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Arturo Zavala-Río & Griselda I. Zamora-Gómez

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Local-homogeneity-based global continuous control for mechanical systems with constrained inputs: finite-time and exponential stabilization

Arturo Zavala-Río* and Griselda I. Zamora-Gómez

Instituto Potosino de Investigación Científica y Tecnológica, División de Matemáticas Aplicadas, San Luis Potosí, Mexico

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A global continuous control scheme for the finite-time or (local) exponential stabilization of mechanical systems with constrained inputs is proposed. The approach is formally developed within the theoretical framework of local homogeneity. This has permitted to solve the formulated problem not only guaranteeing input saturation avoidance but also giving a wide range of design flexibility. The proposed scheme is characterized by a Saturating-Proportional-Derivative type term with generalized saturating and locally-homogeneous structure that permits multiple design choices on both aspects. The work includes a simulation implementation section where the veracity of the so-cited argument claiming that finite-time stabilizers are faster than asymptotical ones is studied. In particular, a way to carry out the design so as to indeed guarantee faster stabilization through finite-time controllers (beyond their finite-time convergence) is shown.

Keywords: finite-time stabilization; local homogeneity; mechanical systems; constrained inputs; saturation

1. Introduction

Continuous control aiming at the finite-time convergence of an equilibrium being (simultaneously) rendered stable has been a topic of increasing interest in the last decades. Inspired by the seminal work of Haimo (1986), several researchers have devoted efforts to settle down a suitable underlying analytical framework for the subject. Important contributions in this direction are those due to Bhat & Bernstein (1995, 1997, 1998, 2000, 2005), by formally stating a precise definition of *finite-time stability* that gathers both the (Lyapunov) stability and finite-time convergence, thoroughly developing Lyapunov-based criteria for its determination, and clearly characterizing its relationship with homogeneous vector fields. This latter characterization has been particularly attractive in view of its simplicity: for a homogeneous vector field with asymptotically stable equilibrium at the origin, verifying negativity of the homogeneity degree suffices to conclude finite-time stability (of the origin). This naturally leads to the idea of involving homogeneity in control design to readily achieve finite-time stabilization. Nevertheless, such a strategy is tied to the requirements imposed by homogeneity, which is (conventionally) a *global* property. For instance, in a coordinate-dependent framework, a vector field with bounded components cannot be homogeneous (Bhat & Bernstein, 2005). Consequently, within such a framework, the referred strategy cannot be applied under bounded input constraints. Nevertheless, such a design restriction has been proven to be relaxed through alternative notions of homogeneity (Zavala-Río & Fantoni, 2014).

*Email: azavala@ipicyt.edu.mx

Based on the theoretical framework of local-homogeneity (details are given in Section 2), this work proposes a bounded continuous control design method for constrained-input mechanical systems, guaranteeing global stabilization with either finite-time or (local) exponential convergence. The choice upon the type of convergence is simply stated through a design parameter involved in the control scheme. Such a choice is made possible through a suitable extension (stated in this paper) of the theoretical framework of local homogeneity; interesting enough, within the design context developed in this work, such an extension permits exponential stabilization through unconventional control structures. The finite-time stabilization choice of the proposed approach —achieved through bounded inputs— remains however the main motivation and original goal of the present work. This is motivated by the advantages of finite-time controllers that are generally claimed in relation to asymptotic ones —such as faster convergence and improved robustness to uncertainties (Hong, Wang & Cheng, 2006; Huang, Lin & Yang, 2005; Qian & Li, 2005)— as well as their conceptual suitability for certain tasks such as *consensus* (Wang & Xiao, 2010) and *formation* (Xiao, Wang, Chen & Gao, 2009) of multi-agent systems.

An initial work on finite-time continuous control for mechanical manipulators was presented in (Hong, Xu & Huang, 2002) assuming unconstrained inputs. The controller design adopted Proportional (P) and Derivative (D) type actions with two options on the structure: one of them compensating for the whole system dynamics and the other one only for the gravity terms. The closed-loop analysis was developed based on the conventional analytical framework of homogeneity. Although a variation of the latter structure with the P and D type actions included (each of them separately) within conventional saturation functions was further contemplated, no formal closed-loop analysis was presented for this case, which does not fit within the analytical framework where the unconstrained versions were developed (as previously explained).

Another work oriented to the finite-time control of robotic manipulators, disregarding input constraints, appeared later in (Zhao, Li, Zhu & Gao, 2010). The scheme proposed therein is structured aiming at the compensation for the whole system nominal dynamics. The rest of the synthesis is developed applying the *backstepping* design technique, by viewing the velocity vector variable as a virtual (artificial) input to achieve finite-time control of the positions, and the (generalized) force input vector to impose a closed-loop continuous dynamics that guarantees finite-time stabilization of the consequent error variables. The design is then complemented through a *Lyapunov-redesign* type procedure that results in the addition of a control term in charge to reject system-uncertainty perturbations, which *a priori* renders discontinuous the resulting control law. Alternative approximations of certain control terms are suggested in order to avoid discontinuities and singularities implied by the developed approach, expecting close-enough (to the desired position) stabilization through their replacement. Although a bounded version of the developed controller is also contemplated by involving conventional saturation functions, no further analysis is included for this case, which is claimed to be left for future research.

A more recent finite-time continuous stabilization scheme for mechanical systems was proposed in (Sanyal & Bohn, 2015) similarly assuming unconstrained inputs. The approach is based on the definition of a manifold where the system is proven to converge to the zero (desired) state in a finite time T_1. A suitable closed loop form ensuring convergence of the system variables to such a manifold in a finite time T_2 is then found. The control law is then designed through *exact dynamic compensation* so as to impose the closed-loop form found in the precedent step. Nevertheless, the extension of such an approach to the constrained input case was not developed.

The continuous approach proposed in this work is designed so as to continually exert bounded non-linear corrective actions on the position and velocity errors through a Saturating-Proportional-Derivative (SPD) structure adopting a generalized form. Such a PD type structure has the advantage to keep the main (beneficial) features of PD type controllers, such as the intuitive sense on the role of the P and D control gains. The proposed algorithm further includes gravity compensation but no additional term of the system dynamics needs to be compensated. Furthermore, the generalized form of the SPD term does not only include the SP-SD type algorithm found in (Zavala-Río

& Fantoni, 2014) as a very particular case, but actually permits to adopt multiple particular saturating structures, which gives an additional degree of design flexibility. This is made possible under the consideration of special continuous functions that suitably shape the P and D terms, each of them separately and both resulting actions together, which in turn extends the choices on the required locally-homogeneous structure. The study includes simulation results through a 2 degree-of-freedom (DOF) manipulator model. These show finite-time control implementations avoiding input saturation, which are compared with analog exponential stabilization simulations, particularly focusing in corroborating the so-cited argument claiming that finite-time controllers achieve faster stabilization than asymptotic ones. The design flexibility provided by the local homogeneity framework will prove to be very useful to achieve finite-time control structures that indeed guarantee (in addition to input saturation avoidance) faster stabilization, beyond the finite-time nature of the convergence. The proposed SPD control scheme does not only adopt the well-known advantages of finite-time stabilizers (over asymptotic ones) but also finds potential applications at all tasks where finite-time stabilization adopts conceptual suitability.

2. Preliminaries

Let $X \in \mathbb{R}^{m\times n}$ and $y \in \mathbb{R}^n$. Throughout this work, X_{ij} denotes the element of X at its i^{th} row and j^{th} column, X_i represents the i^{th} row of X and y_i stands for the i^{th} element of y. 0_n represents the origin of $\mathbb{R}^n$ and I_n the $n \times n$ identity matrix. We denote $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ for scalars, and $\mathbb{R}^n_{>0} = \{x \in \mathbb{R}^n : x_i > 0,\ i = 1,\dots,n\}$ and $\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n : x_i \geq 0,\ i = 1,\dots,n\}$ for vectors. $\|\cdot\|$ stands for the standard Euclidean norm for vectors and induced norm for matrices. An $(n-1)$-dimensional sphere of radius $c > 0$ on $\mathbb{R}^n$ is denoted S^{n-1}_c, *i.e.* $S^{n-1}_c = \{x \in \mathbb{R}^n : \|x\| = c\}$. We will consider the sign function to be zero at zero, *i.e.*

$$\operatorname{sign}(\varsigma) = \begin{cases} \frac{\varsigma}{|\varsigma|} & \text{if } \varsigma \neq 0 \\ 0 & \text{if } \varsigma = 0 \end{cases}$$

and denote $\operatorname{sat}(\cdot)$ the standard (unitary) saturation function, *i.e.* $\operatorname{sat}(\varsigma) = \operatorname{sign}(\varsigma)\min\{|\varsigma|, 1\}$.

2.1 *Mechanical systems*

Consider the n-DOF fully-actuated frictionless mechanical system dynamics (Brogliato, Lozano, Maschke & Egeland, 2007, §6.1)

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \tag{1}$$

where q, $\dot{q}$, $\ddot{q} \in \mathbb{R}^n$ are the position (generalized coordinates), velocity, and acceleration vectors, $H(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$ is the Coriolis and centrifugal effect matrix, $g(q) = \nabla\mathcal{U}(q)$ with $\mathcal{U} : \mathbb{R}^n \to \mathbb{R}$ being the potential energy function of the system, and $\tau \in \mathbb{R}^n$ is the external input (generalized) force vector. Some well-known properties characterizing the terms of such a dynamical model are recalled here (Brogliato et al., 2007, §6.1.2) (Ortega, Loría, Nicklasson & Sira-Ramírez, 1998, §2.3). Subsequently, we denote $\dot{H}$ the rate of change of H, *i.e.* $\dot{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ with $\dot{H}_{ij}(q,\dot{q}) = \frac{\partial H_{ij}}{\partial q}(q)\dot{q}$, $i,j = 1,\dots,n$.

Property 1: $H(q)$ is a continuously differentiable positive definite symmetric matrix function.

Property 2: The Coriolis and centrifugal effect matrix satisfies:

2.1. $\dot{q}^T\left[\frac{1}{2}\dot{H}(q,\dot{q}) - C(q,\dot{q})\right]\dot{q} = 0$, $\forall (q,\dot{q}) \in \mathbb{R}^n \times \mathbb{R}^n$;
2.2. $C(x,y)z = C(x,z)y$, $\forall x,y,z \in \mathbb{R}^n$.

Remark 1: Observe from Property 2.2 that $C(q,a\dot{q})b\dot{q} = C(q,b\dot{q})a\dot{q} = C(q,ab\dot{q})\dot{q} = C(q,\dot{q})ab\dot{q}$, $\forall q,\dot{q} \in \mathbb{R}^n$, $\forall a,b \in \mathbb{R}$.

In this work, we consider the (realistic) bounded input case, where the absolute value of each input τ_i is constrained to be smaller than a given saturation bound $T_i > 0$, *i.e.* $|\tau_i| \leq T_i$, $i = 1,\ldots,n$. More precisely, letting u_i represent the control variable (controller output) relative to the i^{th} degree of freedom, we have that

$$\tau_i = T_i \text{sat}(u_i/T_i) \tag{2}$$

Further assumptions are stated next.

Assumption 1: *The conservative (generalized) force vector $g(q)$ is bounded, or equivalently, every one of its elements, $g_i(q)$, $i = 1,\ldots,n$, satisfies $|g_i(q)| \leq B_{gi}$, $\forall q \in \mathbb{R}^n$, for some positive constant B_{gi}.*

Assumption 2: $T_i > B_{gi}$, $\forall i \in \{1,\ldots,n\}$.

Assumption 1 applies *e.g.* for robot manipulators having only revolute joints (Kelly, Santibáñez & Loría, 2005, §4.3). Assumption 2 renders it possible to hold the system at any desired equilibrium configuration $q_d \in \mathbb{R}^n$.

2.2 *Local homogeneity, finite-time stability and δ-exponential stability*

Definitions and results stated in this subsection are strongly related to *family of dilations* δ_ε^r, defined as $\delta_\varepsilon^r(x) = \left(\varepsilon^{r_1}x_1,\ldots,\varepsilon^{r_n}x_n\right)^T$, $\forall x \in \mathbb{R}^n$, $\forall \varepsilon > 0$, with $r = (r_1,\ldots,r_n)^T$, where the *dilation coefficients* $r_1,\ldots,r_n$ are positive scalars.

Definition 1: (Zavala-Río & Fantoni, 2014) A function $V : \mathbb{R}^n \to \mathbb{R}$, resp. vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, is *locally homogeneous of degree α* with respect to the family of dilations δ_ε^r —or equivalently, it is said to be *locally r-homogeneous of degree α*— if there exists an open neighborhood of the origin $D \subset \mathbb{R}^n$ —referred to as the *domain of homogeneity*— such that, for every $x \in D$ and all $\varepsilon \in (0,1]$: $\delta_\varepsilon^r(x) \in D$ and

$$V(\delta_\varepsilon^r(x)) = \varepsilon^\alpha V(x) \tag{3}$$

resp.

$$f_i(\delta_\varepsilon^r(x)) = \varepsilon^{\alpha+r_i} f_i(x) \tag{4}$$

$i = 1,\ldots,n$.[1]

Definition 2: (Kawski, 1990; M'Closkey & Murray, 1997) Given $r \in \mathbb{R}^n_{>0}$, a continuous map $\mathbb{R}^n \to \mathbb{R} : x \mapsto \|x\|_r$ is called a *homogeneous norm* with respect to the family of dilations δ_ε^r —or equivalently, it is said to be an *r-homogeneous norm*— if, for every $x \in \mathbb{R}^n$, $\|x\|_r \geq 0$ with $\|x\|_r = 0 \iff x = 0_n$ and $\|\delta_\varepsilon^r(x)\|_r = \varepsilon\|x\|_r$ for all $\varepsilon > 0$. In particular, an r-homogeneous p-norm ($p \geq 1$) is defined as $\|x\|_{r,p} = \left[\sum_{i=1}^n |x_i|^{p/r_i}\right]^{1/p}$.

[1]The authors recently got aware that the notion of local homogeneity and related results appeared in the literature before (Zavala-Río & Fantoni, 2014), proposed by Orlov (2005). More details about this are given in Appendix A.

Remark 2: Subsequently, in this work, an r-homogeneous norm $\|\cdot\|_r$ will conventionally be considered to refer to an r-homogeneous p-norm with $p > \max_i\{r_i\}$.

Definition 3: An *r-homogeneous $(n-1)$-sphere* of radius $c > 0$ is the set $S_{r,c}^{n-1} = \{x \in \mathbb{R}^n : \|x\|_r = c\}$.

Consider an n-th order autonomous system

$$\dot{x} = f(x) \tag{5}$$

where $f : \mathcal{D} \to \mathbb{R}^n$ is continuous on an open neighborhood of the origin $\mathcal{D} \subset \mathbb{R}^n$ and $f(0_n) = 0_n$, and let $x(t; x_0)$ represent the system solution with initial condition $x(0; x_0) = x_0$.

Definition 4: (Bhat & Bernstein, 2005) The origin is said to be a *finite-time stable* equilibrium of system (5) if it is Lyapunov stable and there exist an open neighborhood of the origin, $\mathcal{N} \subset \mathcal{D}$, being positively invariant with respect to (5), and a positive definite function $T : \mathcal{N} \to \mathbb{R}_{\geq 0}$, called the *settling-time function*, such that $x(t; x_0) \neq 0_n$, $\forall t \in \big[0, T(x_0)\big)$, $\forall x_0 \in \mathcal{N} \setminus \{0_n\}$, and $x(t; x_0) = 0_n$, $\forall t \geq T(x_0)$, $\forall x_0 \in \mathcal{N}$. The origin is said to be a *globally finite-time stable* equilibrium if it is finite-time stable with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$.[2]

Remark 3: Note, from Definition 4, that the origin is a globally finite-time stable equilibrium of system (5) if and only if it is globally asymptotically stable and finite-time stable.

Theorem 1: *(Zavala-Río & Fantoni, 2014) Consider system* (5) *with $\mathcal{D} = \mathbb{R}^n$. Suppose that f is a locally r-homogeneous vector field of degree α with domain of homogeneity $D \subset \mathbb{R}^n$. Then, the origin is a globally finite-time stable equilibrium of system* (5) *if and only if it is globally asymptotically stable and $\alpha < 0$.*

The next definition is stated under the additional consideration that, for some $r \in \mathbb{R}^n_{>0}$, f in (5) is locally r-homogeneous with domain of homogeneity $D \subset \mathcal{D}$.

Definition 5: (Kawski, 1990; M'Closkey & Murray, 1997) The equilibrium point $x = 0_n$ of (5) is *δ-exponentially stable*[3] with respect to the homogeneous norm $\|\cdot\|_r$ if there exist a neighborhood of the origin, $\mathcal{V} \subset D$, and constants $a \geq 1$ and $b > 0$ such that $\|x(t; x_0)\|_r \leq a\|x_0\|_r e^{-bt}$, $\forall t \geq 0$, $\forall x_0 \in \mathcal{V}$.

Remark 4: Observe that Definition 5 becomes equivalent to the usual definition of exponential stability when the standard dilation is concerned, *i.e.* when $r_i = 1$, $i = 1, \ldots, n$.[4]

The next lemma is a trivial extension to the local homogeneity context of (Kawski, 1990, Lemma 2.4). Analogously to (Kawski, 1990, Lemma 2.4), it is stated under the additional consideration that solutions of (5) with $x_0 \in D$ remain unique (while belonging to D).[5]

[2] Definition 4 states an equilibrium-related finite-time stability concept that gathers Lyapunov stability with finite-time convergence, which is at the basis of the analytical setting underlying this work. It should be noted that alternative unrelated notions of finite-time stability, characterizing different aspects on the system performance, have also appeared in the literature, *e.g.* (Amato, Ariola & Cosentino, 2005; Dorato, 2006). In these latter references, for instance, such designation has been used to describe systems whose trajectories remain in a prescribed region during a finite-time interval.

[3] We adopt the dilation-related designation stated in (Kawski, 1990) for Definition 5, *i.e.* *δ-exponential stability*. In (M'Closkey & Murray, 1997), the same definition is alternatively designated as *ρ-exponential stability*, with ρ referring to the involved r-homogeneous norm, in accordance to the notation stated therein.

[4] Non-equivalence among the usual definition of exponential stability and Definition 5 with a non-standard dilation is illustrated in (M'Closkey & Murray, 1997, §III.C). An analog illustration is developed in (Kawski, 1990, §2) as an example to show non-equivalence of δ-exponential stability cases with different dilation coefficient vectors r.

[5] Another version of (Kawski, 1990, Lemma 2.4) is stated in (M'Closkey & Murray, 1997, Lemma 1) where no restriction on the uniqueness of solutions is considered. It is further concluded from (M'Closkey & Murray, 1997, §III.E) that the solutions of autonomous systems $\dot{x} = f(x)$ with r-homogeneous vector field being locally Lipschitz on $\mathbb{R}^n \setminus \{0_n\}$ are unique.

Lemma 1: *Suppose that f in (5) is a locally r-homogeneous vector field of degree $\alpha = 0$ with domain of homogeneity $D \subset \mathcal{D}$. Then, the origin is a δ-exponentially stable equilibrium if and only if it is asymptotically stable.*

Observe that the assumptions of Lemma 1 imply the existence of a neighborhood of the origin $\mathcal{V} \subset D$ such that $x_0 \in \mathcal{V} \implies x(t;x_0) \in D$, $\forall t \geq 0$. The proof of Lemma 1 is thus analogous to the one developed in (Hahn, 1967, §57) for the special case of $r = (r_1, \ldots, r_n)^T$ with $r_i = 1$, $i = 1, \ldots, n$.[6]

Remark 5: Let us note that if a vector field f is locally r-homogeneous of degree $\alpha = 0$ with dilation coefficients $r_i = r_0$, $\forall i \in \{1, \ldots, n\}$, for some $r_0 > 0$, then f is locally r^*-homogeneous of degree $\alpha = 0$ with dilation coefficients $r_i^* = r_0^*$, $\forall i \in \{1, \ldots, n\}$, for any $r_0^* > 0$. Indeed, observe that if, for every $x \in D$, $f(\varepsilon^{r_0} x) = \varepsilon^{r_0} f(x)$, $\forall \varepsilon \in (0, 1]$, then, by taking $\epsilon = \varepsilon^{r_0/r_0^*}$, we have that $f(\epsilon^{r_0^*} x) = \epsilon^{r_0^*} f(x)$, $\forall \epsilon \in (0, 1]$. Consequently, if f in (5) is locally r-homogeneous of degree $\alpha = 0$ with dilation coefficients $r_i = r_0$, $\forall i \in \{1, \ldots, n\}$, for some $r_0 > 0$, then (under the consideration of Remark 4) the origin turns out to be exponentially stable if and only if it is δ-exponentially stable.

Consider an n-th order autonomous system of the form

$$\dot{x} = f(x) + \hat{f}(x) \tag{6}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\hat{f} : \mathbb{R}^n \to \mathbb{R}^n$ are continuous vector fields such that $f(0_n) = \hat{f}(0_n) = 0_n$. The next result is an extended version of (Zavala-Río & Fantoni, 2014, Lemma 3.2).

Lemma 2: *Suppose that, for some $r \in \mathbb{R}^n_{>0}$, f in (6) is a locally r-homogeneous vector field of degree $\alpha < 0$, resp. $\alpha = 0$, with domain of homogeneity $D \subset \mathbb{R}^n$, and that 0_n is a globally asymptotically, resp. δ-exponentially, stable equilibrium of $\dot{x} = f(x)$. Then, the origin is a finite-time, resp. δ-exponentially, stable equilibrium of system (6) if*

$$\lim_{\varepsilon \to 0^+} \frac{\hat{f}_i(\delta_\varepsilon^r(x))}{\varepsilon^{\alpha + r_i}} = 0 \tag{7}$$

$i = 1, \ldots, n$, $\forall x \in S_c^{n-1}$, resp. $\forall x \in S_{r,c}^{n-1}$, for some $c > 0$ such that $S_c^{n-1} \subset D$, resp. $S_{r,c}^{n-1} \subset D$.

Proof. See Appendix B. □

Remark 6: Notice that the condition required by Lemma 2 may be equivalently verified through the satisfaction of

$$\lim_{\varepsilon \to 0^+} \left\| \varepsilon^{-\alpha} \mathrm{diag}\left[\varepsilon^{-r_1}, \ldots, \varepsilon^{-r_n}\right] \hat{f}(\delta_\varepsilon^r(x)) \right\| = 0 \tag{8}$$

$\forall x \in S_c^{n-1}$ (resp. $S_{r,c}^{n-1}$). In other words, (7) is fulfilled for all $i = 1, \ldots, n$ and all $x \in S_c^{n-1}$ (resp. $S_{r,c}^{n-1}$) if and only if (8) is satisfied for all $x \in S_c^{n-1}$ (resp. $S_{r,c}^{n-1}$).

2.3 *Scalar functions with particular properties*

Definition 6: A continuous scalar function $\sigma : \mathbb{R} \to \mathbb{R}$ will be said to be:

(1) *positively upper-bounded* (by M^+) if $\sigma(\varsigma) \leq M^+$, $\forall \varsigma \in \mathbb{R}$, for some positive constant M^+;
(2) *negatively lower-bounded* (by $-M^-$) if $\sigma(\varsigma) \geq -M^-$, $\forall \varsigma \in \mathbb{R}$, for some positive constant M^-;

[6] One further concludes from (Hahn, 1967, §57) that asymptotic stability when $\alpha > 0$ is not δ-exponential (*i.e.* δ-exponential stability is a property that can only take place when $\alpha = 0$).

(3) *bounded* (by M) if $|\sigma(\varsigma)| \leq M$, $\forall \varsigma \in \mathbb{R}$, for some positive constant M;
(4) *strictly passive* if $\varsigma\sigma(\varsigma) > 0$, $\forall \varsigma \neq 0$;
(5) *strongly passive* if $|\sigma(\varsigma)| \geq \kappa \big| a\,\mathrm{sat}(\varsigma/a)\big|^b = \kappa\big(\min\{|\varsigma|, a\}\big)^b$, $\forall \varsigma \in \mathbb{R}$, for some positive constants κ, a and b.

Remark 7: Let us note that a non-decreasing strictly passive function σ is strongly passive. Indeed, notice that the strictly passive character of σ implies the existence of a sufficiently small $a > 0$ such that $|\sigma(\varsigma)| \geq \kappa|\varsigma|^b$, $\forall |\varsigma| \leq a$, for some positive constants κ and b, while from its nondecreasing character we have that $|\sigma(\varsigma)| \geq |\sigma(\mathrm{sign}(\varsigma)a)| \geq \kappa a^b$, $\forall |\varsigma| \geq a$, and thus $|\sigma(\varsigma)| \geq \kappa\big(\min\{|\varsigma|, a\}\big)^b = \kappa\big|a\,\mathrm{sat}(\varsigma/a)\big|^b$, $\forall \varsigma \in \mathbb{R}$.

The following statement is straightforward.

Lemma 3: *Let $\sigma : \mathbb{R} \to \mathbb{R}$, $\sigma_0 : \mathbb{R} \to \mathbb{R}$ and $\sigma_1 : \mathbb{R} \to \mathbb{R}$ be strongly passive functions and k be a positive constant. Then:*

(1) $\int_0^\varsigma \sigma(k\nu)d\nu > 0$, $\forall \varsigma \neq 0$;
(2) $\int_0^\varsigma \sigma(k\nu)d\nu \to \infty$ as $|\varsigma| \to \infty$;
(3) $\sigma_0 \circ \sigma_1$ is strongly passive.

Lemma 4: *Let $\sigma_0 : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function, $\sigma_2 : \mathbb{R} \to \mathbb{R}$ be strictly passive, and k be a positive constant. Then: $\varsigma_2\big[\sigma_0(\varsigma_1 + \sigma_2(k\varsigma_2)) - \sigma_0(\varsigma_1)\big] > 0$, $\forall \varsigma_2 \neq 0$, $\forall \varsigma_1 \in \mathbb{R}$.*

Proof. See Appendix C. □

3. A generalized SPD-type stabilizer

Consider the following SPD-type controller

$$u(q, \dot{q}) = -s_0\big(s_1(K_1\bar{q}) + s_2(K_2\dot{q})\big) + g(q) \tag{9}$$

where $\bar{q} = q - q_d$, for any constant (desired equilibrium position) $q_d \in \mathbb{R}^n$; $K_1 = \mathrm{diag}[k_{11}, \ldots, k_{1n}]$ and $K_2 = \mathrm{diag}[k_{21}, \ldots, k_{2n}]$ with $k_{1j} > 0$, $k_{2j} > 0$, $\forall j \in \{1, \ldots, n\}$; and for any $x \in \mathbb{R}^n$, $s_i(x) = \big(\sigma_{i1}(x_1), \ldots, \sigma_{in}(x_n)\big)^T$, $i = 0, 1, 2$, with, for each $j \in \{1, \ldots, n\}$, σ_{0j} being a strictly increasing strictly passive function, while σ_{1j} and σ_{2j} are non-decreasing strictly passive, all three being locally Lipschitz-continuous on $\mathbb{R} \setminus \{0\}$ and such that

$$B_j \triangleq \max\left\{ \lim_{\varsigma\to\infty} \sigma_{0j}\big(\sigma_{1j}(\varsigma) + \sigma_{2j}(\varsigma)\big)\ ,\ \lim_{\varsigma\to-\infty} -\sigma_{0j}\big(\sigma_{1j}(\varsigma) + \sigma_{2j}(\varsigma)\big) \right\} < T_j - B_{gj} \tag{10}$$

Remark 8:

(1) Note that by (10) (under the consideration of the monotonic characteristics of σ_{ij}, $i = 0, 1, 2$), we have that —for each $j \in \{1, \ldots, n\}$— either:
 (a) σ_{0j} is bounded (whether σ_{1j} and/or σ_{2j} are/is bounded or not), or
 (b) σ_{1j} and σ_{2j} are both bounded (whether σ_{0j} is bounded or not), or
 (c) σ_{0j} is positively upper-bounded, resp. negatively lower-bounded, and σ_{ij}, $i = 1, 2$, are both negatively lower-bounded, resp. positively upper-bounded (whether σ_{ij}, $i = 0, 1, 2$, —all together, any of them or any combination of them— are bounded or not).
(2) Let us notice that —for each $j \in \{1, \ldots, n\}$— if σ_{1j} and σ_{2j} are both positively upper-bounded, σ_{0j} does not need to be defined on $\big(\lim_{\varsigma\to\infty}[\sigma_{1j}(\varsigma) + \sigma_{2j}(\varsigma)]\ ,\ \infty\big)$. Similarly, if

σ_{1j} and σ_{2j} are both negatively lower-bounded, σ_{0j} does not need to be defined on $\big(-\infty\,,\,\lim_{\varsigma\to-\infty}[\sigma_{1j}(\varsigma)+\sigma_{2j}(\varsigma)]\big)$.

Proposition 1: *Consider system* (1)-(2) *in closed loop with the proposed control law* (9)*. Thus, for any positive definite diagonal matrices K_1 and K_2, global asymptotic stability of the closed-loop trivial solution $\bar{q}(t)\equiv 0_n$ is guaranteed with $|\tau_j(t)|=|u_j(t)|<T_j$, $j=1,\ldots,n$, $\forall t\geq 0$.*

Proof. Observe that —for every $j\in\{1,\ldots,n\}$— by (10), we have that, for any $(q,\dot{q})\in\mathbb{R}^n\times\mathbb{R}^n$:

$$\begin{aligned}|u_j(q,\dot{q})| &= \big|-\sigma_{0j}\big(\sigma_{1j}(k_{1j}\bar{q}_j)+\sigma_{2j}(k_{2j}\dot{q}_j)\big)+g_j(q)\big|\\ &\leq \big|\sigma_{0j}\big(\sigma_{1j}(k_{1j}\bar{q}_j)+\sigma_{2j}(k_{2j}\dot{q}_j)\big)\big|+|g_j(q)|\\ &\leq B_j+B_{gj}<T_j\end{aligned}$$

From this and (2), one sees that $T_j>|u_j(q,\dot{q})|=|u_j|=|\tau_j|$, $\forall(q,\dot{q})\in\mathbb{R}^n\times\mathbb{R}^n$, which shows that, along the system trajectories, $|\tau_j(t)|=|u_j(t)|<T_j$, $j=1,\ldots,n$, $\forall t\geq 0$. This proves that, under the proposed scheme, the input saturation values, T_j, are never reached. Hence, the closed-loop dynamics takes the form

$$H(q)\ddot{q}+C(q,\dot{q})\dot{q}=-s_0\big(s_1(K_1\bar{q})+s_2(K_2\dot{q})\big)$$

By defining $x_1=\bar{q}$ and $x_2=\dot{q}$, the closed-loop dynamics adopts the $2n$-order state-space representation

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= H^{-1}(x_1+q_d)\Big[-C(x_1+q_d,x_2)x_2-s_0\big(s_1(K_1x_1)+s_2(K_2x_2)\big)\Big]\end{aligned}$$

By further defining $x=(x_1^T,x_2^T)^T$, these state equations may be rewritten in the form of system (6) with

$$f(x)=\begin{pmatrix}x_2\\ -H^{-1}(q_d)s_0\big(s_1(K_1x_1)+s_2(K_2x_2)\big)\end{pmatrix} \tag{11a}$$

$$\hat{f}(x)=\begin{pmatrix}0_n\\ -H^{-1}(x_1+q_d)C(x_1+q_d,x_2)x_2-\mathcal{H}(x_1)s_0\big(s_1(K_1x_1)+s_2(K_2x_2)\big)\end{pmatrix} \tag{11b}$$

where

$$\mathcal{H}(x_1)=H^{-1}(x_1+q_d)-H^{-1}(q_d) \tag{12}$$

Thus, the closed-loop stability property stated through Proposition 1 is corroborated by showing that $x=0_{2n}$ is a globally asymptotically stable equilibrium of the state equation $\dot{x}=f(x)+\hat{f}(x)$, which is proven through the following theorem (whose formulation proves to be convenient for subsequent developments and proofs).

Theorem 2: *Under the stated specifications, the origin is a globally asymptotically stable equilibrium of both the state equation $\dot{x}=f(x)$ and the (closed-loop) system $\dot{x}=f(x)+\hat{f}(x)$, with $f(x)$ and $\hat{f}(x)$ defined through Eqs.* (11)*.*

Proof. For every $\ell \in \{0,1\}$, let us define the continuously differentiable scalar function

$$V_\ell(x_1,x_2) = \frac{1}{2}x_2^T H(\ell x_1 + q_d)x_2 + \int_{0_n}^{x_1} s_0^T\big(s_1(K_1 r)\big)dr$$

where $\int_{0_n}^{x_1} s_0^T\big(s_1(K_1 r)\big)dr = \sum_{j=1}^n \int_0^{x_{1j}} \sigma_{0j}\big(\sigma_{1j}(k_{1j}r_j)\big)dr_j$. From Property 1, Lemma 3 and Remark 7 (whence one corroborates the strongly passive character of σ_{ij}, $i=0,1$, $j=1,\dots,n$), $V_\ell(x_1,x_2)$, $\ell=0,1$, are concluded to be positive definite and radially unbounded. Further, for every $\ell \in \{0,1\}$, the derivative of V_ℓ along the trajectories of $\dot{x} = f(x) + \ell\hat{f}(x)$, is obtained as

$$\begin{aligned}
\dot{V}_\ell(x_1,x_2) &= x_2^T H(\ell x_1 + q_d)\dot{x}_2 + \frac{\ell}{2}x_2^T \dot{H}(x_1+q_d)x_2 + s_0^T\big(s_1(K_1x_1)\big)x_2 \\
&= x_2^T\Big[-\ell C(x_1+q_d,x_2)x_2 - s_0\big(s_1(K_1x_1)+s_2(K_2x_2)\big)\Big] \\
&\quad + \frac{\ell}{2}x_2^T\dot{H}(x_1+q_d)x_2 + x_2^T s_0\big(s_1(K_1x_1)\big) \\
&= -x_2^T\Big[s_0\big(s_1(K_1x_1)+s_2(K_2x_2)\big) - s_0\big(s_1(K_1x_1)\big)\Big] \\
&= -\sum_{j=1}^n x_{2j}\Big[\sigma_{0j}\big(\sigma_{1j}(k_{1j}x_{1j}) + \sigma_{2j}(k_{2j}x_{2j})\big) - \sigma_{0j}\big(\sigma_{1j}(k_{1j}x_{1j})\big)\Big]
\end{aligned}$$

where, in the case of $\ell=1$, Property 2.1 has been applied. Note, from Lemma 4, that $\dot{V}_\ell(x_1,x_2) \leq 0$, $\forall (x_1,x_2) \in \mathbb{R}^n \times \mathbb{R}^n$, with $Z_\ell \triangleq \{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n : \dot{V}_\ell(x_1,x_2)=0\} = \{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n : x_2 = 0_n\}$. Further, from the system dynamics $\dot{x} = f(x)+\ell\hat{f}(x)$ —under the consideration of the strictly passive character of σ_{1j}, $j=1,\dots,n$, Property 1 and the positive definiteness of K_1— one sees that $x_2(t)\equiv 0_n \implies \dot{x}_2(t)\equiv 0_n \implies s_0\big(s_1(K_1x_1(t))\big)\equiv 0_n \iff s_1(K_1x_1(t))\equiv 0_n \iff x_1(t)\equiv 0_n$ (which shows that $(x_1,x_2)(t)\equiv(0_n,0_n)$ is the only system solution completely remaining in Z_ℓ), and corroborates that at any $(x_1,x_2)\in\{(\bar{q},\dot{q})\in Z_\ell : \bar{q}\neq 0_n\}$, the resulting unbalanced force term $-s_0\big(s_1(K_1x_1)\big)$ acts on the closed-loop dynamics, forcing the system trajectories to leave Z_ℓ, whence $\{(0_n,0_n)\}$ is concluded to be the only invariant set in Z_ℓ, $\ell=0,1$. Therefore, by the invariance theory (Michel, Hou & Liu, 2008, §7.2) —more precisely by (Michel et al., 2008, Corollary 7.2.1)— $x=0_{2n}$ is concluded to be a globally asymptotically stable equilibrium of both the state equation $\dot{x}=f(x)$ and the (closed-loop) system $\dot{x}=f(x)+\hat{f}(x)$. □

4. Finite-time and exponential stabilization

Proposition 2: *Consider the proposed control scheme under the additional consideration that, for every $j\in\{1,\dots,n\}$, σ_{ij}, $i=1,2$, are locally r_i-homogeneous of degree $\alpha_j>0$ —i.e. $r_{1j}=r_1$, $r_{2j}=r_2$ and $\alpha_{1j}=\alpha_{2j}=\alpha_j>0$ for all $j\in\{1,\dots,n\}$— with domain of homogeneity $D_{ij} = \{\varsigma\in\mathbb{R} : |\varsigma| < L_{ij}\in(0,\infty]\}$ and σ_{0j} is locally α_j-homogeneous of degree $\alpha_0 = 2r_2-r_1$ —i.e. $\alpha_{0j}=\alpha_0=2r_2-r_1$ for all $j\in\{0,\dots,n\}$— with domain of homogeneity $D_{0j}=\{\varsigma\in\mathbb{R}:|\varsigma|< L_{0j}\in(0,\infty]\}$, for some dilation coefficients $r_i>0$, $i=1,2$, such that $\alpha_0=2r_2-r_1>0$. Thus, for any positive definite diagonal matrices K_1 and K_2, $|\tau_j(t)|=|u_j(t)|<T_j$, $j=1,\dots,n$, $\forall t\geq 0$, and the closed-loop trivial solution $\bar{q}(t)\equiv 0_n$ is:*

(1) globally finite-time stable if $r_2<r_1$;
(2) globally asymptotically stable with (local) exponential stability if $r_2=r_1$.

Proof. Since the proposed control scheme is applied —with all its previously stated specifications— Proposition 1 holds and consequently $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \dots, n$, $\forall t \geq 0$. Then, all that remains to be proven is that the additional considerations give rise to the specific stability properties claimed in items 1 and 2 of the statement. In this direction, let $\hat{r}_i = (r_{i1}, \dots, r_{in})^T$, $i = 1, 2$, $r = (\hat{r}_1^T, \hat{r}_2^T)^T$, $\hat{r}_0 = (\alpha_1, \dots, \alpha_n)^T$, $\hat{\alpha}_0 = (\alpha_{01}, \dots, \alpha_{0n})^T$, $D \triangleq \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n : K_i x_i \in D_{i1} \times \cdots \times D_{in}\,,\ i = 1, 2\,,\ s_1(K_1 x_1) + s_1(K_2 x_2) \in D_{01} \times \cdots \times D_{0n}\} = \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n : |x_{1j}| < L_{1j}/k_{1j}\,,\ |x_{2j}| < L_{2j}/k_{2j}\,,\ |\sigma_{1j}(k_{1j}x_{1j}) + \sigma_{2j}(k_{2j}x_{2j})| < L_{0j}\,,\ j = 1, \dots, n\}$, and consider the previously defined state (vector) variables and the consequent closed-loop state-space representation $\dot{x} = f(x) + \hat{f}(x)$, with f and $\hat{f}$ as defined through Eqs. (11). Since D defines an open neighborhood of the origin, there exists $\rho > 0$ such that $B_\rho \triangleq \{x \in \mathbb{R}^{2n} : \|x\| < \rho\} \subset D$. Moreover, for every $x \in B_\rho$ and all $\varepsilon \in (0, 1]$, we have that $\delta_\varepsilon^r(x) \in B_\rho$ (since $\|\delta_\varepsilon^r(x)\| < \|x\|$, $\forall \varepsilon \in (0, 1)$), and, for every $j \in \{1, \dots, n\}$,

$$f_j(\delta_\varepsilon^r(x)) = \varepsilon^{r_{2j}} x_{2j} = \varepsilon^{r_2} x_{2j} = \varepsilon^{(r_2 - r_1) + r_1} x_{2j} = \varepsilon^{(r_2 - r_1) + r_{1j}} f_j(x)$$

and[7]

$$\begin{aligned} f_{n+j}(\delta_\varepsilon^r(x)) &= -H_j^{-1}(q_d) s_0\big(s_1(K_1 \delta_\varepsilon^{\hat{r}_1}(x_1)) + s_2(K_2 \delta_\varepsilon^{\hat{r}_2}(x_2))\big) \\ &= -H_j^{-1}(q_d) s_0\big(s_1(\varepsilon^{r_1} K_1 x_1) + s_2(\varepsilon^{r_2} K_2 x_2)\big) \\ &= -H_j^{-1}(q_d) s_0\big(\delta_\varepsilon^{\hat{r}_0}\big(s_1(K_1 x_1)\big) + \delta_\varepsilon^{\hat{r}_0}\big(s_2(K_2 x_2)\big)\big) \\ &= -H_j^{-1}(q_d) s_0\big(\delta_\varepsilon^{\hat{r}_0}\big(s_1(K_1 x_1) + s_2(K_2 x_2)\big)\big) \\ &= -H_j^{-1}(q_d) \delta_\varepsilon^{\hat{\alpha}_0}\big(s_0\big(s_1(K_1 x_1) + s_2(K_2 x_2)\big)\big) \\ &= -\varepsilon^{\alpha_0} H_j^{-1}(q_d) s_0\big(s_1(K_1 x_1) + s_2(K_2 x_2)\big) \\ &= \varepsilon^{(r_2 - r_1) + r_{2j}} f_{n+j}(x) \end{aligned} \tag{13}$$

whence one concludes that f is a locally r-homogeneous vector field of degree $\alpha = r_2 - r_1$, with domain of homogeneity B_ρ. Hence, by Theorems 1 and 2, Lemma 1 and Remark 5, the origin of the state equation $\dot{x} = f(x)$ is concluded to be a globally finite-time stable equilibrium if $r_2 < r_1$, and a globally asymptotically stable equilibrium with (local) exponential stability if $r_2 = r_1$. Thus, by Theorem 2, Lemma 2, and Remarks 3 and 6, the origin of the closed-loop system $\dot{x} = f(x) + \hat{f}(x)$ is concluded to be a globally finite-time stable equilibrium provided that $r_2 < r_1$, and a globally asymptotically stable equilibrium with (local) exponential stability provided that $r_2 = r_1$, if

$$\begin{aligned} \mathcal{L}_0 &\triangleq \lim_{\varepsilon \to 0^+} \Big\| \varepsilon^{-\alpha} \mathrm{diag}[\varepsilon^{-r_{11}}, \dots, \varepsilon^{-r_{1n}}, \varepsilon^{-r_{21}}, \dots, \varepsilon^{-r_{2n}}] \hat{f}(\delta_\varepsilon^r(x)) \Big\| \\ &= \lim_{\varepsilon \to 0^+} \Big\| \varepsilon^{-\alpha} \mathrm{diag}[\varepsilon^{-r_{21}}, \dots, \varepsilon^{-r_{2n}}] \big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \dots, \hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T \Big\| \\ &= \lim_{\varepsilon \to 0^+} \Big\| \varepsilon^{-\alpha - r_2} \big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \dots, \hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T \Big\| \\ &= \lim_{\varepsilon \to 0^+} \varepsilon^{r_1 - 2r_2} \Big\| \big[\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \dots, \hat{f}_{2n}(\delta_\varepsilon^r(x))\big]^T \Big\| \end{aligned} \tag{14}$$

for all $x \in S_c^{2n-1} = \{x \in \mathbb{R}^{2n} : \|x\| = c\}$ (resp. $x \in S_{r,c}^{2n-1} = \{x \in \mathbb{R}^{2n} : \|x\|_r = c\}$), for some $c > 0$ such that $S_c^{2n-1} \subset D$ (resp. $S_{r,c}^{2n-1} \subset D$). Hence, from (11b), under the consideration of Property

[7] Observe, for every $x \in B_\rho$ and all $\varepsilon \in (0, 1]$, that $\sigma_{ij}(k_{ij}\varepsilon^{r_{ij}} x_{ij}) = \sigma_{ij}(\varepsilon^{r_i} k_{ij} x_{ij}) = \varepsilon^{\alpha_j} \sigma_{ij}(k_{ij} x_{ij})$, $i = 1, 2$, $j = 1, \dots, n$ $\iff$ $s_i(K_i \delta_\varepsilon^{\hat{r}_i}(x_i)) = s_i(\varepsilon^{r_i} K_i x_i) = \delta_\varepsilon^{\hat{r}_0}\big(s_i(K_i x_i)\big)$, $i = 1, 2$, and $\sigma_0(\varepsilon^{\alpha_j} \cdot) = \varepsilon^{\alpha_{0j}} \sigma_0(\cdot) = \varepsilon^{\alpha_0} \sigma_0(\cdot)$, $j = 1, \dots, n$ $\iff$ $s_0(\delta_\varepsilon^{\hat{r}_0}(\cdot)) = \delta_\varepsilon^{\hat{\alpha}_0}\big(s_0(\cdot)\big) = \varepsilon^{\alpha_0} s_0(\cdot)$.

2.2 and Remark 1, we have, for all such $x \in S_c^{2n-1}$ (resp. $x \in S_{r,c}^{2n-1}$):

$$\begin{aligned}
&\left\| \left[\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \ldots, \hat{f}_{2n}(\delta_\varepsilon^r(x))\right]^T \right\| \\
&\quad = \left\| -H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, \varepsilon^{r_2}x_2)\varepsilon^{r_2}x_2 - \mathcal{H}(\varepsilon^{r_1}x_1)s_0\big(s_1(\varepsilon^{r_1}K_1x_1) + s_2(\varepsilon^{r_2}K_2x_2)\big) \right\| \\
&\quad \le \left\| -H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)\varepsilon^{2r_2}x_2 \right\| + \left\| \mathcal{H}(\varepsilon^{r_1}x_1)s_0\big(\delta_\varepsilon^{\hat{r}_0}\big(s_1(K_1x_1) + s_2(K_2x_2)\big)\big) \right\|
\end{aligned}$$

whence, through a procedure similar to the one developed to obtain (13), we get

$$\begin{aligned}
&\left\| \left[\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \ldots, \hat{f}_{2n}(\delta_\varepsilon^r(x))\right]^T \right\| \\
&\quad \le \varepsilon^{2r_2}\left\| H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2 \right\| + \varepsilon^{2r_2 - r_1}\left\| \mathcal{H}(\varepsilon^{r_1}x_1)s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \right\|
\end{aligned}$$

and consequently, from (14), we get

$$\begin{aligned}
\mathcal{L}_0 &\le \lim_{\varepsilon\to 0^+} \varepsilon^{r_1}\left\| H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2 \right\| + \lim_{\varepsilon\to 0^+} \left\| \mathcal{H}(\varepsilon^{r_1}x_1)s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \right\| \\
&\le \left\| H^{-1}(q_d)C(q_d, x_2)x_2 \right\| \lim_{\varepsilon\to 0^+} \varepsilon^{r_1} + \left\| s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \right\| \lim_{\varepsilon\to 0^+} \left\| \mathcal{H}(\varepsilon^{r_1}x_1) \right\| \\
&\le \left\| s_0\big(s_1(K_1x_1) + s_2(K_2x_2)\big) \right\| \cdot \left\| \mathcal{H}(0_n) \right\| = 0
\end{aligned}$$

(note, from (12), that $\|\mathcal{H}(0_n)\| = \|H^{-1}(q_d) - H^{-1}(q_d)\| = 0$), which completes the proof. □

Corollary 1: *Consider the proposed control scheme taking σ_{ij}, $i = 0, 1, 2$, $j = 1, \ldots, n$, such that*

$$\sigma_{ij}(\varsigma) = \mathrm{sign}(\varsigma)|\varsigma|^{\beta_{ij}} \quad \forall |\varsigma| \le L_{ij} \in (0, \infty) \tag{15}$$

with constants β_{ij} such that

$$\beta_{1j} > 0 \quad , \quad \beta_{2j} = \gamma\beta_{1j} \quad , \quad \beta_{0j} = \frac{2-\gamma}{\gamma\beta_{1j}} \tag{16}$$

for a constant $\gamma \in (0, 2)$. Thus, for any positive definite diagonal matrices K_1 and K_2, $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \ge 0$, and the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is:

(1) globally finite-time stable if $1 < \gamma < 2$;
(2) globally asymptotically stable with (local) exponential stability if $\gamma = 1$.

Proof. Note that for any continuous nondecreasing function σ_{ij} fulfilling (15), we have $\sigma_{ij}(\varsigma) \ge \min\{\varsigma^{\beta_{ij}}, L_{ij}^{\beta_{ij}}\} > 0$, $\forall \varsigma > 0$, and $\sigma_{ij}(\varsigma) \le -\min\{|\varsigma|^{\beta_{ij}}, L_{ij}^{\beta_{ij}}\} < 0$, $\forall \varsigma < 0$, and consequently $\varsigma\sigma_{ij}(\varsigma) \ge |\varsigma|\min\{|\varsigma|^{\beta_{ij}}, L_{ij}^{\beta_{ij}}\} > 0$, $\forall \varsigma \ne 0$. This shows that, for every $j \in \{1, \ldots, n\}$, a strictly increasing σ_{0j} satisfying (15) is a strictly increasing strictly passive function, and nondecreasing σ_{ij}, $i = 1, 2$, fulfilling (15) are nondecreasing strictly passive functions. Note further that, given any $r_{ij} > 0$, for every $\varsigma \in (-L_{ij}, L_{ij})$: $\varepsilon^{r_{ij}}\varsigma \in (-L_{ij}, L_{ij})$ and $\sigma_{ij}(\varepsilon^{r_{ij}}\varsigma) = \mathrm{sign}(\varepsilon^{r_{ij}}\varsigma)|\varepsilon^{r_{ij}}\varsigma|^{\beta_{ij}} = \varepsilon^{r_{ij}\beta_{ij}}\mathrm{sign}(\varsigma)|\varsigma|^{\beta_{ij}} = \varepsilon^{r_{ij}\beta_{ij}}\sigma_{ij}(\varsigma)$, $\forall \varepsilon \in (0, 1]$. Hence, under the consideration of expressions (16), for every $j \in \{1, \ldots, n\}$, we have, for any $r_{1j} = r_1 > 0$, that taking $r_{2j} = r_2 = r_1/\gamma$ and $r_{0j} = r_1\beta_{1j}$, σ_{ij}, $i = 1, 2$, are locally r_i-homogeneous of degree $\alpha_{2j} = r_2\beta_{2j} = r_1\beta_{1j} = \alpha_{1j} = \alpha_j$ with domain of homogeneity $D_{ij} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{ij}\}$, and σ_{0j} is locally α_j-homogeneous of degree $\alpha_{0j} = \alpha_0 = (2-\gamma)r_1/\gamma$ with domain of homogeneity $D_{0j} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{0j}\}$, while under the additional condition on γ, we have that $0 < \gamma < 2 \iff (0 < \gamma) \wedge (0 < 2 - \gamma) \implies 0 < (2-\gamma)r_1/\gamma = \alpha_0 \iff 0 < 2r_2 - r_1 = \alpha_0$. The requirements of Proposition 2 are thus concluded to be satisfied with $r_2 < r_1 \iff 1 < \gamma < 2$ and $r_2 = r_1 \iff \gamma = 1$. □

Remark 9: Since the results of this section depart from the application of the proposed control scheme, the cases of Proposition 2 with $r_2 > r_1$ and Corollary 1 with $\gamma \in (0, 1)$ are particular cases of Proposition 1 where the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is globally asymptotically stable but not (locally) exponentially stable (in accordance to Footnote 6).

5. Simulation results

The proposed scheme was implemented through computer simulations considering the model of a 2-DOF mechanical manipulator corresponding to the experimental robotic arm used in (Zavala-Río and Santibáñez, 2006). For such a robot, the various terms characterizing the system dynamics in Eq. (1) are given by

$$H(q) = \begin{pmatrix} 2.351 + 0.168\cos q_2 & 0.102 + 0.084\cos q_2 \\ 0.102 + 0.084\cos q_2 & 0.102 \end{pmatrix}$$

$$C(q,\dot{q}) = \begin{pmatrix} -0.084\dot{q}_2 \sin q_2 & -0.084(\dot{q}_1 + \dot{q}_2)\sin q_2 \\ 0.084\dot{q}_1 \sin q_2 & 0 \end{pmatrix}$$

$$g(q) = \begin{pmatrix} 38.465\sin q_1 + 1.825\sin(q_1+q_2) \\ 1.825\sin(q_1+q_2) \end{pmatrix}$$

Assumption 1 is thus satisfied with $B_{g1} = 40.29$ Nm and $B_{g2} = 1.825$ Nm. Furthermore, the input saturation bounds are $T_1 = 150$ Nm and $T_2 = 15$ Nm for the first and second links respectively, whence one can corroborate that Assumption 2 is fulfilled too.

For the application of the proposed design methodology, let us define the functions

$$\sigma_u(\varsigma;\beta,a) = \text{sign}(\varsigma)\max\{|\varsigma|^\beta, a|\varsigma|\} \tag{17a}$$

$$\sigma_{bh}(\varsigma;\beta,a,M) = \text{sign}(\varsigma)\min\{|\sigma_u(\varsigma;\beta,a)|, M\} \tag{17b}$$

$$\sigma_{bs}(\varsigma;\beta,a,M,L) = \begin{cases} \sigma_u(\varsigma;\beta,a) & \text{if } |\varsigma| \le L \\ \text{sign}(\varsigma)\sigma_{bs}^+(|\varsigma|;\beta,a,M,L) & \text{if } |\varsigma| > L \end{cases} \tag{17c}$$

where

$$\sigma_{bs}^+(\varsigma;\beta,a,M,L) = \sigma_u(L;\beta,a) + (M - \sigma_u(L;\beta,a))\tanh\left(\frac{\sigma_u(\varsigma;\beta,a) - \sigma_u(L;\beta,a)}{M - \sigma_u(L;\beta,a)}\right)$$

for constants $\beta > 0$, $a \in \{0,1\}$, $M > 0$, and $L > 0$ such that $\sigma_u(L;\beta,a) < M$. Figure C1 shows examples.

5.1 *Finite-time vs exponential stabilization*

The goal of this subsection is to show finite-time control implementations (through the proposed scheme) on the considered 2-DOF manipulator model, and to compare them with analog exponential stabilization tests. In particular, we are interested in corroborating the so-cited argument claiming that finite-time controllers achieve faster stabilization than asymptotic ones. Let us note that through the incorporation of exponential regulation implementations in the comparison, the fastest and more desirable type of asymptotic stabilization is being considered.

Based on the functions in Eqs. (17), we define —for every $j = 1,2$— those involved in the implementations performed in this subsection as

$$\sigma_{0j}(\varsigma) = \sigma_{bs}(\varsigma; \beta_0, a_{0j}, M_{0j}, L_{0j}) \tag{18a}$$

$$\sigma_{ij}(\varsigma) = \sigma_u(\varsigma; \beta_i, a_{ij}) \qquad i = 1,2 \tag{18b}$$

Following the design procedure in accordance to Corollary 1, we fixed $\gamma = 3/2$, $\beta_1 = 1/3$, $\beta_2 = 1/2$ and $\beta_0 = 1$ for the finite-time control implementations, and $\gamma = \beta_1 = \beta_2 = \beta_0 = 1$ for the exponential stabilization tests. Let us note that through these definitions we have $B_j = M_{0j}$, $j = 1,2$ (see (10)). Thus, by fixing $M_{01} = 100$ and $M_{02} = 13$ [Nm], the inequalities from expression (10) are satisfied. We further fixed $L_{0j} = 0.9M_{0j}$, $j = 1,2$. All the implementations were run taking the desired configuration at $q_d = \left(\pi/4 \;\; \pi/2\right)^T$ [rad] and initial conditions as $q(0) = \dot{q}(0) = 0_2$. For comparison purposes among the implementations, for every closed-loop response, we got (from the simulation data) the ϱ-stabilization time t^s_ϱ, defined as $t^s_\varrho \triangleq \inf\{t_s \geq 0 : \|x(t)\| \leq \varrho \;\; \forall t \geq t_s\}$, where $x \triangleq \left(\bar{q}^T \;\; \dot{q}^T\right)^T$.

Figure C2

shows results obtained taking, for every $j = 1,2$: $a_{ij} = 0$, $\forall i \in \{0,1,2\}$, $k_{1j} = 1$ [Nm/rad] and $k_{2j} = 1$ [Nms/rad]. One sees that the finite-time controller did give rise to faster responses, which is corroborated from ϱ-stabilization times, obtained for $\varrho = 0.01$ as $t^s_{0.01} = 8.31$ s for the finite-time control implementation and $t^s_{0.01} = 21.61$ s for the exponential stabilization test. Keeping $a_{ij} = 0$, $\forall i \in \{0,1,2\}$, $\forall j \in \{1,2\}$, further simulations were run increasing the control gain values to $k_{1j} = 10$ [Nm/rad] and $k_{2j} = 10$ [Nms/rad], $j = 1,2$. The results are shown in Figure 1,

whence one sees that, this time, it is the exponential stabilizer that gives rise to faster closed loop reactions (transient responses with shorter rise times). This seems to be confirmed through preliminary ϱ-stabilization time estimations, obtained for $\varrho = 0.01$ as $t^s_{0.01} = 6.55$ s for the finite-time control implementation and $t^s_{0.01} = 5.35$ s for the exponential stabilization test. A detailed inspection gives a clue on the reaction-speed differences so far observed, which are explained as follows. While the exponential stabilizers remain Lipschitz-continuous, the finite-time controllers loose Lipschitz-continuity at the origin. For instance, the σ_{ij}, $i,j = 1,2$, functions of the implemented algorithm keep a unitary slope around zero in the exponential stabilization case while they adopt a vertical slope at zero for the finite-time controller. Consequently, in the finite-time control case, there is a region around zero where each one of the corresponding control force components is magnified by an additional (non-linear) gain induced by the involved functions (see for instance Figure C1). In the case of the implemented finite-time stabilizer, by denoting ς_{ij}, $i,j = 1,2$, the corresponding arguments of σ_{ij}, such a region is characterized as $\{|\varsigma_{ij}| \leq 1,\ i,j = 1,2\} \triangleq D_M$. Outside this region, the involved functions have a reductive effect on their arguments in the finite-time control case, as may be corroborated for instance through Figure C1. Thus, when the closed-loop trajectories are such that the arguments of the involved functions, ς_{ij}, remain most of the time within the referred region, D_M, the corresponding (P and D type) control force components act with higher intensity in the finite-time case, forcing the resulting (error-variable) trajectories to ultimately reach any neighborhood of the origin faster. On the contrary, when the closed-loop trajectories spend most of the transient time outside such a region, D_M, slower (transient) reactions take place through the finite-time controller. Figures 2

and 3

show the variation of the arguments of the functions σ_{ij}, $i, j = 1, 2$, obtained from both implementations. One sees that with unitary gains (Figure 2), such arguments remained most of the time within the referred region D_M, explaining the quicker convergence of the trajectories obtained with the finite-time controller. On the contrary, with the higher gains (Figure 3), the referred arguments remained most of the transient time outside D_M, which explains the faster (transient) reaction of the trajectories obtained with the exponential controller. Nevertheless, further ϱ-stabilization times obtained in this latter case for $\varrho = 0.001$ (resp. $\varrho = 0.0001$) gave $t^s_{0.001} = 7.17$ s (resp. $t^s_{0.0001} = 7.37$ s) for the finite-time controller and $t^s_{0.001} = 7.63$ s (resp. $t^s_{0.0001} = 9.91$ s) for the exponential stabilizer, which seems to show that $\|x(t)\|$ finishes up by converging quicker to zero through finite-time control. This is expectable from the finite-time convergence, which forces the trajectories to exactly reach the equilibrium at the settling (finite) time, *versus* the asymptotic *infinite-time* attraction, which implies (*divergently*) longer time intervals to get to smaller neighborhoods of the origin. However, from a practical viewpoint, ϱ-stabilization times estimated for $\varrho = 0.01$ could be enough to determine that closed loop trajectories *practically* reached the equilibrium, giving rise to the possibility to have closed-loop implementations where exponential stabilizers be considered to (*practically*) achieve stabilization faster than finite-time controllers. This may be of particular interest under uncertain dynamics, perturbation terms or unmodelled phenomena that entail steady-state errors, in view of which exact stabilization cannot be guaranteed.

The observed reaction-speed differences may be avoided through different selections on the P and D type action related functions. For instance, further simulations were run with the same control gain combinations, already tested, but this time taking $a_{ij} = 1$, $\forall i \in \{0, 1, 2\}$, $\forall j \in \{1, 2\}$. Such a choice keeps the same *nonlinear* gains on D_M but avoids differences on the involved functions among the tested stabilizers outside D_M (see Figure C1). Figures 4

and 5

show the new results. One sees that, with unitary gains (Figure 4), the resulting closed loop responses seem to be close to those of the corresponding preceding test, *i.e.* to those shown in Figure C2 (the results corresponding to the exponential stabilizer were identical). This was expectable since, analogously to the corresponding precedent test, the arguments of the referred functions remained within D_M most of the time. The ϱ-stabilization time for $\varrho = 0.01$ gave $t^s_{0.01} = 8.27$ s for the finite-time controller (a little smaller than in the precedent case, where $t^s_{0.01} = 8.31$ s was obtained), keeping a ratio with respect to the estimation obtained for the exponential stabilizer, $t^s_{0.01} = 21.61$ s, close to that of the corresponding precedent test. Furthermore, with the higher gains (Figure 5), the trajectories obtained with the finite-time controller are observed to be close to those obtained with the exponential stabilizer (which remained identical). This is not surprising since, as expected, most of the transient time, the arguments of the involved function remained outside D_M (where the P and D action related functions keep the same form for both stabilizers). The ϱ-stabilization time for $\varrho = 0.01$ gave $t^s_{0.01} = 4.77$ s for the finite-time controller —against $t^s_{0.01} = 5.35$ s for the exponential stabilizer— showing that, contrarily to the corresponding precedent case, this time, the finite-time controller may indeed be concluded to achieve faster convergence either through *practical* criteria (and not just in view of its finite-time nature). It is worth emphasizing that such a way to ensure faster stabilization —as well as input saturation avoidance— through finite-time control was achieved thanks to the design flexibility permitted within the framework of local homogeneity, which allows to involve functions that are not forced to keep the homogeneity property globally but may rather adopt suitable changes.

5.2 *Multiple saturating structure*

Another aspect from the proposed scheme that is worth exploring concerns the design flexibility permitted on the choice of the saturating structure. We present an alternative test where two finite-time controllers that keep the same control gains but adopt different saturating structures are compared. It is worth pointing out that the proposed design methodology does not force to keep the same saturating structure at every one of the controlled degree of freedom but rather permits different choices among them. However, for our comparison purposes, the saturating structures are chosen different among the controllers but are kept the same among the controlled degrees of freedom for each one of the implemented stabilizer.

One of the implemented finite-time controllers adopts the same saturating structure of the precedent subsection, *i.e.* it involves the functions defined through Eqs. (18). Since this stabilizer uses, at every controlled degree of freedom, a single saturation function that includes both the P and D actions, it will be referred to as the SPD controller. The alternative finite-time stabilizer is structured taking, for every $j = 1, 2$:

$$\sigma_{0j}(\varsigma) = \sigma_u(\varsigma; \beta_0, a_{0j})$$

$$\sigma_{ij}(\varsigma) = \sigma_{bh}(\varsigma; \beta_i, a_{ij}, M_{ij}) \qquad i = 1, 2$$

Since this stabilizer uses a saturation function for each one of the P and D actions (separately), it will be referred to as the SP-SD controller.

For both —the SPD and SP-SD— finite-time controllers, we keep the same values taken in the precedent subsection for the parameters γ and β_i, $i = 0, 1, 2$, *i.e.* $\gamma = 3/2$, $\beta_1 = 1/3$, $\beta_2 = 1/2$ and $\beta_0 = 1$. Let us note that with these values, for the SP-SD algorithm we have $B_j = M_{1j} + M_{2j}$, $j = 1, 2$ (see (10)). Thus, by fixing $M_{i1} = 50$ and $M_{i2} = 6.5$ [Nm], $i = 1, 2$, the inequalities in expression (10) are satisfied. For the SPD algorithm we kept the same values of the parameters M_{0j} and L_{0j} taken in the precedent subsection, *i.e.* $M_{01} = 100$, $M_{02} = 13$ [Nm] and $L_{0j} = 0.9M_{0j}$, $j = 1, 2$. Both controllers were implemented taking $a_{ij} = 1$, $i = 0, 1, 2$, $j = 1, 2$, $k_{11} = 200$, $k_{12} = 20$ [Nm/rad], $k_{21} = 200$ and $k_{22} = 20$ [Nms/rad]. The implementation was run taking the same desired configuration and initial conditions taken in the precedent tests, *i.e.* $q_d = \begin{pmatrix}\pi/4 & \pi/2\end{pmatrix}^T$ [rad] and $q(0) = \dot{q}(0) = 0_2$.

Figure 6

shows the results obtained from the implementations. Observe that the SPD controller achieved faster finite-time stabilization. Such a result corroborates the usefulness of the structural variety offered by the proposed approach in searching for performance improvement.

5.3 *The control signals*

Observe that in all the previous simulation results (shown in both previous subsections) the control signals are corroborated to remain within the pre-specified ranges avoiding input saturation. In particular, the contrast on the convergence among the finite-time and exponential control signals is visible from the simulation results shown in Subsection 5.1. As for those shown in Subsection 5.2, one observes how the SPD structure permits a quicker control reaction than the SP-SD one, which explains the observed performance differences. Further exploration on the control signals and closed-loop system robustness against perturbations is left for future work.

6. Conclusions

Global regulation of mechanical systems with input constraints guaranteeing finite-time or exponential stabilization has been made possible through local homogeneity. A control scheme based on such a recent concept has been thoroughly developed and formally proposed, leaving the designer the election on the mentioned types of convergence through a simple parameter. It keeps a generalized SPD-type form permitting multiple saturating and locally-homogeneous structures. The work has been complemented through a simulation implementation section where it has not only been possible to illustrate the application of the proposed method and confirm the analytical results but also to study the veracity of the so-cited argument claiming that finite-time controllers achieve faster stabilization than asymptotic ones. This was actually shown to depend on the specific locally homogeneous functions involved in the SPD term of the controller and the precision used to *practically* evaluate the stabilization time. Furthermore, a way to define such functions has been shown through which finite-time controllers indeed prove to be faster than asymptotical stabilizers. This was made possible thanks to the design flexibility permitted within the framework of local homogeneity, which allows to involve functions that are not forced to keep the homogeneity property globally but may rather adopt suitable changes. Future work will consider robustness issues under uncertainties.

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Appendix A.

The work in (Orlov, 2005) already stated important extensions to the theory of homogeneity and finite-time stability. In particular, related definitions and results were extended therein to the framework of differential inclusions. Within such a context, two definitions stating a notion of local homogeneity were presented, both reproduced next.

Definition 7: (Orlov, 2005, Definition 2.9) The differential inclusion $\Sigma_{di} : \dot{x} = \Phi(x,t)$ (the differential equation $\Sigma_{de} : \dot{x} = \varphi(x,t)$ or the uncertain system $\Sigma_{us} : \dot{x} = \varphi(x,t) + \psi(x,t)$, $|\psi_i(x,t)| \leq M_i$, $i = 1,\ldots,n$) is called locally homogeneous of degree $q \in \mathbb{R}$ with respect to dilation $(r_1,\ldots,r_n)$, where $r_i > 0$, $i = 1,\ldots,n$, if there exist a constant $c_0 > 0$, called a lower estimate of the homogeneity parameter, and a ball $B_\delta \subset \mathbb{R}^n$, called a homogeneity ball, such that any solution $x(\cdot)$ of Σ_{di} (respectively, that of Σ_{de} or Σ_{us}), evolving within the ball B_δ, generates a parameterized set of solutions $x^c(\cdot)$ with components

$$x_i^c(t) = c^{r_i} x_i(c^q t)$$

and parameter $c \geq c_0$.

Definition 8: (Orlov, 2005, Definition 2.10) A piecewise continuous function $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is called locally homogeneous of degree $q \in \mathbb{R}$ with respect to dilation $(r_1,\ldots,r_n)$, where $r_i > 0$, $i = 1,\ldots,n$, if there exist a constant $c_0 > 0$ and a ball $B_\delta \subset \mathbb{R}^n$ such that

$$\varphi_i(c^{r_1}x_1,\ldots,c^{r_n}x_n,c^{-q}t) = c^{q+r_i}\varphi_i(x_1,\ldots,x_n,t)$$

for all $c \geq c_0$ and almost all $(x,t) \in B_\delta \times \mathbb{R}$.

Furthermore, a local-homogeneity-based criterion for global finite-time stability, analog to the one presented in (Zavala-Río & Fantoni, 2014, Theorem 3.1) (reproduced in Section 2, above, as Theorem 1), was also previously presented in (Orlov, 2005, Theorem 3.1). Since the latter was stated within the context of differential inclusions, the respective proofs follow completely different analytical procedures.

In view of the different formulations among the definitions of local homogeneity stated in (Orlov, 2005) and that (Zavala-Río & Fantoni, 2014), the verification of their fulfilment would imply different analytical procedures. For this reason, and since this work does not deal with differential inclusions, the results in this paper are developed within the analytical context of (Zavala-Río & Fantoni, 2014).

Interesting enough, an *approximation* approach of homogeneity has also been alternatively referred to as *local homogeneity* in (Efimov & Perruquetti, 2010). More precisely, the referred *approximation* notion is defined therein as follows.

Definition 9: (Efimov & Perruquetti, 2010, Definition 2) The function $g : \mathbb{R}^n \to \mathbb{R}$, $g(0_n) = 0$, is called (r,λ_0,g_0)-homogeneous ($r_i > 0$, $i = \overline{1,n}$; $g_0 : \mathbb{R}^n \to \mathbb{R}$, $g_0(0_n) = 0$) if for any $x \in S_1^{n-1}$

$$\lim_{\lambda \to \lambda_0} \lambda^{-d_0} g(\delta_\lambda^r(x)) - g_0(x) = 0$$

for some $d_0 \geq 0$. The system $\dot{x} = f(x)$ is called (r,λ_0,f_0)-homogeneous ($r_i > 0$, $i = \overline{1,n}$; $f_0 : \mathbb{R}^n \to \mathbb{R}^n$, $f_0(0_n) = 0_n$) if for any $x \in S_1^{n-1}$

$$\lim_{\lambda \to \lambda_0} \lambda^{-d_0} \mathrm{diag}[\lambda^{-r_1},\ldots,\lambda^{-r_n}] f(\delta_\lambda^r(x)) - f_0(x) = 0$$

for some $d_0 \geq -\min_{1\leq i\leq n} r_i$.

Such an approximation approach actually includes the one considered in Eq. (6), which would be a special case of Definition 9 for $\lambda_0 = 0$. This special case, expressed through Eq. (6), had been previously considered to state the preservation of asymptotic stability (at the origin) in (Rosier, 1992) and of finite-time stability in (Hong, Huang & Xu, 2006), both within the conventional context of homogeneity, and of finite-time stability within the framework of local homogeneity in (Zavala-Río & Fantoni, 2014). An extension of the latter, that includes preservation of exponential stability, is presented in Section 2, above, as Lemma 2, whose formulation will prove to be useful within the analytical context developed in this paper.

Appendix B.

Reference (Zavala-Río & Fantoni, 2014) states and proves the *finite-time* stability version of Lemma 2. We thus prove here the part of the statement concerning δ-*exponential* stability.

Let $D_c = \{x \in D : \|x\|_r < c\}$. Consider the following variable transformation

$$\begin{pmatrix} y \\ \rho \end{pmatrix} : D_c \setminus \{0_n\} \to S^{n-1}_{r,c} \times (0, c)$$
$$x \mapsto \begin{pmatrix} \begin{bmatrix} \frac{c^{r_1} x_1}{\|x\|_r^{r_1}} & \frac{c^{r_2} x_2}{\|x\|_r^{r_2}} & \cdots & \frac{c^{r_n} x_n}{\|x\|_r^{r_n}} \end{bmatrix}^T \\ \|x\|_r \end{pmatrix} \tag{B1a}$$

whose inverse is

$$x : S^{n-1}_{r,c} \times (0, c) \to D_c \setminus \{0_n\}$$
$$(y, \rho) \mapsto \delta^r_{\rho/c}(y) \tag{B1b}$$

The vector function defining y in (B1a) projects every point of $D_c \setminus \{0_n\}$ onto the r-homogeneous $(n-1)$-sphere $S^{n-1}_{r,c}$, and every point y on $S^{n-1}_{r,c}$ is the projection of the whole dilation segment (or *ray segment through* y (Zavala-Río & Fantoni, 2014)) $\delta^r_\varepsilon(y)$ $\forall \varepsilon \in (0, 1)$, or equivalently, $\delta^r_{\rho/c}(y)$ $\forall \rho \in (0, c)$.[8] From Eqs. (B1), the system dynamics in the (y, ρ)-coordinates on $S^{n-1}_{r,c} \times (0, c)$, taking $\alpha = 0$, is obtained, after basic developments, as

$$\dot{y} = \Big[I_n - \mathrm{diag}[r_1, \ldots, r_n] y \psi(y) \Big] \Big[f(y) + \mathrm{diag}[(c/\rho)^{r_1}, \ldots, (c/\rho)^{r_n}] \hat{f}(\delta^r_{\rho/c}(y)) \Big] \tag{B2a}$$

$$\dot{\rho} = \Big[\psi(y) f(y) + \hat{F}(y, \rho) \Big] \rho \tag{B2b}$$

where

$$\hat{F}(y, \rho) = \sum_{i=1}^{n} \psi_i(y) (c/\rho)^{r_i} \hat{f}_i\big(\delta^r_{\rho/c}(y)\big)$$

[8] In the conventional analytical context of homogeneity, where $D = \mathbb{R}^n$, D_c and $(0, c)$ in expressions (B1) are respectively replaced by $\mathbb{R}^n$ and $\mathbb{R}_{>0}$, and c is conventionally taken as $c = 1$, the vector function defining y in (B1a) projects every point of $\mathbb{R}^n \setminus \{0_n\}$ onto the r-homogeneous $(n-1)$-sphere $S^{n-1}_{r,c}$ (Kawski, 1990; M'Closkey & Murray, 1997), and every point y on $S^{n-1}_{r,c}$ is the projection of the whole dilation (or *ray through* y (Aeyels & de Leenheer, 2002)) $\delta^r_\varepsilon(y)$ $\forall \varepsilon > 0$.

and $\psi(y) = [\psi_1(y) \;\; \psi_2(y) \;\; \cdots \;\; \psi_n(y)]$, with

$$\psi_i(y) = \frac{\mathrm{sign}(y_i)}{c^p\, r_i}|y_i|^{\frac{p}{r_i}-1}$$

$i = 1, \ldots, n$. Integration yields $\rho(t) = \rho(0)e^{\varphi(t)+\hat{\varphi}(t)}$, with $\varphi(t) = \int_0^t \psi(y(\varsigma))f(y(\varsigma))d\varsigma$ and $\hat{\varphi}(t) = \int_0^t \hat{F}(y(\varsigma), \rho(\varsigma))d\varsigma$ (as long as $\rho(t) < c$). Since in case $\hat{f}(x) \equiv 0_n$ the origin is a δ-exponentially stable equilibrium, there are constants $a \geq 1$, $b > 0$ and $c_0 \in (0, c/a)$ such that $\rho(t) = \rho(0)e^{\varphi(t)} \leq a\rho(0)e^{-bt}$, $\forall t \geq 0$, $\forall \rho(0) \in (0, c_0)$. Now, by continuity of $\psi(y)$ (recall Remark 2) and the compactness of $S_{r,c}^{n-1}$, $\psi(y)$ happens to be bounded on $S_{r,c}^{n-1}$, or equivalently, there exists a positive constant B_ψ such that $|\psi_i(y)| \leq B_\psi$, $i = 1, \ldots, n$, $\forall y \in S_{r,c}^{n-1}$. Further, by (7) (with $\alpha = 0$), there exists $c_1 \in (0, c_0)$ such that, for every $y \in S_{r,c}^{n-1}$: $\left|(c/\rho)^{r_i}\hat{f}_i\big(\delta_{\rho/c}^r(y)\big)\right| \leq \frac{b}{2nB_\psi}$, $i = 1, \ldots, n$, $\forall \rho \in (0, c_1)$. Thus, for every $y \in S_{r,c}^{n-1}$ and all $\rho \in (0, c_1)$, $\hat{F}(y, \rho) \leq \sum_{i=1}^n |\psi_i(y)|\,|(c/\rho)^{r_i}\hat{f}_i(\delta_{\rho/c}^r(y))| \leq \sum_{i=1}^n B_\psi \frac{b}{2nB_\psi} = b/2$, and consequently $\rho(t) = \rho(0)e^{\varphi(t)+\hat{\varphi}(t)} \leq a\rho(0)e^{-bt+\int_0^t (b/2)d\varsigma} = a\rho(0)e^{-(b/2)t}$, $\forall t \geq 0$, $\forall \rho(0) \in (0, c_1/a)$, or equivalently $\|x(t; x_0)\|_r \leq a\|x_0\|_r e^{-(b/2)t}$, $t \geq 0$, $\forall \|x_0\|_r \in (0, c_1/a)$, which concludes the proof.

Appendix C.

Let $\varsigma_0, \varsigma_1, \varsigma_2 \in \mathbb{R}$. Since σ_0 is strictly increasing, we have that $\sigma_0(\varsigma_0) > \sigma_0(\varsigma_1) \iff \varsigma_0 > \varsigma_1$ and $\sigma_0(\varsigma_0) < \sigma_0(\varsigma_1) \iff \varsigma_0 < \varsigma_1$. From this and the strictly passive character of σ_2 we have, by letting $\varsigma_0 = \varsigma_1 + \sigma_2(k\varsigma_2)$, that $\sigma_0(\varsigma_1 + \sigma_2(k\varsigma_2)) - \sigma_0(\varsigma_1) > 0 \iff \sigma_2(k\varsigma_2) > 0 \iff \varsigma_2 > 0$ and $\sigma_0(\varsigma_1 + \sigma_2(k\varsigma_2)) - \sigma_0(\varsigma_1) < 0 \iff \sigma_2(k\varsigma_2) < 0 \iff \varsigma_2 < 0$, $\forall \varsigma_1 \in \mathbb{R}$, whence it follows that $\varsigma_2\big[\sigma_0(\varsigma_1 + \sigma_2(k\varsigma_2)) - \sigma_0(\varsigma_1)\big] > 0$, $\forall \varsigma_2 \neq 0$, $\forall \varsigma_1 \in \mathbb{R}$.

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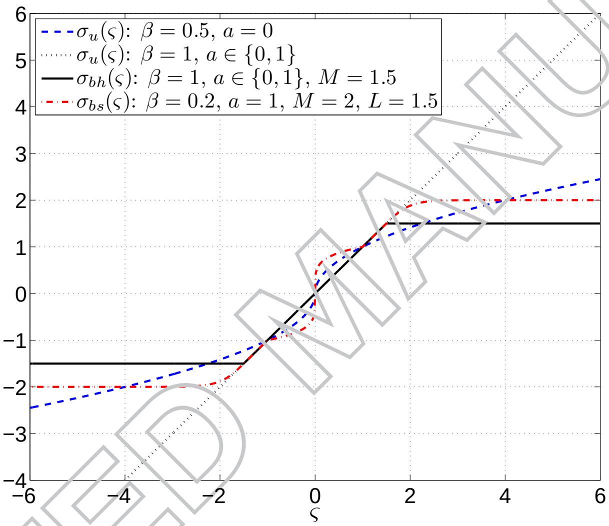

Figure 1. Examples of $\sigma_u(\varsigma;\beta,a)$, $\sigma_{bh}(\varsigma;\beta,a,M)$ and $\sigma_{bs}(\varsigma;\beta,a,M,L)$

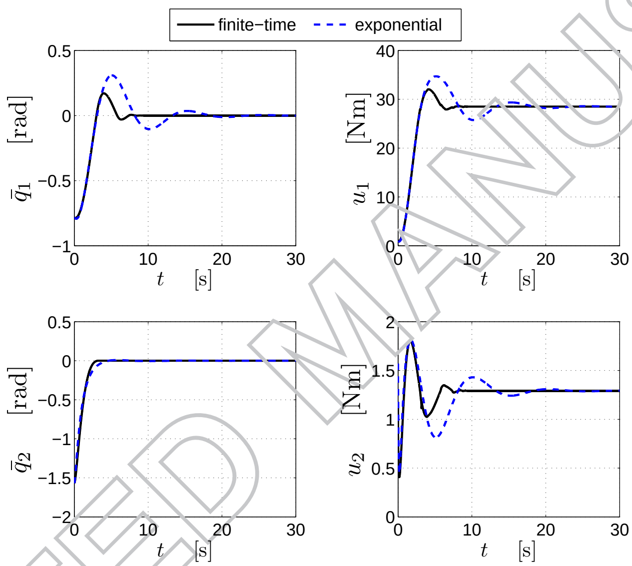

Figure 2. Results with $a_{ij} = 0$, $\forall i \in \{0, 1, 2\}$, $\forall j \in \{1, 2\}$, $k_{1j} = 1$ [Nm/rad] and $k_{2j} = 1$ [Nms/rad], $j = 1, 2$: position errors (left) and control signals (right)

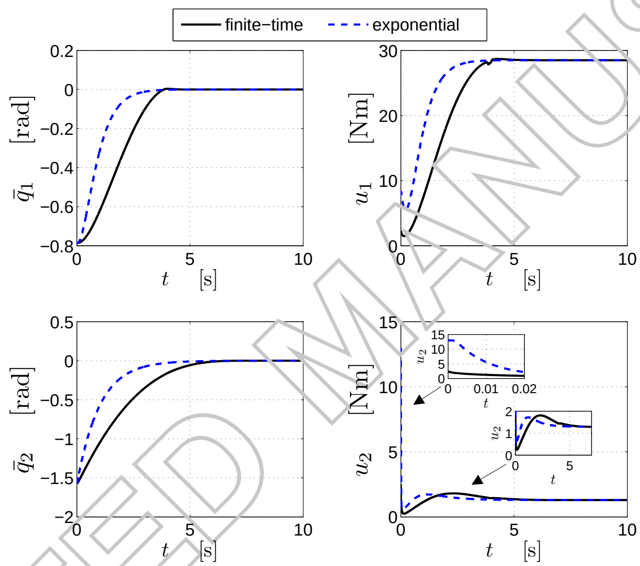

Figure 3. Results with $a_{ij} = 0$, $\forall i \in \{0, 1, 2\}$, $\forall j \in \{1, 2\}$, $k_{1j} = 10$ [Nm/rad] and $k_{2j} = 10$ [Nms/rad], $j = 1, 2$: position errors (left) and control signals (right)

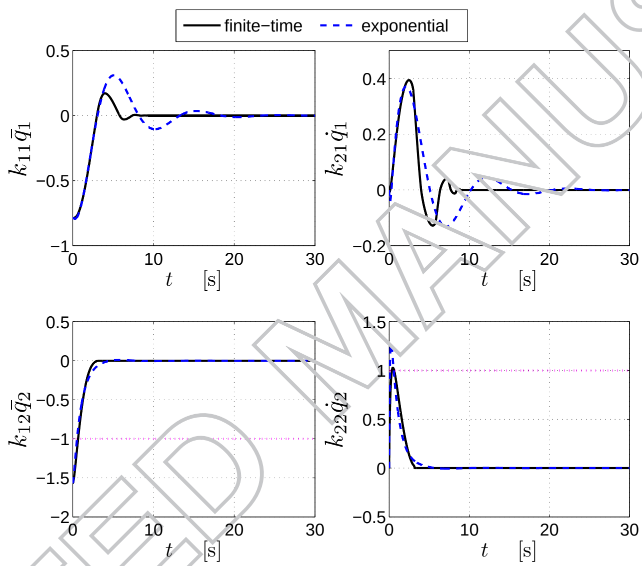

Figure 4. Variation of the arguments of σ_{ij}, $i, j = 1, 2$, obtained from the implementations with $k_{1j} = 1$, [Nm/rad] and $k_{2j} = 1$ [Nms/rad], $j = 1, 2$

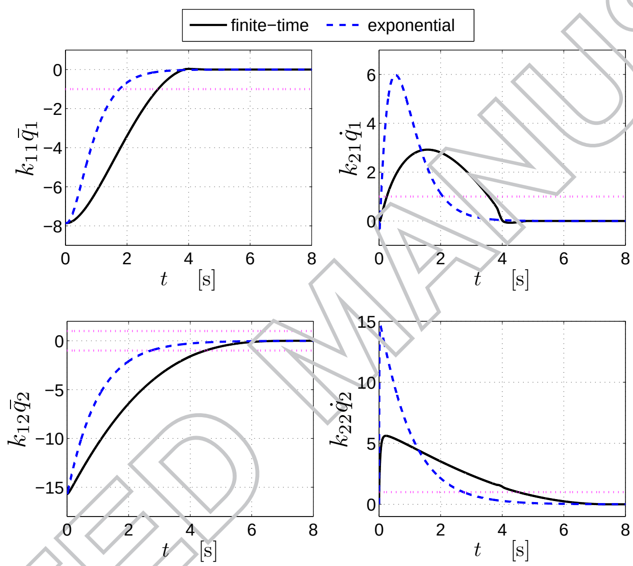

Figure 5. Variation of the arguments of σ_{ij}, $i, j = 1, 2$, obtained from the implementations with $k_{1j} = 10$, [Nm/rad] and $k_{2j} = 10$ [Nms/rad], $j = 1, 2$

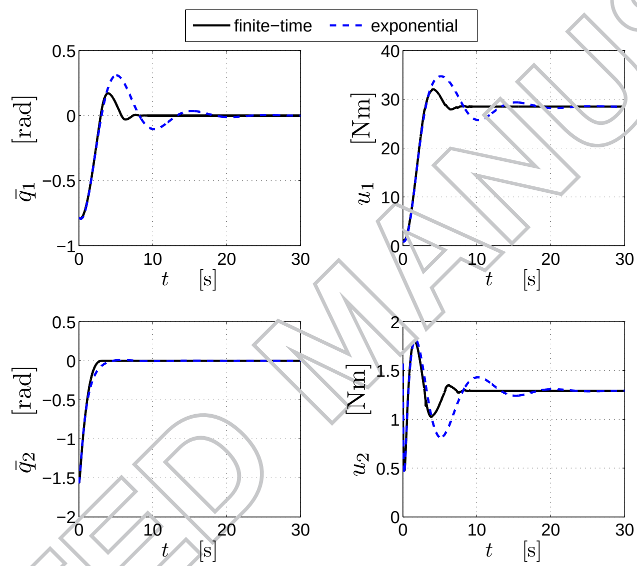

Figure 6. Results with $a_{ij} = 1$, $\forall i \in \{0, 1, 2\}$, $\forall j \in \{1, 2\}$, $k_{1j} = 1$ [Nm/rad] and $k_{2j} = 1$ [Nms/rad], $j = 1, 2$: position errors (left) and control signals (right)

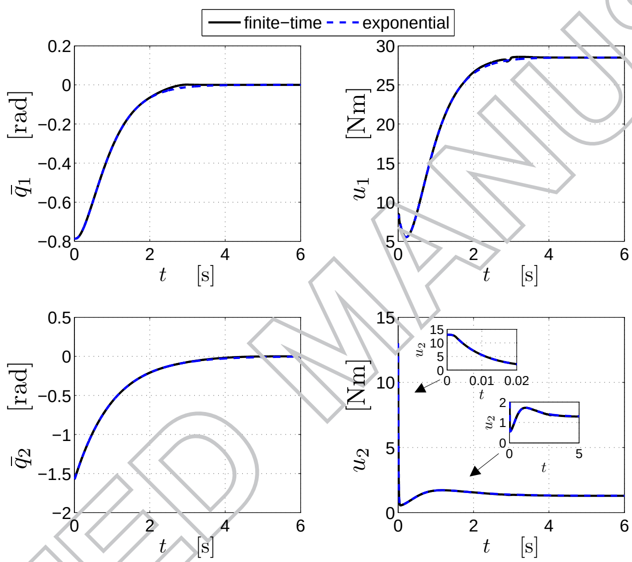

Figure 7. Results with $a_{ij} = 1$, $\forall i \in \{0, 1, 2\}$, $\forall j \in \{1, 2\}$, $k_{1j} = 10$ [Nm/rad] and $k_{2j} = 10$ [Nms/rad], $j = 1, 2$: position errors (left) and control signals (right)

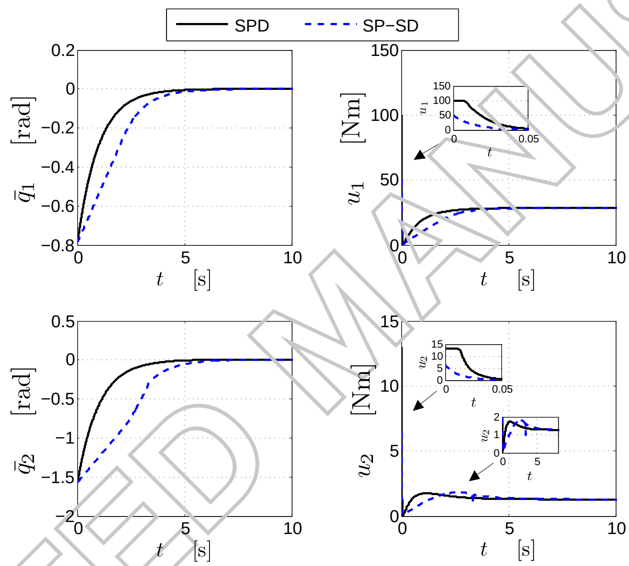

Figure 8. SPD *vs* SP-SD finite-time controllers: position errors (left) and control signals (right)