Synchronization in complex networks under structural evolution

A. Anzo[a], J. G. Barajas-Ramírez[a]

[a]*IPICyT, División de Matemáticas Aplicadas,*
Camino a la Presa San José 2055
Col. Lomas 4a Secc. C.P.78216,
San Luis Potosí, S.L.P., México.
Email: {andres.anzo; jgbarajas}@ipicyt.edu.mx

Abstract

We investigate the effects of structural evolution on the stability of synchronized behavior in complex networks. By structural evolution we mean processes that change the topology of the network. In particular, we consider structural evolution as two simultaneous processes: on the one hand, the topology changes according to an arbitrary switching law among a set of admissible patterns of connection; on the other hand, the strength of connection evolves according to an adaptive law. Our results show that by constraining the admissible patterns of connection, and using an adaptive law based on the difference between the nodes, we can guarantee the stability of the synchronized solution of the network despite structural changes. Additionally, we extend our results by considering alternative structural evolution processes, namely, a node-based adaptive strategy and a resetting switching law. We illustrate our results with numerical simulation.

Keywords: Complex networks, Adaptive processes, Synchronization, Switching topology.

1. Introduction

The study of complex networks has attracted considerable attention from different areas of science and technology. This is mainly due to its potential applicability to the analysis, modeling, and control of real-world networks like the Internet, power grids, transportation networks, among many others (see [1, 2, 3, 4] and reference therein). One of the most significant aspects of network science is the structural analysis of networks. In particular, the interplay between its structural features and dynamical properties. In this sense, one of the main topics of research is the effect of network topology on the synchronization of dynamical networks [5, 6, 7]. A basic hypothesis in the majority of these studies is that the network topology is fixed. However, constant topological change is an intrinsic part of real-world networks, that is, connections and nodes are either lost or added to the network constantly, therefore networks are under constant structural change.

Different models have been proposed to describe structural change in complex networks. One of the earliest is the Barabási-Albert model, which describes network growth as a two step iterative algorithm, namely, growth and preferential attachment [8]. Another way to model structural change is to describe its effects on the network's functionality. Examples of this type of modeling include [9, 10, 11] where the robustness-fragility feature and cascading failures of complex networks are investigated. However, these models consider structural change as an external event, that occurs in a sudden and isolated manner.

In this paper, we consider structural change as an intrinsic part of the network dynamics, which we refer to as structural evolution. In particular, we model structural evolution as two simultaneous processes: one describing changes in the pattern of connection, and another describing the way in which connection strength is updated. The first process is modeled as a switching law, where the structural change is generated by switching among a set of admissible patterns of connection. The second process changes the connection strength as an adaptive law, which is defined in terms of the differences between nodes. Our main concern is to determine the effects of structural evolution in the synchronization of the network.

Synchronization of networks with switching topology has been investigated by different authors. For example, Stilwell et al. consider that with a sufficiently fast switching law the stability of the synchronized solution can be determined from the network's average model [12]. Belykh et al derive conditions for synchronization in networks with stochastically switching topology [13]. Another example is the work by Jia and Tang, which investigates the consensus problem in networks where agents are connected and disconnected intermittently [14]. In the sense of our current concern, the works by Hill and collaborators are particularly important; in them the synchronization of networks with switching topology is studied using Lyapunov stability analysis, the authors show that synchronization is achieved for simultaneously triangularizable topologies [15, 16].

Synchronization on network with adaptive connection strength also has been extensively studied in the literature. A simple method for adaptive synchronization of networks is to adjust all the connections strengths at the same time to the same value, using an adaptive law based on the synchronization error [17]. However, this global approach requires dynamical information of the entire network. In this context, in [23] a distributed adaptive strategy is proposed which enable the network to achieve consensus via the updating just a small number of connections strengths. An alternative strategy is to change the connection strength of each node with its neighbors, this is usually called a node-based adaptation strategy [18, 19, 20]. Furthermore, an edge-based adaptation strategy can be defined, in which an adaptive law is defined for each connection in the network in terms of the error between the nodes it connects [21, 22]. On the other hand, in [17, 24, 25] using the direct Lyapunov method, the stability of adaptive synchronization was investigated for these strategies. For our current concern, a particulary significant result is the snapping model proposed by DeLellis et al. [26], in which the adaptive law is defined based on a double-well potential function, such that under very specific dynamical conditions, connections can be effectively cut. However, the snapping model is markedly different from our proposed structural evolution model, since we are describing structural change as two simultaneous dynamical processes intrinsical to the nature of the network.

As shown above, the effects of structural change in the synchronization of dynamical networks have been usually investigated separately. In this paper we propose to study the synchronization of networks under structural evolution, which is modeled as two simultaneous processes. Our main result shows that by constraining the admissible pattern of connections, to a set of connected graphs with the same number of nodes and links; then, an adaptive law defined in terms of synchronization errors can guarantee the synchronization of a network. We extend our results to alternative versions of the structural evolution processes. That is, we investigate the synchronization of a network where the coupling strength is updated under a node-based adaptive strategy; additionally, we consider networks where at each switching event, the nodes that change their connections are reset to zero.

The remainder of the paper is organized as follows: In Section 2, we present some necessary mathematical preliminaries. In Section 3, we analyzed the synchronization of a network under structural evolution, using Lyapunov stability theory. Synchronization of networks under a node-based adaptive strategy and a resetting switching law are investigated in Section 4. Numerical illustrations of our results are given in Section 5. Finally, we present the conclusions or our work.

2. Preliminaries

The dynamics of a network with N linearly coupled identical dynamical systems under the structural evolution is given by:

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} g_{ij}(X,t)x_j(t), \quad i = 1, \ldots, N. \tag{1}$$

where $x_i(t) = [x_i^1(t), \ldots, x_i^n(t)]^\top \in \mathbf{R}^n$ is the state variable of the i-th node; $X = [x_1(t)^\top, x_2(t)^\top, ..., x_N(t)^\top]^\top \in \mathbf{R}^{Nn}$ and $f : \mathbf{R}^n \times R_+ \to \mathbf{R}^n$ is a nonlinear function which describes the dynamics of a single isolated node. The connection between the i-th and j-th nodes is expressed by the matrix function $g_{ij}(X,t) \in \mathbf{R}^{n\times n}$, which in general can be seen as consisting of two time varying parts: a matrix $W_{ij}(t) = diag([w_{ij}^1(t), \ldots, w_{ij}^n(t)]) \in \mathbf{R}^{n\times n}$ describing the strength of connection between the components of the i-th and j-th nodes at the time instant t, and a zero-one valued scalar function $a_{ij}(t) \in \{0,1\}$ which is one when the nodes are connected and zero otherwise ($i \neq j$).

The network described by (1) includes many other instances of dynamical networks that have been studied in the literature. In particular, if we let the matrix function be $g_{ij}(X,t) = ca_{ij}\Gamma$, with c the uniform coupling strength, a_{ij} a constant 0-1 scalar function describing whether the nodes are connected, and Γ a constant matrix of inner coupling between nodes, the basic dynamical network model studied in [5, 6, 7] is obtained. By letting the matrix function be time-varying we can have $g_{ij}(X,t) = ca_{ij}(t)\Gamma$, which corresponds to the case where the pattern of connection changes over time as in [27]. Furthermore, we can have the case where the pattern of connection changes by switching among a set of admissible connection topologies, like in [15, 16]. Alternatively, the time-varying matrix function can be $g_{ij}(X,t) = a_{ij}W_{ij}(t)$, in this case the pattern of connection remains fixed while the inner coupling matrix changes over time; a particularly interesting variation of this network model is the case where the weights are allowed to change according to an adaptive law, as in [17, 18, 20, 26]

In this paper we consider that two processes occur simultaneously: The pattern of connection changes according to a switching law $\sigma(X,t)$, while at the same time, the connection weights are updated following an adaptive law $\phi(X,t)$, that is, the coupling matrix function between the i-th and j-th nodes is given by:

$$g_{ij}(X,t) = a_{ij}(\sigma,t)W_{ij}(\sigma,\phi,t). \tag{2}$$

Here for simplicity of notation, the dependencies of the switching and adaptive laws are not written.

It's worth remarking that the coupling matrix function in (2), can be seen as a model of the real-world situation where, simultaneously, the pattern of connection and the connection weights change. We can take as an example of such systems a power grid, where a spontaneous failure causes the system to switch its pattern of connection while at the same time the power ow on the transmission lines changes. Other examples of this type system include the Internet, trading networks, transportation networks, among many others.

We assume that at each time instant only one connection pattern ($A(\sigma,t)$ $= \{a_{ij}(\sigma,t)\} \in \mathbf{R}^{N\times N}$) is selected from a set of admissible structures $\mathcal{A}$, which has m as its cardinality. Further, we assume that the switching law is a piecewise-constant and continuous-from-the-right function, such that

$$\sigma(X,t) = \tau, \text{ for } t \in [t_k, t_{k+1}), \tag{3}$$

where $k \in \mathbf{Z}_+$, $\tau \in I = \{1,2,\ldots,m\}$, with I the index set of $\mathcal{A}$. Accordingly the switching events occur at the time instant t_k, such that $A(\tau,t_k)$ is the active pattern of connection for the time increment $\mathcal{D}_k = t_{k+1} - t_k$, without loss of generality, we will assume the time increments between switching events to be equal and constant ($\mathcal{D}_k = T$, $\forall k$ with $T > 0$)

Notice that in general, the set of admissible structures $\mathcal{A}$ may include all possible patterns of connection. That is, we can switch between patterns of connection that are radically different. As such, we may have situations where from one instant to the next, the number of nodes and links in the network reduces or increases significantly. To avoid such situations, we constrain the set of admissible structures in the following way. All the elements of $\mathcal{A}$ correspond to connected networks with a given number of nodes and connections. With these restrictions, we ensure that changes will not be too extreme and that in every time instant all nodes remain connected to the network.

There are two classes of switching laws. The first is arbitrary switching, in which at any given time we can select any admissible structure without restrictions. Alternatively, we can have constrained switching, that is, given a current structure the following one is determined by a set of conditions. In this work will focus on the effect of arbitrary switching in the stability of the synchronized state of the network.

In [12, 15, 16] the synchronization problem for networks under arbitrary switching was considered and some conditions to achieve synchronization were established. However, in these works the connection are taken to be unweighted. Unlike the previously investigated situation, we consider that a complementary evolution process occurs simultaneously with the switching topology. Namely, after every switching event the connection strength change its value according to an adaptive law, which is described as follows:

$$W_{ij}(\sigma,\phi,t) = \begin{cases} \text{if} \quad a_{ij}(\sigma,t_k^-) = 1 \text{ and } a_{ij}(\sigma,t_k) = 1 \\ \qquad (a) W_{ij}(t_k) = W_{ij}(t_k^-), \text{ and} \\ \qquad (b) \dot{W}_{ij}(t) = \phi_{ij}(X,t) \\ \\ \text{if} \quad a_{ij}(\sigma,t_k^-) = 0 \text{ and } a_{ij}(\sigma,t_k) = 1 \\ \qquad (a) W_{ij}(t_k) = \epsilon I_n, \text{ and} \\ \qquad (b) \dot{W}_{ij}(t) = \phi_{ij}(X,t) \\ \\ \text{if} \quad a_{ij}(\sigma,t_k^-) = 1 \text{ and } a_{ij}(\sigma,t_k) = 0 \\ \qquad (a) W_{ij}(t_k) = 0, \text{ and} \\ \qquad (b) \dot{W}_{ij}(t) = 0 \\ \\ \text{if} \quad a_{ij}(\sigma,t_k^-) = 0 \text{ and } a_{ij}(\sigma,t_k) = 0 \\ \qquad (a) W_{ij}(t_k) = W_{ij}(t_k^-), \text{ and} \\ \qquad (b) \dot{W}_{ij}(t) = 0. \end{cases} \tag{4}$$

for $t \in [t_k, t_{k+1})$, with $\epsilon > 0$ a small positive scalar, and I_q the q-dimensional identity matrix. We use the notation t_k^- to indicate the instant previous to the k-th switching event.

Equation (4) has two components: (a) are the values of the connection strength at time instant t_k; and (b) is the adaptive law which updates the values of the connection strength. We consider that the adaptive law is either zero or is given by

$$\phi_{ij}(X,t) = \alpha[diag(|x_i^1(t) - x_j^1(t)|, ..., |x_i^n(t) - x_j^n(t)|)], \tag{5}$$

with α the adaptation constant.

Notice, that with each switching event the pattern of connection changes from $A(\tau', t)$ to $A(\tau, t)$ ($\tau', \tau \in I$), where if the i-th and j-th nodes remain connected ($a_{ij}(\sigma, t_k^-) = a_{ij}(\sigma, t_k) = 1$), the connection strength matrix retains its value and keeps being updated according to the adaptive law $\phi_{ij}(X, t)$. Conversely, if the i-th and j-th nodes were disconnected and become connected at the switching event ($a_{ij}(\sigma, t_k^-) = 0$ and $a_{ij}(\sigma, t_k) = 1$), the connection strength matrix has a small nonnegative initial value (ϵI_n) and is updated according to the adaptive law $\phi_{ij}(X, t)$. While if the nodes become disconnected, the strength of connection is set to zero and remains at zero for the entire time interval.

The topology of the network in (1), under the switching and adaptive laws described above, is expressed by the coupling matrix function $G(t) = \{g_{ij}(X, t)\} \in \mathbf{R}^{Nn \times Nn}$ which represents both the patterns of connection and the connection strengths of the network at the time instant t. In particular, we assume that the network structure evolves over time in such a way, that for any fixed time $t_1 > t_0$, $G(t_1)$ can be considered a constant matrix. Furthermore, if the eigenvalues of $G(t)$ are $\lambda_1(t), \lambda_2(t), ..., \lambda_{Nn}(t)$, then $\lambda_i(t)$ is either $\lambda_i(t) \neq 0$ or $\lambda_i(t) = 0$ for $i = 1, ..., Nn$ and for all $t > t_0$ with $t \in \mathcal{D}_k$ for any $k \in \mathbf{Z}_+$.

With respect to the coupling topology of the network we assume that connections are bidirectional, then at any time t, we have $W_{ij}(\sigma, \phi, t) = W_{ji}(\sigma, \phi, t)$, and $a_{ij}(\sigma, t) = a_{ji}(\sigma, t)$ for $\forall i$, $\forall j$ with $i \neq j$. Additionally, for any t the diagonal elements are $W_{ii}(\sigma, \phi, t) = -\sum_{j=1, j\neq i}^{N} W_{ij}(\sigma, \phi, t)$, and $a_{ii}(\sigma, t) = 1$ for $i = 1, 2, ..., N$. Such that,

$$g_{ii}(X, t) = -\sum_{\substack{j=1 \\ j\neq i}}^{N} g_{ij}(X, t) = -\sum_{\substack{j=1 \\ j\neq i}}^{N} a_{ij}(\sigma, t) W_{ij}(\sigma, \phi, t), \tag{6}$$

for $i = 1, \ldots, N$, $\forall t, \forall k$.

Under the restrictions on $\mathcal{A}$, in the admissible patterns of connection there are no isolated nodes, and assuming that all nodes are connected through all their state components (if $a_{ij}(\sigma, t) = 1$ then $w_{ij}^l(t) \neq 0$, for $l = 1, 2, ..., n$ and $\forall t$) we have that at each $t \geq t_0$, $G(t)$ is symmetric and its sum by columns or rows is zero. Then, from the Gerschgorin's circle theorem we have that the eigenvalues of $G(t)$, $\lambda_i(t)$ ($i = 1, ..., Nn$) are less or equal to 0, with 0 an eigenvalue of n multiplicity.

Definition 1:

The switching and adaptive network (1) is said to (asymptotically) achieve (identical complete) synchronization, under the structural evolution described by (3) and (4), if as $t \to \infty$ the states of the network $(X(t))$ tend to the synchronization manifold $\Omega(t) = \{x(t) \in \mathbf{R}^n : x_1(t) = x_2(t) \cdots = x_N(t)\}$.

It is worth remarking when all the nodes move at unison $(x_1(t) = x_2(t) = \ldots = x_N(t))$, by condition (6), the last term of the right hand side of (1) becomes zero. Additionally, since all nodes have identical vector fields; the network dynamics move in a synchronized manner along the diagonal n-dimensional manifold $\Omega(t)$. On the synchronization manifold, one can say that the trajectories of the network equation (1) collapse to those of a single isolated node $s(t)$, which satisfies (7). Therefore, when the network is synchronized the solution to (1) has N identical components $s(t)$, which we write as $S(t) = [s^\top(t), \ldots, s^\top(t)]^\top \in \mathbf{R}^{Nn}$. Furthermore, $s(t)$ can be an equilibrium point, a periodic orbit or even a chaotic trajectory.

In the following section we derive conditions for synchronization under arbitrary switching with adaptive weights.

3. Stability of the synchronized solution

For our dynamical network model (1) under the structural evolution described by (3) and (4), assuming that (6) is satisfied uniformly in time, the synchronized solution corresponds to that of an isolated node, *i.e.*,

$$\dot{s}(t) = f(s(t)). \tag{7}$$

In what follows we will consider that the nodes are chaotic dynamical systems and $s(t)$ corresponds to a stable chaotic solution. Further, we assume that the nodes in the network satisfy the QUAD condition [17, 24, 26], that is,

Definition 2:

The function $f : \mathbf{R}^n \times R_+ \to \mathbf{R}^n$ is QUAD if and only if, for any $x, y \in \mathbf{R}^n$

$$(x-y)^\top [f(x) - f(y)] - (x-y)^\top \Delta (x-y) \leq -\bar{w}(x-y)^\top (x-y), \tag{8}$$

where $\Delta \in \mathbf{R}^{n\times n}$ is an arbitrary diagonal matrix and $\bar{w}$ is a positive scalar.

It is worth noting that many benchmark chaotic systems satisfy the QUAD condition [22, 25]. Additionally, in some instances *e.g.* Chua's circuit, we have $\Delta = 0_n$ in that case the QUAD restriction becomes $(x-y)^\top[f(x) - f(y)] \leq -\bar{w}(x-y)^\top(x-y)$ [24].

The stability of the synchronized solution is equivalent to the stability of the error $e_i(t) = x_i(t) - s(t)$, for $i = 1, 2, ..., N$ about its zero solution. From (1) and (7) the error dynamics are given by

$$\dot{e}_i(t) = \tilde{f}(x_i, s, t) + \sum_{j=1}^{N} g_{ij}(X, t)e_j(t), \text{ for } i = 1, 2, ..., N; \tag{9}$$

where $\tilde{f}(x_i, s, t) = f(x_i(t)) - f(s(t))$.

The conditions for the asymptotical stability of the error dynamics, under an arbitrary switching law described by (3) with adaptive connection strengths that update according to (4), are given in the following result:

Theorem 1: Consider the system (9).Under the assumption that the nodes in the network satisfy the QUAD condition described in (8), and that (6) is satisfied uniformly in time. The synchronized solution $S(t) = [s^\top(t), ..., s^\top(t)]^\top \in \mathbf{R}^{Nn}$ with $s(t) \in \mathbf{R}^n$ as in (7), will be asymptotically stable if there exist a symmetric positive definite matrix $P = P^\top > 0 \in \mathbf{R}^{Nn \times Nn}$ and a sufficiently large positive constant $\beta > 0$, such that

$$G^\top(t)P + PG(t) + 2P\bar{\Delta} \text{ and} \tag{10}$$

$$-[\beta I_n - g_{ij}(X, t)] \tag{11}$$

both are uniformly in time negative definite matrices $\forall i$, $\forall j$ $(i \neq j)$, where $\bar{\Delta} = \Delta \otimes I_N \in \mathbf{R}^{Nn \times Nn}$ with $\Delta \in \mathbf{R}^n$ the diagonal matrix from condition (8) and $\otimes$ the Kronecker product.

Proof:

Let $E(t) = [e_1^\top(t), ..., e_N^\top(t)]^\top \in \mathbf{R}^{Nn}$ be the error vector of the entire network. Then, the error dynamics (9) can be rewritten as:

$$\dot{E}(t) = F(X, S, t) + G(t)E(t), \tag{12}$$

where $F(X, S, t) = [\tilde{f}^\top(x_1, s, t), ..., \tilde{f}^\top(x_N, s, t)]^\top \in \mathbf{R}^{Nn}$ and $G(t) = \{g_{ij}(X, t)\}$ $= \{a_{ij}(\sigma, t)W_{ij}(\sigma, \phi, t)\} \in \mathbf{R}^{Nn \times Nn}$ evolves over time as described in the previous section.

Consider the Lyapunov function candidate

$$V(t) = E^\top(t)PE(t) + \frac{1}{2\alpha} \sum_{i=1, i\neq j}^{N} \sum_{j=1, j\neq i}^{N} [\beta I_n - g_{ij}(X,t)]^\top [\beta I_n - g_{ij}(X,t)], \quad (13)$$

where β a positive scalar and P a symmetric and positive definite $Nn \times Nn$ matrix.

The time derivative of $V(t)$ along the trajectories of the error dynamics (12) is given by

$$\begin{aligned} \dot{V}(t) = \; & 2E^\top(t)PF(X,S,t) \\ & + E^\top(t)[G^\top(t)P + PG(t)]E(t) \\ & - \tfrac{1}{\alpha} \textstyle\sum_{i=1,i\neq j}^{N} \sum_{j=1,j\neq i}^{N} [\beta I_n - g_{ij}(X,t)]^\top \dot{g}_{ij}(X,t). \end{aligned} \quad (14)$$

In order to proof that $\dot{V}(t)$ is uniformly negative definitive we add and subtract $2E^\top(t)P\bar{\Delta}E(t)$ to (14), with $\bar{\Delta} \in \mathbf{R}^{Nn\times Nn}$ a diagonal matrix.

$$\begin{aligned} \dot{V}(t) = \; & 2[E^\top(t)PF(X,S,t) - E^\top(t)P\bar{\Delta}E(t)] \\ & + E^\top(t)[G^\top(t)P + PG(t) + 2P\bar{\Delta}]E(t) \\ & - \tfrac{1}{\alpha} \textstyle\sum_{i=1,i\neq j}^{N} \sum_{j=1,j\neq i}^{N} [\beta I_n - g_{ij}(X,t)]^\top \dot{g}_{ij}(X,t). \end{aligned}$$

Notice that if the nodes in the network satisfy the QUAD condition and considering that P is a positive definite symmetric matrix, from (8), we have that

$$E^\top(t)PF(X,S,t) - E^\top(t)P\bar{\Delta}E(t) \leq -\bar{w}E^\top(t)PE(t) \quad (15)$$

where $\bar{\Delta} = \Delta \otimes I_N$ with $\Delta \in \mathbf{R}^n$ the diagonal matrix that satisfies condition (8) for the nodes in the network. Then, we have

$$\begin{aligned} \dot{V}(t) \leq \; & -2\bar{w}E^\top(t)PE(t) \\ & + E^\top(t)[G^\top(t)P + PG(t) + 2P\bar{\Delta}]E(t) \\ & - \tfrac{1}{\alpha} \textstyle\sum_{i=1,i\neq j}^{N} \sum_{j=1,j\neq i}^{N} [\beta I_n - g_{ij}(X,t)]^\top \dot{g}_{ij}(X,t) \end{aligned} \quad (16)$$

The first terms of the right-hand side of (16) is uniformly negative definitive, while the second becomes the condition in (10). For the last term of the right-hand side of (16), we have from (2) that $\dot{g}_{ij}(X,t) = a_{ij}(\sigma,t)\dot{W}_{ij}(\sigma,\phi,t)$, and from (4),

$$\dot{W}_{ij} = \begin{cases} \alpha(diag([|x_i^1 - x_j^1|, ..., |x_i^n - x_j^n|])) & \text{if } a_{ij}(\sigma,t_k) = 1 \\ 0 & \text{if } a_{ij}(\sigma,t_k) = 0. \end{cases} \quad (17)$$

By the restrictions on the admissible patterns of connections $\mathcal{A}$, we have no isolated nodes, therefore, at least one of the terms in the sums will be different from zero. Then, we have that $\dot{g}_{ij}(X, t)$ is uniformly positive definite. Furthermore, $g_{ij}(X, t)$ is bounded for all time, then is possible to find a constant β sufficiently large such that the matrix $[\beta I_n - g_{ij}(X, t)]$ is uniformly positive definite, this becomes the condition in (11) ■.

Notice that by the restrictions placed on the structural evolution processes, the symmetric and diffusive properties of $G(t)$ are preserved both during the switching events between admissible patterns of connections and as the strengths are being updated. Therefore, one way to interpret the matrix P which satisfies condition (10) is a matrix describing a common quadratic Lyapunov function for all the admissible patterns of connection, where the adaptive law automatically adjust the connection strengths such that synchronization is achieved.

In the following section, we investigate the effects of slightly different descriptions of the evolution processes on the stability of the synchronized solution.

4. Alternative structural evolution processes

In the previous section we made two significant assumptions: 1) All the connection strengths update their value simultaneously according to the adaptive law (4); this usually is referred to as a link-based adaptive strategy [17]. 2) We assumed that at every switch instant, the dynamical state of every node remained unchanged, that is, it had the same value as in the instant prior to the switching event. In this section we extend the results of Section 3 for two structural evolution scenarios, in the first, the connection strength of our switching topology network is updated using a node-based adaptive strategy; while in our second scenario, the states of the nodes associated with structural change are reset to zero on every switching events.

4.1. Synchronization with a node-based adaptive strategy

An alternative to the adaptive strategy described by (4) is the so-called node-based strategy, which consist in varying the strength of all the connections associated to a node at the same time to the same value, that is, the coupling strength of the i-th node, is $c_i(t)$. In this context, the matrix function describing the connection between the i-th and j-th nodes becomes:

$$g_{ij}(X, t) = c_i(\sigma, \phi_i, t) a_{ij}(\sigma, t)\Gamma. \tag{18}$$

Then, the state equation of our network under structural evolution is given by

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} c_i(\sigma, \phi_i, t) a_{ij}(\sigma, t) \Gamma x_j(t), \quad i = 1, \ldots, N, \tag{19}$$

where $\Gamma = diag\{\gamma_1, \ldots, \gamma_n\} \in \mathbf{R}^{n \times n}$ is a constant matrix, with $\gamma_i = 1$ if the connected nodes are coupled through their i-th component, and $\gamma_i = 0$ otherwise. As before, we assume that the set of admissible structures $\mathcal{A} = \{ A(\tau, t) \, | \tau \in I\}$, with $I = \{1, \ldots, m\}$ the index set, is composed by patterns of connection with the same number of nodes and links; additionally, the patterns of connection have no isolated nodes. The switching law is arbitrary, piecewise-constant, and continuous-from-the-right, such that if $\sigma(t) = \tau$, with $\tau \in I$ for $t \in [t_k, t_{k+1})$ and $k \in \mathbf{Z}_+$, then $A(\tau, t) = \{a_{ij}(\tau, t)\} \in \mathbf{R}^{N \times N}$ is the active pattern of connection and the 0-1 scalar function $a_{ij}(\sigma, t)$ describes whether the nodes are connected in the active pattern of connection at the time instant t.

In the node-based adaptive strategy, the connection strength associated to the i-th node is updated according to the adaptive law

$$\phi_i(X, \sigma, t) = \alpha \sum_{j=1, j \neq i}^{N} a_{ij}(\sigma, t) \| x_i(t) - x_j(t) \|, \tag{20}$$

where α is the adaptation constant and $|| \cdot ||$ is the euclidian norm. Notice that $\phi_i(X, \sigma, t)$ changes as the pattern of connection changes, however, due to the restriction on the set of admissible structures the sum for any i is only zero when the nodes are synchronized. Furthermore, as the pattern of connection changes by switching only the intensity of the adaptive law changes. Then, the dynamics of the connection strength for each node are given by

$$c_i(\sigma, \phi, t_k) = \begin{cases} \text{if} & d_i(t_k) = d_i(t_k^-) \\ \text{(a)} & c_i(\sigma, \phi, t_k) = c_i(\sigma, \phi_i, t_k^-), \\ \text{(b)} & \dot{c}_i(\sigma, \phi_i, t) = \phi_i(X, \sigma, t), \\ & \\ \text{if} & d_i(t_k) \neq d_i(t_k^-) \\ \text{(a)} & c_i(\sigma, \phi, t_k) = \epsilon, \\ \text{(b)} & \dot{c}_i(\sigma, \phi_i, t) = \phi_i(X, \sigma, t), \end{cases} \tag{21}$$

for $i = 1, \ldots, N$, $t \in [t_k, t_{k+1})$ with $k \in \mathbf{Z}_+$, where $\epsilon > 0$ and $d_i(t)$ is the node degree of the i-th node at time instant t. Equation (21.a) indicates that at each switching instant t_k the initial condition of each coupling strength retains its previous value if its node degree remains the same, but it is ϵ if its node degree change; and (21.b) describes how the coupling strength updates during the corresponding time interval.

The topology of the network in (19) is described by the time-varying coupling matrix $C(t) = \{\rho_{ij}(t)\} \in \mathbf{R}^{N \times N}$, where $\rho_{ij}(t) = c_i(\sigma, \phi_i, t)a_{ij}(\sigma, t)$. We assume the connections are bidirectional and that the diagonal elements of the coupling matrix are given by

$$\rho_{ii}(t) = -\sum_{\substack{j=1 \\ j \neq i}}^{N} \rho_{ij}(t) = -c_i(\sigma, \phi_i, t) \sum_{\substack{j=1 \\ j \neq i}}^{N} a_{ij}(\sigma, t), \tag{22}$$

for $i = 1, 2, ..., N$, with $t \in [t_k, t_{k+1})$ and $k \in \mathbf{Z}_+$.

Due to the node-based nature of the connection strength adaptation strategy the coupling matrix is not symmetric, yet assuming that (22) is uniformly satisfied and that at each time instant $C(t)$ can be seen as a constant matrix. Using similar arguments as before, we have that $C(t)$ is diffusive and its eigen-spectrum $\lambda_i(t)$, with $(i = 1, ..., N)$ are less or equal to 0, with 0 an eigenvalue of multiplicity one. Then, from Theorem 1, we derive the following corollary for the synchronization of network (19) under structural evolution with a node-based adaptive strategy.

Corollary 1: Assume that the nodes in the network (19) satisfy the QUAD condition described in (8), and that (22) is uniformly satisfied. Then, the synchronized solution is $S(t) = [s^\top(t), ..., s^\top(t)]^\top \in \mathbf{R}^{Nn}$ with $s(t) \in \mathbf{R}^n$ as described in (7) is asymptotically stable if there exist a symmetric positive definite matrix $P = P^\top > 0 \in \mathbf{R}^{N \times N}$ and a sufficiently large positive constant $\beta > 0$, such that

$$C^\top(t)P + PC(t) + 2P\bar{\Delta} \tag{23}$$

is uniformly negative definite matrix, and

$$\beta \geq c_i(\sigma, \phi, t_k), \quad \forall i. \tag{24}$$

Proof:

Let $E(t) \in \mathbf{R}^{Nn}$ be the error vector of the entire network obtained from (19) and (7), then we write the error dynamics in matrix form

$$\dot{E}(t) = F(X, S, t) + \tilde{C}(t)E(t), \tag{25}$$

where $F(X,S,t) \in \mathbf{R}^{Nn}$ and $\tilde{C}(t) = C(t) \otimes I_n$

The Lyapunov function candidate is taken to be

$$V(t) = E^\top(t)\tilde{P}E(t) + \frac{1}{2}\sum_{i=1}^{N}(\beta - c_i(\sigma,\phi_i,t))^2, \tag{26}$$

where β a positive scalar and $\tilde{P} = P \otimes I_n$, with P a symmetric and positive definite $N \times N$ matrix.

The time derivative of $V(t)$ along the trajectories of the error dynamics (25) are given by

$$\begin{aligned} \dot{V}(t) = \;& 2E^\top(t)\tilde{P}F(X,S,t) \\ & + E^\top(t)[\tilde{C}^\top(t)\tilde{P} + \tilde{P}\tilde{C}(t)]E(t) \\ & - \textstyle\sum_{i=1}^{N}(\beta - c_i(\sigma,\phi_i,t))\dot{c}_i(\sigma,\phi_i,t). \end{aligned}$$

In order to prove that $\dot{V}(t)$ is uniformly negative definitive we add and subtract $2E^\top(t)\tilde{P}\bar{\Delta}E(t)$ to (14), with $\bar{\Delta} = \Delta \otimes I_N \in \mathbf{R}^{Nn \times Nn}$ a diagonal matrix that satisfies condition (8). Then, we have

$$\begin{aligned} \dot{V}(t) \leq \;& -2\bar{w}E^\top(t)\tilde{P}E(t) \\ & + E^\top(t)[\tilde{C}^\top(t)\tilde{P} + \tilde{P}\tilde{C}(t) + 2\tilde{P}\bar{\Delta}]E(t) \\ & - \textstyle\sum_{i=1}^{N}(\beta - c_i(\sigma,\phi_i,t))\dot{c}_i(\sigma,\phi_i,t). \end{aligned} \tag{27}$$

The first terms of the right-hand side of (27) is uniformly negative definitive, while the second becomes the condition in (23). For the last term of the right-hand, we have from (20) that $\dot{c}_i(\sigma,\phi_i,t) \leq 0$ for all time and for any $\sigma(t) = \tau \in I = \{1,\ldots,m\}$, then it is possible to find a constant β sufficiently large such that $\beta - c_i(\sigma,\phi_i,t)$ is uniformly positive definite, this becomes the condition in (24) ■.

4.2. Synchronization under a resetting switching law

For many real-world networks topology switching requires additional preparations. For example, in a power grid the transition from one configuration to another requires that diverse isolation and start-up protocols be followed for security reasons. Taking inspiration from such situations, we investigate an alternative version of the connection change process in our model of structural evolution. We consider that at each switching event the states of the nodes that change their connections are reset to zero.

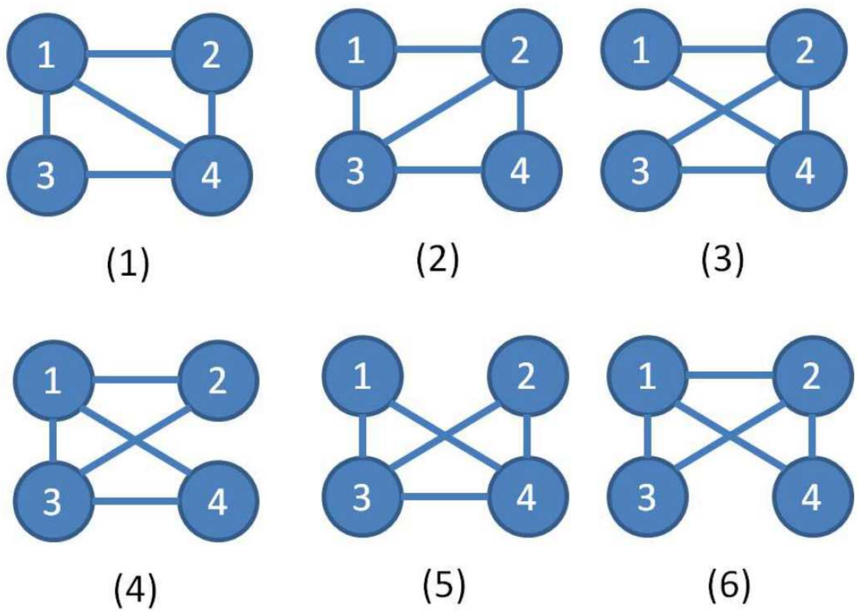

Figure 1: The set of admissible structures ($\mathcal{A}$) with four nodes and five links.

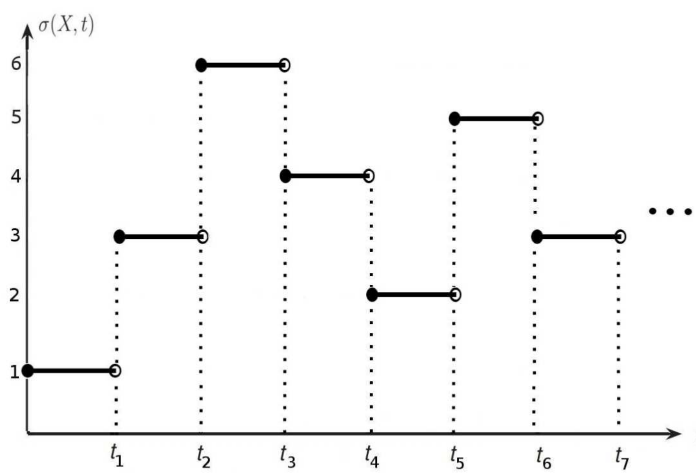

Figure 2: Example of an arbitrary switching law.

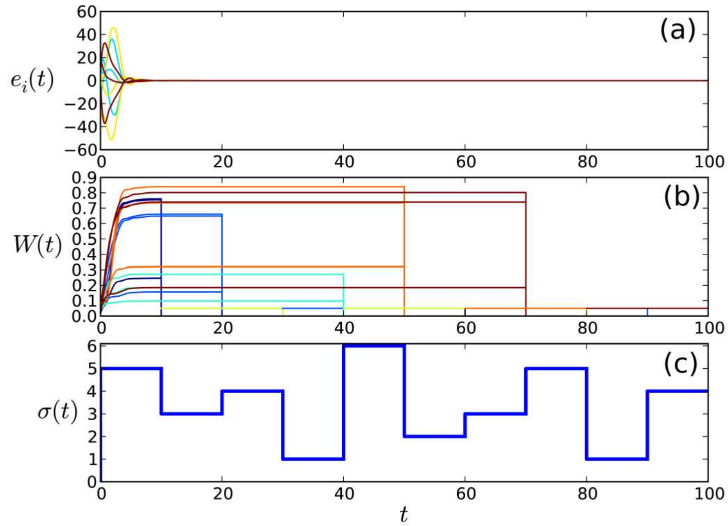

Figure 3: Network of Chua's circuits with arbitrary switching law (3) and edge-based adaptive strategy (4). (a) Evolution of synchronization errors, (b) connections strengths, and (c) switching law.

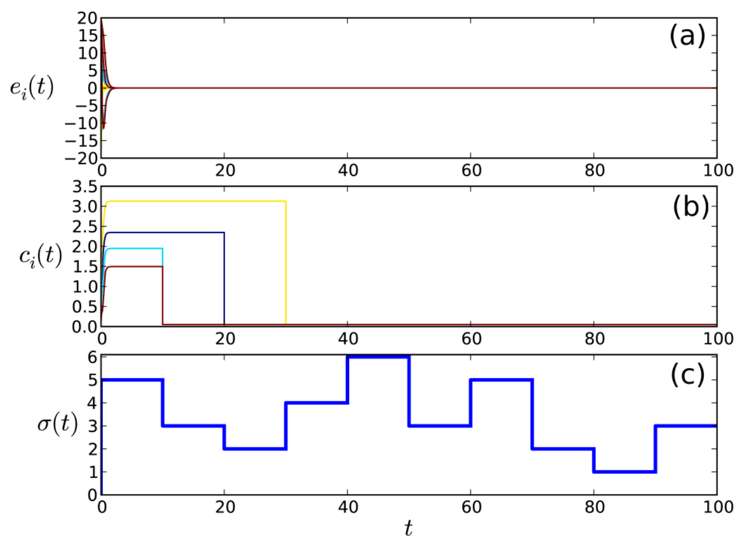

Figure 4: Network of Chua's circuits with arbitrary switching law (3) and node-based adaptive strategy (20). (a) Evolution of synchronization errors, (b) connections strengths, and (c) switching law.

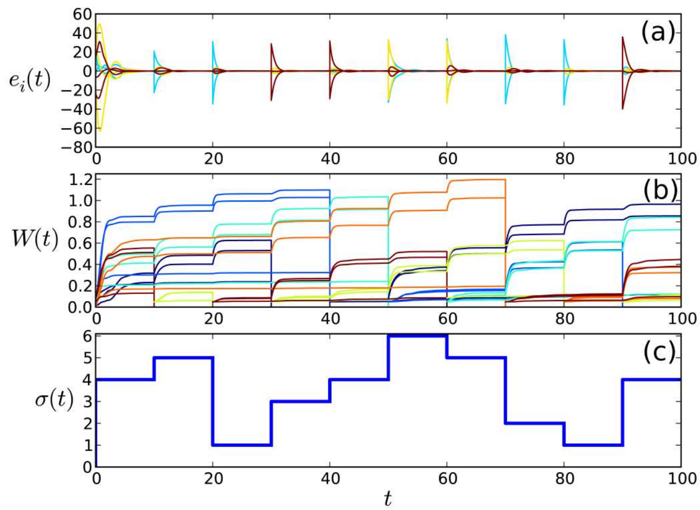

Figure 5: Network of Chua's circuits with resetting switching law (28) and edge-based adaptive strategy (4). (a) Evolution of synchronization errors, (b) connections strengths, and (c) switching law.

The resetting switching process is included in the network dynamics as a perturbation that occurs at the switching instants t_k ($k \in \mathbf{Z}_+$) on the nodes that change their neighborhood. To establish which nodes need to be reset to change the pattern of connection, we use the graph theory concept of node degree, which is defined as the number links that connect a node with its neighbors. We denote with $d_i(t)$ the node degree of the i-th node at time instant t. Then, the resetting process is given by

$$\eta_i(t) = \psi_i(t_k)\delta(t - t_k), \quad i = 1, \dots, N; \tag{28}$$

where $\delta(\cdot)$ is the delta function and

$$\psi_i(t_k) = \begin{cases} -x_i(t_k) & \text{if } d_i(t_k^-) \neq d_i(t_k), \\ 0 & \text{if } d_i(t_k^-) = d_i(t_k). \end{cases} \tag{29}$$

The state equation of the network under resetting switching topology and adaptive connection strength is given by

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} g_{ij}(X, t)x_j(t) + \eta_i(t), \quad i = 1, \dots, N; \tag{30}$$

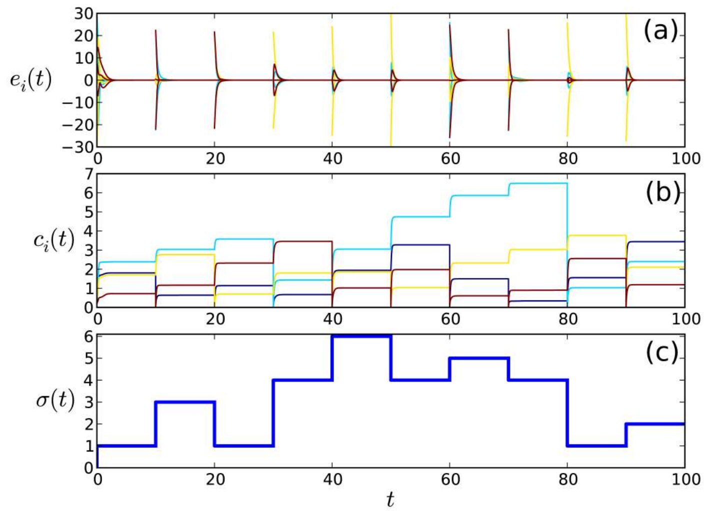

Figure 6: Network of Chua's circuits with resetting switching law (3) and node-based adaptive strategy (20). (a) Evolution of synchronization errors, (b) connections strengths, and (c) switching law.

where the structure of the network evolves according to the switching $\sigma(X,t)$, and the adaptive $\phi(X,t)$ law described in (3) and (4)-(5), respectively; with the additional resetting process $\eta(t)$ described in (28)-(29).

The resetting process can be seen as a vanishing perturbation of the dynamical network described in (1) or the network in (19) in the case of a node-based adaptive strategy for the connection strength. However, notice that from (29) we have that the resetting perturbation is bounded ($||\eta_i(t)|| \leq \gamma ||x_i(t_k)||$) and is applied only in the switching instants t_k, such that, $\eta_i(t) = 0$ $\forall t \in (t_k, t_{k+1})$ with $k \in \mathbf{Z}_+$. Then, under the same assumptions of the previous results, namely, QUAD nodes and structural evolution that uniformly preserves the diffusive nature of the coupling matrix $G(t) = \{g_{ij}(X,t)\}$; the results of Theorem 1 and Corollary 1 hold. As such, if the conditions in (10)-(11) or (23)-(24) are satisfied, respectively; then, the network synchronizes asymptotically.

5. Illustrative examples

We consider a dynamical network where each node is a Chua's circuit given by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} a[-y(t) - NL(x(t))] \\ x(t) - y(t) + z(t) \\ by(t) \end{bmatrix}; \tag{31}$$

where $NL(x(t)) = m_0 x(t) + 0.5(m_1 - m_0)(|x(t)+1| - |x(t)-1|)$, with $a = 4.916$, $b = 3.641$, $m_0 = -0.07$ and $m_1 = 1.5$. As shown in [24], (31) satisfies the QUAD condition with $\Delta = 0$.

For illustrative purposes, we consider a network with four nodes and five links. Then, the set of admissible structures $\mathcal{A}$ (Figure 1) has a cardinality of six, with the index set $I = \{1, 2, 3, 4, 5, 6\}$. The switching law that determines the active pattern of connection $A(\tau, t)$ ($\tau \in I$) is described in (3). Figure 2 shows an example of the time evolution of our switching law $\sigma(X, t)$, where the time interval between switching events is $\mathcal{D}_k = 10$, $\forall k$.

First, we consider that the connection strength updates according to an edge-based adaptive strategy as described in (4), with $\alpha = 0.01$ and $\epsilon = 0.05$. As a second example, we consider a dynamical network where the connection strength updates according to the node-based adaptive strategy described in (20). The simulations results are shown in Figures (3) and (4), respectively. In both cases, synchronization is achieved very fast and is preserved despite the changes in the pattern of connections. With respect to the connection strengths we can see that after a few switching events, all the strengths take the value ϵ for the rest of the simulation, the reason for this is that since the nodes are synchronized, the adaptive law $\phi(X, t)$ is zero.

To illustrate the fact that in real-world networks changes in the pattern of connections affects the connection strengths, we propose the resetting switching law described in (28). In Figures (5) and (6) we present the simulations results for a network of Chua's circuits under (28) with edge-based and node-based adaptive strategies, respectively. We can note that in both cases the synchronization is achieved in the first time instants. Even more, we can observe that in every switching event the synchronization is lost by the effect of the perturbations, and because the adaptive law, the network achieve again synchronization before the next switched event. With respect to the connection strength, we can note that it is bounded and increase it value at every switching event in order to keep the synchronous behavior.

6. Conclusion

In this paper we investigate the effects of structural evolution on the synchronization of complex networks. We model structural evolution as an arbitrary switching law that changes the pattern of connection, and an adaptive law that updates the connection strengths. Our theoretical and numerical results show that, by constrained the set of admissible structures and with an adaptive law based on the differences between nodes, synchronization is achieved. In particular, we observe that once synchronization is achieved; even if the pattern of connections changes, the strengths are not affected. Furthermore, the connection strength can be set arbitrary small and still synchronization is preserved. This motivated us to propose a resetting switching law, which introduces vanishing perturbations in the network at each switching instant. In this case, the adaptive law provides the necessary adjustments in connection strengths in order to recover the synchronized state of the network. The proposed structural evolution model combines previously used methods to describe network evolution with the emphasis on considering change as an intrinsic part of the nature of complex networks. As such, we believe that taking into consideration these processes of change and their effects on the emergence of self-organized phenomena, such as synchronization, is of significant importance in the study of complex networks and warrants further investigation.

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