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Continuous control for fully-damped mechanical systems with input constraints: finite-time and exponential tracking

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***Abstract*—A motion continuous control scheme for fully-damped mechanical systems with constrained inputs is proposed. It gives the freedom to choose among finite-time and (local) exponential convergence through a simple design parameter. The control objective is achieved from any initial conditions, for desired trajectories that can be physically tracked avoiding actuator saturation and loss of motion error dissipation, globally induced through the aid of the natural damping terms explicitly considered in the open-loop dynamics. The stability analysis is based on a strict Lyapunov function and is formally developed within an appropriate analytical framework that takes into account the time-varying character naturally adopted by the closed loop. Simulation tests are further included.**

***Index Terms*—Uniform finite-time tracking, motion continuous control, mechanical systems, input constraints, strict Lyapunov function.**

I. Introduction

FINITE-TIME control through continuous feedback has been a research topic of increasing interest in the last years. Such an intriguing topic has attracted attention on its need for a suitable analytical framework around its conceptualization and characterization. In this direction, important contributions have been developed for autonomous systems in the works of [2], [3], by stating a precise definition of a finite-time stable equilibrium, a Lyapunov-function-based criterion for its determination, and a useful characterization for homogeneous vector fields.

Finite-time stability and stabilization for time-varying vector fields has evolved more slowly and is still in progress. Important extensions and generalizations of the previously cited works from Bhat and Bernstein have been developed for instance in [9] by stating precise definitions and Lyapunov-type characterizations for non-autonomous systems. Uniform stability has been very recently studied within the framework of homogeneity in [13] where, in particular, the characterization of global uniform finite-time stability has been extended to time-varying vector fields. These contributions show the complexity entailed in the non-autonomous case in relation to the previously cited time-invariant case. For instance, the existence of a homogeneous Lyapunov function characterized for autonomous vector fields in [12] does not apply for time-varying ones, and a similar extension for the latter case does not exist. Consequently, results based on such a fundamental work of [12], like the finite-time-stability-preservation *approximation* approach of [5], do not apply in the non-autonomous case. Stability/stabilization studies in the time-varying context shall take into account such important analytical limitations and consequently entail a more complex analysis.

Finite-time continuous control of mechanical systems has been treated for instance in [4], [6], [14], [16]. These works mainly give rise to divers finite-time regulators and are consequently developed within the framework of autonomous systems. Once we move on to the tracking control problem, which naturally implies a time-varying closed-loop dynamics, the stability analysis suffers from the above mentioned impossibility to involve analytical tools exclusively addressed to time-invariant vector fields, and shall consequently be developed within the framework of non-autonomous systems, for instance through the use of a suitable *strict* Lyapunov function. Strict Lyapunov functions have hardly been very recently constructed in [4] to support finite-time control of robot manipulators disregarding input constraints, leaving the more complex tracking-under-bounded-input case unsolved.

This work gives a solution to the —up to our knowledge— open problem of (uniform) finite-time tracking continuous control of constrained-input mechanical systems, under the consideration of linear damping terms in the open-loop dynamics. The proposed approach actually gives the freedom to choose the type of trajectory convergence, among finite-time and exponential, through a simple control parameter. The stability analysis is based on a suitable strict Lyapunov function, and is formally developed within an appropriate analytical framework that takes into account the inherent time-varying nature of the closed loop. The design relies on the consideration of the natural damping terms, which are directly involved in the characterization of the subset of desired trajectories for which the control objective is achieved from any initial conditions (by ensuring motion error dissipation *globally*, as will be made clear later on in Remark 3.4). Such a characterization further restricts the choice to desired motions generating open-loop (reaction and inherent force/torque) terms whose addition remains within the

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actuator bounds; those transgressing such a restriction would not even be physically possible to be accurately tracked. The control synthesis thus guarantees the formulated goal through control signals evolving within pre-specified bound values, in order to avoid actuator saturation, which leads to one of the main conceptual differences in the Lyapunov function design with respect to the approach in [4]. Furthermore, following the energy shaping plus damping injection methodology, the approach in [4] focuses on providing sufficient conditions on the closed-loop potential and dissipation energy functions and, consequently, the corresponding Lyapunov analysis involves such conditions leaving the referred energy functions implicitly represented, while explicit generalized control expressions and corresponding design requirements are given here to solve the considered constrained-input problem, and all the analytical aspects supporting such a design methodology are thoroughly developed. Moreover, although the strict Lyapunov function involved here is inspired from [4], the closed-loop analysis follows a different procedure, involving complementary analytical tools. It is worth adding that, while the proposed approach includes the constant desired trajectory case, the closed-loop autonomous nature in such a *regulation* case has permitted in [16] a different analytical treatment leading to a more generalized control structure giving rise to more degrees of design flexibility in benefit of closed-loop performance improvement (as shown in [16]).

II. Preliminaries

Let $X, Y \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$. Throughout this work, X_{ij} denotes the element of X at its i^{th} row and j^{th} column and x_i stands for the i^{th} element of x. As conventionally, with $m = n$, $X > 0$, resp. $X \geq 0$, denotes that X is positive definite, resp. semidefinite, and $X > Y$, resp. $X \geq Y$, that $X - Y$ is positive definite, resp. semidefinite. 0_n represents the origin of $\mathbb{R}^n$ and I_n the $n \times n$ identity matrix. We denote $\mathbb{R}^n_{>0} = \{x \in \mathbb{R}^n : x_i > 0,\ i = 1,\dots,n\}$ and $\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n : x_i \geq 0,\ i = 1,\dots,n\}$, while $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ will be used when $n = 1$. For a subset $A \subset \mathbb{R}^n$, ∂A stands for its boundary. $\|\cdot\|$ will conventionally denote the standard Euclidean norm, *i.e.* the 2-norm for vectors and induced 2-norm for matrices. Other p-norms will be denoted $\|\cdot\|_p$. An n-dimensional closed ball and an $(n-1)$-dimensional sphere, both of radius $c > 0$, are denoted $\mathcal{B}^n_c$ and $\mathcal{S}^{n-1}_c$, respectively, *i.e.* $\mathcal{B}^n_c = \{x \in \mathbb{R}^n : \|x\| \leq c\}$ and $\mathcal{S}^{n-1}_c = \{x \in \mathbb{R}^n : \|x\| = c\}$. Let $\mathcal{A}$ and $\mathcal{E}$ be subsets (with non-empty interior) of some vector spaces $\mathbb{A}$ and $\mathbb{E}$, respectively. For any integer $m \geq 0$, we denote $\mathcal{C}^m(\mathcal{A};\mathcal{E})$ the set of continuous functions from $\mathcal{A}$ to $\mathcal{E}$, being m times continuously differentiable when m is strictly positive (with differentiability at any point on the boundary of $\mathcal{A}$ meant as the limit from the interior of $\mathcal{A}$). Consider a function $h \in \mathcal{C}^2(\mathbb{R}_{\geq 0};\mathcal{E})$. The first- and second-order rates of change of h are respectively denoted $\dot{h}$ and $\ddot{h}$. For $f \in \mathcal{C}^1(\mathbb{R}^n;\mathbb{R})$ and $g : \mathbb{R}^n \to \mathbb{R}^n$, we denote $D_g f$ the directional derivative of f along g, *i.e.* $D_g f(x) = \frac{\partial f}{\partial x} g(x)$. We will consider the sign function —$\text{sign}(\cdot)$— to be zero at zero, and denote $\text{sat}(\cdot)$ the standard (unitary) saturation function, *i.e.* $\text{sat}(\varsigma) = \text{sign}(\varsigma)\min\{|\varsigma|, 1\}$. Fundamental facts that will be involved in this study are *Young's inequality*, *i.e.* for any $\phi, \psi \in (1,\infty)$ such that $\frac{1}{\phi} + \frac{1}{\psi} = 1$ and any $a, b \in \mathbb{R}_{\geq 0}$: $ab \leq \frac{a^\phi}{\phi} + \frac{b^\psi}{\psi}$; *Hölder inequality*, *i.e.* for any $\phi, \psi \in [1,\infty]$ such that $\frac{1}{\phi} + \frac{1}{\psi} = 1$ and any $x, y \in \mathbb{R}^n$: $|x^T y| \leq \|x\|_\phi \|y\|_\psi$; and the following properties of p-norms.

Lemma 2.1: For any $x \in \mathbb{R}^n$, $\|x\|_p$ is nonincreasing in p.

Proof. See Appendix A. □

Remark 2.1: By equivalence of p-norms, for any $\|\cdot\|_\phi$ and $\|\cdot\|_\psi$, with $\phi \neq \psi$, there exist constants $\bar{c}_{\phi,\psi} > c_{\phi,\psi} > 0$ such that $c_{\phi,\psi}\|x\|_\psi \leq \|x\|_\phi \leq \bar{c}_{\phi,\psi}\|x\|_\psi$, $\forall x \in \mathbb{R}^n$. In particular, by Lemma 2.1, $c_{\phi,\psi} = 1$ if $\phi < \psi$ and $\bar{c}_{\phi,\psi} = 1$ if $\phi > \psi$. △

A. Mechanical systems

Consider the n-degree-of-freedom (DOF) fully-actuated mechanical system dynamics with linear damping effects

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + g(q) = \tau \tag{1}$$

where q, $\dot{q}$, $\ddot{q} \in \mathbb{R}^n$ are the position (generalized coordinates), velocity, and acceleration vectors, $H(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$ is the Coriolis and centrifugal effect matrix defined through the Christoffel symbols of the first kind, $F \in \mathbb{R}^{n\times n}$ is the (*a priori* symmetric positive semidefinite) effect matrix, $g(q) = \nabla\mathcal{U}(q)$ with $\mathcal{U} : \mathbb{R}^n \to \mathbb{R}$ being the potential energy function of the system, and $\tau \in \mathbb{R}^n$ is the external input (generalized) force vector. Some well-known properties characterizing the terms of such a dynamical model are recalled here. Subsequently, we denote $\dot{H}$ the rate of change of H; more precisely $\dot{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ with $\dot{H}_{ij}(q,\dot{q}) = \frac{\partial H_{ij}}{\partial q}(q)\dot{q}$, $i, j = 1,\dots,n$.

Property 2.1: $H(q)$ is a continuously differentiable positive definite symmetric matrix function, and actually $H(q) \geq \mu_m I_n$ —whence $\|H(q)\| \geq \mu_m$— $\forall q \in \mathbb{R}^n$, for some $\mu_m > 0$.

Property 2.2: The Coriolis and centrifugal effect matrix defined through the Christoffel symbols of the 1st kind satisfies:

2.2.1. $\dot{H}(q,\dot{q}) = C(q,\dot{q}) + C^T(q,\dot{q})$, $\forall q, \dot{q} \in \mathbb{R}^n$, and consequently $z^T\left[\frac{1}{2}\dot{H}(x,y) - C(x,y)\right]z = 0$, $\forall x, y, z \in \mathbb{R}^n$;
2.2.2. $C(w, x+y)z = C(w,x)z + C(w,y)z$, $\forall w, x, y, z \in \mathbb{R}^n$;
2.2.3. $C(x,y)z = C(x,z)y$, $\forall x, y, z \in \mathbb{R}^n$;
2.2.4. $\|C(x,y)\| \leq \vartheta(x)\|y\|$, $\forall x, y \in \mathbb{R}^n$, for some $\vartheta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$.

We consider the bounded input case, where the absolute value of each input τ_i is constrained to be smaller than a given saturation bound $T_i > 0$, *i.e.* $|\tau_i| \leq T_i$, $i = 1,\dots,n$. More precisely, letting u_i represent the control variable (controller output) relative to the i^{th} degree of freedom, we have that

$$\tau_i = T_i \text{sat}(u_i/T_i) \tag{2}$$

Assumption 2.1: The inertia matrix is bounded, *i.e.* $\|H(q)\| \leq \mu_M$, $\forall q \in \mathbb{R}^n$, for some $\mu_M \geq \mu_m > 0$.

Assumption 2.2: $\vartheta(\cdot)$ in Property 2.2.4 is bounded; consequently $\|C(x,y)\| \leq k_C\|y\|$, $\forall x, y \in \mathbb{R}^n$, for some $k_C \geq 0$.

Assumption 2.3: The conservative (generalized) force vector $g(q)$ is bounded, or equivalently, every one of its elements, $g_i(q)$, $i = 1,\dots,n$, satisfies $|g_i(q)| \leq B_{gi}$, $\forall q \in \mathbb{R}^n$, for some $B_{gi} > 0$.

Assumption 2.4: The damping effect matrix F is symmetric positive definite, and consequently $f_m\|x\|^2 \leq x^T F x \leq f_M\|x\|^2$, $\forall x \in \mathbb{R}^n$, for some constants $f_M \geq f_m > 0$.

Assumption 2.5: $T_i > B_{gi}$, $\forall i \in \{1, \ldots, n\}$.

Assumptions 2.1–2.3 apply *e.g.* for manipulators having only revolute joints [7]. Assumption 2.4 is coherent with the dissipative nature of the damping term $F\dot{q}$ in (1) [10].

B. Uniform finite-time stability

Consider an n-th order non-autonomous system

$$\dot{x} = f(t, x) \tag{3}$$

where $f : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}^n$ is continuous, $\mathcal{D} \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0_n$, and $f(t, 0_n) = 0_n$, $\forall t \geq 0$. We denote $x(t; t_0, x_0)$ —or simply $x(t)$ whenever convenient or clear from the context— a solution of (3) with initial condition $x(t_0; t_0, x_0) = x_0 \in \mathcal{D}$ at initial time $t_0 \geq 0$, and $\mathcal{S}(t_0, x_0)$ is the set of all solutions $x(t; t_0, x_0)$ starting from $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathcal{D}$.

Definition 2.1: [9] The equilibrium point $x = 0_n$ of (3) is

- *weakly finite-time stable* if:
  1) it is Lyapunov stable;
  2) for each $t_0 \geq 0$, there exists $\bar{\delta} = \bar{\delta}(t_0) > 0$ such that, if $\|x_0\| < \bar{\delta}$ then, for all $x(t) \in \mathcal{S}(t_0, x_0)$:
     a) $x(t)$ is defined for all $t \geq t_0 \geq 0$;
     b) $\exists\, T \in [0, \infty)$ such that $x(t) = 0_n$, $\forall t \geq t_0 + T$.

  $T_0[x(t; t_0, x_0)] \triangleq \inf\{T \geq 0 : x(t; t_0, x_0) = 0_n\ ,\ \forall t \geq t_0 + T\}$ is called the settling time of $x(t; t_0, x_0)$.
- *finite-time stable* if, in addition to items 1 and 2 above:
  3) $T_0(t_0, x_0) \triangleq \sup_{x(t)\in\mathcal{S}(t_0,x_0)} T_0[x(t)] < \infty$.

  $T_0(t_0, x_0)$ is called the settling time with respect to initial conditions (at (t_0, x_0)).

Remark 2.2: If f in (3) is locally Lipschitz-continuous in x on $\mathcal{D} \setminus \{0_n\}$ (uniformly in t on $\mathbb{R}_{\geq 0}$) then, by uniqueness of solutions, the settling time of a solution $x(t; t_0, x_0)$ and the settling time with respect to initial conditions at (t_0, x_0) are the same, *i.e.* $T_0(t_0, x_0) = T_0[x(t; t_0, x_0)]$. △

Definition 2.2: [9] The equilibrium point $x = 0_n$ of (3) is *uniformly finite-time stable* if:

1) it is uniformly asymptotically stable;
2) it is finite-time stable;
3) there exists a positive definite continuous function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that the settling time with respect to initial conditions satisfies $T_0(t_0, x_0) \leq \varphi(\|x_0\|)$.

Theorem 2.1: [9] Let $V : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that $W_1(x) \leq V(t, x) \leq W_2(x)$ and $\dot{V}(t, x) \leq -\nu\big(V(t, x)\big)$, $\forall (t, x) \in \mathbb{R}_{\geq 0} \times \mathcal{D}$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on $\mathcal{D}$, $\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f$, and $\nu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a positive definite continuous function such that $\int_0^{\varpi} \frac{dz}{\nu(z)} < \infty$, for some $\varpi > 0$. Then $x = 0_n$ is uniformly finite-time stable.

Remark 2.3: With $\nu(z) = cz^{\bar{\alpha}}$, for any $c > 0$ and $\bar{\alpha} \in (0, 1)$, we have $\int_0^{\varpi} \frac{dz}{\nu(z)} = \frac{\varpi^{1-\bar{\alpha}}}{c(1-\bar{\alpha})} < \infty$, for any $\varpi \in (0, \infty)$. This special case generates a natural or direct extension to time-varying vector fields of the celebrated Lyapunov-type criterion stated for autonomous systems in [2]. △

Remark 2.4: The stability properties stated through Definitions 2.1 and 2.2 are *global* if $\mathcal{D} = \mathbb{R}^n$ and items 2a-2b in Definition 2.1 are satisfied for any $x(t_0) = x_0 \in \mathbb{R}^n$. Moreover, one notes from Definition 2.2 that an equilibrium may be concluded to be *globally uniformly finite-time stable* if it is globally uniformly asymptotically stable and uniformly finite-time stable. △

C. Locally homogeneous functions

This work involves the notion of *locally homogeneous function* [15, Definition 2.1], stated in terms of *family of dilations* δ_ϵ^r, defined as $\delta_\epsilon^r(x) = (\epsilon^{r_1} x_1, \ldots, \epsilon^{r_n} x_n)^T$, $\forall x \in \mathbb{R}^n$, $\forall \epsilon > 0$, where $r = (r_1, \ldots, r_n)^T$, with the dilation coefficients r_i, $i = 1, \ldots, n$, being positive real numbers.

Lemma 2.2: [15] Suppose that, for every $i = 1, 2$, V_i is a continuous scalar function being locally r-homogeneous of degree $\alpha_i > 0$, with domain of homogeneity $\mathcal{D}_i$. Suppose further that V_1 is positive definite on $\mathcal{D}_1$. Let $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2$ and consider an $(n-1)$-dimensional sphere $\mathcal{S}_c^{n-1}$ of some radius $c > 0$ such that $\mathcal{S}_c^{n-1} \subset \mathcal{D}$. Then, for every $x \in \mathcal{D}$: $c_1[V_1(x)]^{\alpha_2/\alpha_1} \leq V_2(x) \leq c_2[V_1(x)]^{\alpha_2/\alpha_1}$, with $c_1 = [\min_{z\in\mathcal{S}_c^{n-1}} V_2(z)] \cdot [\max_{z\in\mathcal{S}_c^{n-1}} V_1(z)]^{-\alpha_2/\alpha_1}$ and $c_2 = [\max_{z\in\mathcal{S}_c^{n-1}} V_2(z)] \cdot [\min_{z\in\mathcal{S}_c^{n-1}} V_1(z)]^{-\alpha_2/\alpha_1}$.

Remark 2.5: Observe that if V_2 happens to be positive —resp. negative— definite, then c_1 and c_2 in Lemma 2.2 are both positive —resp. negative— constants. △

D. Scalar functions with particular properties

Definition 2.3: A continuous scalar function $\sigma : \mathbb{R} \to \mathbb{R}$ will be said to be:

1) *strictly passive* if $\varsigma\sigma(\varsigma) > 0$, $\forall \varsigma \neq 0$;
2) *strongly passive* —for (κ, a, b)— if it is a strictly passive function satisfying $|\sigma(\varsigma)| \geq \kappa\big|b\,\mathrm{sat}(\varsigma/b)\big|^a = \kappa\big(\min\{|\varsigma|, b\}\big)^a$, $\forall \varsigma \in \mathbb{R}$, for some positive constants κ, a and b.
3) *bounded strongly passive* —for $(\kappa, a, b, \bar{\kappa}, \bar{a}, \bar{b})$— if it is a strongly passive function for (κ, a, b) such that $|\sigma(\varsigma)| \leq \bar{\kappa}\big|\bar{b}\,\mathrm{sat}(\varsigma/\bar{b})\big|^{\bar{a}} = \bar{\kappa}\big(\min\{|\varsigma|, \bar{b}\}\big)^{\bar{a}}$, $\forall \varsigma \in \mathbb{R}$, for some positive constants κ, a, b, $\bar{\kappa}$, $\bar{a}$ and $\bar{b}$.

Lemma 2.3: Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a strongly passive function for (κ, a, b) and k be a positive constant. Then, for all $\varsigma \in \mathbb{R}$:

$$\int_0^{\varsigma} \sigma(kz)dz \geq S(\varsigma) = \begin{cases} \frac{\kappa k^a}{1+a}|\varsigma|^{1+a} & \forall|\varsigma| \leq \frac{b}{k} \\ \kappa b^a\Big(|\varsigma| - \frac{ab}{k(1+a)}\Big) & \forall|\varsigma| > \frac{b}{k} \end{cases} \tag{4}$$

Lemma 2.3 arises from Definition 2.3 whence, since $|\sigma(\varsigma)| \geq \kappa|b\,\mathrm{sat}(\varsigma/b)|^a$, we have that $\int_0^{\varsigma} \sigma(kz)dz \geq \int_0^{\varsigma} \mathrm{sign}(z)\kappa|b\,\mathrm{sat}(kz/b)|^a dz = S(\varsigma)$.

Lemma 2.4: For every $j = 1, \ldots, n$, let σ_j be a strongly passive function for (κ, a, b), k_j be a positive constant, $k_m = \min_j\{k_j\}$, $k_M = \max_j\{k_j\}$ and, for any $x \in \mathbb{R}^n$ and $c > 0$, $S_0(x; a, c) = \|x\|\big(\min\{\|x\|, c\}\big)^a$. Then

1) $\sum_{j=1}^n \int_0^{x_j} \sigma_j(k_j z_j)dz_j \geq \frac{\kappa k_m^a}{1+a} S_0(x; a, b/k_M)$, $\forall x \in \mathbb{R}^n$;
2) $\sum_{j=1}^n x_j\sigma_j(k_j x_j) \geq \kappa k_m^a S_0(x; a, b/k_M)$, $\forall x \in \mathbb{R}^n$.

Proof. See Appendix B. □

Remark 2.6: Note that for a bounded strongly passive function σ for some $(\kappa, a, b, \bar{\kappa}, \bar{a}, \bar{b})$, we have $\kappa\big(\min\{|\varsigma|, b\}\big)^a \leq |\sigma(\varsigma)| \leq \bar{\kappa}\big(\min\{|\varsigma|, \bar{b}\}\big)^{\bar{a}} \leq \bar{\kappa}|\varsigma|^{\bar{a}}$, $\forall \varsigma \in \mathbb{R}$. △

Lemma 2.5: Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a bounded strongly passive function for $(\kappa, a, b, \bar{\kappa}, \bar{a}, \bar{b})$ and k be a positive constant. Then, in addition to (4), $\int_0^{\varsigma} \sigma(kz)dz \leq \frac{\bar{\kappa}k^{\bar{a}}}{1+\bar{a}}|\varsigma|^{1+\bar{a}}$, $\forall \varsigma \in \mathbb{R}$.

Lemma 2.5 arises from Remark 2.6 whence, since $|\sigma(\varsigma)| \leq \bar{\kappa}|\varsigma|^{\bar{a}}$, we have that $\int_0^{\varsigma} \sigma(kz)dz \leq \int_0^{\varsigma} \mathrm{sign}(z)\bar{\kappa}|kz|^{\bar{a}}dz$. The next lemma arises directly from Lemma 2.5 and Remark 2.1.

Lemma 2.6: For every $j = 1, \ldots, n$, let σ_j be a bounded strongly passive function for $(\kappa, a, b, \bar{\kappa}, \bar{a}, \bar{b})$, k_j be a positive constant, $k_m = \min_j\{k_j\}$ and $k_M = \max_j\{k_j\}$. Then, in addition to item 1 of Lemma 2.4, $\sum_{j=1}^n \int_0^{x_j} \sigma_j(k_j z_j)dz_j \leq \frac{\bar{\kappa}k_M^{\bar{a}}\bar{c}_{1+\bar{a},2}^{1+\bar{a}}}{1+\bar{a}}\|x\|^{1+\bar{a}}$, $\forall x \in \mathbb{R}^n$.

III. The proposed control scheme

We begin by characterizing —based on Assumptions 2.1–2.5— a set of desired trajectories $q_d(t)$ for which the proposed scheme will prove to guarantee the considered tracking objective avoiding input saturation and for any initial condition.

Assumption 3.1: $q_d \in \mathcal{C}^2(\mathbb{R}_{\geq 0}; \mathbb{R}^n)$ such that $\|\dot{q}_d(t)\| \leq B_{dv}$ and $\|\ddot{q}_d(t)\| \leq B_{da}$, $\forall t \geq 0$, for sufficiently small (positive) bound values B_{dv} and B_{da} such that $\mu_M B_{da} + k_C B_{dv}^2 + f_M B_{dv} < T_j - B_{gj}$, $\forall j \in \{1, \ldots, n\}$, and $B_{dv} < f_m/k_C$.

We propose the following control law

$$
\begin{aligned}
u(t, q, \dot{q}) = &-s_1(K_1\bar{q}) - s_2(K_2\dot{\bar{q}}) + H(q)\ddot{q}_d(t) \\
&+ C\big(q, \dot{q}_d(t)\big)\dot{q}_d(t) + F\dot{q}_d(t) + g(q) \quad (5)
\end{aligned}
$$

where $\bar{q} = q - q_d(t)$; $K_i = \mathrm{diag}[k_{i1}, \ldots, k_{in}]$ with $k_{ij} > 0$, $\forall i \in \{1, 2\}$, $\forall j \in \{1, \ldots, n\}$; and for any $x \in \mathbb{R}^n$, $s_i(x) = \big(\sigma_{i1}(x_1), \ldots, \sigma_{in}(x_n)\big)^T$, $i = 1, 2$, with, for each $j \in \{1, \ldots, n\}$, σ_{ij} being a bounded strongly passive function for some $(\kappa_i, a_i, b_i, \bar{\kappa}_i, a_i, b_i) \in \mathbb{R}^6_{>0}$, both $(i = 1, 2)$ being locally Lipschitz-continuous on $\mathbb{R}\backslash\{0\}$ and such that $a_1 \in (0, 1]$, $a_2 = \frac{2a_1}{1+a_1} \in (0, 1]$, and

$$
\begin{aligned}
B_j \triangleq &\sup_{(\varsigma_1,\varsigma_2)\in\mathbb{R}^2} \big|\sigma_{1j}(\varsigma_1) + \sigma_{2j}(\varsigma_2)\big| \\
&< T_j - \mu_M B_{da} - k_C B_{dv}^2 - f_M B_{dv} - B_{gj} \quad (6)
\end{aligned}
$$

Proposition 3.1: Consider system (1)-(2) in closed loop with the proposed control law (5), under Assumptions 2.1–2.5 and 3.1. Thus, for any positive definite diagonal matrices K_1 and K_2, $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq t_0 \geq 0$, and the closed-loop trivial solution $\bar{q}(t) \equiv 0_n$ is:

1) globally uniformly finite-time stable if $a_1 \in (0, 1)$;
2) globally uniformly asymptotically stable with (local) exponential stability if $a_1 = 1$.

Proof. The proof is divided into four stages.

1st stage: input saturation avoidance and closed-loop dynamics. Observe that —for every $j \in \{1, \ldots, n\}$— by Assumptions 2.1–2.4 and 3.1, and the satisfaction of (6), we have, for any $(t, q, \dot{q}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n$: $|u_j(t, q, \dot{q})| \leq |\sigma_{1j}(k_{1j}\bar{q}_j) + \sigma_{2j}(k_{2j}\dot{\bar{q}}_j)| + \|H(q)\|\|\ddot{q}_d(t)\| + \|C\big(q, \dot{q}_d(t)\big)\|\|\dot{q}_d(t)\| + \|F\|\|\dot{q}_d(t)\| + |g_j(q)| \leq B_j + \mu_M B_{da} + k_C B_{dv}^2 + f_M B_{dv} + B_{gj} < T_j$. From this and (2), one sees that $T_j > |u_j(t, q, \dot{q})| = |u_j| = |\tau_j|$, $\forall (t, q, \dot{q}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n$, which shows that, along the system trajectories, $|\tau_j(t)| = |u_j(t)| < T_j$, $j = 1, \ldots, n$, $\forall t \geq t_0 \geq 0$. This proves that, under the proposed scheme, the input saturation values, T_j, are never attained. Thus, the closed-loop dynamics becomes

$$
\begin{aligned}
H(q)\ddot{\bar{q}} + C(q, \dot{q})\dot{\bar{q}} + C\big(q, \dot{q}_d(t)\big)\dot{\bar{q}} + F\dot{\bar{q}}& \\
= -s_1(K_1\bar{q}) - s_2(K_2\dot{\bar{q}})& \quad (7)
\end{aligned}
$$

where Property 2.2.3 has been used.

2nd stage: energy function. Let us consider the continuously differentiable energy function $V_0(t, \bar{q}, \dot{\bar{q}}) = \frac{1}{2}\dot{\bar{q}}^T H\big(\bar{q}+q_d(t)\big)\dot{\bar{q}} + \int_{0_n}^{\bar{q}} s_1^T(K_1 z)dz$, where $\int_{0_n}^{\bar{q}} s_1^T(K_1 z)dz = \sum_{j=1}^n \int_0^{\bar{q}_j} \sigma_{1j}(k_{1j}z_j)dz_j$. From Property 2.1, Assumption 2.1 and Lemmas 2.4 and 2.6:

$$
W_{01}(\bar{q}, \dot{\bar{q}}) \leq V_0(t, \bar{q}, \dot{\bar{q}}) \leq W_{02}(\bar{q}, \dot{\bar{q}}) \quad (8)
$$

with $W_{01}(\bar{q}, \dot{\bar{q}}) \triangleq \frac{\mu_m}{2}\|\dot{\bar{q}}\|^2 + \frac{\kappa_1 k_{1m}^{a_1}}{1+a_1}S_1(\bar{q})$ and $W_{02}(\bar{q}, \dot{\bar{q}}) \triangleq \frac{\mu_M}{2}\|\dot{\bar{q}}\|^2 + \frac{\bar{\kappa}_1 k_{1M}^{a_1}\bar{c}^{1+a_1}}{1+a_1}\|\bar{q}\|^{1+a_1}$, where $S_1(\bar{q}) = S_0(\bar{q}; a_1, b_1/k_{1M})$, $k_{1m} = \min_j\{k_{1j}\}$, $k_{1M} = \max_j\{k_{1j}\}$ and $\bar{c} = \bar{c}_{1+a_1,2}$. The derivative of V_0 along the closed-loop system trajectories is obtained, after basic developments, as $\dot{V}_0(t, \bar{q}, \dot{\bar{q}}) = -\dot{\bar{q}}^T s_2(K_2\dot{\bar{q}}) - \dot{\bar{q}}^T C\big(q, \dot{q}_d(t)\big)\dot{\bar{q}} - \dot{\bar{q}}^T F\dot{\bar{q}}$, where $H(q)\ddot{\bar{q}}$ has been replaced by its equivalent expression from the closed-loop dynamics (7) and Property 2.2.1 has been used. Further, by Assumptions 2.2, 2.4, 3.1, and Lemma 2.4:

$$
\begin{aligned}
\dot{V}_0(t, \bar{q}, \dot{\bar{q}}) &\leq -\sum_{j=1}^n \dot{\bar{q}}_j\sigma_{2j}(k_{2j}\dot{\bar{q}}_j) - (f_m - k_C B_{dv})\|\dot{\bar{q}}\|^2 \\
&\leq -\kappa_2 k_{2m}^{a_2} S_2(\dot{\bar{q}}) - d\|\dot{\bar{q}}\|^2 \leq -\eta\|\dot{\bar{q}}\|^{1+a_2} \quad (9)
\end{aligned}
$$

where $S_2(\dot{\bar{q}}) = S_0(\dot{\bar{q}}; a_2, b_2/k_{2M})$, $k_{2m} = \min_j\{k_{2j}\}$, $k_{2M} = \max_j\{k_{2j}\}$, $d = f_m - k_C B_{dv} > 0$ (by Assumption 3.1) and $\eta = \min\left\{\kappa_2 k_{2m}^{a_2}\ ,\ d\Big(\frac{b_2}{k_{2M}}\Big)^{1-a_2}\right\}$.[1] The expressions so far obtained will prove to be useful next.

3rd stage: global uniform asymptotic stability. Let us now define the scalar function

$$
V(t, \bar{q}, \dot{\bar{q}}) = V_0^{\beta}(t, \bar{q}, \dot{\bar{q}}) + \varepsilon\rho^T(\bar{q})H\big(\bar{q} + q_d(t)\big)\dot{\bar{q}} \quad (10)
$$

where V_0 is as defined in the previous stage, $\beta = (3 + a_1)/[2(1 + a_1)]$, ε is a positive constant, and $\rho(\bar{q}) = h(\bar{q}; b_1/k_{1M})\bar{q}$, with $h \in \mathcal{C}^0\big(\mathbb{R}^n \times \mathbb{R}_{>0}; (0, 1]\big)$ being continuously differentiable on $\mathbb{R}^n \setminus \{0_n\}$, uniformly on $\mathbb{R}_{>0}$, and such that, for any $c > 0$, ρ is a continuously differentiable function satisfying

$$
\|\rho(x)\| = h(x; c)\|x\| \leq \min\{\|x\|, c\} \quad (11)
$$

$\forall x \in \mathbb{R}^n$, and

$$
-h(x; c) < D_x h(x; c) < 0 \quad (12)
$$

[1]Observe that for all $\|\dot{\bar{q}}\| \leq b_2/k_{2M}$, we have that $\kappa_2 k_{2m}^{a_2} S_2(\dot{\bar{q}}) + d\|\dot{\bar{q}}\|^2 \geq \kappa_2 k_{2m}^{a_2} S_2(\dot{\bar{q}}) = \kappa_2 k_{2m}^{a_2}\|\dot{\bar{q}}\|^{1+a_2} \geq \eta\|\dot{\bar{q}}\|^{1+a_2}$, and for all $\|\dot{\bar{q}}\| > b_2/k_{2M}$ that $\kappa k_{2m}^{a_2} S_2(\dot{\bar{q}}) + d\|\dot{\bar{q}}\|^2 \geq d\|\dot{\bar{q}}\|^2 = d\|\dot{\bar{q}}\|^{1-a_2}\|\dot{\bar{q}}\|^{1+a_2} \geq d\Big(\frac{b_2}{k_{2M}}\Big)^{1-a_2}\|\dot{\bar{q}}\|^{1+a_2} \geq \eta\|\dot{\bar{q}}\|^{1+a_2}$.

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$\forall x \neq 0_n$; an example of a family of functions h with such properties is $h(x;c) = c/\big[c^\varpi + \|x\|^\varpi\big]^{1/\varpi}$ for any $\varpi > 0$.[2]

Remark 3.1: In view of (12), h is decreasing on any radial direction, and consequently (since $h : \mathbb{R}^n \times \mathbb{R}_{>0} \to (0,1]$) $h(x;c) \to \omega$ as $\|x\| \to \infty$ for some non-negative constant ω, while, on any compact connected neighborhood of the origin $\Upsilon \subset \mathbb{R}^n$, h is lower-bounded by a positive value $h_{m,\Upsilon}$, or more precisely: $1 \geq h(0_n;c) \geq h(x;c) \geq \inf_{x\in\Upsilon} h(x;c) \triangleq h_{m,\Upsilon} = \inf_{x\in\partial\Upsilon} h(x;c) > \omega \geq 0$, $\forall x \in \Upsilon$. △

Remark 3.2: Observe, that $\frac{\partial\rho}{\partial x}(x) = \frac{\partial}{\partial x}\big[h(x;c)x\big] = h(x;c)I_n + x\frac{\partial h}{\partial x}(x;c)$, whence we get that $x^T\frac{\partial\rho}{\partial x}(x)x = x^T\big[h(x;c)I_n + x\frac{\partial h}{\partial x}(x;c)\big]x = h(x;c)x^Tx + x^Tx\frac{\partial h}{\partial x}(x;c)x = \|x\|^2\big[h(x;c) + D_x h(x;c)\big]$ wherefrom, in view of (12) (whence we have that $0 < h(x;c) + D_x h(x;c) < h(x;c) < 1$, $\forall x \neq 0_n$), one sees that $0 < x^T\frac{\partial\rho}{\partial x}(x)x < \|x\|^2$, $\forall x \neq 0_n$, and consequently $0 < \frac{\partial\rho}{\partial x}(x) \leq I_n$, $\forall x \in \mathbb{R}^n$, which implies that $\big\|\frac{\partial\rho}{\partial x}(x)\big\| \leq 1$, $\forall x \in \mathbb{R}^n$. △

Remark 3.3: Let $h_1(\bar q) = h(\bar q; b_1/k_{1M})$. Useful facts on ρ that will be subsequently invoked are

$$\|\rho(\bar q)\|^{1+a_1} \leq h_1(\bar q)S_1(\bar q) \leq S_1(\bar q) \tag{13}$$

$$\|\rho(\bar q)\|^2 \leq \left(\tfrac{b_1}{k_{1M}}\right)^{1-a_1} h_1(\bar q)S_1(\bar q) \leq \left(\tfrac{b_1}{k_{1M}}\right)^{1-a_1} S_1(\bar q) \tag{14}$$

$\forall \bar q \in \mathbb{R}^n$. Indeed, based on the properties of ρ and h_1 (particularly (11) and $h_1(\bar q) \in (0,1]$, $\forall \bar q \in \mathbb{R}^n$), we have, for all $\bar q \in \mathcal{B}^n_{b_1/k_{1M}}$, that $\|\rho(\bar q)\|^{1+a_1} = [h_1(\bar q)]^{1+a_1}\|\bar q\|^{1+a_1} = h_1^{a_1}(\bar q)h_1(\bar q)S_1(\bar q) \leq h_1(\bar q)S_1(\bar q) \leq S_1(\bar q)$, and for all $\bar q \notin \mathcal{B}^n_{b_1/k_{1M}}$ that $\|\rho(\bar q)\|^{1+a_1} = \|\rho(\bar q)\|^{a_1}\|\rho(\bar q)\| \leq (b_1/k_{1M})^{a_1}h_1(\bar q)\|\bar q\| = h_1(\bar q)S_1(\bar q) \leq S_1(\bar q)$, corroborating (13). On the other hand, by (11) and (13), we have that $\|\rho(\bar q)\|^2 = \|\rho(\bar q)\|^{1-a_1}\|\rho(\bar q)\|^{1+a_1} \leq (b_1/k_{1M})^{1-a_1}h_1(\bar q)S_1(\bar q)$, $\forall \bar q \in \mathbb{R}^n$, corroborating (14). △

We will show that, for a sufficiently small value of ε, V in (10) is a suitable Lyapunov function through which the proof will be completed; in particular, this will be proven with

$$\varepsilon < \varepsilon_0 \triangleq \min\{\varepsilon_{1,\mu_m}\,,\ \varepsilon_{1,\kappa_1 k_{1m}^{a_1}}\,,\ \varepsilon_{2,h_{1m}}\,,\ \varepsilon_{3,h_{1m}}\} \tag{15a}$$

where

$$\varepsilon_{1,\hbar} = \frac{1}{\mu_M}\left(\frac{3+a_1}{1+a_1}\cdot\frac{\hbar}{2}\right)^\beta\ ,\ \varepsilon_{3,\hbar} = \frac{\hbar\kappa_1 k_{1m}^{a_1}\beta\eta\left(\frac{\mu_m}{2}\right)^{\beta-1}}{2\hbar\kappa_1 k_{1m}^{a_1}\upsilon_1 + \upsilon_2^2} \tag{15b}$$

$$\varepsilon_{2,\hbar} = \frac{\beta\eta\left(\frac{\mu_m}{2}\right)^{\beta-1}}{\bar c\bar\kappa_2 k_{2M}^{a_2} a_2}\left[\frac{\hbar\kappa_1 k_{1m}^{a_1}(1+a_1)}{2\bar c\bar\kappa_2 k_{2M}^{a_2}}\right]^{1/a_1} \tag{15c}$$

$$h_{1m} \triangleq \inf_{\bar q\in\mathcal{B}^n_{b_1/k_{1M}}} h_1(\bar q) = \inf_{\bar q\in\partial\mathcal{B}^n_{b_1/k_{1M}}} h_1(\bar q) \in (0,1) \tag{16}$$

(recall Remark 3.1) and

$$\upsilon_1 = \frac{k_C b_1}{k_{1M}} + \mu_M\ ,\ \upsilon_2 = (2k_C B_{dv} + f_M)\left(\frac{b_1}{k_{1M}}\right)^{\frac{1-a_1}{2}} \tag{17}$$

[2]Letting $\bar\rho_\varpi(x) = \bar h_\varpi(x;c)x$, $\varpi > 0$, with $\bar h_\varpi(x;c) \triangleq c/\big[c^\varpi + \|x\|^\varpi\big]^{1/\varpi}$, one verifies after basic developments that $D_x\bar h_\varpi(x;c) = -\bar h_\varpi(x;c)(\|\bar\rho_\varpi(x)\|/c)^\varpi$, whence one corroborates that $-\bar h_\varpi(x;c) < D_x\bar h_\varpi(x;c) < 0$, $\forall x \neq 0_n$.

With such a goal in mind, let us begin by noting, from (10) and (8), that

$$V \geq W_{01}^\beta(\bar q,\dot{\bar q}) - \varepsilon\mu_M\left(\|\rho(\bar q)\|^{1/\beta}\|\dot{\bar q}\|^{1/\beta}\right)^\beta \tag{18}$$

$$\geq W_{01}^\beta(\bar q,\dot{\bar q}) - \varepsilon\mu_M\left(\frac{2}{3+a_1}\|\rho(\bar q)\|^{1+a_1} + \frac{1+a_1}{3+a_1}\|\dot{\bar q}\|^2\right)^\beta \tag{19}$$

$$\geq W_{01}^\beta(\bar q,\dot{\bar q}) - \varepsilon\mu_M\left(\frac{2}{3+a_1}S_1(\bar q) + \frac{1+a_1}{3+a_1}\|\dot{\bar q}\|^2\right)^\beta \tag{20}$$

$$\geq W_{01}^\beta(\bar q,\dot{\bar q}) - W_{10}^\beta(\bar q,\dot{\bar q}) \triangleq W_1(\bar q,\dot{\bar q}) \tag{21}$$

with $W_{10}(\bar q,\dot{\bar q}) = (\varepsilon\mu_M)^{1/\beta}\left(\frac{2}{3+a_1}S_1(\bar q) + \frac{1+a_1}{3+a_1}\|\dot{\bar q}\|^2\right)$, where Assumption 2.1 has been applied to get (18), Young's inequality [with $\phi = (3+a_1)/2$ and $\psi = (3+a_1)/(1+a_1)$ (in accordance to the notation used in Section II)] to obtain (19), and Remark 3.3 (more specifically inequality (13)) to get (20). Notice further that $W_{01}^\beta(\bar q,\dot{\bar q}) - W_{10}^\beta(\bar q,\dot{\bar q}) > 0 \iff W_{01}^\beta(\bar q,\dot{\bar q}) > W_{10}^\beta(\bar q,\dot{\bar q}) \iff W_{01}(\bar q,\dot{\bar q}) > W_{10}(\bar q,\dot{\bar q}) \iff W_{01}(\bar q,\dot{\bar q}) - W_{10}(\bar q,\dot{\bar q}) > 0$. Hence, by proving that $W_{01}(\bar q,\dot{\bar q}) - W_{10}(\bar q,\dot{\bar q}) > 0$, $\forall(\bar q,\dot{\bar q}) \neq (0_n,0_n)$, positive definiteness of $W_1(\bar q,\dot{\bar q})$ in (21) is concluded. In this direction, let us define $\kappa_{mv} \triangleq \left(\frac{\mu_m}{2}\right) - (\varepsilon\mu_M)^{1/\beta}\left(\frac{1+a_1}{3+a_1}\right)$ and $\kappa_{mp} \triangleq \left(\frac{\kappa_1 k_{1m}^{a_1}}{1+a_1}\right) - (\varepsilon\mu_M)^{1/\beta}\left(\frac{2}{3+a_1}\right)$, and let us further note that, from Eqs. (15), one may corroborate, after basic developments, that $\varepsilon < \varepsilon_0 \leq \varepsilon_{1,\mu_m} \implies \kappa_{mv} > 0$ and $\varepsilon < \varepsilon_0 \leq \varepsilon_{1,\kappa_1 k_{1m}^{a_1}} \implies \kappa_{mp} > 0$. From this and the expressions defining $W_{01}(\bar q,\dot{\bar q})$ and $W_{10}(\bar q,\dot{\bar q})$, we have $W_{01}(\bar q,\dot{\bar q}) - W_{10}(\bar q,\dot{\bar q}) = \kappa_{mv}\|\dot{\bar q}\|^2 + \kappa_{mp}S_1(\bar q) > 0$, $\forall(\bar q,\dot{\bar q}) \neq (0_n,0_n)$, whence positive definiteness of $W_1(\bar q,\dot{\bar q})$ is concluded. Furthermore, from previous arguments, one sees that $\kappa_{mv} = \left(\frac{\mu_m}{2}\right) - (\varepsilon\mu_M)^{1/\beta}\left(\frac{1+a_1}{3+a_1}\right) > 0 \iff \bar\kappa_{mv} \triangleq \left(\frac{\mu_m}{2}\right)^\beta - \varepsilon\mu_M\left(\frac{1+a_1}{3+a_1}\right)^\beta > 0$ and $\kappa_{mp} = \left(\frac{\kappa_1 k_{1m}^{a_1}}{1+a_1}\right) - (\varepsilon\mu_M)^{1/\beta}\left(\frac{2}{3+a_1}\right) > 0 \iff \bar\kappa_{mp} \triangleq \left(\frac{\kappa_1 k_{1m}^{a_1}}{1+a_1}\right)^\beta - \varepsilon\mu_M\left(\frac{2}{3+a_1}\right)^\beta > 0$. From this and (21), one sees, for every $j = 1,\ldots,n$, that $\lim_{|\dot{\bar q}_j|\to\infty} W_1(0_n,\dot{\bar q}) = \lim_{|\dot{\bar q}_j|\to\infty} \bar\kappa_{mv}|\dot{\bar q}_j|^{2\beta} = \infty$ on $\{\dot{\bar q} \in \mathbb{R}^n : \dot{\bar q}_\ell = 0\ \forall\ell \neq j\}$, and $\lim_{|\bar q_j|\to\infty} W_1(\bar q,0_n) = \lim_{|\bar q_j|\to\infty} \bar\kappa_{mp}(b_1/k_{1M})^{a_1\beta}|\bar q_j|^\beta = \infty$ on $\{\bar q \in \mathbb{R}^n : \bar q_\ell = 0\ \forall\ell \neq j\}$. Hence, under the consideration of its positive definiteness, $W_1(\bar q,\dot{\bar q})$ is additionally concluded to be radially unbounded [8, p. 115]. Furthermore, from (8) and the properties of ρ (particularly (11)), one gets

$$V(t,\bar q,\dot{\bar q}) \leq \left(\frac{\mu_M}{2}\|\dot{\bar q}\|^2 + \frac{\bar\kappa_1 k_{1M}^{a_1}\bar c^{1+a_1}}{1+a_1}\|\bar q\|^{1+a_1}\right)^\beta + \varepsilon\mu_M\|\bar q\|\|\dot{\bar q}\| \triangleq W_2(\bar q,\dot{\bar q}) \tag{22}$$

Since $W_2(\bar q,\dot{\bar q}) \geq W_1(\bar q,\dot{\bar q})$, $\forall(\bar q,\dot{\bar q}) \in \mathbb{R}^n\times\mathbb{R}^n$, and $W_2(0_n,0_n) = W_1(0_n,0_n) = 0$, the time-invariant function W_2 is corroborated to be a positive definite (radially unbounded) function. Therefore, from the conclusions so far drawn on W_1 and W_2, V is concluded to be a positive definite, radially unbounded and decrescent function. Its derivative along the closed-loop system trajectories is obtained, after basic developments, as $\dot{V}(t,\bar{q},\dot{\bar{q}}) = \beta V_0^{\beta-1}(t,\bar{q},\dot{\bar{q}})\dot{V}_0(t,\bar{q},\dot{\bar{q}}) - \varepsilon\rho^T(\bar{q})C\big(q,\dot{q}_d(t)\big)\dot{\bar{q}} - \varepsilon\rho^T(\bar{q})F\dot{\bar{q}} - \varepsilon\rho^T(\bar{q})s_1(K_1\bar{q}) - \varepsilon\rho^T(\bar{q})s_2(K_2\dot{\bar{q}}) + \varepsilon\dot{\bar{q}}^TC(q,\dot{\bar{q}})\rho(\bar{q}) + \varepsilon\dot{\bar{q}}^TC\big(q,\dot{q}_d(t)\big)\rho(\bar{q}) + \varepsilon\dot{\bar{q}}^TH(q)\rho'(\bar{q})\dot{\bar{q}}$, where $H(q)\ddot{\bar{q}}$ has been replaced by its equivalent expression from the closed-loop dynamics (7), Properties 2.2.1 and 2.2.2 have been used and $\rho'(\bar{q}) = \frac{\partial\rho}{\partial\bar{q}}(\bar{q})$. At this point, it is important to note (for its subsequent use in the analysis) that, from Eqs. (15) and (16), one may corroborate that $\varepsilon < \varepsilon_0 \leq \varepsilon_{2,h_{1m}} \implies \gamma_m < \gamma_{M,h_{1m}}$, with

$$\gamma_m \triangleq \frac{\varepsilon\bar{c}\bar{\kappa}_2k_{2M}^{a_2}a_2}{\beta\eta\big(\frac{\mu_m}{2}\big)^{\beta-1}}\ ,\ \gamma_{M,\hbar} \triangleq \left(\frac{\hbar\kappa_1k_{1m}^{a_1}(1+a_1)}{2\bar{c}\bar{\kappa}_2k_{2M}^{a_2}}\right)^{\frac{1}{a_1}} \tag{23}$$

We proceed to analyze the terms of $\dot{V}(t,\bar{q},\dot{\bar{q}})$.

First term. From (8) and (9) (recalling that $\beta = (3+a_1)/[2(1+a_1)]$ and $a_2 = 2a_1/(1+a_1)$), we get: $\beta V_0^{\beta-1}(t,\bar{q},\dot{\bar{q}})\dot{V}_0(t,\bar{q},\dot{\bar{q}}) \leq -\beta W_{01}^{\beta-1}(\bar{q},\dot{\bar{q}})\eta\|\dot{\bar{q}}\|^{1+a_2} \leq -\beta\eta\big(\frac{\mu_m}{2}\|\dot{\bar{q}}\|^2\big)^{\beta-1}\|\dot{\bar{q}}\|^{1+a_2} \leq -\beta\eta\big(\frac{\mu_m}{2}\big)^{\beta-1}\|\dot{\bar{q}}\|^2$.

Second, third, sixth, seventh and eighth terms. From Assumptions 2.1, 2.2, 2.4 and 3.1, the properties of ρ (through inequality (11) and Remark 3.2), Remark 3.3 (particularly inequality (14)), and Eqs. (17), we get: $-\varepsilon\rho^T(\bar{q})C\big(q,\dot{q}_d(t)\big)\dot{\bar{q}} - \varepsilon\rho^T(\bar{q})F\dot{\bar{q}} + \varepsilon\dot{\bar{q}}^TC(q,\dot{\bar{q}})\rho(\bar{q}) + \varepsilon\dot{\bar{q}}^TC\big(q,\dot{q}_d(t)\big)\rho(\bar{q}) + \varepsilon\dot{\bar{q}}^TH(q)\rho'(\bar{q})\dot{\bar{q}} \leq 2\varepsilon k_CB_{dv}\|\rho(\bar{q})\|\|\dot{\bar{q}}\| + \varepsilon f_M\|\rho(\bar{q})\|\|\dot{\bar{q}}\| + \varepsilon k_C\Big(\frac{b_1}{k_{1M}}\Big)\|\dot{\bar{q}}\|^2 + \varepsilon\mu_M\|\dot{\bar{q}}\|^2 \leq \varepsilon\upsilon_2\big[h_1(\bar{q})S_1(\bar{q})\big]^{1/2}\|\dot{\bar{q}}\| + \varepsilon\upsilon_1\|\dot{\bar{q}}\|^2$.

Fourth term. From the definition of ρ and Lemma 2.4, we get: $-\varepsilon\rho^T(\bar{q})s_1(K_1\bar{q}) = -\varepsilon h_1(\bar{q})\bar{q}^Ts_1(K_1\bar{q}) \leq -\varepsilon\kappa_1k_{1m}^{a_1}h_1(\bar{q})S_1(\bar{q})$.

Fifth term. From Hölder and Young's inequalities (both with $\phi = 1+a_1$ and $\psi = 2/a_2$), the definition of s_2, Remarks 2.1 (recalling that $\bar{c} = \bar{c}_{1+a_1,2}$), 2.6 and 3.3, and the consideration of a positive constant $\gamma \in (\gamma_m,\gamma_{M,h_{1m}})$ (recall (23)), we have (recalling that $a_2 = 2a_1/(1+a_1)$) that $-\varepsilon\rho^T(\bar{q})s_2(K_2\dot{\bar{q}}) \leq \varepsilon\big|\rho^T(\bar{q})s_2(K_2\dot{\bar{q}})\big| \leq \varepsilon\|\rho(\bar{q})\|_{1+a_1}\|s_2(K_2\dot{\bar{q}})\|_{2/a_2} \leq \varepsilon\bar{c}\|\rho(\bar{q})\|\bar{\kappa}_2\|K_2\dot{\bar{q}}\|^{a_2} \leq \varepsilon\bar{c}\bar{\kappa}_2k_{2M}^{a_2}\Big(\gamma^{a_2/2}\big[h_1(\bar{q})S_1(\bar{q})\big]^{1/(1+a_1)}\Big)\Big(\gamma^{-a_2/2}\|\dot{\bar{q}}\|^{a_2}\Big) \leq \varepsilon\bar{c}\bar{\kappa}_2k_{2M}^{a_2}\Big(\frac{\gamma^{a_1}}{1+a_1}h_1(\bar{q})S_1(\bar{q}) + \frac{a_2}{2}\gamma^{-1}\|\dot{\bar{q}}\|^2\Big)$.

Thus, from the expressions obtained above, we get

$$\begin{aligned}\dot{V} \leq &-\left[\beta\eta\Big(\frac{\mu_m}{2}\Big)^{\beta-1} - \varepsilon\upsilon_1 - \frac{\varepsilon\bar{c}\bar{\kappa}_2k_{2M}^{a_2}a_2\gamma^{-1}}{2}\right]\|\dot{\bar{q}}\|^2\\ &+\varepsilon\upsilon_2\big[h_1(\bar{q})S_1(\bar{q})\big]^{1/2}\|\dot{\bar{q}}\|\\ &-\varepsilon\left(\kappa_1k_{1m}^{a_1} - \frac{\bar{c}\bar{\kappa}_2k_{2M}^{a_2}\gamma^{a_1}}{1+a_1}\right)h_1(\bar{q})S_1(\bar{q})\end{aligned} \tag{24}$$

which may be rewritten as

$$\begin{aligned}\dot{V} \leq &-\frac{1}{2}\begin{pmatrix}\big[h_1(\bar{q})S_1(\bar{q})\big]^{1/2}\\ \|\dot{\bar{q}}\|\end{pmatrix}^TQ_1\begin{pmatrix}\big[h_1(\bar{q})S_1(\bar{q})\big]^{1/2}\\ \|\dot{\bar{q}}\|\end{pmatrix}\\ &-\varepsilon k_{mp,1}h_1(\bar{q})S_1(\bar{q}) - \frac{k_{mv}}{2}\|\dot{\bar{q}}\|^2 \triangleq W_3(\bar{q},\dot{\bar{q}})\end{aligned} \tag{25}$$

where

$$Q_\hbar = \begin{pmatrix}\varepsilon\hbar\kappa_1k_{1m}^{a_1} & -\varepsilon\upsilon_2\\ -\varepsilon\upsilon_2 & \beta\eta\big(\frac{\mu_m}{2}\big)^{\beta-1} - 2\varepsilon\upsilon_1\end{pmatrix}$$

$k_{mp,\hbar} \triangleq \frac{\hbar\kappa_1k_{1m}^{a_1}}{2} - \frac{\bar{c}\bar{\kappa}_2k_{2M}^{a_2}\gamma^{a_1}}{1+a_1}$ and $k_{mv} \triangleq \beta\eta\big(\frac{\mu_m}{2}\big)^{\beta-1} - \varepsilon\bar{c}\bar{\kappa}_2k_{2M}^{a_2}a_2\gamma^{-1}$. Furthermore, from (23), one may corroborate after basic developments that $\gamma_m < \gamma < \gamma_{M,h_{1m}} < \gamma_{M,1} \implies k_{mp,1} > 0$ and $k_{mv} > 0$, and from Eqs. (15) that $\varepsilon < \varepsilon_0 \leq \varepsilon_{3,h_{1m}} < \varepsilon_{3,1} \implies Q_1 > 0$, whence $W_3(\bar{q},\dot{\bar{q}})$ in (25) is concluded to be negative definite. Hence, V in (10) is a strict Lyapunov function proving that the trivial solution $\bar{q}(t) \equiv 0_n$ of the closed-loop system is globally uniformly asymptotically stable [8, Corollary 3.3].

4th stage: uniform finite-time/exponential stability. Thus, under the consideration of Remark 2.4, all that remains to be proven is that the trivial solution is uniformly finite-time stable if $a_1 \in (0,1)$, or (locally) exponentially stable if $a_1 = 1$. With this goal in mind, we retake V in (10) and analyze its derivative along the closed-loop system trajectories on $\mathbb{R}_{\geq 0} \times \mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$. More precisely, one sees from Remark 3.1 and (16) that, on $\mathbb{R}_{\geq 0} \times \mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$, (24) takes the form $\dot{V}(t,\bar{q},\dot{\bar{q}}) \leq -\Big[\beta\eta\big(\frac{\mu_m}{2}\big)^{\beta-1} - \varepsilon\upsilon_1 - \frac{\varepsilon\bar{c}\bar{\kappa}_2k_{2M}^{a_2}a_2\gamma^{-1}}{2}\Big]\|\dot{\bar{q}}\|^2 + \varepsilon\upsilon_2\|\bar{q}\|^{(1+a_1)/2}\|\dot{\bar{q}}\| - \varepsilon\Big(h_{1m}\kappa_1k_{1m}^{a_1} - \frac{\bar{c}\bar{\kappa}_2k_{2M}^{a_2}\gamma^{a_1}}{1+a_1}\Big)\|\bar{q}\|^{1+a_1}$, which may be rewritten as

$$\begin{aligned}\dot{V}(t,\bar{q},\dot{\bar{q}}) \leq &-\frac{1}{2}\begin{pmatrix}\|\bar{q}\|^{(1+a_1)/2}\\ \|\dot{\bar{q}}\|\end{pmatrix}^TQ_{h_{1m}}\begin{pmatrix}\|\bar{q}\|^{(1+a_1)/2}\\ \|\dot{\bar{q}}\|\end{pmatrix}\\ &-\varepsilon k_{mp,h_{1m}}\|\bar{q}\|^{1+a_1} - \frac{k_{mv}}{2}\|\dot{\bar{q}}\|^2 \triangleq W_4(\bar{q},\dot{\bar{q}})\end{aligned} \tag{26}$$

Furthermore, from (23), one may corroborate, after basic developments, that $\gamma_m < \gamma < \gamma_{M,h_{1m}} \implies k_{mp,h_{1m}} > 0$ and, from Eqs. (15), that $\varepsilon < \varepsilon_0 \leq \varepsilon_{3,h_{1m}} \implies Q_{h_{1m}} > 0$, whence $W_4(\bar{q},\dot{\bar{q}})$ in (26) is concluded to be negative definite (on $\mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$). Furthermore, by defining $\bar{r}_i = (r_{i1},\ldots,r_{in})^T$, $i = 1,2$, with $r_{1j} = \alpha_0/(1+a_1)$ and $r_{2j} = \alpha_0/2$ for all $j = 1,\ldots,n$ and any positive constant α_0, and $\bar{r} = (\bar{r}_1^T\ \bar{r}_2^T)^T$, one can see that, for every $(\bar{q},\dot{\bar{q}}) \in \mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$ and all $\epsilon \in (0,1]$, we have on the one hand that $\|\delta_\epsilon^{\bar{r}_1}(\bar{q})\| \leq \|\bar{q}\| \leq b_1/k_{1M}$ and $\|\delta_\epsilon^{\bar{r}_2}(\dot{\bar{q}})\| \leq \|\dot{\bar{q}}\| \leq b_2/k_{2M}$, and consequently $\delta_\epsilon^{\bar{r}}(\bar{q},\dot{\bar{q}}) \in \mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$, and on the other hand, from (22) and (26), after basic developments, that $W_4(\delta_\epsilon^{\bar{r}}(\bar{q},\dot{\bar{q}})) = W_4(\delta_\epsilon^{\bar{r}_1}(\bar{q}),\delta_\epsilon^{\bar{r}_2}(\dot{\bar{q}})) = W_4(\epsilon^{r_1}\bar{q},\epsilon^{r_2}\dot{\bar{q}}) = \epsilon^{\alpha_0}W_4(\bar{q},\dot{\bar{q}})$ and $W_2(\delta_\epsilon^{\bar{r}}(\bar{q},\dot{\bar{q}})) = W_2(\delta_\epsilon^{\bar{r}_1}(\bar{q}),\delta_\epsilon^{\bar{r}_2}(\dot{\bar{q}})) = W_2(\epsilon^{r_1}\bar{q},\epsilon^{r_2}\dot{\bar{q}}) = \epsilon^{\alpha_0\beta}W_2(\bar{q},\dot{\bar{q}})$, *i.e.* that W_2 and W_4 are locally $\bar{r}$-homogeneous of degree $\alpha_2 = \alpha_0\beta$ and $\alpha_4 = \alpha_0$, respectively, with (common) domain of homogeneity $\mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$. Hence, by Lemma 2.2 and Remark 2.5, there exists a positive constant c such that $W_4(\bar{q},\dot{\bar{q}}) \leq -c[W_2(\bar{q},\dot{\bar{q}})]^{\alpha_4/\alpha_2}$, $\forall(\bar{q},\dot{\bar{q}}) \in \mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$ and, consequently, from (22) and (26), we have that $\dot{V}(t,\bar{q},\dot{\bar{q}}) \leq -c[V(t,\bar{q},\dot{\bar{q}})]^{1/\beta}$, $\forall(t,\bar{q},\dot{\bar{q}}) \in \mathbb{R}_{\geq 0} \times \mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$, with $\frac{1}{\beta} = \frac{2(1+a_1)}{3+a_1} \leq 1$. Moreover, since $a_1 \in (0,1) \implies 1/\beta \in (0,1)$, by Theorem 2.1 and Remark 2.3, item 1 of the proposition is proven. On the other hand, since $a_1 = 1 \implies 1/\beta = 1$, item 2 of the proposition follows from [8, Proof of Corollary 3.4]. □

Remark 3.4: One notes from the second stage of the proof (see particularly (9)) that motion error dissipation is injected

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by the Saturating-Derivative (SD) type control term s_2, while the motion error damping term $F\dot{\bar{q}}$ is in charge to dominate the *damping-indefinite residual* third term (from left to right) in (7), $C\big(q,\dot{q}_d(t)\big)\dot{\bar{q}}$, thus rendering a damping compound effect. The referred domination effect is included in the control strategy in view of the impossibility of the bounded term s_2 to dominate the referred unbounded residual term when this generates force/torque values beyond the limits of the SD type control term. The motion error damping term $F\dot{\bar{q}}$ thus proves —in the third stage of the proof— to be useful to render the uniform asymptotic stability of the closed-loop trivial solution ($\bar{q}(t) \equiv 0_n$) global. Locally, s_2 actually suffices to provide damping enough to guarantee the finite-time/exponential tracking. Indeed, suppose that the last inequality in Assumption 3.1 ($B_{dv} < f_m/k_C$) is omitted, permitting further that $F \geq 0$; observe that this includes the naturally undamped case $F = 0$. One sees (from footnote 1) that if $d \geq 0$ then (9) holds on $\mathbb{R}_{\geq 0} \times \mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$ with $\eta = \kappa_2 k_{2m}^{a_2}$ and consequently the fourth stage of the proof holds, while if $d < 0$ then (9) holds on $\mathbb{R}_{\geq 0} \times \mathcal{B}^n_{b_1/k_{1M}} \times \mathcal{B}^n_{b_2/k_{2M}}$ with $\eta = \kappa_2 k_{2m}^{a_2} + d(b_2/k_{2M})^{1-a_2}$ and consequently, for suitable control parameters (for instance, sufficiently high control gains K_2) such that $\eta > 0$, *i.e.* $\kappa_2 k_{2m}^{a_2} k_{2M}^{1-a_2} > (k_C B_{dv} - f_m) b^{1-a_2}$ (or even $\kappa_2 k_{2m}^{a_2} k_{2M}^{1-a_2} > k_C B_{dv} b^{1-a_2} \geq (k_C B_{dv} - f_m) b^{1-a_2}$), the fourth stage of the proof holds as well. $\triangle$

IV. Simulation results

The proposed scheme was implemented through computer simulations considering the model of a 2-DOF mechanical manipulator corresponding to the experimental robotic arm used in [1], where the accurate expressions of $H(q)$, $C(q,\dot{q})$, $g(q)$ and F can be consulted (they are omitted here in view of space limitations). For such a robot, Property 2.1 and Assumptions 2.1–2.4 are satisfied with $\mu_m = 0.088$ kg m^2, $\mu_M = 2.533$ kg m^2, $k_C = 0.1422$ kg m^2, $f_m = 0.175$ kg m^2/s, $f_M = 2.288$ kg m^2/s, $B_{g1} = 40.29$ Nm and $B_{g2} = 1.825$ Nm. Furthermore, the input saturation bounds are $T_1 = 150$ Nm and $T_2 = 15$ Nm for the first and second links respectively, whence one can corroborate that Assumption 2.5 is fulfilled too. For the sake of simplicity, units are subsequently omitted.

For the implementation of the proposed scheme, we define $\sigma_{ij}(\varsigma) = \mathrm{sign}(\varsigma)\min\{|\varsigma|^{a_i}, M_{ij}\}$, $i,j = 1,2$, for constants $a_i > 0$ and $M_{ij} > 0$. For each $i = 1,2$, such functions prove to be bounded strongly passive functions for $(\kappa_i, a_i, b_i, \bar{\kappa}_i, a_i, b_i)$, where $b_i = \min\{b_{i1}, b_{i2}\}$, $\kappa_i \leq 1$ and $\bar{\kappa}_i \geq b_i^{-a_i}\max\{M_{i1}, M_{i2}\}$, with —for every $j = 1,2$— $b_{ij} = M_{ij}^{1/a_i}$. Following the proposed design procedure, we fixed $a_1 = 1/2$ and $a_2 = 2/3$ for the finite-time control implementations, and $a_1 = a_2 = 1$ for the exponential tracking tests. Let us note that by the defined functions σ_{ij}, we have $B_j = M_{1j} + M_{2j}$, $j = 1,2$ (see (6)). On the other hand, the simulations were run taking initial conditions at $q(0) = [-\pi/4\,,\,-\pi/2]$, $\dot{q}(0) = [-5\pi\,,\,-5\pi]$, and the desired trajectory as $q_d(t) = \left[\frac{7\pi}{4} + \sin(t)\;,\;\frac{\pi}{4} + \cos(t)\right]^T$, for which $B_{dv} = 1$ and $B_{da} = 1$. From this and the above-listed values of the parameters characterizing Property 2.1 and Assumptions 2.1–2.4, one can corroborate that Assumption 3.1 is satisfied too. Moreover, from the considered desired trajectory, one sees that (6) is satisfied provided that $M_{11} + M_{21} < 104.74$ and $M_{12} + M_{22} < 8.21$. Hence, for all the simulations, we took $M_{11} = M_{21} = 50$ and $M_{12} = M_{22} = 4$.

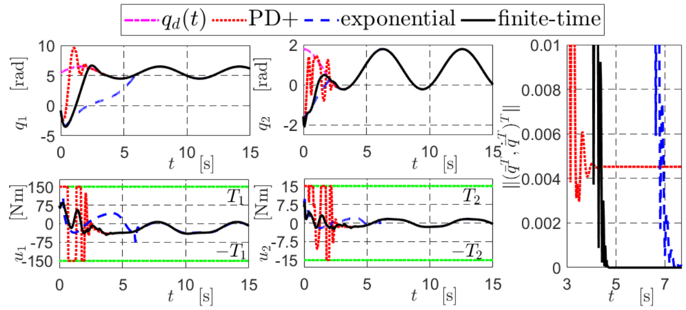

Fig. 1. Position responses (↑), control signals (↓) and $\left\|\left(\bar{q}^T, \dot{\bar{q}}^T\right)^T(t)\right\|$ (→)

As a suggestion from an anonymous reviewer, we further included a *standard* unbounded tracking controller in the comparison. More specifically, we implemented the classical PD+ controller, from [11], given as $u(t,q,\dot{q}) = -K_1\bar{q} - K_2\dot{\bar{q}} + H(q)\ddot{q}_d(t) + C(q,\dot{q})\dot{q}_d(t) + g(q)$.

Fig. 1 shows results obtained taking control gains $K_1 = \mathrm{diag}[800, 50]$ and $K_2 = \mathrm{diag}[28, 2]$ for all the implemented controllers. The tracking objective is observed to be achieved avoiding input saturation through the proposed scheme (for both the finite-time and exponential versions) while the PD+ controller did undergo input saturation at several time intervals during the transient (which is generally undesirable in practice, as pointed out for instance in [1] and references therein). One further observes that the finite-time controller accurately achieves zero tracking error in less than 5 seconds, while with the exponential algorithm the system error trajectories keep approaching zero after 7 seconds; the transient time of the PD+ controlled system trajectories are observed to be practically over in less than 5 seconds too but a post-transient error is perceived, which arises in view of the omission of the system damping term ($F\dot{q}$ or $F\dot{q}_d(t)$) in the PD+ control expression, in view of which $q_d(t)$ is not an equilibrium trajectory of the PD+ controlled (closed-loop) system. As a matter of fact, analog post-transient errors could generally arise in practice due to model inaccuracies, but with an acceptable model that includes the linear components of the damping terms, the proposed scheme is expected to give rise to smaller post-transient errors. Work in this direction, concerning robustness aspects of the proposed approach, is intended to be done in the future. It is further worth observing that while the proposed approach achieves the control objective giving rise to smooth control signals and closed loop trajectories, the PD+ algorithm gives rise to control signals and system trajectories that are highly varying during the transient, demanding great efforts to the actuators and producing an important overshoot.

V. Conclusions

In this work, the —up to our knowledge— open problem of finite-time tracking continuous control of constrained-input mechanical systems has been solved through a strict Lyapunov function, under the consideration of linear damping terms in the open-loop dynamics. While the construction of such a strict Lyapunov function and the corresponding analytical support constitute by themselves an innovative analytical finding, the proposed approach gathers a series of appealing properties by: keeping a continuous structure; giving the freedom to choose among finite-time and (local) exponential convergence through a simple design parameter; guaranteeing the control objective for any initial conditions; avoiding input saturation along the closed loop trajectories, and keeping a Saturating-Proportional (SP) Saturating-Derivative (SD) type action based structure with no tuning restriction on the control gains. Nevertheless, there are further aspects of the proposal that should still be refined in order to enhance its applicability and theoretical value, such as the achievement of the global character for under-damped and undamped mechanical systems, as well as the relaxation of the conditions on the desired trajectory characterization and the requirements imposed to the bounds of the SP and SD actions, which remain conservative in view of the *worst-case* procedure followed in their derivation.

Appendix A
Proof of Lemma 2.1

Since $\|0_n\|_p = 0$ for any p-norm ($p \geq 1$), it is clear that $\left[\frac{\partial}{\partial p}\|x\|_p\right]_{x=0_n} = 0$. For $x \neq 0_n$: $\frac{\partial}{\partial p}\|x\|_p = \frac{\partial}{\partial p}\left[\sum_{i=1}^n |x_i|^p\right]^{1/p} = \frac{\|x\|_p^{1-p}}{p}\left[\sum_{i=1}^n |x_i|^p \ln|x_i|\right] - \frac{\|x\|_p}{p^2}\ln\|x\|_p^p = \frac{\|x\|_p^{1-p}}{p^2}\left[\sum_{i=1}^n |x_i|^p \ln|x_i|^p - \sum_{i=1}^n |x_i|^p \ln\|x\|_p^p\right] = \frac{\|x\|_p^{1-p}}{p^2}\left[\sum_{i=1}^n |x_i|^p \ln\frac{|x_i|^p}{\|x\|_p^p}\right] \leq 0$, since $0 < \frac{|x_i|^p}{\|x\|_p^p} \leq 1 \iff \ln\frac{|x_i|^p}{\|x\|_p^p} \leq 0$, $i = 1, \ldots, n$.

Appendix B
Proof of Lemma 2.4

Item 1. Departing from Lemma 2.3, we have that $\sum_{j=1}^n \int_0^{x_j} \sigma_j(k_j z_j)dz_j \geq \sum_{j=1}^n S(x_j)$, $\forall x \in \mathbb{R}^n$. From this and Lemma 2.1 we get, for all $\|x\| \leq b/k_M$, that $\sum_{j=1}^n S(x_j) = \frac{\kappa}{1+a}\sum_{j=1}^n k_j^a |x_j|^{1+a} \geq \frac{\kappa k_m^a}{1+a}\sum_{j=1}^n |x_j|^{1+a} = \frac{\kappa k_m^a}{1+a}\|x\|_{1+a}^{1+a} \geq \frac{\kappa k_m^a}{1+a}\|x\|^{1+a} = \frac{\kappa k_m^a}{1+a}S_0(x; b/k_M)$, and for all $\|x\| > b/k_M$ we have, for every $j = 1, \ldots, n$, that $\|x\| \geq |x_j| \iff \|x\|^{1-a} \geq |x_j|^{1-a} \implies |x_j|^a \geq \|x\|^a \frac{|x_j|}{\|x\|} \geq (b/k_M)^a \frac{|x_j|}{\|x\|} \implies \kappa k_j^a |x_j|^a \geq \kappa k_j^a (b/k_M)^a \frac{|x_j|}{\|x\|} \geq \kappa k_m^a (b/k_M)^a \frac{|x_j|}{\|x\|} \geq \frac{\kappa k_m^a}{1+a}(b/k_M)^a \frac{|x_j|}{\|x\|}$, and on the other hand that $\kappa b^a \geq \kappa b^a (k_m/k_M)^a \frac{|x_j|}{\|x\|} \geq \frac{\kappa k_m^a}{1+a}(b/k_M)^a \frac{|x_j|}{\|x\|}$, and consequently $\min\{\kappa k_j^a |x_j|^a, \kappa b^a\} \geq \frac{\kappa k_m^a}{1+a}(b/k_M)^a \frac{|x_j|}{\|x\|}$, whence we get that $D_x\left[\sum_{j=1}^n S(x_j)\right] = \sum_{j=1}^n |x_j| \min\{\kappa k_j^a |x_j|^a, \kappa b^a\} \geq \sum_{j=1}^n |x_j| \frac{\kappa k_m^a}{1+a}(b/k_M)^a \frac{|x_j|}{\|x\|} = \frac{\kappa k_m^a}{1+a}(b/k_M)^a \|x\| = D_x\left[\frac{\kappa k_m^a}{1+a}S_0(x; b/k_M)\right] \implies \sum_{j=1}^n S(x_j) \geq \frac{\kappa k_m^a}{1+a}S_0(x; b/k_M)$.

Item 2. Departing from Definition 2.3, we have that $\sum_{j=1}^n x_j \sigma_j(k_j x_j) \geq \sum_{j=1}^n |x_j| \kappa\big(\min\{|k_j x_j|, b\}\big)^a = \kappa\sum_{j=1}^n |x_j| k_j^a \min\{|x_j|^a, (b/k_j)^a\} \geq \kappa k_m^a \sum_{j=1}^n |x_j| \min\{|x_j|^a, (b/k_M)^a\}$, $\forall x \in \mathbb{R}^n$. From this and Lemma 2.1 we get, for all $\|x\| \leq b/k_M$, that $\kappa k_m^a \sum_{j=1}^n |x_j| \min\{|x_j|^a, (b/k_M)^a\} = \kappa k_m^a \sum_{j=1}^n |x_j|^{1+a} = \kappa k_m^a \|x\|_{1+a}^{1+a} \geq \kappa k_m^a \|x\|^{1+a} = \kappa k_m^a S_0(x; b/k_M)$, and for all $\|x\| > b/k_M$ we have, for every $j = 1, \ldots, n$, that $\|x\| \geq |x_j| \iff \|x\|^{1-a} \geq |x_j|^{1-a} \implies |x_j|^a \geq \|x\|^a \frac{|x_j|}{\|x\|} \geq (b/k_M)^a \frac{|x_j|}{\|x\|} \implies (1+a)|x_j|^a \geq |x_j|^a \geq (b/k_M)^a \frac{|x_j|}{\|x\|}$, and on the other hand that $(b/k_M)^a \geq (b/k_M)^a \frac{|x_j|}{\|x\|}$, and consequently that $\min\{(1+a)|x_j|^a, (b/k_M)^a\} \geq (b/k_M)^a \frac{|x_j|}{\|x\|}$, whence we get that $D_x\left[\kappa k_m^a \sum_{j=1}^n |x_j| \min\{|x_j|^a, (b/k_M)^a\}\right] = D_x\left[\kappa k_m^a \sum_{j=1}^n \min\{|x_j|^{1+a}, (b/k_M)^a |x_j|\}\right] = \kappa k_m^a \sum_{j=1}^n |x_j| \min\{(1+a)|x_j|^a, (b/k_M)^a\} \geq \kappa k_m^a \sum_{j=1}^n |x_j| (b/k_M)^a \frac{|x_j|}{\|x\|} = \kappa k_m^a (b/k_M)^a \|x\| = D_x\left[\kappa k_m^a S_0(x; b/k_M)\right] \implies \kappa k_m^a \sum_{j=1}^n |x_j| \min\{|x_j|^a, (b/k_M)^a\} \geq \kappa k_m^a S_0(x; b/k_M)$.

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