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Stability analysis of orbital modes for a generalized Lane-Emden equation

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We present a stability analysis of the standard nonautonomous systems type for a recently introduced generalized Lane-Emden equation which is shown to explain the presence of some of the structures observed in the atomic spatial distributions of magnetically-trapped ultracold atomic clouds. A Lyapunov function is defined which helps us to prove that stable spatial structures in the atomic clouds exist only for the adiabatic index $\gamma = 1 + 1/n$ with even n . In the case when n is odd we provide an instability result indicating the divergence of the density function for the atoms. Several numerical solutions, which according to our stability analysis are stable, are also presented.

Keywords: stability analysis, generalized Lane-Emden equation, nonautonomous system, Lyapunov function, numerical orbital modes

I. THE GENERALIZED LANE-EMDEN EQUATION

In a recent article [13], Rodrigues *et al.* use the condition of hydrostatic equilibrium in the set of the fluid continuity and momentum (Navier-Stokes) equations and a Poisson-like equation to determine the equation of state of a laser-cooled gas within a magneto-optical trap. The authors obtain what they call the *generalized* Lane-Emden equation for the confined atomic profiles, which is also derived in the context of astrophysical fluids [4], and is given by

$$\gamma \frac{1}{\zeta^2} \frac{d}{d\zeta} \left(\zeta^2 \theta^{\gamma-2} \frac{d\theta}{d\zeta} \right) + 1 - \Omega \theta = 0. \quad (1)$$

The adiabatic index γ relates the pressure and density of atoms by the relation $p = C_\gamma \rho^\gamma$, where $C_\gamma = \frac{p(0)}{\rho(0)^\gamma}$, and $p(0)$, $\rho(0)$ are the pressure and density at the center of the atomic cloud while the density at some distance r from the center is given by $\rho(r) = \rho(0)\theta(r)$. Using radial symmetry, $\theta(\zeta)$ is the nondimensional density of atoms which depends on the nondimensional distance ζ from the center of the cloud given by $\zeta = \frac{r}{a_\gamma}$, with $a_\gamma = \sqrt{\frac{C_\gamma}{3m\omega_0^2} \rho(0)^{\frac{\gamma-1}{2}}}$, and trap frequency $\omega_0^2 = \frac{\kappa}{m}$ [13]. The nondimensional constant Ω is the ratio of multiple scattering induced radiation pressure to trapping forces, is defined by $\Omega = \frac{Q\rho(0)}{3m\omega_0^2}$, with Q being the square of effective electric charge [11], and is the equivalent of the plasma frequency [10]. The stability of solutions depends on Ω , as we will see in the following section, with stable solutions found when $0 < \Omega < 1$. Terças *et al.* [16] derived the same equation which models the polytropic equilibrium of a magneto-optical trap that describes the crossover between the two limiting cases: temperature-dominated ($\Omega \rightarrow 0$), and multiple-scattering-dominated traps ($\Omega \rightarrow 1$).

For $\gamma = 2$, (1) is linear

$$\theta_{\zeta\zeta} + \frac{2}{\zeta} \theta_\zeta + \frac{1 - \Omega \theta}{2} = 0, \quad (2)$$

and may be roughly considered as a repulsive electrostatic counterpart of the classic Lane-Emden equation in astrophysics with solutions representing the Newton-Poisson gravitational potential of stars, considered

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as spheres filled with polytropic gas [6]. For other values of γ , (1) has an additional operatorial term of the form $\gamma(\gamma - 2)\theta\zeta^2/\theta$, and also two nonoperatorial terms instead of one.

When the effects of the multiple scattering can be neglected, and thermal effects dominate $\Omega \rightarrow 0$, and for $\gamma \neq 1$ the atomic density is given by

$$\theta(\zeta) = \left[(1 - \gamma) \left(c_1 + \frac{c_2}{\zeta} + \frac{\zeta^2}{6\gamma} \right) \right]^{\frac{1}{\gamma-1}}. \quad (3)$$

For a bounded density at the origin $c_2 = 0$, and using $\theta(0) = \theta_0 > 0$ the solution to (3) corresponding to the density of atoms having no effective charge is

$$\theta(\zeta) = \left[\theta_0^{\gamma-1} - \frac{\gamma-1}{6\gamma} \zeta^2 \right]^{\frac{1}{\gamma-1}}. \quad (4)$$

The case $\gamma = 1$ corresponds to an isothermal gas as discussed in [13], and yields the Maxwell-Boltzmann equilibrium where (1) reduces to the separable equation

$$\frac{d}{d\zeta} \left(\zeta^2 \frac{d \ln \theta}{d\zeta} \right) = -\zeta^2. \quad (5)$$

By two quadratures, and assuming the same initial conditions, the atomic density has the Gaussian profile

$$\theta(\zeta) = \theta_0 e^{-\frac{\zeta^2}{6}}. \quad (6)$$

This solution can also be found by taking the limit $\gamma \rightarrow 1$ in (4), and describes a trap in the temperature-limited regime when scattering effects are negligible [12].

On the other hand, when the effects of multiple scattering dominate then $\Omega \rightarrow 1$. For an isobaric process, $\gamma = 0$, the density of atoms is given by

$$\theta(\zeta) = \frac{1}{\Omega} H(\xi - \xi_0), \quad (7)$$

where $H(\xi - \xi_0)$ is the Heaviside step function, and $\xi_0 = \frac{3}{\sqrt[3]{4\pi\Omega}}$ is the Lane-Emden radius of the traps, which was also observed in the experiments of [7]. This solution is known as the *water-bag* equilibrium profile [12, 18].

For the aforementioned case $\gamma = 2$, which can be taken into account in the astrophysics of dark matter halos with possible substantial pressure compared to the atom density of the *dark* atoms [3, 4], the polytropic equation of state is $p = C_2 \rho^2$. Then (2) can be written in self-adjoint form

$$\frac{d}{d\zeta} \left(\zeta^2 \frac{d\theta}{d\zeta} \right) + \zeta^2 \frac{1 - \Omega\theta}{2} = 0 \quad (8)$$

and if we use $\psi = \frac{1 - \Omega\theta}{2}$, and $A = -\frac{\Omega}{2}$, it becomes

$$\frac{d}{d\zeta} \left(\zeta^2 \frac{d\psi}{d\zeta} \right) + A\zeta^2 \psi = 0. \quad (9)$$

In the latter format, this equation is close to the ‘electrostatic Lane-Emden equation’, known as the Thomas-Fermi equation, associated with the field of the statistical distribution of electrons in heavy atoms which takes into account their repulsive electrostatic interactions [17]

$$\frac{d}{d\zeta} \left(\zeta^2 \frac{d\psi}{d\zeta} \right) + B\zeta^2 \psi^{\frac{3}{2}} = 0, \quad (10)$$

where $B = -8\sqrt{2}/3\pi$, see equation (1.2) in [17].

By assuming initial conditions $\theta(0) = \theta_0$, and $\theta_\zeta(0) = 0$, the atom density is given by

$$\theta(\zeta) = \frac{1}{\Omega} \left[1 + (\theta_0\Omega - 1) \operatorname{shc} \left(\sqrt{\frac{\Omega}{2}} \zeta \right) \right], \quad (11)$$

where shc denotes the cardinal hyperbolic sinus function defined by

$$\text{shc}(x) := \begin{cases} \frac{\sinh(x)}{x}, & \text{for } x \neq 0, \\ 1, & \text{for } x = 0. \end{cases}$$

The boundary of the halo ζ_M , can be found numerically by assuming that on the boundary we must have zero atom density, thus $\text{shc}\left(\sqrt{\frac{\Omega}{2}}\zeta_M\right) = \frac{1}{1-\theta_0\Omega}$, provided that $0 < \Omega < \frac{1}{\theta_0}$.

Since (1) cannot be solved analytically for other values of γ , an important issue that we address in the rest of this paper is the stability analysis of solutions that are obtained numerically. We develop this stability analysis using a generalized Lyapunov function for the corresponding nonautonomous system of differential equations in Section 2. If the positive departure from unity of the adiabatic index is parametrized by $1/n$, where $n \in \mathbb{N}$, then we find that stable solutions exist only for even n , while for odd n all the solutions are unstable. The numerical solutions are presented in Section 3 as a proof of the results obtained in the previous section, and the paper ends with a conclusion section.

II. THE STABILITY ANALYSIS

For any $\gamma \neq 1$, we write (1) in self-adjoint form as

$$\frac{d}{d\zeta} \left(\zeta^2 \frac{d}{d\zeta} \theta^{\gamma-1} \right) = \frac{\gamma-1}{\gamma} \zeta^2 (\Omega\theta - 1). \quad (12)$$

Letting $\theta = z^{\frac{1}{\gamma-1}}$ we obtain

$$\frac{d}{d\zeta} \left(\zeta^2 \frac{dz}{d\zeta} \right) = \frac{\gamma-1}{\gamma} \zeta^2 \left(\Omega z^{\frac{1}{\gamma-1}} - 1 \right), \quad (13)$$

and by using $\gamma = 1 + 1/n$ with $n \in \mathbb{N}$, we get

$$\frac{d}{d\zeta} \left(\zeta^2 \frac{dz}{d\zeta} \right) = \frac{\zeta^2}{n+1} (\Omega z^n - 1). \quad (14)$$

Eq. (14) can be written as a nonautonomous first order system

$$\frac{dy}{d\zeta} = f(\zeta, y) \quad (15)$$

using

$$y = \begin{pmatrix} z \\ z_\zeta \end{pmatrix}, \quad \text{and} \quad f(z, z_\zeta) = \left(\frac{1}{n+1} (\Omega z^n - 1) - \frac{2z_\zeta}{\zeta} \right).$$

There is only one real critical point of the system (15) for n odd given by $\bar{y}_0^T = (\frac{1}{\Omega^{1/n}}, 0)$ while for n even there are two $\bar{y}_{1,2}^T = (\mp \frac{1}{\Omega^{1/n}}, 0)$. All the critical points are shifted to the origin by the substitution $x^T = y^T - \bar{y}^T = (x_1, x_2)$, then (15) becomes

$$x_\zeta = f(\zeta, x + \bar{y}) = \left(\frac{1}{n+1} \left[\Omega \left(x_1 \mp \frac{x_2}{\Omega^{1/n}} \right)^n - 1 \right] - \frac{2x_2}{\zeta} \right). \quad (16)$$

First, we define the notion of stability for a general nonlinear nonautonomous system

$$x_\zeta = f(\zeta, x), \quad \zeta \geq \zeta_0, \quad (17)$$

where $\zeta_0 \geq 0$ and $x(\zeta_0) = x_0 \in \mathbb{R}^n$. We assume $f : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous and satisfies $f(\zeta, 0) = 0$, $\forall \zeta \geq \zeta_0$.

Definition II.1. (9) The system (17) is said to be:

(i) *Lyapunov stable* if for each $\epsilon > 0$, there is $\delta(\epsilon, \zeta_0) > 0$ such that

$$\|x(\zeta_0)\| < \delta \Rightarrow \|x(\zeta)\| < \epsilon, \quad \forall \zeta \geq \zeta_0 \geq 0. \quad (18)$$

(ii) *Uniformly stable* if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$, independent of ζ_0 , such that (18) is satisfied.

(iii) *Asymptotically stable* if it is *Lyapunov stable*, and there is a constant $l = l(\zeta_0) > 0$ such that $x(\zeta) \rightarrow 0$ as $\zeta \rightarrow \infty$, for all $\|x(\zeta_0)\| < l$.

(iv) *Uniformly asymptotically stable* if it is *uniformly stable*, and there is a constant $l > 0$ independent of ζ_0 , such that for all $\|x(\zeta_0)\| < l$, $x(\zeta) \rightarrow 0$ as $\zeta \rightarrow \infty$, uniformly in ζ_0 . This means that for each $\eta > 0$ there is $T = T(\eta) > 0$ such that

$$\|x(\zeta)\| < \eta, \quad \forall \zeta \geq \zeta_0 + T(\eta), \quad \forall \|x(\zeta_0)\| < l. \quad (19)$$

(v) *Exponentially stable* if there exist positive constants l, k , and p such that

$$\|x(\zeta)\| \leq k \|x(\zeta_0)\| e^{-p(\zeta - \zeta_0)}, \quad \forall \|x(\zeta_0)\| < l. \quad (20)$$

Before attempting the stability problem for the nonlinear system (16), it behooves us to first study the stability of the linearization of (16) about $\bar{x}^T = (0, 0)$. Supposing n is even, we use the left equilibrium $\bar{y}_1^T = (-\frac{1}{\Omega^{1/n}}, 0)$, and we set $A(\zeta) = \nabla_x f(\zeta, x)|_{x=\bar{x}}$, then

$$A(\zeta) = \left(\begin{array}{cc} 0 & 1 \\ \frac{\Omega}{1+1/n} (x_1 - \frac{1}{\Omega^{1/n}})^{n-1} & -\frac{2}{\zeta} \end{array} \right) \Bigg|_{x=\bar{x}} = \left(\begin{array}{cc} 0 & 1 \\ -\frac{1}{1+1/n} \Omega^{1/n} & -\frac{2}{\zeta} \end{array} \right). \quad (21)$$

Using (21) we then consider the corresponding linear distance-varying system to (16),

$$x_\zeta = A(\zeta)x(\zeta), \quad x(\zeta_0) = x_0. \quad (22)$$

To establish the stability of (22), we make use of a linear matrix inequality developed in [5]. We restate the result below for completeness.

Theorem II.2. (Theorem 1 in [5].) Consider the system (22). Suppose there exists a positive definite differentiable matrix function $P: \mathbb{R}^+ \rightarrow \mathbb{R}^{n \times n}$ and a continuous function $g: \mathbb{R}^+ \rightarrow \mathbb{R}$ such that

$$A^T P + PA + P_\zeta \leq g(\zeta)P \quad (23)$$

with $\int_{\zeta_0}^{\infty} g(\zeta) d\zeta = -\infty$, then the system (22) is asymptotically stable.

Theorem II.3. The origin is asymptotically stable for the system (22).

Proof. To apply the above result to the linear system (22) let $P(\zeta) = \begin{pmatrix} a(\zeta) & b(\zeta) \\ b(\zeta) & c(\zeta) \end{pmatrix}$, then

$$A^T P + PA + P_\zeta = \begin{pmatrix} a_\zeta - \frac{\Omega^{1/n}}{1+1/n} b & a - \frac{2}{\zeta} b + b_\zeta - \frac{\Omega^{1/n}}{1+1/n} c \\ a - \frac{2}{\zeta} b + b_\zeta - \frac{\Omega^{1/n}}{1+1/n} c & 2b - \frac{4}{\zeta} c + c_\zeta \end{pmatrix}. \quad (24)$$

By setting $b = 0$, $a = \frac{1}{\zeta}$, $c = \frac{1+1/n}{\Omega^{1/n}\zeta}$, and $g(\zeta) = -\frac{1}{\zeta}$ the matrix inequality (23) becomes

$$\begin{pmatrix} -\frac{1}{\zeta^2} & 0 \\ 0 & -\frac{5(1+1/n)}{\Omega^{1/n}\zeta^2} \end{pmatrix} \leq \begin{pmatrix} -\frac{1}{\zeta^2} & 0 \\ 0 & -\frac{1+1/n}{\Omega^{1/n}\zeta^2} \end{pmatrix}, \quad (25)$$

hence for $\zeta_0 > 0$, (25) is satisfied for $\zeta \geq \zeta_0$. Furthermore, the matrix function

$$P(\zeta) = \begin{pmatrix} \frac{1}{\zeta} & 0 \\ 0 & \frac{1+1/n}{\Omega^{1/n}\zeta} \end{pmatrix},$$

is differentiable. For the positive definiteness of $P(\zeta)$ we compute $x^T P x$,

$$x^T P x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} \frac{1}{\zeta} & 0 \\ 0 & \frac{1+1/n}{\Omega^{1/n}\zeta} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\zeta} \left(x_1^2 + \frac{1+1/n}{\Omega^{1/n}} x_2^2 \right) > 0,$$

for x_1 and x_2 both nonzero. Therefore the distance-varying system (22) is locally asymptotically stable. ■

It is only of interest to us to study the left equilibrium point $\bar{y}_1^T = (-\frac{1}{\Omega^{1/n}}, 0)$ for stability, as the equilibrium point $\bar{y}_2^T = (\frac{1}{\Omega^{1/n}}, 0)$ will be shown to be unstable (see Theorem II.10), for $n \in \mathbb{N}$. Before we state the stability/instability results for the nonlinear system (16) we first need to introduce some terminology that is used in [15].

Definition II.4. ([15]) A scalar continuous function $V(x)$ is said to be *locally positive definite* if $V(0) = 0$ and, in a ball $B_r(0)$

$$x \neq 0 \implies V(x) > 0. \quad (26)$$

If $V(0) = 0$ and the above property holds over the whole state space, then $V(x)$ is said to be *globally positive definite*.

Related concepts can be defined analogously, in a local or global sense. A function $V(x)$ is *negative definite* if $-V(x)$ is positive definite; $V(x)$ is positive *semi-definite* if $V(0) = 0$ and $V(x) \geq 0$ for $x \neq 0$; $V(x)$ is *negative semi-definite* if $-V(x)$ is positive semi-definite. For our purposes we will need the notion of positive definiteness for a scalar valued distance-varying function.

Definition II.5. ([15]) A scalar distance-varying function $V(x, \zeta)$ is *locally positive definite* if $V(0, \zeta) = 0$, and there exists a distance-invariant positive definite function $V_0(x)$ such that

$$\forall \zeta \geq \zeta_0, \quad V(x, \zeta) \geq V_0(x). \quad (27)$$

Thus, a distance-variant function is locally positive definite if it dominates a distance-invariant locally positive function. The notions of negative definite, semi-definite, and negative semi-definite can be defined similarly.

Next, we introduce the notion of a decrescent function.

Definition II.6. ([15]) A scalar function $V(x, \zeta)$ is said to be *decrescent* if $V(0, \zeta) = 0$, and if there exists a distance-invariant positive definite function $W(x)$ such that

$$\forall \zeta \geq 0, \quad V(x, \zeta) \leq W(x). \quad (28)$$

In other words, $V(x, \zeta)$ is decrescent if it is dominated by a distance-invariant positive definite function.

With the above definitions introduced we now state the classical Lyapunov stability theorem for non-autonomous systems.

Theorem II.7. ([15]) If, in a ball $B_r(0)$ around the equilibrium point 0, there exists a scalar function $V(x, \zeta)$ with continuous partial derivatives such that

- (i) V is positive definite,
- (ii) V_ζ is negative semi-definite.

Then the equilibrium point 0 is stable in the sense of Lyapunov. If, furthermore,

- (iii) V is decrescent,

then the origin is uniformly stable.

With the stability of the linear system established, we turn our attention to the nonlinear system (16). We seek to prove the stability of the origin for (16) in the case where n is even and $\bar{y}_2^T = (-\frac{1}{\Omega^{1/n}}, 0)$, leading to the system (16) with the minus sign (-).

Defining the Lyapunov function

$$V(x) = -\frac{2\Omega}{(n+1)^2} \left[\left(x_1 - \frac{1}{\Omega^{1/n}} \right)^{n+1} + \frac{1}{\Omega^{1+1/n}} \right] + \frac{2}{n+1} x_1 + x_2^2, \quad (29)$$

we then calculate the derivative of V

$$\begin{aligned} V_\zeta &= -\frac{2\Omega x_2}{n+1} \left(x_1 - \frac{1}{\Omega^{1/n}} \right)^n + \frac{2x_2}{n+1} + \frac{2x_2}{n+1} \left[\Omega \left(x_1 - \frac{1}{\Omega^{1/n}} \right)^n - 1 \right] - \frac{4x_2^2}{\zeta} \\ &= -\frac{4x_2^2}{\zeta} \leq 0. \end{aligned} \quad (30)$$

If $x_1 < \frac{1}{\Omega^{1/n}}$, then $-x_1\Omega^{1/n} \geq -1$. This allows us to utilize the Bernoulli inequality $(1+u)^s \geq 1+su$, where $u \geq -1$, $s \in \mathbb{R}^+ \setminus (0, 1)$.

Theorem II.8. The origin is a uniformly Lyapunov stable equilibrium point of system (16) for n even.

Proof. We start by establishing that $V(x)$ is a locally positive definite function for $x \in B_r(0)$ where $r = \frac{2}{\Omega^{1/n}}$, and $\zeta \geq \zeta_0$. We first consider the case where $x_1 \leq \frac{1}{\Omega^{1/n}}$, and we apply the Bernoulli inequality above to obtain

$$\begin{aligned} V(x) &= \frac{2\Omega}{(n+1)^2} \frac{1}{\Omega^{1+1/n}} \left(1 - x_1\Omega^{1/n} \right)^{n+1} - \frac{2}{(n+1)^2\Omega^{1/n}} + \frac{2}{n+1} x_1 + x_2^2 \\ &\geq \frac{2}{(n+1)^2} \frac{1}{\Omega^{1/n}} \left(1 - (n+1)x_1\Omega^{1/n} \right) - \frac{2}{(n+1)^2\Omega^{1/n}} + \frac{2}{n+1} x_1 + x_2^2 \\ &= -\frac{2}{n+1} x_1 + \frac{2}{n+1} x_1 + x_2^2 = x_2^2 \geq 0. \end{aligned}$$

To consider the case where $\frac{1}{\Omega^{1/n}} \leq x_1 \leq \frac{2}{\Omega^{1/n}}$ we note that the function

$$W(u) = -\frac{2\Omega}{(n+1)^2} \left[\left(u - \frac{1}{\Omega^{1/n}} \right)^{n+1} + \frac{1}{\Omega^{1+1/n}} \right] + \frac{2}{n+1} u$$

satisfies

$$W_u(u) = -\frac{2\Omega}{n+1} \left(u - \frac{1}{\Omega^{1/n}} \right)^n + \frac{2}{n+1}.$$

We then see that $W_u(u) \geq 0$ for $x_1 \in [\frac{1}{\Omega^{1/n}}, \frac{2}{\Omega^{1/n}}]$, and since $W(\frac{1}{\Omega^{1/n}}) = \frac{2}{n+1} \left(1 - \frac{1}{n+1} \right) > 0$, we conclude that $W(u) \geq 0$ on $[\frac{1}{\Omega^{1/n}}, \frac{2}{\Omega^{1/n}}]$ which in turn implies that $V(x)|_{B_r(0)} \geq 0$.

Since V is distance-invariant it follows that V is a locally positive definite function, moreover it is evident that V is also a decrescent function. Therefore using Theorem III.7 together with $V(0) = 0$, and $V_\zeta \leq 0$ we conclude that the origin is uniformly Lyapunov stable. ■

Theorem II.9. The equilibrium point 0 of system (16) is asymptotically stable when n is even.

Proof. To study the asymptotic stability of the origin, we note that on the domain $D = \{x_1 \leq \frac{2}{\Omega^{1/n}}\}$ we have that $V|_D$ is positive definite and $V_\zeta|_D \leq 0$. Unlike in the autonomous case where LaSalle's invariance theorem can be used to find a maximally invariant set that the trajectories converge to, in the nonautonomous case it is less clear how to define such a set. In fact since $-V_\zeta$ is not a locally positive definite function, Theorem 8.4 in [9] does not directly apply. We can still use the ideas from Theorem 8.4 to construct an invariant set in D . If we consider the closed ball of radius $r = \frac{2}{\Omega^{1/n}}$ centered at the origin, then $B_r(0) \subset D$. We introduce the following distance invariant set $\mathcal{B}_\alpha = \{x \in B_r(0) \mid V(x) \leq \alpha\}$, where

$\alpha < \min_{\|x\|=r} V(x)$. To calculate $\min_{\|x\|=r} V(x)$ we proceed as follows: on $\overline{B_r(0)}$ we can write $V(x_1) = -\frac{2\Omega}{(n+1)^2} \left[\left(x_1 - \frac{1}{\Omega^{1/n}} \right)^{n+1} + \frac{1}{\Omega^{1+1/n}} \right] + \frac{2}{n+1} x_1 + r^2 - x_1^2$ with the derivative $V_{x_1} = -\frac{2}{n+1} (1 - \Omega^{1/n} x_1)^n + \frac{2}{n+1} - 2x_1$. We then see that $V_{x_1}(0) = 0$, and note that $V_{x_1 x_1} = \frac{2\Omega^{1/n}}{1+1/n} (1 - \Omega^{1/n} x_1)^{n-1} - 2$. If $-\frac{2}{\Omega^{1/n}} \leq x_1 \leq \frac{1}{\Omega^{1/n}} \left[1 - \left(\frac{1+1/n}{\Omega^{1/n}} \right)^{\frac{1}{n-1}} \right]$ then $V_{x_1 x_1} \geq 0$, and for $x_1 \geq \frac{1}{\Omega^{1/n}} \left[1 - \left(\frac{1+1/n}{\Omega^{1/n}} \right)^{\frac{1}{n-1}} \right]$ then $V_{x_1 x_1} \leq 0$. Thus, $x_1 = 0$ is the only critical point. Since $V(\frac{2}{\Omega^{1/n}}, 0)$ is less than both $V(0, 0)$ and $V(-\frac{2}{\Omega^{1/n}}, 0)$ then $\min_{\|x\|=r} V(x) = V(\frac{2}{\Omega^{1/n}}, 0)$.

Returning to the set \mathcal{B}_α , we have that $\mathcal{B}_\alpha \subset B_r(0) \subset D$ for all $\zeta \geq \zeta_0$ which implies that \mathcal{B}_α is bounded. Since $V_\zeta \leq 0$ in D we have that for $\zeta_0 > 0$ the solution starting at (x_0, ζ_0) stays in \mathcal{B}_α for all $\zeta \geq \zeta_0$, hence the solution is bounded for all $\zeta \geq \zeta_0$. Moreover, from (30) we have

$$V(x(\zeta)) - V(x(\zeta_0)) = -4 \int_{\zeta_0}^{\zeta} \frac{x_2^2(s)}{s} ds. \quad (31)$$

Since V is bounded below on D , and $V_\zeta \leq 0$ we have that $\lim_{\zeta \rightarrow \infty} V(x(\zeta), \zeta) = V_\infty$ exists, and $V_\infty \leq V(x(\zeta_0), \zeta_0)$. Taking $\zeta \rightarrow \infty$ in (31) yields

$$V(x(\zeta_0)) - V_\infty = 4 \lim_{\zeta \rightarrow \infty} \int_{\zeta_0}^{\zeta} \frac{x_2^2(s)}{s} ds. \quad (32)$$

Next, we claim that $\lim_{\zeta \rightarrow \infty} x_2^2(\zeta) = 0$. By contradiction, suppose that $\lim_{\zeta \rightarrow \infty} x_2^2(\zeta) = L$ where $L \neq 0$ (note that $\lim_{\zeta \rightarrow \infty} x_2(\zeta)$ exists since V_∞ exists). Then for $\epsilon = \frac{L}{2}$ there exists $\zeta_1 > 0$ such that $x_2^2(\zeta) > L/2$ for $\zeta \geq \zeta_1$. This implies that

$$\lim_{\zeta \rightarrow \infty} \int_{\zeta_1}^{\zeta} \frac{x_2^2(s)}{s} ds \geq \frac{L}{2} \lim_{\zeta \rightarrow \infty} \int_{\zeta_1}^{\zeta} \frac{1}{s} ds = \infty. \quad (33)$$

Eq. (33) implies that $V(x(\zeta_0)) - V_\infty$ is divergent which is a contradiction, therefore $\lim_{\zeta \rightarrow \infty} x_2(\zeta) = 0$. Moreover, $x_{2\zeta}$ is uniformly continuous since

$$x_{2\zeta} = \frac{\Omega}{1+1/n} \left(x_1 - \frac{1}{\Omega^{1/n}} \right)^{n-1} x_2 - \frac{2}{\zeta(n+1)} \left[\Omega \left(x_1 - \frac{1}{\Omega^{1/n}} \right)^n - 1 \right] + \frac{6x_2}{\zeta^2}$$

is bounded because x_1 and x_2 are bounded functions. Thus, from Barbalat's lemma (15) we see that $\lim_{\zeta \rightarrow \infty} x_{2\zeta} = 0$ which implies from (16) that

$\lim_{\zeta \rightarrow \infty} \Omega \left(x_1 - \frac{1}{\Omega^{1/n}} \right)^n - 1 = 0$, and hence $\lim_{\zeta \rightarrow \infty} x_1(\zeta) = 0$. ■

At this point, we have established that the origin is asymptotically stable for (16). With this in mind, it is of interest to estimate the basin of attraction for the origin by using the ideas from the proof of Theorem II.9. We start with noting that the constant α from the above proof satisfies $\alpha < V(\frac{2}{\Omega^{1/n}}, 0) = \frac{4n}{\Omega^{1/n}(n+1)^2}$. Setting $\alpha = \frac{4n}{\Omega^{1/n}(n+1)^2} - \delta$ for δ sufficiently small, the set \mathcal{B}_α can be expressed as $\mathcal{B}_\delta = \left\{ x \in B_{\frac{2}{\Omega^{1/n}}}(0) \mid V(x) \leq \frac{4n}{\Omega^{1/n}(n+1)^2} - \delta \right\}$. Therefore, for an initial condition of the form $x_0^T = (\frac{1}{\Omega^{1/n}} - \epsilon, 0)$ for $\epsilon > 0$ arbitrarily small, we can find $\delta > 0$ such that \mathcal{B}_δ is a positively invariant set containing x_0 and hence the flow to (16) with initial condition x_0 converges asymptotically to the origin.

To continue our stability analysis, we consider the equilibrium point $\bar{y}_0^T = (\frac{1}{\Omega^{1/n}}, 0)$, which is present as the only equilibrium when n is odd, and is one of two equilibria for n even. It is our intention to establish \bar{y}_0 as unstable. To accomplish this we apply one of the instability theorems found in (15). The notion of instability used here is taken from (8), that is the equilibrium of the system (16) is called unstable if it is not stable. To this end, there exists an $\epsilon > 0$ arbitrarily small such that no $\delta > 0$ exists ensuring Lyapunov stability. This means there exists a x_0 arbitrarily close to 0 and a sequence $\zeta_n \rightarrow \infty$ so that $|x(\zeta_n)| \geq \epsilon$ for all $\zeta_n \geq \zeta_0$, where $x(\zeta_0) = x_0$.

Theorem II.10. (15) If, in a certain neighborhood \mathcal{B}_0 of the origin, there exists a continuously differentiable, decreascent scalar function $V(x, \zeta)$ such that

- (i) $V(0, \zeta) = 0, \quad \zeta \geq \zeta_0,$
- (ii) $V(x, \zeta_0)$ can assume strictly positive values arbitrarily close to the origin,
- (iii) $V_\zeta(x, \zeta)$ is positive definite (locally in \mathcal{B}_0),

then the equilibrium point 0 at distance ζ_0 is unstable.

Recall for n odd, the only real equilibrium point is $\bar{y}_0^T = (\frac{1}{\Omega^{1/n}}, 0)$, and when n is even the real equilibrium point lying on the negative horizontal axis is $\bar{y}_1^T = (-\frac{1}{\Omega^{1/n}}, 0)$. For the odd case the system obtained by shifting $x = y - \bar{y}_0$ is (16) with the (+) sign. In what follows we show that the origin is unstable in the case where n is odd.

Theorem II.11. The equilibrium point 0 of (16) is unstable for odd n .

Proof. We apply the above instability Theorem (II.10). We define the following scalar valued Lyapunov function

$$V(x, \zeta) = \left(\Omega \left(x_1 + \frac{1}{\Omega^{1/n}} \right)^n - 1 \right) x_2 + \frac{(n+1)x_2^2}{\zeta}, \quad (34)$$

noting that $V(0, \zeta) = 0$, and also that $V(x, \zeta)$ is a decrescent function. Furthermore, for $x_1 = 0$ and $x_2 \neq 0$, $V(x, \zeta)$ is strictly positive, hence condition (ii) in Theorem (II.10) is satisfied. For the derivative we obtain

$$\begin{aligned} V_\zeta &= \frac{1}{n+1} \left(\Omega \left(x_1 + \frac{1}{\Omega^{1/n}} \right)^n - 1 \right)^2 + n\Omega x_2^2 \left(x_1 + \frac{1}{\Omega^{1/n}} \right)^{n-1} - \frac{5(n+1)x_2^2}{\zeta^2} \\ &= \frac{1}{n+1} \left(\Omega \left(x_1 + \frac{1}{\Omega^{1/n}} \right)^n - 1 \right)^2 + x_2^2 \left(n\Omega \left(x_1 + \frac{1}{\Omega^{1/n}} \right)^{n-1} - \frac{5(n+1)}{\zeta^2} \right). \end{aligned}$$

Since $(x_1 + \frac{1}{\Omega^{1/n}})^{n-1} \geq (-\frac{1}{2\Omega^{1/n}} + \frac{1}{\Omega^{1/n}})^{n-1} = \frac{1}{2^{n-1}\Omega^{1-1/n}}$, then by taking $\zeta_0^2 = 1 + \frac{5(1+1/n)2^{n-1}\Omega^{1-1/n}}{\Omega}$ we have that

$$\begin{aligned} V_\zeta &\geq \frac{1}{n+1} \left(\Omega \left(x_1 + \frac{1}{\Omega^{1/n}} \right)^n - 1 \right)^2 + x_2^2 \left(n\Omega \left(x_1 + \frac{1}{\Omega^{1/n}} \right)^{n-1} - \frac{5(n+1)}{\zeta_0^2} \right) \\ &= V_0(x) \geq 0, \end{aligned} \quad (35)$$

provided $\|x\| < r = \frac{2}{\Omega^{1/n}}$. So $V_0(x)$ is a distance-invariant locally positive definite function, therefore (iii) is satisfied, and Theorem (II.10) implies that 0 is unstable at ζ_0 . Thus, we can conclude that 0 is unstable for all $\zeta \geq \zeta_0$. ■

III. ORBITAL MODES

In this section, we discuss the effects of n and Ω on the solutions of (1). The formation of stable orbital modes in the density function was observed in rotary clouds of atoms [1, 2, 14, 18], and was previously believed to be a consequence of rotation in the cloud. The analysis of the previous section reveals bounded solutions when n is even which are periodic as long as the starting initial conditions are to the left of the unstable positive fixed point \bar{y}_2^T , and are inside the basin of attraction \mathcal{B}_α . Since we use the initial condition $\theta(\zeta_0) = 1$, and to be inside of the basin we require that $1 < \frac{1}{\Omega^{1/n}}$. This yields to $0 < \Omega < 1$, which was also claimed numerically, and by linear analysis by [16]. The solutions are unbounded when n is odd since there is only one unstable equilibrium given by \bar{y}_0^T , or if the initial conditions are outside of the basin of attraction \mathcal{B}_α . In [13] and [16], the authors discuss orbital modes for the cases $\gamma = 1, 2, 4$. However, the case $\gamma = 1$ is discussed in Section 1, and $\gamma = 2$ follows from the conclusion of Theorem (II.11). It is relevant to note that the case $\gamma = \frac{3}{2}$ is studied in [12] of which the stability follows from Theorem (II.9).

The left panel of Fig. 1 depicts a sublevel set for the function $V(x + \bar{y}_1)$, which are the translations of the level sets of the Lyapunov function $V(x)$. The right panel corresponds to an asymptotically stable solution to (15) with $n = 2$, $\Omega = 0.5$, and left stable fixed point $\bar{y}_1^T = (-\frac{1}{\Omega^{1/n}}, 0)$. We start from initial condition $x_0^T = (\frac{1}{\Omega^{1/n}} - \epsilon, 0)$ located to the left of the right unstable fixed point $\bar{y}_2^T = (\frac{1}{\Omega^{1/n}}, 0)$. The shaded region in the right panel represents the sublevel set $\{x \in \mathbb{R}^2 \mid V(x + \bar{y}_1) \leq \alpha\}$, we take the bounded component of this set, then translate it to the right by \bar{y}_1 to obtain \mathcal{B}_α which is used to estimate the basin of attraction for the origin of system (16).

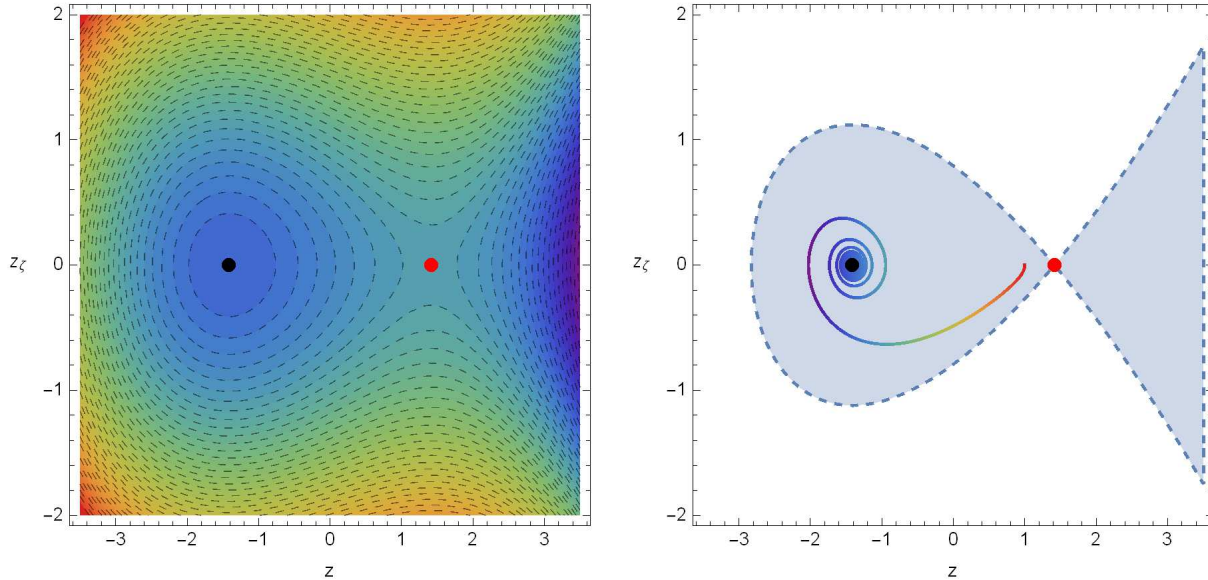


FIG. 1: Left panel: translations of the level sets of the Lyapunov function $V(x)$ given by (29). Right panel: basin of attraction for periodic solutions of (16). The fixed points are indicated by the black and red dots, left fixed point is always stable while the right fixed point is always unstable. Simulations must start inside of the basin of attraction which require that $0 < \Omega < 1$. Also, the necessary conditions to have two fixed points is that n is even. A typical trajectory starts close and to the left of the unstable fixed point and converges to the stable fixed point.

The left panel of Fig. 2 shows the solutions $z(\zeta)$, $z_c(\zeta)$ of (15) with the red dashed line representing the right (unstable) equilibrium, and the black dashed line the left (stable) equilibrium point. The right figure depicts the density function $\theta(\zeta)$ of (1), with $\gamma = \frac{3}{2}$, $\Omega = 0.5$, and initial conditions $\theta(\zeta_0) = 1$, $\theta_c(\zeta_0) = 0$. Since $\zeta = 0$ is a singular point of (1), for numerical simulations, we used $\zeta_0 = 0.001$. When $n = 2, 4, 6$ which gives $\gamma = \frac{3}{2}, \frac{5}{4}, \frac{7}{6}$, and starting with the same initial conditions, we obtain periodic solutions that are depicted in Fig. 3. The boundary ζ^* is given by $\theta(\zeta) = 0$, is increasing as γ is increasing, and represents the nondimensional radius for which the density of atoms is zero.

For a fixed adiabatic index, and with the number of atoms that increases then Ω increases, thus the central core density increases. This was observed experimentally when a single ring turns into a ring with a central core, causing the cloud to rotate faster [2], as we can see in Fig. 4.

IV. CONCLUSION

A detailed stability analysis of a generalized Lane-Emden equation recently elaborated in the physics of trapped atomic clouds, and previously in astrophysics has been presented. The linearized form of the corresponding nonautonomous first order system has been shown to be locally asymptotically stable, and the Lyapunov indirect approach has been used to construct a Lyapunov function for the nonlinear system. When n is even there are two equilibrium points $\bar{y}_1 < \bar{y}_2$, and we showed that the left equilibrium point \bar{y}_1 is asymptotically stable. Using the Lyapunov function we provide an estimate for the basin of attraction

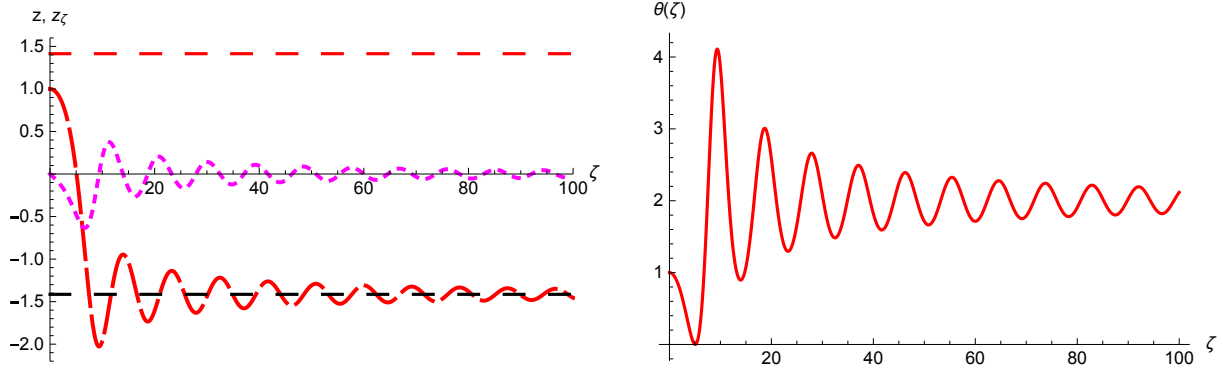


FIG. 2: Left panel: solutions $z(\zeta)$, $z_\zeta(\zeta)$ of (15), with the red dashed line representing the right (unstable) fixed point, and the black dashed line the left (stable) fixed point. The red dashed function $z(\zeta)$ and the magenta dotted function $z_\zeta(\zeta)$ represent the trajectory from the right panel of Fig. 1. Right panel: density function $\theta(\zeta) = z^2$ of (1), with $\gamma = \frac{3}{2}$, $\Omega = 0.5$ of the solution $z(\zeta)$ from the left panel.

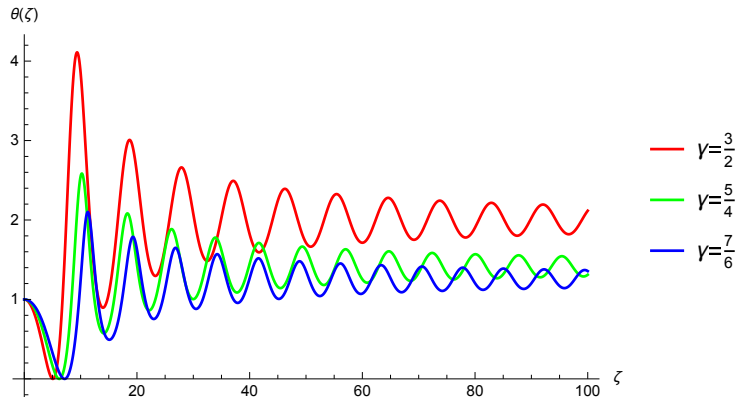


FIG. 3: Stable periodic solutions $\theta(\zeta)$ of (1) with $\Omega = 0.5$, and three values of the parameter $\gamma = \frac{3}{2}$, $\frac{5}{4}$, and $\frac{7}{6}$. The top curve is the solution from the right panel of Fig. 2.

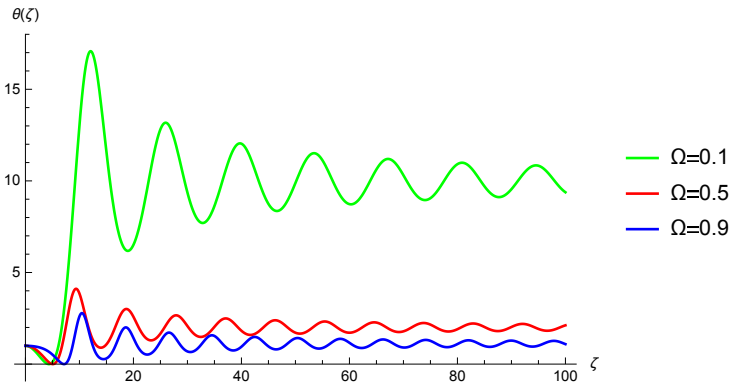


FIG. 4: Stable periodic solutions $\theta(\zeta)$ of (1) with $\gamma = \frac{3}{2}$, and three values of the parameter $\Omega = 0.1, 0.5$, and 0.9 . The middle curve is the solution from the right panel of Fig. 2.

in which initial conditions should be picked that yield periodic solutions. For the other equilibrium point \bar{y}_2 , we provide an instability result showing that in the case where n is odd this equilibrium is unstable,

and when n is even the same equilibrium $\bar{y}_0 = \bar{y}_2$ is also unstable.

This analysis yields a special set of solutions for the density of atoms of the generalized Lane-Emden equation obtained when n is even, and initial conditions are inside a region of attraction to the left of the stable equilibrium. These periodic solutions (orbital modes) have been observed experimentally previously in rotating ultracold atomic clouds, and are demonstrated analytically to be stable.

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