Electronic version of an article published as International Journal of Modern Physics C, 28 (01), 1750008 (2017)
https://doi.org/10.1142/S0129183117500085

International Journal of Modern Physics C

Generation of Chaotic Attractors Without Equilibria Via Piecewise Linear Systems

R.J. Escalante-González and E. Campos-Cantón

División de Matemáticas Aplicadas, Instituto Potosino de Investigación Científica y Tecnológica A. C., Camino a la Presa San José 2055, Col. Lomas 4 Sección, C.P. 78216, San Luis Potosí, S.L.P., México.
rodolfo.escalante@ipicyt.edu.mx

Received Day Month Year
Revised Day Month Year

In this paper we present a mechanism of generation of a class of switched dynamical system without equilibrium points that generates a chaotic attractor. The switched dynamical systems are based on piecewise linear (PWL) systems. The theoretical results are formally given through a theorem and corollary which give necessary and sufficient conditions to guaranty that a linear affine dynamical system has no equilibria. Numerical results are according with the theory.

Keywords: dynamical systems; systems without equilibrium; linear systems.

PACS Nos.: 11.25.Hf, 123.1K

1. Introduction

Recently the construction of non linear systems with chaos has been studied, trying to find the simplest systems with complex behavior, this is by looking for the combination of quantity and type of nonlinear terms in the differential equations that describe the system. This has lead to systematic methods for finding chaotic systems, as found in Ref. 1 where is shown a system without equilibria, and a methodology to construct a system with any number of equilibria was found by adding symmetry to the system with only one stable point, but curiously a one-scroll chaotic attractor is generated. It also has lead to the design of chaotic and hyperchaotic electronic circuits as the presented in Ref. 2.

Something interesting about the chaotic systems is the behavior near the equilibria. Several 3-dimensional systems that exhibit chaos have a saddle-foci equilibrium point and those based on PWL systems have been taken to generate multiscroll attractors.[3,4] However a system with different kind of equilibria has been reported by Yang and Chen in Ref. 5 which presents a saddle equilibrium point and two stable node-foci equilibria, this system connects the originals Lorenz and Chen systems. An interesting chaotic system with an unique stable node-focus equilibrium point was presented in Ref. 6.

The trend was to generate chaotic systems without equilibrium point, as the system reported by Sprott.[7] In the same spirit in Ref. 8 three methods have been used to produce seventeen three-dimensional chaotic systems based on quadratic terms to yield nonlinearities which present no equilibria. The methods can be classified as follows:

- Adding a constant term to a nonhyperbolic system, adjusting and simplifying the parameter and coefficients of other terms.
- Looking at cases where equilibria of a parametric system are imaginary and adjusting and simplifying the parameter and coefficients of the terms.
- Adding a constant to each of the derivatives in known chaotic systems and looking for solutions where the numerically calculated equilibria do not exist.

In Ref. 9 a simple three-dimensional autonomous system based on Sprott D system is reported. This system presents the coexistence of chaotic attractors with two saddle-foci, non-hyperbolic equilibrium or no equilibria through a constant controller, which can adjust the type of chaotic attractors.

Nowadays, there is a trend to bring the chaotic integer-order systems to chaotic fractional-order systems. This is the case of the fractional-order systems without equilibria presented in Ref. 10 which has been developed starting from the corresponding integer-order system.[8]

These systems without equilibrium points could be helpful in the deterministic modeling of dynamics which seems to have no equilibria in reality as the Brownian motion, also could contribute in the construction of pseudorandom number generators and new communication schemes as those reported in Ref. 11,12.

The previous results about chaotic dynamical systems without equilibrium point have been constructed by means of nonlinear functions based on cuadratic terms or the multiplication of their states, and also using the absolute value and sing functions.

As it is well known, it is not possible to obtain complex behavior with an affine linear system of the form $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B$. So it is necessary to use piece wise-linear (PWL) systems, for example the well known Chua system which has three linear parts. So the aim is to generate a PWL system with chaotic behavior without equilibrium point using the method for generating an strange attractor proposed in Ref. 13 using 3-D unstable dissipative system which are stable in two components but unstable in the other one (UDS Type II) and a switching law. This PWL system without equilibria can help to revealing some new mysterious features of chaos due to simplicity.

In this paper we give conditions to construct a system without equilibria based on PWL systems. Section 2 contains the results about the generation of dynamical systems without equilibrium point based on affine transformations. In Section 3, we present a methodology to generate a dynamical system in order to yield a strange attractor without equilibria using a switching law and two affine transformations.

Section 4 contains some examples of dynamical systems without equilibria based on the mechanism presented in previous sections. Finally in Section 5 are the conclusions of the work.

2. Dynamical systems without equilibrium based on affine transformation

We start by considering a dynamical system given by the following form:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B, \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n\times n}$ is a linear operator and $B \in \mathbb{R}^n$ is a constant vector. The equilibria of the system is determined by $\dot{\boldsymbol{x}} = 0$. So, when the vector $B = 0$, the equilibria of the system is given by the solution of the homogeneous systems of equations of the form $A\boldsymbol{x} = 0$ and the systems always present at least one equilibrium point, i. e., if $Rank(A) = n$ then the equilibrium point is at the origin $\boldsymbol{x} = 0$; but if $Rank(A) < n$ then there are an infinite number of equilibria.

When vector $B \neq 0$ we get non homogeneous systems of equations of the form $A\boldsymbol{x} = -B$. There are two cases, the former is when the matrix A is full rank then the system is definite and has only an equilibrium point at $\boldsymbol{x}^* = -A^{-1}B$. The second case is when the matrix A is not full rank then there are two possibilities: the system is inconsistent and has no equilibrium or it is indefinite and has an infinity number of equilibria. Notice that for the second case the matrix A does not have inverse matrix, this is a necessary condition but is not enough for guaranty systems without equilibrium point.

The following Theorem establishes necessary and sufficient conditions for the construction of a linear affine dynamical system without equilibrium points:

Theorem 1. *Given a dynamical system based on affine transformation of the form $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B$ where $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector, $B \in \mathbb{R}^n$ is a nonzero constant vector and $A \in \mathbb{R}^{n\times n}$ is a linear operator, the system posses no equilibrium point if and only if:*

- *A is not invertible.*
- *$B \neq 0$ is linearly independent of the set of vectors comprised by columns of the operator A.*

Proof. ($\Leftarrow$) Let us begin by considering the case where the matrix A is not invertible and the nonzero vector B is linearly independent of the column vectors of the operator A.
If A is not invertible then $det(A) = 0$, thus this implies that A has at least a linearly dependent column vector, which means the set of column vectors of A cannot

span R^n. So the vector B is linearly independent of the column vectors of A and $B \notin rangeA$. Then there is not an $\boldsymbol{x}$ that satisfies $A\boldsymbol{x} = B$, so this is a contradiction. We can conclude that the dynamical system given by $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B$ does not have equilibrium point.

($\Rightarrow$) Now let us consider that the system posses no equilibrium point then $\dot{\boldsymbol{x}} \neq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$. So there is a contradiction in $A\boldsymbol{x} = B$ which means $B \notin rangeA$, also as $B \in \mathbb{R}^n$ this means $dim(rangeA) < n$. As $rangeA$ is the span of the column vectors of the matrix A implies that the number of linearly independent columns of A is less then n thus $det(A) = 0$ which implies that A is not invertible, also as vector B cannot be represented as a linear combination of the columns of A it should be linearly independent of them. □

Corollary 1. *Given a dynamical system based on affine transformation of the form* $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B$ *where* $\boldsymbol{x} \in \mathbb{R}^n$ *is the state vector,* $B \in \mathbb{R}^n$ *is a nonzero constant vector and* $A \in \mathbb{R}^{n\times n}$ *is a linear operator, the system posses no equilibrium point if and only if:*

- $rank([A, B]) > rank(A)$.

Proof. ($\Leftarrow$) Let A be not invertible and $rank([A, B]) > rank(A)$. Due to $[A, B]$ is a rectangular matrix of $n \times n + 1$ and the maximum rank possible is n then $rank(A) < n$ thus $det(A) = 0$ and A is not invertible. The rank of $[A, B]$ is equal to the number of linearly independent column vectors of $[A, B]$ and the rank of A is equal the number of linearly independent column vectors of A, then B must be linearly independent of the column vectors of A in order to fulfill $rank([A, B]) > rank(A)$. Then conditions in Theorem 1 are fulfilled and the system posses no equilibria.
($\Rightarrow$) Now consider the system posses no equilibria, due to Theorem 1 A is not invertible and B linearly independent of the column vectors of A. As A is not invertible $rank(A) < n$, as B is linearly independent of the column vectors of A then $rank([A, B]) = rank(A) + 1$ thus $rank([A, B]) > rank(A)$. □

3. Chaotic Attractor without equilibria based on PWL systems

There are two methods to obtain a PWL system without equilibrium points: The former is by getting a system in which the equilibrium points are always in a different domain of the current state so the system can be said to have no equilibrium; the second is by ensuring there exist no such point due to a contradiction in the finding of the equilibrium point for all the linear parts. This paper is focused in the last method.

Consider a dynamical system given by Eq. (1), whose A matrix is not full rank and vector $B = 0$, which means the system has a continuous of equilibrium points. For vector $B \neq 0$ if this vector makes the system fulfills conditions in Theorem 1 then the dynamical system (1) doesn't have equilibria.

This allows us to compare the dynamical system (1) with and without equilibria for $B = 0$ and $B \neq 0$, respectively, and considering the same matrix A. As the system losses its equilibria due to vector $B \neq 0$ the vector field in those points are not longer zero and the magnitude and direction of the vector field depends on the linear operator A and the vector B. To illustrate this, consider the dynamical system given by (1) with vector $B = 0$ and matrix A as follows:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & a_{22} \end{bmatrix}, \tag{2}$$

with $a_{22} \neq 0$ and $\mathbf{x} = (x_1, x_2)^T$, note that the matrix A is singular and $rank(A) = 1$. The x_1 and x_2 axes correspond to the eigenspaces associated to the eigenvalues 0 and a_{22}, respectively. The x_1 axis is the continuous of equilibrium points and could be seen as a set of α limit points, and the x_2 axis is a stable eigenspace for $a_{22} < 0$ and unstable eigenspace for $a_{22} > 0$.

Now for the case $B \neq 0$, without loss of generality when $B = (b_1, 0)^T$, there are two cases $b_1 > 0$ and $b_1 < 0$, note that both cases make the system fulfills the second condition of Theorem 1 since $rank(-[A, B]) = 2$. The new vector field and the trajectory behavior are toward right side or toward left side for $b_1 > 0$ and $b_1 < 0$, respectively. Thus the dynamical system (1) does not have equilibria.

As it can be seen after changing the vector $B = 0$ to $B \neq 0$, the dynamical system (1) fulfilling conditions of Theorem 1 the points in the state space which were equilibrium points for $B = 0$ are no longer equilibrium points in the cases $B \neq 0$ and furthermore the vector field change is easy to follow.

It is important to observe that the equilibria could be located in a different way and the idea still apply, the vector field change depends on the matrix A and the vector B. For example, another possible location of the equilibria consider the system given by (1) with matrix A as follows:

$$A = \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix}. \tag{3}$$

The eigenvalues of the matrix A are a_{11} and 0, in this case the continuous of equilibria is located along x_2 axis. Keeping this matrix A and using the same vectors B proposed before both conditions in Theorem 1 are fulfilled.

Hitherto we have studied the dynamics of the system (1) fulfilling conditions of Theorem 1. However the goal is to generate chaotic attractors, so it is important to consider switched system based on the previous theory.

For our proposed method with PWL systems we are going to restrict the construction to systems of the form given by (1) in $\mathbb{R}^3$ which fulfill conditions in Theorem 1 and whose matrix A has eigenvalues $\lambda_i, i = 1, 2, 3$, where two of them are complex conjugate with negative real part and one of them zero i.e. the systems have a continuous of equilibria for $B = 0$. We could see these systems as a set of parallel systems in $\mathbb{R}^2$ which present a stable focus, as is shown in Fig. 1.

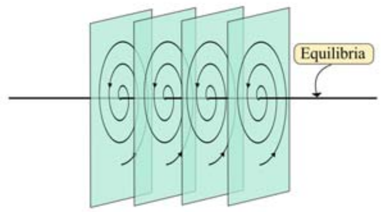


Fig. 1. Linear affine system with an eigenvalue $\lambda = 0$.

However if we consider the vecto $B \neq 0$ such that the dynamical system (1) fulfills conditions in Theorem 1 we obtain something similar to Fig. 2.

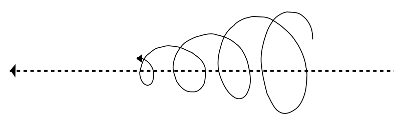


Fig. 2. Linear affine system without equilibria.

In Ref. 13 two kind of unstable dissipative systems (UDS's) were defined as follows:

Definition 1. A system given by $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ in $\mathbb{R}^3$ with eigenvalues $\lambda_i, i = 1, 2, 3$, is said to be an UDS Type I, if $\Sigma_{i=1}^{3}\lambda_i < 0$ and one eigenvalue λ_i is negative real and the other two are complex conjugate with a positive real part.[13]

Definition 2. A system given by $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ in $\mathbb{R}^3$ with eigenvalues $\lambda_i, i = 1, 2, 3$, is said to be an UDS Type II, if $\Sigma_{i=1}^{3}\lambda_i < 0$ and one eigenvalue is positive real and the other two are complex conjugate with a negative real part.[13]

Also in Ref. 13 a way to generate an attractor is proposed by using switched system based on two UDS Type II and a switching surface located between the two equilibrium points. Although the linear system without equilibria is not a UDS Type II we can appreciate some similarities in the sense that its behavior is the same as a UDS Type II in half the state space. Exploiting these observations we propose the construction of a PWL system without equilibria that generates a chaotic attractor.

Definition 3. Consider two system of the form given by Eq. (1) in $\mathbb{R}^3$ with domains $D_j \subset \mathbb{R}^3$, j=1,2, which fulfill conditions given in Theorem 1 and whose matrix A_i has eigenvalues $\lambda_i, i = 1, 2, 3$, where two of them are complex conjugate with negative real part and one of them is zero. A PWL system without equilibria is given as follows:

$$\dot{\boldsymbol{x}} = \begin{cases} A_1\boldsymbol{x} + B_1, \text{ if } \boldsymbol{x} \in D_1; \\ A_2\boldsymbol{x} + B_2, \text{ if } \boldsymbol{x} \in D_2. \end{cases} \tag{4}$$

Where $D_1 \cup D_2 = \mathbb{R}^3$ and $D_1 \cap D_2 = \emptyset$.

With a correct set of subsystems and an appropriate switching law we can obtain a chaotic attractor. Fig. 3 illustrates the construction.

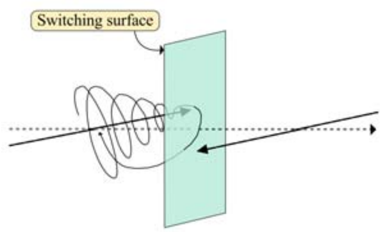


Fig. 3. Mechanism to generate a chaotic attractor.

4. Numerical results

As an example of the construction given in Definition 3, consider the following system whose matrices A_1 and A_2 are equal:

$$\dot{\boldsymbol{x}} = \begin{cases} A\boldsymbol{x} + B_1, \text{ if } x_1 < \sigma; \\ A\boldsymbol{x} + B_2, \text{ if } x_1 \geq \sigma. \end{cases} \tag{5}$$

Where $\sigma = 0$,

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -0.5 & 3 \\ 0 & -3 & -0.5 \end{bmatrix}, B_1 = \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} -1 \\ 10 \\ 0 \end{bmatrix}. \tag{6}$$

The result is a chaotic attractor in $\mathbb{R}^3$ which is shown in Fig. 4. Fig. 5 shows the projections of the strange attractor onto the plane: a) (x_1, x_2), b) (x_1, x_3) and c) (x_2, x_3). Note that in this system the switching surface is located at $x_1 = \sigma$ for $\sigma = 0$ but another value could be chosen for $\sigma \in \mathbb{R}$ and the attractor just get displaced along axis x_1.

Methods to proof a system exhibits chaotic behavior that make use of homoclinic and heteroclinic orbits cannot be applied to this kind of systems since they have not equilibrium points. Nevertheless the maximum Lyapunov exponent for our example was calculated with a modified method based on the methods of Wolf(in Ref. 14)

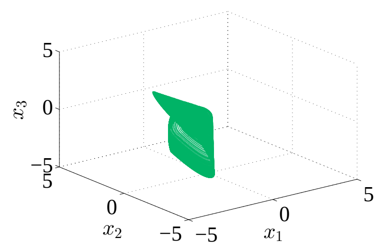
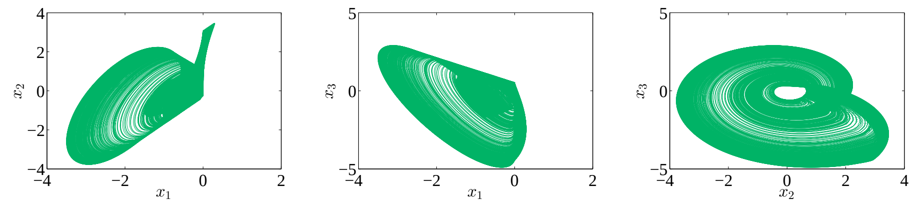


Fig. 4. Chaotic attractor without equilibria in the space.



(a) (b) (c)

Fig. 5. Projections of the chaotic attractor onto the plane: (a) (x_1, x_2), (b) (x_1, x_3) and (c) (x_2, x_3).

and Rosenstein(in Ref. 15), for this, a fourth order Runge-Kutta method was used with a step size of $h = 0.0001$ and the average of 50 trajectories with an orthogonal initial separation of $d_0 = 1e^{-6}$ from a main previously calculated trajectory. The Maximum Lyapunov Exponent (MLE) found is $\lambda = 0.243$ and it is shown in Fig. 6. It is worth mentioning the fact that the exponents calculated from the Jacobian method are the same eigenvalues of the matrix A $(\lambda_{1,2} = -0.5 \pm 3, \lambda_3 = 0)$, this method totally lost the effect of the vectors B in the switched system.

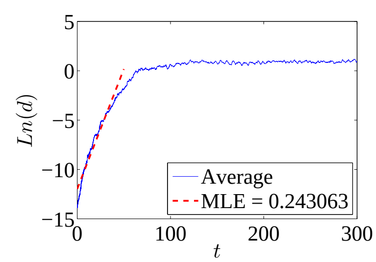


Fig. 6. Calculation of the Maximum Lyapunov Exponent.

As another example of the construction given in Definition 3, consider the following system whose matrices A_1 and A_2 are equal with eigenvalues $\lambda_1 = 0$, and $\lambda_{2,3} = -0.5 \pm 3.4278i$:

$$\dot{\mathbf{x}} = \begin{cases} A\mathbf{x} + B_1, \; if x_1 < 0; \\ A\mathbf{x} + B_2, \; if x_1 \geq 0. \end{cases} \tag{7}$$

$$A = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & 4 \\ 0 & -3 & -1 \end{bmatrix}, B_1 = \begin{bmatrix} 0.5 \\ 15 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} -0.5 \\ 0 \\ 1 \end{bmatrix}. \tag{8}$$

The result is a chaotic attractor in $\mathbb{R}^3$ which is shown in Fig. 7. Fig. 8 shows the projections of the strange attractor onto the plane: a) (x_1, x_2), b) (x_1, x_3) and c) (x_2, x_3).The MLE found is $\lambda = 0.2125$ (Fig. 9).

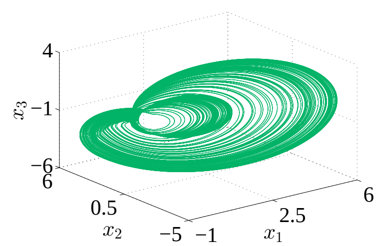


Fig. 7. Chaotic attractor without equilibria in the space.

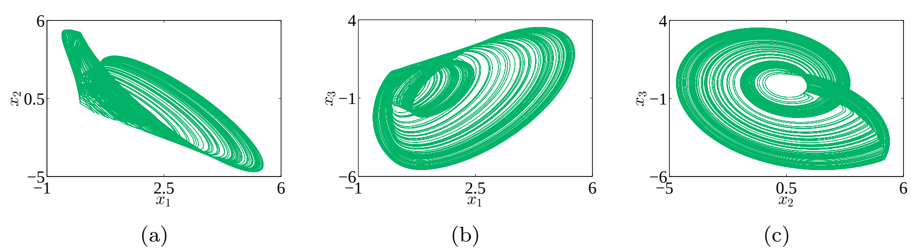


Fig. 8. Projections of the chaotic attractor onto the plane: (a) (x_1, x_2), (b) (x_1, x_3) and (c) (x_2, x_3).

5. Conclusion

In this paper necessary and sufficient conditions are given to construct a dynamical system without equilibria based on linear affine transformations. Also a construction method inspired on the UDS type II systems is proposed to obtain a piecewise linear system without equilibria which generates a chaotic attractor. The switching surface could be placed at any point in the state space which suggests a possible construction of multiple scroll attractors or multiple attractors in the state space,

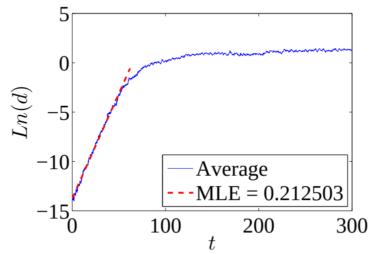


Fig. 9. Calculation of the Maximum Lyapunov Exponent.

but this is under research.

Acknowledgment

R.J. Escalante-González master's degree student of control and dynamical systems at IPICYT thanks CONACYT for the scholarship granted (Register number 337188). E. Campos-Cantón acknowledges CONACYT (Mexico) for the financial support through project No. 181002.

References

1. X. Wang and G. Chen, *Nonlinear Dyn* **71**, pp. 429–436 (2013).
2. C. Li, J. C. Sprott, W. Thio, and H. Zhu, *IEEE Transactions on Circuits and Systems—Part II:Express Briefs* **61**, no. 12, pp. 977–981 (2014).
3. B. Aguirre-Hernández, E. Campos-Cantón, J. A. ópez Renteria, and E. D. González, *Chaos, Solitons and Fractals* **71**, no. 1, pp. 100–106 (2015).
4. E. Campos-Cantón, *International Journal of Modern Physics C* **27-1**, pp. 1 650 008–1–11 (2016).
5. Q. Yang and G. Chen, *Int. J. Bifurcat. Chaos* **18-5**, pp. 1393–1414 (2008).
6. X. Wang and G. Chen, *Commun Nonlinear Sci Numer Simulat* **17**, pp. 1264–1272 (2012).
7. J. Sprott, *The American Physical Society, Physical Review E*, **50-2**, pp. 647–650 (1994).
8. S. Jafari, J. Sprott, and S. M. R. H. Golpayegani, *Physics Letters A* **377**, pp. 699–702 (2013).
9. Z. Wei, Dynamical behaviors of a chaotic system with no equilibria, *Physics Letters A* **376**, pp. 102–108 (2011).
10. D. Cafagna and G. Grassi, *Commun Nonlinear Sci Numer Simulat* **19**, pp. 2919–2927 (2014).
11. G. Huerta-Cuellar and A. N. P. E. Jiménez-López, E. Campos-Cantón, *Commun Nonlinear Sci Numer Simulat* **19**, pp. 2740–2746 (2014).
12. M. García-Martínez, L. J. O. nón García, E. Campos-Cantón, and S. Čelikovský, *Applied Mathematics and Computation* **270**, no. 1, pp. 413–424 (2015).
13. E. Campos-Cantón, R. Femat, and G. Chen, *Chaos* **22**, no. 033121, pp. 1–7 (2012).
14. A. Wolf, J. SWift, H. Swinney, and J. Vastano, *Physica D* **16**, pp. 285–317 (1985).
15. M. Rosenstein, J. Collins, and C. de Luca, *Physica D* **65**, pp. 117–134 (1993).