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Non-classical point of view of the Brownian motion generation via Fractional deterministic model

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In this paper we present a dynamical system based on the Langevin equation without stochastic term and using fractional derivatives that exhibits properties of Brownian motion, *i. e.*, a deterministic model to generate Brownian motion is proposed. The stochastic process is replaced by considering an additional degree of freedom in the second order Langevin equation. Thus it is transformed into a system of three first order linear differential equations, additionally α -fractional derivative are considered which allow us obtain better statistical properties. Switching surfaces are established as a part of fluctuating acceleration. The final system of three α -order linear differential equations does not contain a stochastic term, so the system generates motion in a deterministic way. Nevertheless, from the time series analysis, we found that the behavior of the system exhibits statistics properties of Brownian motion, such as, a linear growth in time of the mean square displacement, a Gaussian distribution. Furthermore, we use the detrended fluctuation analysis to prove the Brownian character of this motion.

Keywords: Fractional Brownian motion; deterministic Brownian motion; unstable dissipative systems; DFA analysis.

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1. Introduction

Brownian motion has been studied since Robert Brown's work on the fertilization process of flowers¹. One of the first to describe Brownian motion as uncorrelated stochastic process was Thorvald N. Thiele², in 1880, in a paper on the method of least squares. At that time the nature of the Brownian motion was uncertain therefore there were many open questions about the particle interactions with their environment. Until 1900, Louis Bachelier, in his PhD thesis applied Brownian motion to the stock and option market fluctuations³. The study of Brownian motion

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was continued by Albert Einstein, who discussed Brownian motion in his work from the point of view of the molecular kinetic theory of heat⁴. It is worth mentioning that Einstein was unaware of the previous work on the subject and he gave the first mathematical description of a free particle Brownian motion. Later, Smoluchowski⁵ brought the solution to the problem and attracted the attention of physicists on the problem.

In 1908, Langevin⁶ obtained the same result as Einstein, using a macroscopically description based on the Newton's second law, *i.e.*, the dynamical model is based on second-order differential equation with a stochastic term. He said that his approach is “infinitely simplest” because it was much simpler than the proposed by Einstein. Since the pioneering work of Langevin to model Brownian motion using a stochastic term in the mathematical model, many papers have been devoted to the description of this phenomenon⁵, where features of this behavior have been defined. According with Van Hove⁷ method Prigogine and Grigolini^{8,9} “believed” that Brownian motion can be derived from deterministic Hamiltonian models of classical mechanics. But this theory, Hamiltonian models of classical mechanics and the Liouville equation, involves grave difficulties as the enormous degrees of freedom and the long times with respect to the duration process. However the first deterministic model of Brownian motion was not proposed until Trefán *et. al.*¹⁰. It is important to mention that two approaches can be distinguished: The former, dynamical models are based on a stochastic term and the second, without a stochastic term, *i.e.* a deterministic Brownian motion.

The idea of deterministic Brownian motion has been discussed in hydrodynamic and chemical reactors with oscillatory behavior, where the dynamic is completely deterministic and it is always referred as “microscopic chaos”. In this sense Trefán *et. al.*¹⁰ and Huerta-Cuellar *et. al.*¹¹ have proposed a Brownian motion models based on Langevin equation. The idea of Trefán *et. al.* consist of replacing the stochastic term by a discrete dynamical system, which generates pseudo-random numbers; the process drives a Brownian particle and has “statistical” properties that differ markedly from the standard assumption of Gaussian statistics, this is because of the discrete dynamical system used by Trefán has a “U-shaped probability” distribution as many chaotic discrete dynamical systems. The approach of Huerta-Cuellar *et. al.* is in the same spirit that Trefán *et. al.* but now the stochastic term is controlled by the jerky equation, *i.e.*, by adding an additional degree of freedom to the Langevin equation it is possible to transform it into a system of three linear differential equations without a stochastic term; this process displays “like-Gaussian probability” distributions of system variables, which differ from the standard assumption. Based on this approach we use fractional derivatives in this work and get a deterministic Brownian motion model with a greatly improved Gaussian probability distributions of all the variables of the system.

The Langevin equation has been used in many areas, such as modeling the evacuation processes¹², photoelectron counting¹³, analyzing the stock market¹⁴, studying the fluid suspensions¹⁵, deuteron-cluster dynamics¹⁶, protein dynamics¹⁷,

self organization in complex systems¹⁸, etc. For other applications of the Langevin equation in physical chemistry and electrical engineering, one can refer to Ref. 19. The classical study of the Brownian motion via the Langevin equation is under the hypothesis that the process is a Markov process, *i.e.*, the random forces modeled by the stochastic term are independent so the process does not have memory. Although the Langevin equation plays an important role in many fields, there are still some behaviors such as anomalous diffusion (superdiffusion and subdiffusion), power law, long-range interactions that the Langevin equation can not well describe them. Therefore, various fractional Langevin equations were proposed in Ref. 19, 20, 21. In this way the fractional Langevin equation can capture the aforementioned features that the Langevin equation can not achieve.

Brownian motion behavior is characterized by specific properties, such as a linear growth in time of the mean square displacement, Gaussian probability distributions of system variables, an exponential in time decay of the positional autocorrelation function. This last property can be characterized by the detrended fluctuation analysis (DFA) which was developed by Peng *et al.*²². So the DFA technique characterizes the correlation properties of a signal. In this paper a deterministic fractional model to generate Brownian motion based on Langevin equation and jerky equation is proposed in the same way that Ref. 11, its behavior is characterized by time series analysis via DFA.

Remark 1. Nowadays the term Brownian motion is used to refer a time series that exhibit the statistical properties of Brownian motion phenomenon. Thus, it is said that a model generates Brownian motion if the time series obtained exhibit the statistical properties of Brownian motion.

This paper is organized as follows: In section 2, some basic concepts of fractional differential equations theory are introduced as differential operators, stability theory in fractional systems and a numerical method to solve fractional systems. In section 3, the proposed model via jerk equation and fractional derivatives is presented with its stability analysis. Section 4 contains the numerical results obtained with the proposed model, and the statistical properties of the time series by means of DFA analysis to confirm the Brownian behavior. Also the maximum Lyapunov exponent is computed. Finally conclusions are drawn in Section 5.

2. Basic concepts in fractional calculus

Fractional derivatives and integrals are generalizations of integer ones. Nevertheless, in literature we can find many different definitions for fractional derivatives Ref. 23, 24, 25, 26 being the Riemann-Liouville and the Caputo definitions the most reported²³. The fractional derivative of Riemann-Liouville is defined as:

$$D_a^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt, \quad (1)$$

and the Caputo definition is described by:

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$$D_0^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \quad (2)$$

with $n = \lceil \alpha \rceil$, and Γ is the Gamma function which is defined as:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \quad (3)$$

For instance, in fractional order systems the stability region depends on the derivative order α as it is depicted in Fig. 1 of Ref. 27. It is important to note that the stability of an equilibrium point can be controled by means of the derivative order α , for example, a saddle hyperbolic equilibrium point of an integer system can be transformed to a stable equilibrium point by changing the system derivative order α . This is an important consideration for designing a mathematical model of Brownian motion because we are interested in unstable dynamics.

A general commensurate fractional order time invariant system is described as follows:

$$D_0^{n_k} x(t) = f(t, x(t), D_0^{n_1} x(t), D_0^{n_2} x(t), \dots, D_0^{n_{k-1}} x(t)), \quad (4)$$

subject to initial conditions

$$x^{(j)}(0) = x_0^{(j)}, \text{ with } j = 0, 1, \dots, \lceil n_k \rceil - 1,$$

where $n_1, n_2, \dots, n_k \in \mathbb{Q}$, such that $n_k > n_{k-1} > \dots > n_1 > 0$, $n_j - n_{j-1} \leq 1$ for all $j = 2, 3, \dots, k$ and $0 < n_1 \leq 1$. Let M be the least common multiple of the denominator of n_1, n_2, \dots, n_k and set $\alpha = 1/M$ and $N = Mn_k$. Then accordingly with theorem 8.1 given in Ref. 23 the equation (4) is equivalent to the following system of equations.

$$\begin{aligned} D_0^\alpha x_0(t) &= x_1(t), \\ D_0^\alpha x_1(t) &= x_2(t), \\ &\vdots \\ D_0^\alpha x_{N-2}(t) &= x_{N-1}(t), \\ D_0^\alpha x_{N-1}(t) &= f(t, x_0(t), x_{n_1/\alpha}(t), \dots, x_{n_{k-1}/\alpha}(t)), \end{aligned} \quad (5)$$

with initial conditions

$$x_j(0) = \begin{cases} x_0^{(j/M)}, & \text{if } j/M \in \mathbb{N} \cup \{0\}, \\ 0, & \text{other case.} \end{cases}$$

Furthermore, this linear time invariant system can be expressed in a matrix form as follows:

$$\frac{d^\alpha \mathbf{x}(t)}{dt^\alpha} = A\mathbf{x}, \quad (6)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n \times n}$ is a linear operator, and α is the fractional commensurate derivative order $0 < \alpha < 1$. The stability of this kind of systems is stated as follows:

- **Asymptotically stable:** The system (6) is asymptotically stable if and only if $|\arg(\lambda)| > \frac{\alpha\pi}{2}$ for all eigenvalues (λ) of matrix A . In this case, the solution $x(t) \rightarrow 0$ as $t \rightarrow \infty$.
- **Stable:** The system (6) is stable if and only if $|\arg(\lambda)| \geq \frac{\alpha\pi}{2}$ for all eigenvalues (λ) of matrix A obeying that the critical eigenvalues must satisfy $|\arg(\lambda)| = \frac{\alpha\pi}{2}$ and have geometric multiplicity of one.

The interest is to have unstable dynamics in order to generate Brownian motion, thus the system is restricted to have at least one eigenvalue in the unstable region, *i.e.*, the system (6) is unstable if and only if $|\arg(\lambda)| < \frac{\alpha\pi}{2}$ for at least one of its eigenvalues (λ) of matrix A .

Additionally, the system given by (6) with equilibrium point at the origin can be generalized by affine linear system as follows:

$$\frac{d^\alpha \mathbf{x}(t)}{dt^\alpha} = A\mathbf{x} + B, \quad (7)$$

where $B \in \mathbb{R}^n$ is a constant vector and $A \in \mathbb{R}^{n \times n}$ is a nonsingular linear operator. Now the equilibrium point $p \equiv (x_1^*, x_2^*, \dots, x_N^*)^T = -A^{-1}B$ of a general commensurate fractional order affine linear system (7) with fractional order $0 < \alpha < 1$, is saddle equilibrium point if its eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_\kappa, \lambda_{\kappa+1}, \dots, \lambda_n$ of its Jacobian matrix evaluated at the equilibrium point fulfill the following condition.

$$\begin{aligned} |\arg(\lambda_i)| &> \frac{\alpha\pi}{2} \quad \text{with } i = 1, 2, \dots, \kappa, \\ |\arg(\lambda_i)| &< \frac{\alpha\pi}{2} \quad \text{with } i = \kappa + 1, \kappa + 2, \dots, n. \end{aligned} \quad (8)$$

Note that we are interested in working with unstable systems, *i.e.*, systems that do not fulfill the locally asymptotic stable condition.

$$\min |\arg(\lambda_i)| > \frac{\alpha\pi}{2}, \quad \text{for } i = 1, 2, \dots, n. \quad (9)$$

2.1. Numerical method to solve fractional differential equations

There are not methods that can provide analytically the exact solution of some fractional differential equation as it also happens in integer-order systems, therefore it is necessary to use numerical methods. The Adams-Bashforth-Moulton (ABM)

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method, a predictor-corrector scheme, was reported in Ref. 28 and it is used to obtain the time evolution of fractional systems. The algorithm is a generalization of the classical Adams-Bashforth-Moulton integrator that is well known for the numerical solution of first-order problems as switching systems²⁷. We now present the method that is well understood and that has been proven to be efficient in many practical applications^{29,30}.

Consider the fractional differential equation that is described by (4) as follows:

$$\begin{aligned} D^\alpha x(t) &= f(t, x(t)), & 0 \leq t \leq T; \\ x^{(k)}(0) &= x_0^{(k)}, & k = 0, 1, \dots, n-1. \end{aligned} \quad (10)$$

We assume the function f is such that a unique solution exists on some interval $[0, T]$, and assume that we are working on a uniform grid $\{t_j = jh : j = 0, 1, \dots, N\}$ with some integer N and $h = T/N$.

The solution of (10) is given by an integral equation of Volterra type as

$$x(t) = \sum_{k=0}^{[\alpha]-1} x_0^k \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} f(z, x(z)) dz, \quad (11)$$

$$x_{k+1} = \sum_{k=0}^{[\alpha]-1} x_0^k \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^k a_{j,k+1} f(t_j, x_j) + a_{k+1,k+1} f(t_{k+1}, x_{k+1}^P) \right), \quad (12)$$

where

$$a_{j,k+1} = \begin{cases} \frac{h^\alpha}{\alpha(\alpha+1)} (k^{\alpha+1} - (k-\alpha)(k+1)^\alpha), & \text{if } j=0; \\ \frac{h^\alpha}{\alpha(\alpha+1)} ((k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}), & \text{if } 1 \leq j \leq k; \\ \frac{h^\alpha}{\alpha(\alpha+1)}, & \text{if } j=k+1. \end{cases} \quad (13)$$

With the predictor structure given as follows.

$$x_{k+1}^P = x(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^k b_{j,k+1} f(t_j, x_j), \quad (14)$$

and

$$b_{j,k+1} = \frac{h^\alpha}{\alpha} ((k+1-j)^\alpha - (k-j)^\alpha). \quad (15)$$

The error of this approximation is given by

$$\max_{j=0,1,\dots,N} |x(t_j) - x_h(t_j)| = O(h^p). \quad (16)$$

3. Deterministic model to generate Brownian motion

The onset of Brownian motion is a suspending particle in fluids. The motion of this particle occurs due to collisions between molecules of the fluid, and considering that in each collision a molecule changes its velocity in small amount. This fact is because of the suspended particle under normal conditions suffers about 10^{21} collisions per second, so the accumulated effect results to be considerable. Each of these collisions is always determined by the last event which is produced by physical interactions in the system. Since it can be thought that each collision produces a kink in the path of the particle, one can not hope to follow the path in any detail, *i. e.*, the details of the path are infinitely fine. Thus the Brownian particle make a fluctuating movement. So stochastic models of Brownian motion follow the average motion of a particle, not a particular path of a particule.

The stochastic theory of Brownian motion of a free particle (in the absence of an external field of force) is generally governed by the Langevin equation as follows.

$$\begin{aligned} \frac{dx}{dt} &= v, \\ \frac{dv}{dt} &= -\gamma \frac{dx}{dt} + A_f(t), \end{aligned} \quad (17)$$

where x denotes the particle position and v its velocity. According to this equation, the influence of the surrounding medium on the particle motion can be splitted into two parts. The first term $-\gamma \frac{dx}{dt}$ stands for the friction applied to the particle, it is assumed that the friction term is according with the Stokes law which states that the friction force $6\pi a\sigma v/m$ decelerates a spherical particle of radius a and mass m . Hence, the friction coefficient is given as follows:

$$\gamma = 6\pi a\sigma/m, \quad (18)$$

where σ denotes the viscosity of the surrounding fluid.

The second term $A_f(t)$ is the fluctuation acceleration which provides a stochastic character of the motion and depends on the fluctuation force $F_f(t)$ as $A_f(t) = F_f(t)/m$, where m is the particle mass.

Two principal assumptions were made about this stochastic term $A_f(t)$ in order to produce Brownian motion:

- $A_f(t)$ is independent of x and v .
- $A_f(t)$ varies extremely fast as compared with the variation of v .

The latter assumption implies that there exists a time interval Δt during which the variations in v are very small. Alternatively, we may say that though $v(t)$ and

$v(t + \Delta t)$ are expected to differ by a negligible amount, no correlation between $A_f(t)$ and $A_f(t + \Delta t)$ exists due to it is a stochastic term.

The idea of use fractional calculus is based in that trajectories with self-similar (fractal) structure in systems that lifetime of any state of regular motion is also random, as is the case of Brownian motion, it becomes possible to describe it by introducing fractional integrals and derivatives. In this sence various fractional Langevin equations have been proposed to generate Brownian motion, see Ref. 19, 20, 21. These models differ from the usual Langevin equation by replacing the derivatives with respect to time by the fractional derivative of α order.

$$\frac{d^\alpha v}{dt^\alpha} = -\gamma \frac{d^\alpha x}{dt^\alpha} + A_f(t), \quad (19)$$

where x denotes the particle position and v its velocity. We can rewrite the fractional Langevin equation (19) in two fractional differential equations by a change of variables,

$$\begin{aligned} D_0^\alpha x &= v, \\ D_0^\alpha v &= -\gamma v + A_f(t), \end{aligned} \quad (20)$$

where D_0^α is the Caputo derivative operator.

In order to generate a deterministic model of Brownian motion an additional freedom degree is added to the system (20). And the stochastic term is avoid by means of replacing the fluctuating acceleration $A_f(t)$ with the jerk equation. The aforementioned statement is reached by considering dynamical systems which are defined by a class of affine linear systems given by:

$$\begin{aligned} \dot{\mathbf{X}} &= \mathbf{A}\mathbf{X} + \mathbf{B}(\mathbf{X}), \\ \mathbf{B}(\mathbf{X}) &= \begin{cases} \mathbf{B}_1, & \text{if } \mathbf{X} \in \mathcal{D}_1; \\ \mathbf{B}_2, & \text{if } \mathbf{X} \in \mathcal{D}_2; \\ \vdots & \vdots \\ \mathbf{B}_k, & \text{if } \mathbf{X} \in \mathcal{D}_k. \end{cases} \end{aligned} \quad (21)$$

The class of affine linear systems considered here, is that that presents oscillations (“one-spiral” trajectories called scroll) around equilibria due to stable and unstable manifolds E^s and E^u , respectively. These manifolds are defined in a way that $\vartheta = \{\vartheta_{1,2,3}\}$ is a set of column eigenvectors such that $\mathbf{A}\vartheta_i = \lambda_i\vartheta_i$ with $i = 1, 2, 3$; $E^s = \text{Span}\{\vartheta_1\}$ and $E^u = \text{Span}\{\vartheta_{2,3}\}$.

If the linear part of the system given by eq. (21) has a saddle equilibrium point, it is possible to generate multi-scrolls by means of a piecewise linear (PWL) system under the following considerations for the vector \mathbf{B} .

The affine vector \mathbf{B} must be driven by a switching function that changes depending on which domain $\mathcal{D}_i \subset \mathbb{R}^3$ with $\mathbb{R}^3 = \cup_{i=1}^k \mathcal{D}_i$ the trajectory is located. The

equilibria of system (21) are given by $\mathbf{X}_i^* = -\mathbf{A}^{-1}\mathbf{B}_i$, with $i = 1, \dots, k$, and each vector \mathbf{B}_i of the system is considered in order to generate a multi-scroll.

The idea of the method lies on defining vectors \mathbf{B}_i in order to assure scrolls of a class of dynamical systems in \mathbb{R}^3 with oscillations within the equilibria. In a way that for any initial condition $\mathbf{X}_0 \in \mathbb{R}^3$, the system given by eq. (21) induces in the phase space \mathbb{R}^3 the flow $(\phi^t)_{t \in \mathbb{R}}$. Thus, each initial condition \mathbf{X}_0 generates a trajectory given by $\phi^t(\mathbf{X}_0) : t \geq 0$ which is oscillating in scrolls after defining at least two vectors \mathbf{B}_1 and \mathbf{B}_2 , as it is described in Ref 32.

This class of systems can display various multi-scroll as a result of a combination of several unstable “one-spiral” trajectories. The number of scrolls depends on the number of vectors B_i , $i = 1, \dots, k$ introduced in the system, thus the equilibrium points are given by $\mathbf{X}_i^* = -\mathbf{A}^{-1}\mathbf{B}_i$ and which trajectories oscillate around them. The characteristic of Brownian motion is that it needs an infinite number of vectors B_i because it is a free movement without limit. This is a consequence of an important feature of this kind of systems. A convenient approach to build the matrix \mathbf{A} and vector \mathbf{B} is based on the linear ordinary differential equation (ODE) given by the jerk form: $\ddot{x} + a_{33}\dot{x} + a_{32}x + a_{31}x + \beta = 0$.

We change the stochastic term A_f for a new variable z which is defined by a third order differential equation as it was reported in Ref. 11. The proposed variable z acts as fluctuating acceleration, and produces a deterministic dynamical motion without stochastic term but the behavior present the statistics features of Brownian motion as it was showed in previous work¹⁰. However, in our model the fluctuation acceleration has a direct dependence on the position, velocity and acceleration due to the jerky equation involved³¹. When a particle is moving in a fluid, friction and collisions with other particles, existing in the environment, necessarily produces changes in the motion velocity and acceleration; all these changes are considered in the jerky equation. Without loss of generality, we define our approach based on a unstable dissipative systems (UDS)^{32,33} as follows:

$$\begin{aligned} D^\alpha x &= v, \\ D^\alpha v &= -\gamma v + z, \\ D^\alpha z &= -a_1 x - a_2 v - a_3 z - a_4(x), \end{aligned} \tag{22}$$

where $a_i \in \mathbb{R}$ are constant parameters, with $i = 1, 2, 3$, and $a_4(x) \in \mathbb{R}$ acts as a constant piecewise function, *i.e.*, a step function.

The first two equations of the fractional Langevin equation (20) are derived from the Langevin equation (17) with a little change: the stochastic term is replaced by a deterministic term in the same spirit that^{32,33}. Now we construct switching surfaces (SW), see Figure 1. Without loss of generality, the SW are defined by perpendicular planes to the x axis, so domains are defined between these SW's which are considered as edges of each domain. In case of real systems, SW can be seen as multi-well potential with short fluctuation escape time where each domain

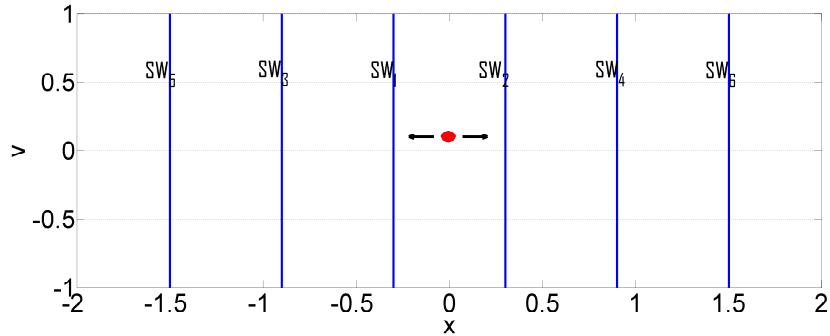


Figure 1. Projection of SW perpendicular on to the plane (x, v) (Blue lines), the red dot depict a Brownian particle that moves along one dimension x . The SW delimit each potential region and when the particle cross it represent a potential change on the particle

defined by SW's preserves its unstable behavior according to lineal part of system. The parameter a_4 is defined as follows:

$$a_4(x) = c_1 \text{round}(x/c_2), \quad (23)$$

with $c_1, c_2 \in \mathbb{R}$ are constants. Here, the $\text{round}(x)$ function will be implemented to generate an infinitely number of SW's in order to get unbounded time series. The function will be defined as follows:

$$\text{round}(x) = \begin{cases} \lceil x - 1/2 \rceil, & \text{for } x < 0; \\ \lfloor x + 1/2 \rfloor, & \text{for } x \geq 0. \end{cases} \quad (24)$$

4. Numerical results

In this section, we numerically investigate the long term behavior of the solutions of equation (22) by considering different derivative orders and the following parameter values: $\gamma = 7 \times 10^{-5}$; $a_1 = 1.5$; $a_2 = 1.2$; $a_3 = 0.1$; $C1 = 0.9$ and $C2 = 0.6$. It is worth to mention that these parameter values are the same to those used to generate Brownian behavior in Ref. 11 by considering an integer order system. To study Brownian motion generated by Eq. (22), we fix the parameters values and explore different derivative orders (α values). The stability of fractional order systems is given by Eq.(9), so the local behavior near the equilibrium point is determined by the Jacobian of the system (22) which has the following spectrum $\Lambda = \{\lambda_1 = -0.8304, \lambda_2 = 0.3651 + 1.2935i, \lambda_3 = 0.3651 - 1.2935i\}$. This spectrum determines the critical value of derivative order $\alpha_c \approx 0.8249$ to get the system (22) to be stable,

$$\alpha < \frac{2}{\pi} \min |\arg(\lambda_i)| \approx 0.8249. \quad (25)$$

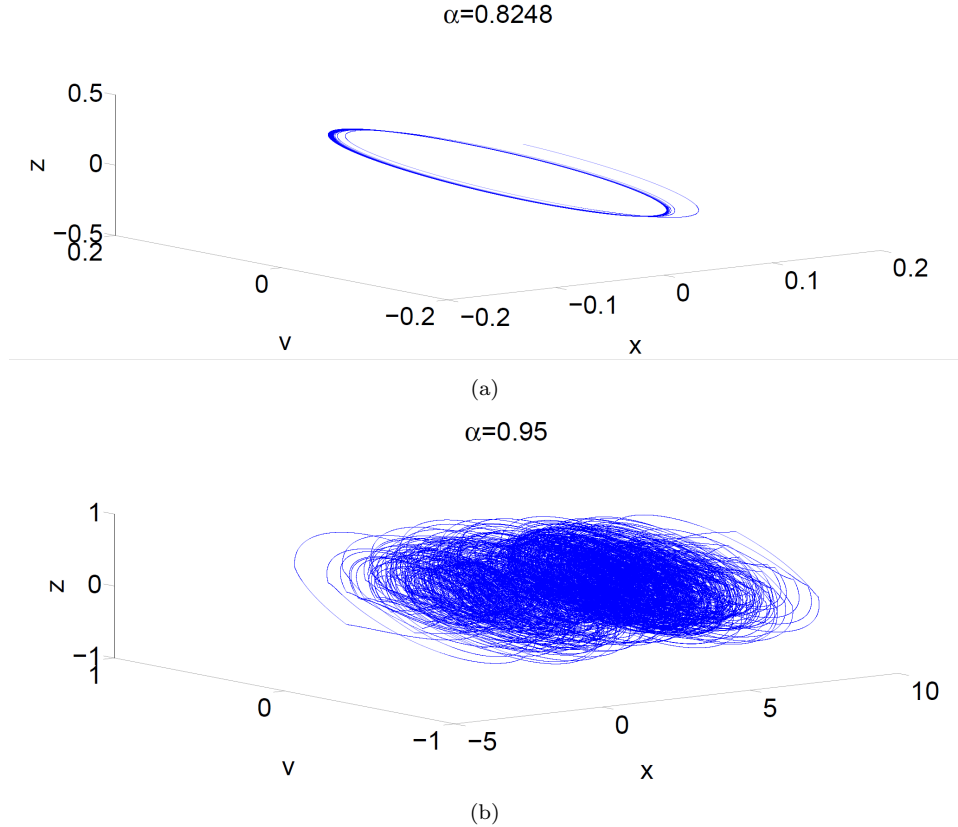


Figure 2. Solutions of the system (22) on phase space for different α values: (a) $\alpha = 0.8248$, (b) $\alpha = 0.9500$.

Accordingly with the aforementioned comments, in order to preserve the system (22) to be unstable to generate oscillatory behavior, we consider $\alpha > \alpha_c$. Figure 2 (a) shows a solution of the system (22) for $\alpha = 0.8248$ and initial condition $(x_0, v_0, z_0)^T = (1.0, 1.0, 1.0)^T$, this derivative order value results in a stable behavior due to $\alpha < \alpha_c$. On the other hand, Figure 2 (b) shows a solution of the system (22) for $\alpha = 0.95$ and the same initial condition, but now this derivative order value results in an unstable behavior due to $\alpha > \alpha_c$. Numerical simulations are performed using the Adams-Bashforth-Moulton algorithm by exploring different α values.

Figure 3 shows a time series of a particle position for $\alpha = 0.95$ where characteristic behavior of Brownian motion can be clearly seen. The trajectory of Brownian motion is determined by initial conditions and the values of the aforementioned parameters. Since there are many steps with short time duration and few steps with long time duration, the predicted mean square displacement in short times is observed.

Figure 4 shows the statistical properties obtained for the time series of the

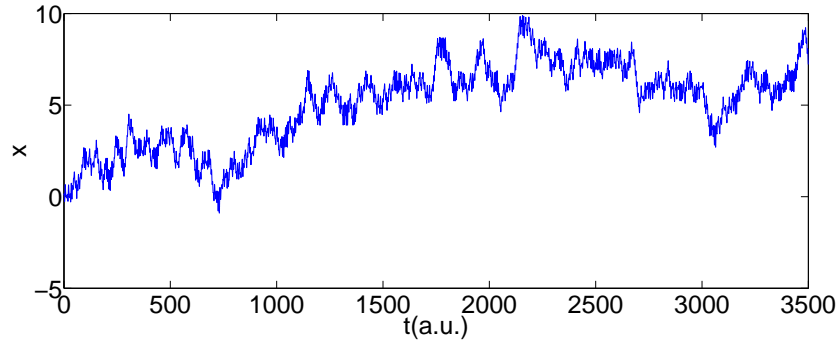


Figure 3. Time series x of deterministic Brownian motion by the proposed model given by (22) with $\alpha = 0.95$.

system (22). Figure 4 (a) displays the linear growth in time of the mean square displacement predicted, regarding with the traditional Brownian motion zero-mean Gaussian probability distributions Figures 4 (b), (c), (d) show the particle probability distributions of displacement, velocity and acceleration, respectively, in which one can see that the obtained distributions of the motion generated by our system have a better Gaussian approximation than that obtained with integer order in Ref. 11,

As it is known, strong sensitivity to initial conditions is an essential characteristic inherent of chaos. In the Brownian motion case also is essential the strong sensibility to initial conditions, and both can be characterized by positive leading Lyapunov exponent. Even though both behaviors can be characterized by Lyapunov exponent the dynamic is completely different; Brownian motion as noise do not form an attractor in the phase space, *i.e.*, they are unbounded trajectories. Brownian and noise trajectories tend to infinity, while the chaotic dynamic generate an attractor localized within a certain area of phase space, so chaotic trajectories are bounded. In figure 5 the maximum Lyapunov exponent obtained, according with Ref. 34, of the proposed system is shown to confirm the noise behavior of Brownian motion.

Finally, the correlation property of the signals generated by (22) is characterized by the detrended fluctuation analysis (DFA) which was developed by Peng *et al.*²². So the DFA technique help us to ensure that the proposed system generates Brownian motion, we apply the DFA evaluation method to the time series obtained with the values of the aforementioned parameters. The DFA is an important tool for the detection of long-range auto-correlations in time series with non-stationarities. The DFA is based on the random walk theory which consists on a scaling analysis. The main advantages of the DFA over many other methods are that it allows the detection of long-range correlations of a signal embedded in seemingly nonstationary time series, and also avoids the spurious detection of apparent long-range correlations that are an artifact of non-stationarity.

The DFA procedure consists on the next four steps

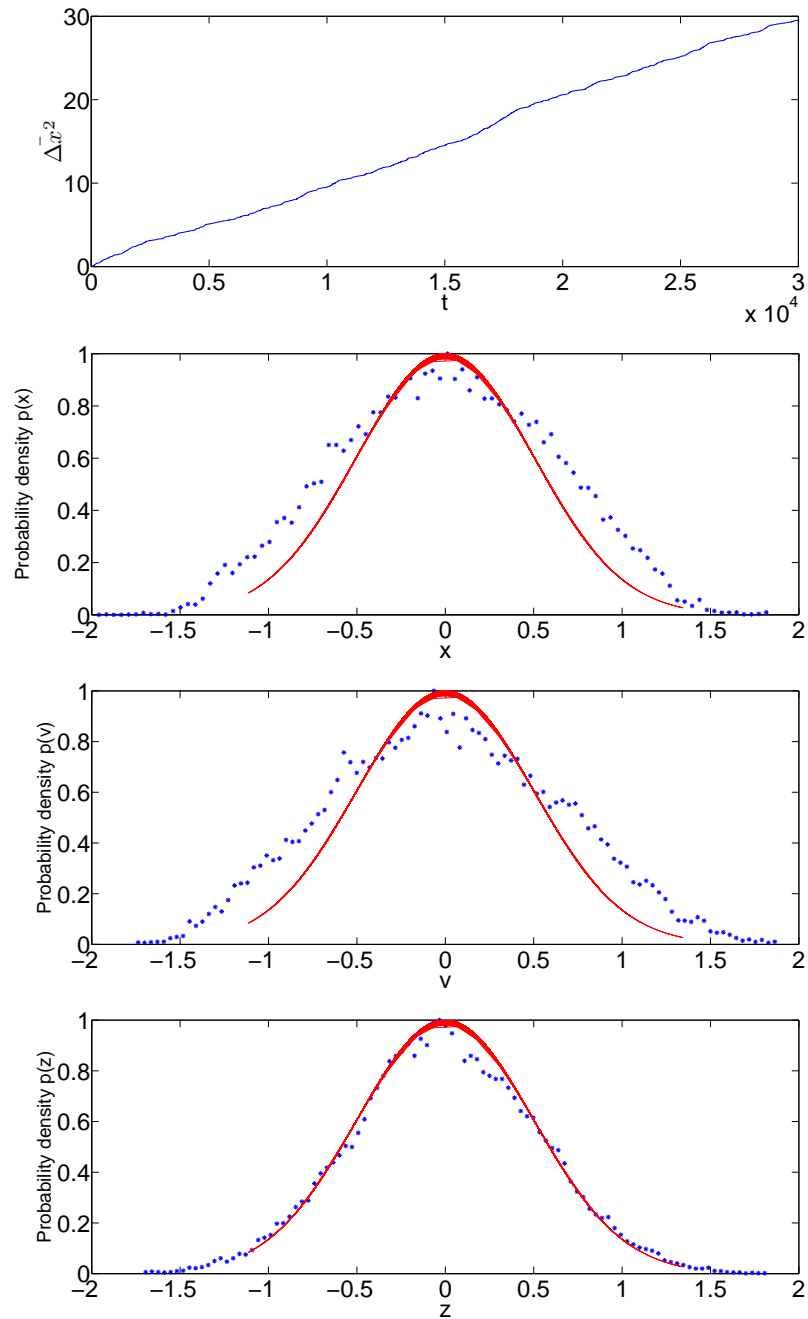


Figure 4. Statistical properties obtained of the system (22). (a) shows a linear growth of the mean square displacement. Probability density obtained from the motion showed by normalized histogram approximation (dotted blue curve), for displacement (b), velocity (c), and acceleration (d), compared with the approximation obtained by evaluate the time series into the Gaussian function (dotted red curve).

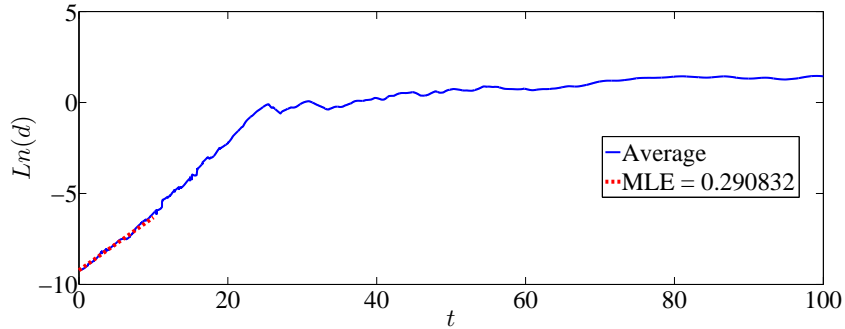


Figure 5. Maximum Lyapunov exponent obtained of the proposed system (22) with $\alpha = 0.95$.

- * Compute the time series mean \bar{x} .
- * The interbeat interval time series (of total length N) is first integrated

$$y(k) = \sum_{i=1}^k [x(i) - \bar{x}]. \quad (26)$$

- * The integrated time series is divided into boxes of equal length n , the local trend is obtained by mean-squares and is removed to each box.
- * The rootmean-square fluctuation of this integrated and detrended time series is calculated by

$$f(n) = \sqrt{\frac{1}{N} \sum_{k=1}^N [y(k) - y_n(k)]^2}. \quad (27)$$

- * The fluctuations can be characterized by a scaling exponent η , the slope of the line relating $\log F(n)$ to $\log n$

$$f_m(n) \sim n^\eta. \quad (28)$$

When the scaling exponent $\eta > 0.5$ three distinct regimes can be defined as follows.

- 1 If $\eta \approx 1$, DFA defines $1/f$ noise.
- 2 If $\eta > 1$, DFA defines a non stationary or unbounded behavior.
- 3 If $\eta \approx 1.5$, DFA defines Brownian motion or Brownian noise.

The scaling law with $\eta = 1.5528$ revealed by the DFA and showed in Figure 6 confirms the Brownian behavior.

5. Concluding remarks

A deterministic dynamical system that exhibits time series with properties of Brownian motion has been presented. This model considers a fractional derivative order

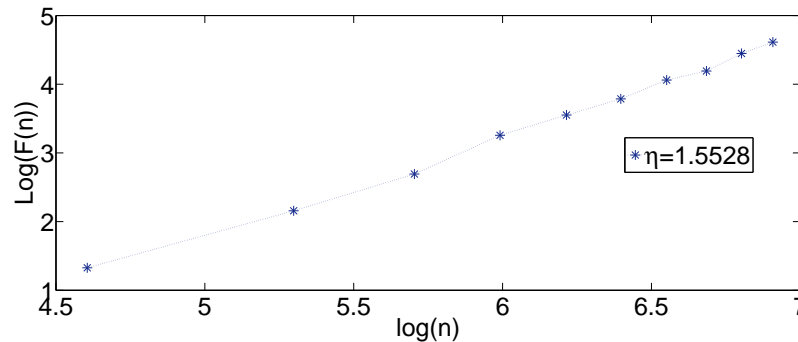


Figure 6. $\eta \approx 1.5$ obtained by DFA indicates the Brownian behavior of the observed motion.

and the jerk equation instead of the stochastic process and integer derivative order in the Langevin equation. These changes modify the Langevin equation by adding an additional degree of freedom, so a three-dimensional model was obtained. By means of considering the fractional derivative order, the statistical properties were improved compared to their integer derivative order. The new variable introduced in the system defined by a third differential equation has a Gaussian probability density distribution which was confirmed with numerical simulations. The statistic analysis of time series obtained with the proposed model displayed typical characteristics of Brownian motion, namely, a linear growth of the mean square displacement, a Gaussian probability density distribution for displacement, velocity and acceleration. Furthermore, the Brownian behavior was confirmed by an approximately 1.5 power law scaling of the fluctuation. These results show that time series obtained of the proposal model fulfill the main characteristics of the Brownian motion via stochastic process.

Based on these results which were obtained by using unstable dissipative system, it can be thought that the methodology presented in this work could be used to construct models under external force fields or behaviors of anomalous diffusion. Additionally the development of adequate realistic models with real experimental time series in order to obtain the Brownian motion in real systems.

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