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An output-feedback global continuous control scheme with desired gravity compensation for the finite-time and exponential regulation of bounded-input robotic systems

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Abstract: A Saturating-Proportional Saturating-Derivative type global continuous control scheme with desired gravity compensation for the finite-time or (local) exponential stabilization of robotic systems with constrained inputs, avoiding velocity variables in the feedback, is presented. The proposed output-feedback controller proves to need a closed-loop analysis with considerably higher degree of complexity, and entail more involved consequent requirements, than in the *on-line* compensation case. Other analytical limitations are further overcome through the developed algorithm. Simulation tests corroborate the efficiency of the proposed approach.

Keywords: Robotic systems, output feedback, finite-time stability, bounded inputs.

1. INTRODUCTION

A global continuous output-feedback scheme for the finite-time and exponential stabilization of mechanical/robotic systems with bounded inputs has been recently proposed and thoroughly motivated in (Zamora-Gómez, Zavala-Río and López-Araujo, 2017). Guaranteeing the corresponding formulated control objective under the explicit consideration of input constraints and the explicit choice on the system trajectory convergence, under the exclusive consideration of position variables in the feedback, are among the main characteristics that distinguish such an approach from continuous finite-time controllers developed for mechanical/robotic systems before its appearance, *e.g.* (Hong, Xu and Huang, 2002; Sanyal and Bohn, 2015; Zhao, Li, Zhu & Gao, 2010) (see for instance (Zamora-Gómez et al., 2017) for a brief description of such previous works). But there is still an important distinction: while the previous works are mainly state-feedback approaches that rely on the *dynamic inversion* technique (except for one of the two controllers presented in (Hong et al., 2002)), and the only output-feedback extension (formulated in (Hong et al., 2002)) is based on (model-based) finite-time observers, the scheme in (Zamora-Gómez et al., 2017) exploits the inherent passive nature of mechanical systems avoiding state reconstruction. This is done by keeping a (saturating) Proportional-Derivative type structure with exclusive compensation of the conservative-force (vector) term; damping is further injected through a (model-free) dynamic dissipation subsystem whose output is involved in the feedback as a *damped-derivative* action. Through such a control scheme, the system model dependence of the designed algorithm is considerably reduced, consequently simplifying the control structure and decreasing the inherent inconveniences of modelling inaccuracies as well as the implied computation burden. But these advantages could still be potentiated by replacing the (unique) on-line compensation term by the conservative-force/gravity term exclusively evaluated at the desired position. Such a *desired gravity compensation* idea was first developed in an unconstrained-input conventional (infinite-time) stabilization framework by (Takegaki and Arimoto, 1981) and, ever since its introduction in the literature, it has been widely appreciated in view of its simplicity and simplification improvements. This constitutes the main motivation of this work which aims at developing a *desired-gravity-compensation* extension of the output-feedback SP-SD-type (Saturating-Proportional Saturating-Derivative) finite-time/exponential stabilization scheme from (Zamora-Gómez et al., 2017) for robotic systems. Far from what one could expect, such a design

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task is not as simple or direct as a simple replacement of the *on-line* compensation term by the *desired* one. Contrarily to the on-line compensation case, in the desired compensation case further design requirements prove to be needed so as to ensure that the control-induced potential energy component *dominates* the open-loop one (in order to guarantee uniqueness of the desired closed-loop equilibrium configuration). This was already pointed out in the unconstrained-input conventional case (Takegaki and Arimoto, 1981), where such a domination goal was shown to be achieved through a P control (vector) term with a(n absolutely) stronger growing rate than that of the open-loop conservative force term in any direction (at every point) on the configuration space; in particular, under the simple consideration of uncoupled linear P and D control actions, this was shown to be achieved by simply fixing P gains higher than the highest (induced) norm value of the Jacobian matrix of the gravity term (assuming that such a Jacobian matrix is bounded). But the solution of the referred uniqueness issue cannot be that simple in the analytical context considered here, in view of the special functions involved in the SP-SD terms to guarantee the achievement of the formulated stabilization goal. This represents an important analytical challenge to which this work succeeds to give a solution.

After the publication of (Zamora-Gómez et al., 2017), continuous output-feedback finite-time stabilization of Euler-Lagrange systems was treated in (Cruz-Zavala, Nuño and Moreno, 2017). In that work, four particular controller cases were presented differing on the type of compensation of the gravity term, among desired and on-line, and on the bounded or unbounded control structure. The bounded controller versions were characterized by the use of specific saturation functions and the application of the control gains to the shaped error correction actions. In particular, such *external weighting* leads the control gains to act on the PD-action bounds, generating the need (at every setting or change on the control gain values) for an additional verification and eventual adjustment on the considered saturation function bounds to guarantee the input saturation avoidance requirements. Further, in the desired compensation case, local exponential stability cannot be concluded through the analytical procedure developed therein. Such limitations are surpassed through the proposal developed in this work, characterized by the direct application of the control gains to the error variables (which avoids the above mentioned control gain tuning readjustment inconvenience) and the more thorough closed-loop analysis and consequent requirement specifications, including the suitable solution given to the proof on its ability to include exponential (in addition to finite-time) stabilization among the control design choices. Simulation tests corroborating the analytical developments are included.

2. PRELIMINARIES

Let $X \in \mathbb{R}^{m\times n}$ and $y \in \mathbb{R}^n$. X_{ij} stands for the element of X at its i^{th} row and j^{th} column, X_i for the i^{th} row of X and y_i for the i^{th} element of y. With $m = n$, $X > 0$ denotes that X is positive definite; for a symmetric matrix X, $\lambda_m(X)$ and $\lambda_M(X)$ respectively stand for its minimum and maximum eigenvalues. 0_n represents the origin of $\mathbb{R}^n$ and I_n the $n \times n$ identity matrix. $\mathbb{R}^n_{>0}$ and $\mathbb{R}^n_{\geq 0}$ denote the set of n-tuples with positive and non-negative entries, respectively. $\|\cdot\|$ stands for the standard Euclidean norm for vectors and induced norm for matrices. Let $S_c^{n-1} = \{x \in \mathbb{R}^n : \|x\| = c\}$: an $(n-1)$-dimensional sphere of radius $c > 0$ on $\mathbb{R}^n$. We denote $D_g f$ the directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ along $g : \mathbb{R}^n \to \mathbb{R}^n$, *i.e.* $D_g f(x) = \frac{\partial f}{\partial x} g(x)$. We consider the sign function $\text{sign}(\cdot)$ to be zero at zero, and denote $\text{sat}(\cdot)$ the standard saturation function, *i.e.* $\text{sat}(\varsigma) = \text{sign}(\varsigma)\min\{|\varsigma|, 1\}$.

2.1 Robotic systems

Consider the n-DOF robot manipulator dynamics

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \tag{1}$$

where $q,\ \dot{q},\ \ddot{q} \in \mathbb{R}^n$ are the position (generalized coordinates), velocity, and acceleration vectors; the inertia matrix $H(q) \in \mathbb{R}^{n\times n}$ is a continuously differentiable positive definite symmetric matrix function, such that $H(q) \geq \mu_m I_n$, $\forall q \in \mathbb{R}^n$, for some $\mu_m > 0$; the Coriolis and centrifugal effect matrix $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$, defined through the Christoffel symbols of the first kind, satisfies $\dot{H}(q,\dot{q}) = C(q,\dot{q}) + C^T(q,\dot{q})$, $\forall q,\dot{q} \in \mathbb{R}^n$, and consequently

$$z^T\left[\frac{1}{2}\dot{H}(x,y) - C(x,y)\right]z = 0 \tag{2a}$$

$\forall x,y,z \in \mathbb{R}^n$, where $\dot{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ with $\dot{H}_{ij}(q,\dot{q}) = \frac{\partial H_{ij}}{\partial q}(q)\dot{q}$, $i,j = 1,\ldots,n$,

$$\|C(x,y)\| \leq \psi(x)\|y\| \tag{2b}$$

for some $\psi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and $C(x,y)z = C(x,z)y$, $\forall x,y,z \in \mathbb{R}^n$, whence we have that

$$C(q,a\dot{q})b\dot{q} = C(q,b\dot{q})a\dot{q} = C(q,ab\dot{q})\dot{q} = C(q,\dot{q})ab\dot{q} \tag{3}$$

$\forall q,\dot{q} \in \mathbb{R}^n$, $\forall a,b \in \mathbb{R}$; $g(q) = \nabla\mathcal{U}_{\text{ol}}(q)$, with $\mathcal{U}_{\text{ol}} : \mathbb{R}^n \to \mathbb{R}$ being the potential energy due to gravity, or equivalently: $\mathcal{U}_{\text{ol}}(q) = \mathcal{U}_{\text{ol}}(q_0) + \int_{q_0}^{q} g^T(z)dz$, for any $q, q_0 \in \mathbb{R}^n$; and $\tau \in \mathbb{R}^n$ is the external input (generalized) force vector.

We consider the bounded input case, where each input τ_i is constrained by a saturation bound $T_i > 0$. More precisely, letting u_i represent the control variable (controller output) relative to the i^{th} degree of freedom, we have that

$$\tau_i = T_i\text{sat}(u_i/T_i) \tag{4}$$

Assumption 2.1. $H(q)$ is bounded, *i.e.* $\|H(q)\| \leq \mu_M$, $\forall q \in \mathbb{R}^n$, for some $\mu_M \geq \mu_m > 0$.

Assumption 2.2. $\psi(\cdot)$ in (2b) is bounded and consequently $\|C(x,y)\| \leq k_C\|y\|$, $\forall x,y \in \mathbb{R}^n$, for some $k_C \geq 0$.

Assumption 2.3. The gravity force vector is a continuously differentiable bounded vector function with bounded Jacobian matrix, or equivalently: $|g_i(q)| \leq B_{gi}$, $i = 1,\ldots,n$, $\forall q \in \mathbb{R}^n$, for some non-negative constant B_{gi}; $\left\|\frac{\partial g}{\partial q}(q)\right\| \leq k_g$, $\forall q \in \mathbb{R}^n$, for some non-negative constant k_g, and consequently $\|g(x) - g(y)\| \leq k_g\|x - y\|$, $\forall x,y \in \mathbb{R}^n$.

Assumption 2.4. $T_i > \eta B_{gi}$, $\forall i \in \{1,\ldots,n\}$, with $\eta \geq 1$.

Assumptions 2.1–2.3 apply *e.g.* for robots having only revolute joints.

Remark 2.1. By the properties of $H(q)$, its inverse matrix, denoted $H^{-1}(q)$, exists and it is a continuously differentiable positive definite matrix function, and actually, under the consideration of Assumption 2.1: $(1/\mu_M)I_n \leq H^{-1}(q) \leq (1/\mu_m)I_n$, $\forall q \in \mathbb{R}^n$. △

2.2 Local homogeneity, finite-time / δ-exponential stability

As in (Zamora-Gómez et al., 2017), this work is developed within the analytical framework of *local homogeneity* (Zavala-Río and Fantoni, 2014), which states a formal analytical platform permitting to handle vector fields with bounded components. Definitions and results in such an analytical context are strongly related to *family of dilations* δ_ε^r, defined as $\delta_\varepsilon^r(x) = \left(\varepsilon^{r_1}x_1, \ldots, \varepsilon^{r_n}x_n\right)^T$, $\forall x \in \mathbb{R}^n$, $\forall \varepsilon > 0$, with $r = (r_1, \ldots, r_n)^T$, where the *dilation coefficients* $r_1, \ldots, r_n$ are positive scalars. Subsequently, an *r-homogeneous norm* —a positive definite continuous function being r-homogeneous of degree 1— (M'Closkey and Murray, 1997; Kawski, 1990), is denoted $\|\cdot\|_r$, and $S_{r,c}^{n-1} = \{x \in \mathbb{R}^n : \|x\|_r = c\}$: an *r-homogeneous* $(n-1)$-*sphere* of radius $c > 0$.

Consider an n-th order autonomous system

$$\dot{x} = f(x) \tag{5}$$

where f is a vector field being continuous on an open neighborhood of the origin $\mathcal{D} \subset \mathbb{R}^n$ and such that $f(0_n) = 0_n$; let $x(t; x_0)$ represent the system solution with initial condition $x(0; x_0) = x_0$. A fundamental concept underlying this work is that of a (globally) *finite-time stable* equilibrium, as defined in (Bhat and Bernstein, 2005).

Remark 2.2. The origin is a globally finite-time stable equilibrium of system (5) if and only if it is globally asymptotically stable and finite-time stable. △

Theorem 2.1. (Zavala-Río and Fantoni, 2014) Consider system (5) with $\mathcal{D} = \mathbb{R}^n$. Suppose f is a locally r-homogeneous vector field of degree α with domain of homogeneity $D \subset \mathbb{R}^n$. Then, the origin is a globally finite-time stable equilibrium of system (5) if and only if it is globally asymptotically stable and $\alpha < 0$.

The next definition is stated under the additional consideration that, for some $r \in \mathbb{R}^n_{>0}$, f in (5) is locally r-homogeneous with domain of homogeneity $D \subset \mathcal{D}$.

Definition 2.1. (Kawski, 1990; M'Closkey and Murray, 1997) The equilibrium point $x = 0_n$ of (5) is *δ-exponentially stable* with respect to the homogeneous norm $\|\cdot\|_r$ if there exist a neighborhood of the origin, $\mathcal{V} \subset D$, and constants $a \geq 1$ and $b > 0$ such that $\|x(t; x_0)\|_r \leq a\|x_0\|_r e^{-bt}$, $\forall t \geq 0$, $\forall x_0 \in \mathcal{V}$.

Remark 2.3. If f in (5) is locally r-homogeneous of degree $\alpha = 0$ with dilation coefficients $r_i = r_0$, $\forall i \in \{1, \ldots, n\}$, for some $r_0 > 0$, then the origin turns out to be exponentially stable (in the usual or standard sense) if and only if it is δ-exponentially stable (Zavala-Río and Zamora-Gómez, 2017, Remark 2.5). △

Consider an n-th order autonomous system of the form

$$\dot{x} = f(x) + \hat{f}(x) \tag{6}$$

where f and $\hat{f}$ are continuous vector fields on $\mathbb{R}^n$ such that $f(0_n) = \hat{f}(0_n) = 0_n$.

Lemma 2.1. (Zavala-Río and Zamora-Gómez, 2017) Assume that, for some $r \in \mathbb{R}^n_{>0}$, f in (6) is a locally r-homogeneous vector field of degree $\alpha < 0$, resp. $\alpha = 0$, with domain of homogeneity $D \subset \mathbb{R}^n$, and that 0_n is a globally asymptotically, resp. δ-exponentially, stable equilibrium of $\dot{x} = f(x)$. Then, the origin is a finite-time, resp. δ-exponentially, stable equilibrium of system (6) if

$$\lim_{\varepsilon \to 0^+} \frac{\hat{f}_i(\delta_\varepsilon^r(x))}{\varepsilon^{\alpha + r_i}} = 0$$

$i = 1, \ldots, n$, $\forall x \in S_c^{n-1}$, resp. $\forall x \in S_{r,c}^{n-1}$, for some $c > 0$ such that $S_c^{n-1} \subset D$, resp. $S_{r,c}^{n-1} \subset D$.

Remark 2.4. The condition required by Lemma 2.1 may be equivalently verified through the fulfilment of

$$\lim_{\varepsilon \to 0^+} \left\| \varepsilon^{-\alpha} \mathrm{diag}\left[\varepsilon^{-r_1}, \ldots, \varepsilon^{-r_n}\right] \hat{f}(\delta_\varepsilon^r(x)) \right\| = 0$$

$\forall x \in S_c^{n-1}$ (resp. $S_{r,c}^{n-1}$). △

2.3 Scalar functions with particular properties

Definition 2.2. A continuous scalar function $\sigma : \mathbb{R} \to \mathbb{R}$ will be said to be:

(1) *bounded* (by M) if $|\sigma(\varsigma)| \leq M$, $\forall \varsigma \in \mathbb{R}$, for some positive constant M;
(2) *strictly passive* if $\varsigma\sigma(\varsigma) > 0$, $\forall \varsigma \neq 0$;
(3) *strongly passive* if it is a strictly passive function satisfying $|\sigma(\varsigma)| \geq \kappa\left|a\,\mathrm{sat}(\varsigma/a)\right|^b = \kappa\left(\min\{|\varsigma|, a\}\right)^b$, $\forall \varsigma \in \mathbb{R}$, for some positive constants κ, a and b.

Remark 2.5. A non-decreasing strictly passive function is strongly passive (Zavala-Río and Zamora-Gómez, 2017, Remark 2.7). △

Remark 2.6. Equivalent characterizations of strictly passive functions are: $\varsigma\sigma(\varsigma) > 0 \iff \mathrm{sign}(\varsigma)\sigma(\varsigma) > 0 \iff \mathrm{sign}(\sigma(\varsigma)) = \mathrm{sign}(\varsigma)$, $\forall \varsigma$. △

Lemma 2.2. (Zavala-Río and Zamora-Gómez, 2017) Let $\sigma : \mathbb{R} \to \mathbb{R}$, $\sigma_0 : \mathbb{R} \to \mathbb{R}$ and $\sigma_1 : \mathbb{R} \to \mathbb{R}$ be strongly passive functions and k be a positive constant. Then:

(1) $\int_0^\varsigma \sigma(k\nu)d\nu > 0$, $\forall \varsigma \neq 0$;
(2) $\int_0^\varsigma \sigma(k\nu)d\nu \to \infty$ as $|\varsigma| \to \infty$;
(3) $\sigma_0 \circ \sigma_1$ is strongly passive.

3. THE PROPOSED CONTROL SCHEME

Consider the following SP-SD type controller with desired gravity compensation

$$u(q, \vartheta) = -s_1(K_1\bar{q}) - s_2(K_2\vartheta) + g(q_d) \tag{7}$$

where $\bar{q} = q - q_d$, for any constant (desired equilibrium configuration) $q_d \in \mathbb{R}^n$; $\vartheta \in \mathbb{R}^n$ is the output vector variable of an auxiliary subsystem defined as

$$\dot{\vartheta}_c = -As_3(\vartheta_c + B\bar{q}) \quad , \quad \vartheta = \vartheta_c + B\bar{q} \tag{8}$$

$K_i = \mathrm{diag}[k_{i1}, \ldots, k_{in}]$, $i = 1, 2$, $A = \mathrm{diag}[a_1, \ldots, a_n]$, $B = \mathrm{diag}[b_1, \ldots, b_n]$, with $k_{ij} > 0$, $a_j > 0$, $b_j > 0$, $\forall j = 1, \ldots, n$; for any $x \in \mathbb{R}^n$, $s_i(x) = \left(\sigma_{i1}(x_1), \ldots, \sigma_{in}(x_n)\right)^T$, $i = 1, 2, 3$, with, for each $j = 1, \ldots, n$, σ_{3j} being a strictly passive function, while σ_{1j} and σ_{2j} are non-decreasing strictly passive functions such that

$$B_j \triangleq \max\left\{ \lim_{\varsigma \to \infty} \left[\sigma_{1j}(\varsigma) + \sigma_{2j}(\varsigma)\right] , \lim_{\varsigma \to -\infty} -\left[\sigma_{1j}(\varsigma) + \sigma_{2j}(\varsigma)\right] \right\} < T_j - B_{gj} \tag{9}$$

all three being locally Lipschitz-continuous on $\mathbb{R} \setminus \{0\}$; and with, for each $j = 1, \ldots, n$, k_{1j} and σ_{1j} additionally required to be such that, for all $\varsigma \in \mathbb{R}$,

$$|\sigma_{1j}(k_{1j}\varsigma)| > \min\left\{k_g|\varsigma| \, , \, 2B_{gj}\right\} \tag{10}$$

A block diagram depicting the proposed control scheme is shown in Fig. 1.

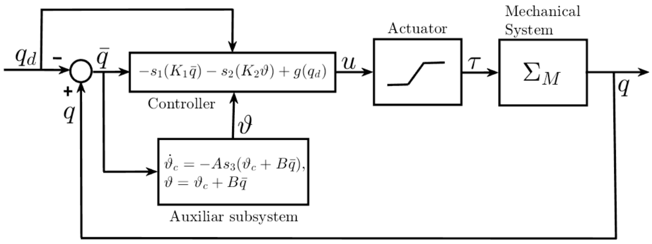

Fig. 1. Block diagram of the proposed control scheme

Remark 3.1. From the above formulation, we have that

$$2B_{gj} < |\sigma_{1j}(k_{1j}\varsigma)| \le B_j < T_j - B_{gj}$$

$\forall|\varsigma| \ge 2B_{gj}/k_g$, whence one sees that Assumption 2.4 with $\eta = 3$ is a necessary condition for the feasibility of the simultaneous fulfilment of (9) and (10). △

Remark 3.2. Inequality (10) implies the existence of constants $\hat{k}_{1j} > k_g$ and $b_j > 2B_{gj}$ such that $\left|\sigma_{1j}(k_{1j}\varsigma)\right| \ge \min\left\{\hat{k}_{1j}|\varsigma|, b_j\right\} > \min\left\{k_g|\varsigma|, 2B_{gj}\right\}$, $\forall\varsigma \neq 0$. △

Proposition 3.1. Consider system (1),(4) in closed loop with the proposed control law (7)-(8), under the above stated Assumptions and design specifications. Thus, global asymptotic stability of the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is guaranteed with $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \ge 0$.

Proof. Observe that —for every $j = 1, \ldots, n$— by (9), we have that, for any $(q, \vartheta) \in \mathbb{R}^n \times \mathbb{R}^n$ and any $q_d \in \mathbb{R}^n$: $|u_j(q, \vartheta)| \le \left|\sigma_{1j}(k_{1j}\bar{q}_j) + \sigma_{2j}(k_{2j}\vartheta_j)\right| + |g_j(q_d)| \le B_j + B_{gj} < T_j$. From this and (4), one sees that $T_j > |u_j(q, \vartheta)| = |u_j| = |\tau_j|$, $\forall(q, \vartheta) \in \mathbb{R}^n \times \mathbb{R}^n$, which shows that, along the system trajectories, $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \ge 0$. Hence, the closed-loop dynamics takes the (equivalent) form

$$H(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = -s_1(K_1\bar{q}) - s_2(K_2\vartheta) + g(q_d)$$
$$\dot{\vartheta} = -As_3(\vartheta) + B\dot{q}$$

By defining $x_1 = \bar{q}$, $x_2 = \dot{q}$, $x_3 = \vartheta$, and $x = (x_1^T, x_2^T, x_3^T)^T$, the closed-loop dynamics adopts the form of (6) with

$$f(x) = \begin{pmatrix} f_{(1)}(x) \\ f_{(2)}(x) \\ f_{(3)}(x) \end{pmatrix} \quad , \quad \hat{f}(x) = \begin{pmatrix} \hat{f}_{(1)}(x) \\ \hat{f}_{(2)}(x) \\ \hat{f}_{(3)}(x) \end{pmatrix} \tag{11}$$

where $f_{(1)}(x) = x_2$, $f_{(2)}(x) = -H^{-1}(q_d)\left[s_1(K_1x_1) + s_2(K_2x_3)\right]$, $f_{(3)}(x) = -As_3(x_3) + Bx_2$, $\hat{f}_{(1)}(x) = \hat{f}_{(3)}(x) = 0_n$, and

$$\hat{f}_{(2)}(x) = -H(x_1 + q_d)[C(x_1 + q_d, x_2)x_2 + g(x_1 + q_d) - g(q_d)] - \mathcal{H}(x_1)[s_1(K_1x_1) + s_2(K_2x_3)] \tag{12}$$

with $\mathcal{H}(x_1) = H^{-1}(x_1 + q_d) - H^{-1}(q_d)$. Thus, the closed-loop stability property stated through Proposition 3.1 is corroborated by showing that $x = 0_{3n}$ is a globally asymptotically stable equilibrium of $\dot{x} = f(x) + \hat{f}(x)$, which is proven through the following theorem.

Theorem 3.1. Under the stated specifications, the origin is a globally asymptotically stable equilibrium of $\dot{x} = f(x) + \ell\hat{f}(x)$ for $\ell \in \{0, 1\}$, with $f(x)$ and $\hat{f}(x)$ stated in (11).

Proof. For every $\ell \in \{0, 1\}$, let us define the continuously differentiable scalar function

$$V_\ell(x_1, x_2, x_3) = \frac{1}{2}x_2^T H(\ell x_1 + q_d)x_2 + \mathcal{U}_\ell(x_1) + \mathcal{I}_2(x_3) \tag{13}$$

where

$$\mathcal{U}_\ell(x_1) \triangleq \mathcal{I}_1(x_1) + \ell\,\mathcal{U}(x_1) \tag{14}$$

$$\mathcal{I}_1(x_1) \triangleq \int_{0_n}^{x_1} s_1^T(K_1z)dz = \sum_{j=1}^n \int_0^{x_{1j}} \sigma_{1j}(k_{1j}z_j)dz_j,$$
$$\mathcal{I}_2(x_3) \triangleq \int_{0_n}^{x_3} s_2^T(K_2z)B^{-1}dz = \sum_{j=1}^n \int_0^{x_{3j}} \frac{\sigma_{2j}(k_{2j}z_j)}{b_j}dz_j,$$

$$\mathcal{U}(x_1) \triangleq \mathcal{U}_{\mathrm{ol}}(x_1 + q_d) - \mathcal{U}_{\mathrm{ol}}(q_d) - g^T(q_d)x_1 \tag{15a}$$
$$= \int_{0_n}^{x_1} \left[g(z + q_d) - g(q_d)\right]^T dz \tag{15b}$$
$$= \int_{0_n}^{x_1} \left[\int_{0_n}^{z} \frac{\partial g}{\partial q}(\bar{z} + q_d)d\bar{z}\right]^T dz \tag{15c}$$

Observe from Eqs. (15) and Assumption 2.3 that

$$\mathcal{U}(x_1) \le \int_{0_n}^{x_1} \left[\int_{0_n}^{z} \left\|\frac{\partial g}{\partial q}(\bar{z} + q_d)\right\| d\bar{z}\right]^T dz$$
$$\le \int_{0_n}^{x_1} k_g z^T dz = \sum_{j=1}^n \int_0^{x_{1j}} k_g z_j dz_j \tag{16}$$

$\forall x_1 \in \mathbb{R}^n$ (from (15c)), and simultaneously that

$$\mathcal{U}(x_1) \le \sum_{j=1}^n \int_0^{x_{1j}} \mathrm{sign}(z_j)\left|g_j(z + q_d) - g_j(q_d)\right| dz_j$$
$$\le \sum_{j=1}^n \int_0^{x_{1j}} \mathrm{sign}(z_j) 2B_{gj} dz_j$$

$\forall x_1 \in \mathbb{R}^n$ (from (15b)). From these inequalities, Eq. (14), the satisfaction of (10) and Remark 3.2, we have that

$$\mathcal{U}_\ell(x_1) \ge \sum_{j=1}^n \int_0^{x_{1j}} \mathrm{sign}(z_j) \min\left\{(\hat{k}_{1j} - \ell k_g)|z_j|\,, (b_j - 2\ell B_{gj})\right\} dz_j$$
$$\ge \sum_{j=1}^n w_{\ell j}(x_{1j}) \triangleq S_\ell(x_1) \tag{17a}$$

with

$$w_{\ell j}(x_{1j}) = \begin{cases} \dfrac{\bar{k}_{\ell j}}{2} x_{1j}^2 & \text{if } |x_{1j}| \le \bar{b}_{\ell j}/\bar{k}_{\ell j} \\ \bar{b}_{\ell j}\left[|x_{1j}| - \bar{b}_{\ell j}/(2\bar{k}_{\ell j})\right] & \text{if } |x_{1j}| > \bar{b}_{\ell j}/\bar{k}_{\ell j} \end{cases} \tag{17b}$$

for some $\hat{k}_{1j} > k_g$ and $b_j > 2B_{gj}$, and any positive constants $\bar{k}_{\ell j} \le \hat{k}_{1j} - \ell k_g$ and $\bar{b}_{\ell j} \le b_j - 2\ell B_{gj}$.

Remark 3.3. One sees from expressions (17) that S_ℓ, $\ell = 0, 1$, are positive definite radially unbounded functions of x_1. Observe further that (involving previous arguments and Remark 2.6)

$$D_{x_1}\mathcal{U}_\ell x_1) = x_1^T\left[s_1(K_1x_1) + \ell\left(g(x_1 + q_d) - g(q_d)\right)\right]$$
$$\ge \sum_{j=1}^n |x_{1j}|\left[\left|\sigma_{1j}(k_{1j}x_{1j})\right| - \ell\left|g_j(x_1 + q_d) - g_j(q_d)\right|\right]$$
$$\ge \sum_{j=1}^n |x_{1j}| \min\left\{\bar{k}_{\ell j}|x_{1j}|\,, \bar{b}_{\ell j}\right\} > 0 \tag{18}$$

$\forall x_1 \neq 0_n$, whence one sees that, for every $\ell = 0, 1$, $\nabla_{x_1}\mathcal{U}_\ell(x_1) = s_1(K_1x_1) + \ell\left[g(x_1 + q_d) - g(q_d)\right] = 0_n \iff x_1 = 0_n$. △

Thus, from (17) and the inertia matrix properties, we get

$$V_\ell(x_1, x_2, x_3) \ge \frac{\mu_m}{2}\|x_2\|^2 + S_\ell(x_1) + \mathcal{I}_2(x_3) \tag{19}$$

whence, under the additional consideration of Lemma 2.2 and Remark 2.5, positive definiteness and radial unboundedness of V_ℓ, $\ell = 0, 1$, is concluded. Further, for every $\ell \in \{0,1\}$, the derivative of V_ℓ along the trajectories of $\dot{x} = f(x)+\ell\hat{f}(x)$, is obtained, after basic developments, as

$$\dot{V}_\ell(x_1,x_2,x_3) = -s_2^T(K_2x_3)B^{-1}As_3(x_3) = -\sum_{j=1}^{n}\frac{a_j}{b_j}\sigma_{2j}(k_{2j}x_{3j})\sigma_{3j}(x_{3j})$$

where, in the case of $\ell = 1$, (2a) has been applied. Note, from the strictly passive character of σ_{2j} and σ_{3j}, $j = 1,\ldots,n$, that $\dot{V}_\ell(x_1,x_2,x_3) \le 0$, $\forall(x_1,x_2,x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, with $Z_\ell \triangleq \{(x_1,x_2,x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : \dot{V}_\ell(x_1,x_2,x_3) = 0\} = \{(x_1,x_2,x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : x_3 = 0_n\}$. Further, from the system dynamics $\dot{x} = f(x) + \ell\hat{f}(x)$ —under the consideration of the positive definiteness of H and Remark 3.3— one sees that $x_3(t) \equiv 0_n \implies \dot{x}_3(t) \equiv 0_n \implies x_2(t) \equiv 0_n \implies \dot{x}_2(t) \equiv 0_n \implies s_1(K_1x_1(t))+\ell\big[g(x_1(t)+q_d)-g(q_d)\big] \equiv 0_n \iff x_1(t) \equiv 0_n$ (which shows that $(x_1,x_2,x_3)(t) \equiv (0_n,0_n,0_n)$ is the only system solution completely remaining in Z_ℓ), and corroborates that at any $(x_1,x_2,x_3) \in Z_\ell \setminus \{(0_n,0_n,0_n)\}$, the resulting unbalanced force terms act on the closed-loop dynamics [$\dot{x} = f(x_1,x_2,0_n) + \ell\hat{f}(x_1,x_2,0_n)$ with $(x_1,x_2) \ne (0_n,0_n)$], forcing the system trajectories to leave Z_ℓ, whence $\{(0_n,0_n,0_n)\}$ is concluded to be the only invariant set in Z_ℓ, $\ell = 0, 1$. Therefore, by the invariance theory (Michel, Hou and Liu, 2008), $x = 0_{3n}$ is concluded to be a globally asymptotically stable equilibrium of both the state equation $\dot{x} = f(x)$ and the (closed-loop) system $\dot{x} = f(x) + \hat{f}(x)$. □

Finite-time/exponential stabilization

Proposition 3.2. Consider the proposed control scheme under the additional consideration that, for every $j = 1,\ldots,n$, σ_{ij}, $i = 1,2$, are locally r_i-homogeneous of (common) degree $\alpha_i = 2r_2 - r_1$ —*i.e.* $r_{1j} = r_1$, $r_{2j} = r_2$ and $\alpha_{1j} = \alpha_1 = 2r_2 - r_1 = \alpha_2 = \alpha_{2j}$ for all $j = 1,\ldots,n$— with dilation coefficients such that $2r_2 - r_1 > 0 > r_2 - r_1$ and domain of homogeneity $D_{ij} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{ij} \in (0,\infty]\}$, and σ_{3j} is locally r_1-homogeneous of degree $\alpha_3 = r_2$ —i.e. $r_{3j} = r_3 = r_1$ and $\alpha_{3j} = \alpha_3 = r_2$ for all $j \in \{1,\ldots,n\}$— with domain of homogeneity $D_{3j} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{3j} \in (0,\infty]\}$. Thus, global finite-time stability of the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is guaranteed with $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1,\ldots,n$, $\forall t \ge 0$.

Proof. Since the proposed control scheme is applied, Proposition 3.1 holds and consequently $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1,\ldots,n$, $\forall t \ge 0$. Then, all that remains to be proven is that the additional considerations give rise to the claimed finite-time stabilization. In this direction, let $\hat{r}_i = (r_{i1},\ldots,r_{in})^T$, $i = 1,2,3$, $r = (\hat{r}_1^T,\hat{r}_2^T,\hat{r}_3^T)^T$, $D \triangleq \{(x_1,x_2,x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : K_ix_i \in D_{i1} \times \cdots \times D_{in}\,,\ i = 1,2,3\} = \{(x_1,x_2,x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : |x_{ij}| < L_{ij}/k_{ij}\,,\ i = 1,2,3,\ j = 1,\ldots,n\}$ —with K_i, $i = 1,2$, as previously defined and $K_3 = \mathrm{diag}[k_{31},\ldots,k_{3n}] = I_n$,— and consider the previously defined state (vector) variables and the consequent closed-loop state-space representation $\dot{x} = f(x)+\hat{f}(x)$, with f and $\hat{f}$ as defined through Eqs. (11). Since D defines an open neighborhood of the origin, there exists $\rho > 0$ such that $B_\rho \triangleq \{x \in \mathbb{R}^{3n} : \|x\| < \rho\} \subset D$. Moreover, for every $x \in B_\rho$ and all $\varepsilon \in (0,1]$, we have that $\delta_\varepsilon^r(x) \in B_\rho$ (since $\|\delta_\varepsilon^r(x)\| < \|x\|$, $\forall\varepsilon \in (0,1)$), and, for every $j = 1,\ldots,n$,

$$f_j(\delta_\varepsilon^r(x)) = \varepsilon^{r_{2j}}x_{2j} = \varepsilon^{(r_2-r_1)+r_{1j}}f_j(x)$$

$$\begin{aligned}
f_{n+j}(\delta_\varepsilon^r(x)) &= -H_j^{-1}(q_d)\big[s_1(\varepsilon^{r_1}K_1x_1) + s_2(\varepsilon^{r_3}K_2x_3)\big]\\
&= -\varepsilon^{2r_2-r_1}H_j^{-1}(q_d)\big[s_1(K_1x_1) + s_2(K_2x_3)\big]\\
&= \varepsilon^{(r_2-r_1)+r_{2j}}f_{n+j}(x)
\end{aligned}$$

$$\begin{aligned}
f_{2n+j}(\delta_\varepsilon^r(x)) &= -As_3(\varepsilon^{r_3}x_3) + \varepsilon^{r_2}Bx_2\\
&= \varepsilon^{r_2}\big[-As_3(x_3) + Bx_2\big]\\
&= \varepsilon^{(r_2-r_1)+r_{3j}}f_{2n+j}(x)
\end{aligned}$$

whence one concludes that f is a locally r-homogeneous vector field of degree $\alpha = r_2 - r_1$, with domain of homogeneity B_ρ. Hence, by Theorems 2.1 and 3.1, the origin of the state equation $\dot{x} = f(x)$ is concluded to be a globally finite-time stable equilibrium since $r_2 < r_1$. Thus, by Theorem 3.1, Lemma 2.1, and Remarks 2.2 and 2.4, the origin of the closed-loop system $\dot{x} = f(x) + \hat{f}(x)$ is concluded to be a globally finite-time stable equilibrium provided that $r_2 < r_1$, if

$$\begin{aligned}
\mathcal{L}_0 &\triangleq \lim_{\varepsilon\to0^+}\Big\|\varepsilon^{-\alpha}\delta_\varepsilon^{-\hat{r}_2}\big(\hat{f}_{(2)}(\delta_\varepsilon^r(x))\big)\Big\| = \lim_{\varepsilon\to0^+}\Big\|\varepsilon^{-\alpha-r_2}\hat{f}_{(2)}(\delta_\varepsilon^r(x))\Big\|\\
&= \lim_{\varepsilon\to0^+}\varepsilon^{r_1-2r_2}\Big\|\hat{f}_{(2)}(\delta_\varepsilon^r(x))\Big\| = 0 \qquad (21)
\end{aligned}$$

for all $x \in S_c^{3n-1} = \{x \in \mathbb{R}^{3n} : \|x\| = c\}$, for some $c > 0$ such that $S_c^{3n-1} \subset D$. Hence, from (12) and (3), we have, for all such $x \in S_c^{3n-1}$:

$$\begin{aligned}
&\Big\|\big[\hat{f}_{(2)}(\delta_\varepsilon^r(x))\Big\|\\
&\le \Big\| - H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)\varepsilon^{2r_2}x_2\Big\|\\
&\quad+\Big\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)\Big\|\Big\|g(\varepsilon^{r_1}x_1 + q_d) - g(q_d)\Big\|\\
&\quad+\Big\|\mathcal{H}(\varepsilon^{r_1}x_1)\big[\varepsilon^{\alpha_1}s_1(K_1x_1) + \varepsilon^{\alpha_2}s_2(K_2x_3)\big]\Big\|\\
&\le \varepsilon^{2r_2}\Big\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2\Big\|\\
&\quad+\Big\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)\Big\|k_g\varepsilon^{r_1}\|x_1\|\\
&\quad+\varepsilon^{2r_2-r_1}\Big\|\mathcal{H}(\varepsilon^{r_1}x_1)\big[s_1(K_1x_1) + s_2(K_2x_3)\big]\Big\|
\end{aligned}$$

and consequently, from (21) (recall that $r_1 > r_2 > 0$):

$$\begin{aligned}
\mathcal{L}_0 &\le \lim_{\varepsilon\to0^+}\varepsilon^{r_1}\Big\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2\Big\|\\
&\quad+k_g\|x_1\|\lim_{\varepsilon\to0^+}\varepsilon^{2(r_1-r_2)}\Big\|H^{-1}(\varepsilon^{r_1}x_1 + q_d)\Big\|\\
&\quad+\lim_{\varepsilon\to0^+}\Big\|\mathcal{H}(\varepsilon^{r_1}x_1)\big[s_1(K_1x_1) + s_2(K_2x_3)\big]\Big\|\\
&\le \Big\|H^{-1}(q_d)C(q_d, x_2)x_2\Big\|\lim_{\varepsilon\to0^+}\varepsilon^{r_1} \qquad (22)\\
&\quad+k_g\|x_1\|\Big\|H^{-1}(q_d)\Big\|\lim_{\varepsilon\to0^+}\varepsilon^{2(r_1-r_2)}\\
&\quad+\Big\|s_1(K_1x_1) + s_2(K_2x_3)\Big\|\lim_{\varepsilon\to0^+}\Big\|\mathcal{H}(\varepsilon^{r_1}x_1)\Big\|\\
&\le \Big\|s_1(K_1x_1) + s_2(K_2x_3)\Big\|\cdot\Big\|\mathcal{H}(0_n)\Big\| = 0
\end{aligned}$$

which completes the proof. □

Corollary 3.1. Consider the proposed control scheme taking σ_{ij}, $i = 1,2,3$, $j = 1,\ldots,n$, such that

$$\sigma_{ij}(\varsigma) = \mathrm{sign}(\varsigma)|\varsigma|^{\beta_{ij}} \quad \forall|\varsigma| \le L_{ij} \in (0,\infty) \qquad (23)$$

with constants $\beta_{ij} = \beta_i$ such that

$$0 < \beta_1 \leq 1 \quad , \quad \beta_2 = \beta_1 \quad , \quad \beta_3 = \frac{1+\beta_1}{2} \tag{24}$$

Thus, $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq 0$, and the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is:

(1) globally finite-time stable if $0 < \beta_1 < 1$;
(2) globally asymptotically stable and (locally) exponentially stable if $\beta_1 = 1$.

Item 1 of Corollary 3.1 is proven by corroborating that, under the stated conditions, for every $j = 1, \ldots, n$ and any $r_1 > 0$, by taking $r_{1j} = r_1$, $r_{2j} = r_2 = (1+\beta_1)r_1/2$ and $r_{3j} = r_3 = r_1$, the requirements of Proposition 3.2 are satisfied with $0 < \beta_1 < 1 \implies r_2 - r_1 < 0 < 2r_2 - r_1$. On the other hand, note that with $r_2 = r_1$ —or analogously $\beta_1 = 1$ in the context of Corollary 3.1— we have that $\varepsilon^{r_2 - r_1} = 1$, $\forall \varepsilon > 0$. Hence, in this case, developments analog to those giving rise to inequalities (22) lead to $\mathcal{L}_0 \leq k_g \|x_1\| \|H^{-1}(q_d)\|$, and consequently, Lemma 2.1 (under the consideration of Remark 2.3) cannot be applied to conclude (local) exponential stability (contrarily to the on-line gravity compensation case of (Zamora-Gómez et al., 2017)). However, while the global asymptotic stability follows from Proposition 3.1, the (local) exponential stability stated through item 2 of Corollary 3.1 is proven by showing that, for sufficiently small and high positive values of ϵ and ϵ_0, respectively,

$$V_2(x_1, x_2, x_3) = V_1(x_1, x_2, x_3) + \epsilon x_1^T H(x_1) x_2 - \epsilon\epsilon_0 x_2^T B^{-1} x_3$$

—with V_1 as defined through Eq. (13)— is a suitable strict Luapunov function of the closed-loop system, on a neighborhood of the origin 0_{3n}. In particular, with

$$\epsilon < \min\{\epsilon_1, \epsilon_2\} \quad , \quad \epsilon_0 > k_C \varrho_1 + \mu_M$$

$$\epsilon_1 = \left[\frac{\bar{k}_{1m} \bar{k}_{2m} \mu_m}{\bar{k}_{2m} \mu_M^2 + \bar{k}_{1m} (\epsilon_0 / b_m)^2} \right]^{1/2}$$

$$\epsilon_2 = \frac{\bar{k}_{1m} \gamma_{22} \tilde{k}_{2m}}{\bar{k}_{1m} \gamma_{22} \gamma_{33} + \bar{k}_{1m} (\gamma_{23}/2)^2 + \gamma_{22} (\gamma_{13}/2)^2}$$

$\bar{k}_{1m} = \min_j\{\bar{k}_{1j}\}$, $\bar{k}_{2m} = \min_j \{k_{2j}/b_j\}$, $b_m = \min_j\{b_j\}$, $\gamma_{22} = \epsilon_0 - k_C \varrho_1 - \mu_M$, $\gamma_{13} = k_{2M} + (k_{1M} + k_g)\epsilon_0/(b_m \mu_m)$, $\gamma_{23} = \epsilon_0 [\bar{a}_M + k_C \varrho_2 / (b_m \mu_m)]$, $\gamma_{33} = \epsilon_0 k_{2M}/(b_m \mu_m)$, $\tilde{k}_{2m} = \min_j \{k_{2j} a_j / b_j\}$, $k_{1M} = \max_j\{k_{1j}\}$, $k_{2M} = \max_j\{k_{2j}\}$, $\bar{a}_M = \max_j\{a_j/bj\}$, $\bar{k}_{2M} = \max_j\{k_{2j}/b_j\}$, any $\varrho_2 > 0$, $\varrho_1 = \max_{x_1 \in \mathcal{Q}_1} \|x_1\|$, $\mathcal{Q}_1 = \mathcal{Q}_{11} \cap \mathcal{Q}_{12}$, $\mathcal{Q}_{11} = \{x_1 \in \mathbb{R}^n : |x_{1j}| \leq \bar{b}_{1j}/\bar{k}_{1j}, j = 1, \ldots, n\}$, $\mathcal{Q}_{12} = \{x_1 \in \mathbb{R}^n : |x_{1j}| \leq L_{1j}/k_{1j}, j = 1, \ldots, n\}$, $\mathcal{Q}_{31} = \{x_3 \in \mathbb{R}^n : |x_{3j}| \leq L_{2j}/k_{2j}, j = 1, \ldots, n\}$, $\mathcal{Q}_{32} = \{x_3 \in \mathbb{R}^n : |x_{3j}| \leq L_{3j}, j = 1, \ldots, n\}$, $\mathcal{Q}_3 = \mathcal{Q}_{31} \cap \mathcal{Q}_{32}$, $\mathcal{B}_2 = \{x \in \mathbb{R}^n : \|x\| \leq \varrho_2\}$,

$$Q_1 = \begin{pmatrix} \bar{k}_{1m} & -\epsilon\mu_M & 0 \\ -\epsilon\mu_M & \mu_m & -\epsilon\epsilon_0/b_m \\ 0 & -\epsilon\epsilon_0/b_m & \bar{k}_{2m} \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} k_{1M} + k_g & \epsilon\mu_M & 0 \\ \epsilon\mu_M & \mu_M & \epsilon\epsilon_0/b_m \\ 0 & \epsilon\epsilon_0/b_m & \bar{k}_{2M} \end{pmatrix}$$

$$Q_3 = \begin{pmatrix} \epsilon\bar{k}_{1m} & 0 & -\epsilon\gamma_{13}/2 \\ 0 & \epsilon\gamma_{22} & -\epsilon\gamma_{23}/2 \\ -\epsilon\gamma_{13}/2 & -\epsilon\gamma_{23}/2 & \tilde{k}_{2m} - \epsilon\gamma_{33} \end{pmatrix}$$

we have, on $\mathcal{Q}_1 \times \mathcal{B}_2 \times \mathcal{Q}_3$, that

$$c_1 \|x\|^2 \leq V_2(x) \leq c_2 \|x\|^2$$
$$\dot{V}_2(x) \leq -c_3 \|x\|^2$$

with $c_1 = \lambda_m(Q_1)/2 > 0$, $c_2 = \lambda_M(Q_2)/2 > 0$ and $c_3 = \lambda_m(Q_3) > 0$, whence we conclude —by (Khalil, 2002, Theorem 4.10)— that the origin $(x_1, x_2, x_3) = (0_n, 0_n, 0_n)$ is a (locally) exponentially stable equilibrium of the closed-loop system. The details omitted in this sketch of the proof of Corollary 3.1 will be thoroughly developed in future communications with more relaxed space limitations.

4. SIMULATION RESULTS

The proposed scheme was tested through computer simulations considering the model of a *Phantom*$^{\text{TM}}$ (model 1.5) haptic interface robot. Thorough technical description and model derivation of such a 3-DOF robotic device are presented in (Cavusoglu and Feygin, 2001) (available at https://www2.eecs.berkeley.edu/Pubs/TechRpts/2001/ERL-01-15.pdf), where H, C and g in (1) are obtained as

$$H(q) = \begin{pmatrix} h_{11}(q) & 0 & 0 \\ 0 & 24.26 & -4.56\sin(q_2 - q_3) \\ 0 & -4.56\sin(q_2 - q_3) & 9.32 \end{pmatrix} \times 10^{-4}$$

$$C(q, \dot{q}) = \begin{pmatrix} c_{11}(q,\dot{q}) & c_{12}(q,\dot{q}) & c_{13}(q,\dot{q}) \\ c_{21}(q,\dot{q}) & 0 & c_{23}(q,\dot{q}) \\ c_{13}(q,\dot{q}) & c_{32}(q,\dot{q}) & 0 \end{pmatrix} \times 10^{-4}$$

$$g(q) = \begin{pmatrix} 0 \\ -162.98\cos(q_2) \\ -737.55\sin(q_3) \end{pmatrix} \times 10^{-4}$$

with

$$h_{11}(q) = 28.33 + 11.32\cos(2q_2) - 3.91\cos(2q_3) + 9.12\cos(q_2)\sin(q_3)$$

$$c_{11}(q,\dot{q}) = -[11.32\sin(2q_2) + 4.56\sin(q_2)\sin(q_3)]\dot{q}_2 + [3.91\sin(2q_3) + 4.56\cos(q_2)\cos(q_3)]\dot{q}_3$$

$$c_{12}(q,\dot{q}) = -c_{21}(q,\dot{q}) = -[11.32\sin(2q_2) + 4.56\sin(q_2)\sin(q_3)]\dot{q}_1$$

$$c_{13}(q,\dot{q}) = -c_{31}(q,\dot{q}) = [3.91\sin(2q_3) + 4.56\cos(q_2)\cos(q_3)]\dot{q}_1$$

$$c_{23}(q,\dot{q}) = 4.56\cos(q_2 - q_3)\dot{q}_3$$

$$c_{32}(q,\dot{q}) = -4.56\cos(q_2 - q_3)\dot{q}_2$$

Assumption 2.3 is thus fulfilled with $B_{g1} = 0$, $B_{g2} = 162.98 \times 10^{-4}$ Nm, $B_{g3} = 737.55 \times 10^{-4}$ Nm and $k_g = 737.55 \times 10^{-4}$ Nm/rad. Input saturation bounds were further valued as $T_j = 1.8$ Nm, $j = 1, 2, 3$, whence Assumption 2.4 can be taken to be fulfilled with $\eta = 3$. For the sake of simplicity, units will be subsequently omitted.

For the application of the proposed design methodology, let us define the functions

$$\sigma_u(\varsigma; \beta, \bar{\alpha}) = \text{sign}(\varsigma) \max\{|\varsigma|^\beta, \bar{\alpha}|\varsigma|\} \tag{25a}$$

$$\sigma_b(\varsigma; \beta, \bar{\alpha}, M) = \text{sign}(\varsigma) \min\{|\sigma_u(\varsigma; \beta, \bar{\alpha})|, M\} \tag{25b}$$

for constants $\beta > 0$, $\bar{\alpha} \in \{0, 1\}$ and $M > 0$. Examples are shown in (Zamora-Gómez et al., 2017, §5).

Based on the functions in Eqs. (25), we define, for every $j = 1, 2$, those involved in the simulations as

$$\sigma_{ij}(\varsigma) = \sigma_b(\varsigma; \beta_i, \bar{\alpha}_{ij}, M_{ij}) \qquad i = 1, 2 \tag{26a}$$

$$\sigma_{3j}(\varsigma) = \sigma_u(\varsigma; \beta_3, \bar{\alpha}_{3j}) \tag{26b}$$

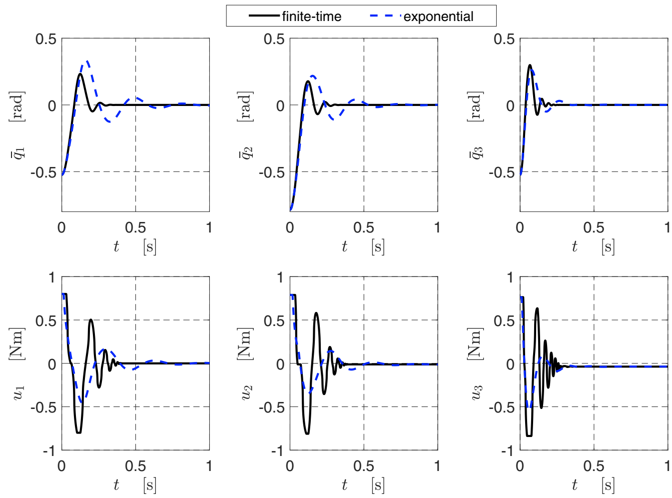

Fig. 2. Finite-time *vs* exponential stabilization

with $\bar{\alpha}_{ij} = 0$, $i = 1, 2, 3$, $j = 1, \ldots, 3$. Conditions on their parameters under which (10) is fulfilled are:

$$k_{1j} > k_g(2B_{gj})^{(1-\beta_1)/\beta_1} \tag{27a}$$

$$M_{1j} > 2B_{gj} \tag{27b}$$

Let us note, from the involved functions, as defined through Eqs. (26), that $B_j = M_{1j} + M_{2j}$, $j = 1, 2$ (see (9)). Thus, by fixing $M_{ij} = 0.4$, $i = 1, 2$, $j = 1, 2$, the inequalities from expressions (9) and (27b) have been simultaneously satisfied. The rest of the control gain/parameter values were chosen taking care that the design requirements were always satisfied. All the implementations were run taking the desired configuration at $q_d = \left(\pi/6 \;\; \pi/4 \;\; \pi/6\right)^T$ [rad] and initial conditions: $q(0) = \dot{q}(0) = (0 \;\; 0 \;\; 0)^T$.

We implemented a test where the aim is to focus on the performance of the finite-time stabilization in contrast to analog exponential regulation implementations. Figure 2 shows results obtained taking $\beta_1 = \beta_2 = 1/2$ and $\beta_3 = 3/4$, for the finite-time controller, and the remaining control gain/parameters were taken, for both (finite-time and exponential) controllers, as: $K_1 = \text{diag}[1, 1, 1]$ (satisfying (27a)), $K_2 = \text{diag}[0.1, 0.1, 0.1]$, $A = \text{diag}[65, 65, 65]$ and $B = \text{diag}[20, 20, 20]$. One sees that both tested controllers achieved the regulation objective avoiding input saturation and with the corresponding types of trajectory convergence. In particular, while the finite-time convergent trajectories attain the desired position fast enough and remain there thereafter, the exponentially convergent responses keep on oscillating entailing a longer practical stabilization.

5. CONCLUSIONS

Global SP-SD-type continuous control of robotic systems with input constraints guaranteeing finite-time or exponential stabilization has been made possible avoiding velocity variables in the feedback and further simplified through desired gravity compensation. Far from what one could have expected, this output-feedback controller is not a simple extension of the on-line compensation case but it has rather proven to need a closed-loop analysis with considerably higher degree of complexity. Simulation results have shown the actual ability of the proposed approach to guarantee the considered types of convergence avoiding input saturation. A more detailed implementation test study focusing on further aspects on the closed-loop performance is intended to be presented on future communications with more relaxed space restrictions. In particular, comparison with observer-based algorithms has been performed in the on-line compensation case in (Zamora-Gómez et al., 2017).

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