear models for operational amplifiers can be described by saturated circuits. We consider the finite-gain of amplifier operational amplifier [22]

$$v_0 = \frac{A_v}{2}\left(\left|v_i + \frac{V_{sat}}{A_v}\right| - \left|v_i - \frac{V_{sat}}{A_v}\right|\right) i(-) = i(+) = 0 \tag{14}$$

where A_v is voltage gain, V_{sat} is positive saturation, $-V_{sat}$ is negative saturation, and the linear region is defined as $-V_{sat} \leq v_0 \leq V_{sat}$, the inverting amplifier is converted into an half-wave rectifier by adding two diodes to generate the vertical voltage shift as shown in Fig. 4, where resistors $Rs1$, $Rs2$ and voltage E control its breakpoint whereas the resistor $Rs3$ and saturation voltage of opamp set by γ.

The circuit to realize the fractional order unstable dissipative system (11) with $\alpha = 0.8$ is displayed in Fig. 4, where the circuit parameters are $Rx = 2.9k\Omega$, $Ry = Rz = 13.8k\Omega$, $Rs1 = 1k\Omega$, $Rs2 = 1M\Omega$, $Rs3 = 25.37k\Omega$, and $D1 = D2 = 1n4001$. Also, in this work the operational amplifier TL081 has been used. Figure 5 shows the circuit simulation of the chaotic attractor of FOUDS with a fractional order $\alpha = 0.8$ and and their respective time response, while the transient part has been eliminated. Finally, the circuit simulation shows that the results are in agreement with numerical simulations.

VI. Conclusion

In this paper, we designed the electronic circuit of a fractional order unstable dissipative system with order $\alpha = 0.8$, moreover, the stability theorem of fractional order systems guarantee that chaos is exhibited in the fractional order unstable dissipative system. Furthermore, based on frequency domain approximation a fractance device is considered to realize the fractional order operator, therefore, the nonlinear function was modeled by PWL approximation. It was shown that voltage saturated function can be synthesized with opamps and diodes by controlling the breakpoint and the vertical voltage shift. Finally, we presented a circuit synthesis of the fractional order unstable dissipative system, as well as SPICE simulations, which agree with the numerical simulations.

Acknowledgment

E. Zambrano-Serrano is a doctoral fellow of CONACYT (Mexico) in the Graduate Program on Control and Dynamical Systems at DMAp-IPICYT. The first author grateful to IPICYT for the scholarship to attendance the International Conference on Modern Circuits and Systems Technologies. J.M. Muñoz-Pacheco thanks "VIEP-BUAP Research Projects 2016" to partially support this research.

References

[1] I. Podlubny, *Fractional Differential Equations Mathematics in Science and Engineering.* New York, NY, USA: Academic Press, 1999.

[2] N. Yang and C. Liu, "A novel fractional-order hyperchaotic system stabilization via fractional sliding-mode control," *Nonlinear Dyn.,* vol. 74, pp. 721-732, Jul. 2013.

[3] B. Li and W. Xie, "Image denoising and enhancement based on adaptive fractional calculus of small probability strategy," *Neurocomputing,* vol. 175, pp. 704-714, Oct. 2015.

[4] M. S. Tavazoei, M. Haeri, S. Jafari, S. Bolouki and Siami, "Some applications of fractional calculus in suppression of chaotic oscillations," *IEEE Trans. Industrial Electronics,* vol. 55, no. 11, pp. 4094-4101, Nov. 2008.

[5] A. Sasso, G. Ducharne and G. Palmieri, "Application of fractional derivative models in linear viscoelastic problems," *Mech. Time Depend Mater.,* vol. 15, pp. 367-387, Sep. 2011.

[6] T. J. Freeborn, "A survey of fractional-order circuit models for biology and biomedicine," *IEEE J. Emerg. Sel. Topics Circuits Syst.,* vol. 3, no, 3, pp. 416-424, Sep. 2014.

[7] S. Ghasemi, A. Tabesh and J. Askari-Masnani, "Application of fractional calculus theory to robust controller design for wind turbine generators," *IEEE J. Emerg. Energy Conversion,* vol. 29, no. 3, pp. 780-787, Sep. 2014.

[8] T. J. Freeborn, B. Maundy and A. S. Elwakil, "Measurement of super capacitor fractional-order model parameters from voltage-excited step response," *IEEE J. Emerg. Sel. Topics Circuits Syst.,* vol. 3, no. 3, pp. 367-376, Sep. 2013.

[9] C. Li and G. Chen, "Chaos and hyperchaos in the fractional-order Rössler equations," *Phys. A,* vol. 341, pp. 55-61, Oct. 2004.

[10] T. Hartley, C. F. Lorenzo and H. K. Hammer, "Chaos in a fractional order Chua's system," *IEEE Trans. Circuits Syst. I Fundam. Theory Appl.,* vol. 42, no. 8, pp. 485-490, Ago. 1995.

[11] J. G. Lu and G. Chen, "A note on the fractional order Chen system," *Chaos Soliton Fractals,* vol. 27, no. 3, pp. 685-688, Feb. 2006.

[12] I. Petras, *Fractional-Order Nonlinear Systems Modeling, Analysis and Simulation.* Berlin, Germany: Springer Berlin Heidelberg, 2011.

[13] A. Kiani-B , K. Fallahi, N. Pariz, and H. Leung, “A chaotic secure communication scheme using fractional chaotic systems based on an extended fractional Kalman filter,” *Commun. Nonlinear Sci. Numer. Simul.,* vol 14, no. 3, pp. 863-879, Mar. 2009.

[14] P. Muthukumar, P. Balasubramaniam, and K. Ratnavelu, “Fast projective synchronization of fractional order chaotic and reverse chaotic systems with its application to an affine cipher using date of birth (DOB),” *Nonlinear Dyn.,* vol. 80, no. 4, pp. 1883-1897, Jul. 2014.

[15] D. Matignon, “Stability result on fractional differential equations with applications to control processing,” *Proc. IMACS-SMC,* Lille, France, 1996, pp. 963-968.

[16] A. Charef, H. H. Sun, Y. Sao, and B, Onaral, ‘Fractal systems as represented by singularity function,” *IEEE Trans. Autom. Control,* vol. 37, no. 9, pp. 1465-1470, Sep. 1992.

[17] M. S. Tavazoei and M. Haeri, “Unreliability of frequency-domain approximation in recognising chaos in fractional-order systems,” *IET Signal Process,* vol. 1, no. 4, pp. 171-181, Dec. 2007.

[18] W. M. Ahmad, and J. C. Sprott, “Chaos in fractional-order autonomous nonlinear systems,” *Chaos Solitons Fractals,* vol, 16, pp. 339-351, Sep. 2002.

[19] E. Zambrano-Serrano, E. Campos-Cantón, and J. M. Muñoz-Pacheco, “Strange attractors generated by a fractional order switching system and its topological horseshoe” *Nonlinear Dyn,* vol. 83, no. 3, pp. 1629-1641, Oct. 2015.

[20] M. Sano and Y. Sawada, “Measurement of the Lyapunov spectrum from a chaotic time series” *Phys. Rev. Lett.,* vol. 55, no. 10, pp. 1082-1085, Sep. 1985.

[21] B. T. Krishna and K. V. V. S. Reddy, “Active and passive realization of fractance device of order 1/2,” *Active Passive Electron. Compon.,* vol. 2008, pp.1-5, 2008.

[22] J. M. Muñoz-Pacheco and E. Tlelo-Cuautle *Electronic Design Automation of Multi-scroll Chaos Generators.* Bentham Science Publishers Ltd, 2010.