Consensus Stability of a Large Scale Robotic Network under Input and Transmission Delays

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Abstract— A large scale network of differential-wheeled robots seeking consensus is considered in the presence of input and transmission delays. Consensus stability of this nonlinear network dynamics holds if its linear part can be shown to achieve consensus stability despite the delays. We present theoretical and computational results on this linear 'large scale' system. From a theoretical point of view, conditions on the amount of input delays are revealed such that stability is guaranteed independent of any transmission delay, and in the case this is not possible, a computationally-efficient algorithm is proposed to reveal the largest transmission delay that the network can accommodate without losing stability.

Index Terms— consensus, stability, differential-wheeled (nonholonomic) robot, time delay, large scale network

I. Introduction

With the rapid advancements in technology, robotic systems are now more affordable, providing new opportunities in building robotic swarms at scales. One direction in this endeavor is to leverage low cost, simpler robots but utilize them in large numbers to perform complex tasks. Coordination of such swarms finds applications from search and rescue to logistics and surveillance [1]–[3]. Such capability however can only be realized by understanding robots' dynamics, their interactions over a network system, and designing controllers for the robots to achieve a particular mission at hand.

Coordination of a network of robots can be seen as a multi-agent system (MAS) in which an underlying graph topology determines robots' neighbors, i.e., which agents communicate with each other and whereby each agent has its own local controller shaped by the information received from its neighbors [4]. One broadly-studied coordination problem in the literature is the problem of consensus, where agents act on information received from their neighbors and seek to reach an agreement in their states, e.g., in their positions [5], [6].

Consensus problem provides a foundation to understand coordination, effects of networks, and distributed controllers. To this end, applications encompass underwater, ground and aerial vehicles in scenarios including consensus, intelligence, resilience, cooperation, and collaboration [7]–[9]. Many theoretical advancements have been attained at the intersection of graph theory and consensus dynamics [5], [10]. Furthermore, the effects of time delays in dynamical systems and MAS is an ongoing research area, with rich results [11]. In MAS specifically, existing literature covers various areas: (i) fast consensus [12], [13], (ii) addressing issues with noise [14]–[16], (iii) the presence of time delays, e.g., [17]–[20], and (iv) manipulating the underlying graphs, graph-related metrics, or delays to improve MAS dynamics [15], [21]–[23].

In the area of ground robots, a well-studied problem is on the coordination of differential-wheeled robots. From a practical point of view, engineering design of these types of robots is accessible and relatively simpler, including available commercial products. From a theoretical perspective, control design of these robots comes with various challenges, including their nonholonomic nature and nonlinearities. Results along these lines include input-output linearization [24], cooperative control [25], and predictor-based controllers [26], [27].

On the other hand, as we are moving to a more connected, large scale MAS, we expect to encounter new technical challenges in the consensus of MAS. Specifically, the network nature of any MAS brings with it the presence of transmission delays $\tau_0 > 0$, that is, information received by an agent at time t was released by another agent at time $t - \tau_0$. Furthermore, as agents are equipped with sophisticated algorithms, this requires more time to compute decisions, which arises in the MAS models as inherent input delay $\tau_{inh} > 0$. While the presence of delays, combined with the nonlinear and nonholonomic nature of differential wheeled robots, brings about theoretical challenges in addressing the stability of the robotic network, in the case of large scale MAS, e.g., those with over $n = 100$ agents, additional challenges also arise in computational efficiency of studying stability and designing distributed controllers.

Existing studies on systems with multiple delays put light on our understanding of the effects of delays on stability, but implementations for large scale systems are yet to be demonstrated. A major challenge in this endeavor is that some studies require expanding system's characteristic equation, which is lengthy for large scale systems, and some others propose to study matrices, e.g., based on LMIs, which can be prohibitive for large scale problems. Possibly due to these reasons, stability

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of large scale systems were so far tackled only under special cases, e.g., when the underlying graph permits decomposing the dynamics [28], however such remedies are not generalizable.

To the best of our knowledge, the above described MAS consensus problem has so far not been studied, and if addressed, will provide a foundation upon which to understand further complex, large scale robotic coordination problems, for example, including the effects of noise, collision avoidance, disturbances, time-varying time delays. Specifically, in this study, we provide a theoretical development for linear stability proofs of a large scale system that is critical to establish for guaranteeing the stability of a network of nonlinear nonholonomic differential-wheeled robots operating under constant transmission and inherent delays. Our approach leverages frequency domain tools to avoid conservatism in the stability analysis and different from current practice, the results here are obtained without having to expand the system characteristic equation, which is otherwise lengthy and thus prohibitive to study due to large scale nature of the problem. This leads to a new computationally-tractable algorithm amenable for parallel computation, which provides the critical delay values (τ_0, τ_{inh}) rendering sustained oscillations in the linear system. These critical delays in turn help determine the regions on the plane $\tau_0 - \tau_{inh}$ where the linear problem is stable, which ultimately helps determine the conditions on the delays for which the nonlinear network dynamics can asymptotically reach consensus. On the theoretical part, the aforementioned linear stability analysis is expanded to understand delay-independent stability. Despite the large size of the given dynamics, a general proof guaranteeing linear stability is presented for any number of robots in the MAS, which then implies the nonlinear MAS will remain stable for any amount of network transmission delay τ_0.

Notations. We write $\lambda_m^\circ = \mathrm{Re}(\lambda_m^\circ) + j\,\mathrm{Im}(\lambda_m^\circ),\ m = \overline{1, M}$ to represent the m-th eigenvalue of the square matrix "$\circ$" where j is the imaginary unit, $\mathrm{Re}(\lambda_m^\circ)$ and $\mathrm{Im}(\lambda_m^\circ)$ are respectively the real and the imaginary parts of λ_m°, and $\overline{1, M} \equiv 1, 2, \ldots, M$. We use $\mathbf{I}_n$ as the $n \times n$ identity matrix, $\mathbf{1}_n$ as the $n \times n$ all-ones matrix, $\mathbb{C}^+$, $\mathbb{C}^-$ and $\mathbb{C}^0$ for the right-half, left-half, and imaginary axis of the complex plane $\mathbb{C}$ and $\bar{\mathbb{C}}^+ = \mathbb{C}^+ \cup \mathbb{C}^0$. The Kronecker product of matrices $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{p\times q}$ is denoted by $\mathbf{A} \otimes \mathbf{B} \in \mathbb{R}^{mp\times nq}$.

II. Preliminaries

We start with definitions and present the dynamics of a single differential-wheeled robot. Next, we introduce the robotic network system and the associated time-delays. Finally, leveraging from prior work, we propose a predictor to compensate against delays in measurements. All the developments in this manuscript assume time-invariant dynamics and serve as a baseline for studying time-varying problems.

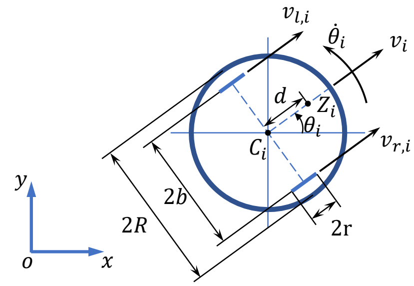

Fig. 1: Differential-wheeled robot with body radius R, wheel radius r, half of wheel axis length b, center point C_i, control point Z_i, orientation θ_i, left wheel angular velocity $\dot{\psi}_{l,i}$, right wheel angular velocity $\dot{\psi}_{r,i}$, left wheel linear velocity $v_{l,i} = r\dot{\psi}_{l,i}$, right wheel linear velocity $v_{r,i} = r\dot{\psi}_{r,i}$, linear velocity v_i at point C_i and angular velocity $\dot{\theta}_i$ about C_i. The distance $d \neq 0$ between C_i and Z_i is a design parameter.

A. Nonholonomic Robot Model and Delayed Measurements

We consider the kinematic differential-wheeled robot shown in Fig.1 with subscript i denoting the robot number. The robot's model is nonlinear and subjected to nonholonomic constraints [29], and moreover it is singular with respect to the center of rotation C_i. To avoid this singularity, a coordinate transformation is applied to shift the control variable to Z_i as shown in the figure.

The model for the i-th robot is given by

$$\dot{Z}_i(t) = \mathbf{\Gamma}(\theta_i(t))V_i(t), \quad \dot{\theta}_i(t) = \mathbf{D}V_i(t), \tag{1}$$

where $Z_i(t) = (Z_{x,i}(t), Z_{y,i}(t))^\top$ and $\theta_i(t)$ are the states of the robot, $V_i(t) = (v_{l,i}(t), v_{r,i}(t))^\top$ denotes the velocities at the center of left and right wheels, $\mathbf{D} = [-1/(2b), 1/(2b)]$ is a constant matrix mapping linear velocities to the angular velocity $\dot{\theta}_i$, and the matrix

$$\mathbf{\Gamma}(\theta_i) := 1/2 \begin{bmatrix} \cos\theta_i + (d/b)\sin\theta_i & \cos\theta_i - (d/b)\sin\theta_i \\ \sin\theta_i - (d/b)\cos\theta_i & \sin\theta_i + (d/b)\cos\theta_i \end{bmatrix},$$

which shifts the coordinates $C_i(t) = (C_{x,i}(t), C_{y,i}(t))^\top$ to $Z_i(t)$, is invertible[1] for $b, d \neq 0$. We also assume the motors of each robot have negligibly small time constants so that velocities can be directly determined by the input, $V_i(t) = U_i(t) = (u_{l,i}(t), u_{r,i}(t))^\top$.

With the above discussions in mind, following [25], [30] we next perform a feedback linearization. To this end, the input to each robot can be proposed[2] as $U_i(t) = \mathbf{\Gamma}^{-1}(\hat{\theta}_i(t))R_i(t)$, where $\hat{\theta}_i(t)$ is the measured orientation and $R_i(t) = [\gamma_{x,i}, \gamma_{y,i}]^\top$ is the desired velocity of the control point Z_i. Next, by separating the linear part from the nonlinear part, the overall robot dynamics can be

[1] Notice that d is a design parameter, and in the case of non-identical robots, $b_i \neq b$, it is possible to select $d_i = d$ such that b_i/d_i remains fixed for all robots $i = 1, \ldots, n$.

[2] This is a decentralized implementation since $\mathbf{\Gamma}^{-1}$ depends only on i.

viewed as

$$\dot{Z}_i(t) = R_i(t) + \mathbf{G}(t, e_{\theta,i})R_i(t), \quad \dot{\theta}_i(t) = \mathbf{D}U_i(t), \tag{2}$$

where $\mathbf{G}(t, e_{\theta,i}) = \mathbf{M}(e_{\theta,i}) - \mathbf{I}_2$, $e_{\theta,i}(t) = \hat{\theta}_i(t) - \theta_i(t)$ is the error between measured orientation and the actual orientation of the robot, and[3]

$$\mathbf{M}(e_{\theta,i}) = \mathbf{\Gamma}(\theta_i)\mathbf{\Gamma}^{-1}(\hat{\theta}_i) = \begin{bmatrix} \cos(e_{\theta,i}) & \sin(e_{\theta,i}) \\ -\sin(e_{\theta,i}) & \cos(e_{\theta,i}) \end{bmatrix}. \tag{3}$$

Furthermore, from [27, Proof of Theorem 1], we know that $\|\mathbf{G}\| = \sqrt{\lambda_{\max}^{\mathbf{G}^\top\mathbf{G}}} = |2\sin(e_{\theta,i}/2)| \leq |e_{\theta,i}|$, for all t. It then follows that $\|\mathbf{G}\| \to 0$ only if $|e_{\theta,i}| \to 0$.

When the measurement $\hat{\theta}_i(t)$ and the state $\theta_i(t)$ are equal to each other, i.e., $|e_{\theta,i}(t)| = 0$, this automatically guarantees that $\mathbf{G}(t, e_{\theta,i}) = 0$ thus the robot dynamics (2) becomes linear. However, robustness against delays is critical since in the presence of inherent delay τ_{inh}, each robot can only access its delayed orientation $\hat{\theta}_i(t) = \theta_i(t - \tau_{inh})$ and hence, in general, the orientation error $e_{\theta,i}(t) = \theta_i(t - \tau_{inh}) - \theta_i(t)$ is not always zero and therefore (2) is nonlinear.

Under the above settings, stability of (2) may be lost for certain values of τ_{inh}. A remedy to this is to utilize a predictor-based framework, see [26], [27], [31], which can be applied to each robot separately to keep the orientation error as small as possible despite the delay τ_{inh}. To this end, the predictor of robot i estimates this robot's orientation via

$$\dot{\hat{\theta}}_i(t) = \mathbf{D}U_i(t) + k_{ob}[\theta_i(t - \tau_{inh}) - \hat{\theta}_i(t - \tau_{inh})], \tag{4}$$

and the estimated orientation $\hat{\theta}_i(t)$ is utilized, instead of the actual measurements, as part of the controller.

Predictor error dynamics obtained by subtracting $\dot{\theta}_i(t)$ in (2) from (4), reads

$$\dot{e}_{\theta,i}(t) = -k_{ob}e_{\theta,i}(t - \tau_{inh}), \tag{5}$$

which can be guaranteed to be asymptotically stable by properly choosing the observer gain k_{ob}. Moreover, this gain can be optimized by a spectrum-based analysis [31] in order to achieve faster decay rate on $e_{\theta,i}(t)$. Once again, as per the decentralized architecture of the control design, if the i-th robot has a different b_i, resulting in a different $\mathbf{D}_i \neq \mathbf{D}$ in (2), then the robot itself can still construct (4) based on the knowledge of $\mathbf{D}_i$ and obtain the same error dynamics (5). As the error $e_{\theta,i}(t)$ becomes sufficiently small in finite time, the robot dynamics (2) will be governed by the following linear system

$$\dot{Z}_i(t) \approx R_i(t). \tag{6}$$

Consequently, one is left with the design of $R_i(t)$ such that robot states $Z_i(t)$ satisfy the objective of achieving consensus.

[3]Thanks to the decentralized architecture of control design, one recovers (3) even if the robots were non-identical, $b_i \neq b, d_i \neq d$ and $\mathbf{\Gamma}(\theta_i; b_i, d_i)$ were different for each robot. For this reason, the following presentation also holds for heterogeneous robots.

Remark 1: Since our focus is on the robustness against delays, in the remainder of the manuscript robot parameters b are assumed to be known and since d is a design parameter it is precisely known. If b has bounded uncertainties, then $\|\mathbf{G}\|$ is bounded. In such cases, the error dynamics would instead read $\dot{e}_{\theta,i}(t) = -k_{ob}e_{\theta,i}(t - \tau_{inh}) + \Delta\mathbf{D}U_i(t)$, where the nominal system (5) is asymptotically stable and input U_i is bounded as per the stability analysis in Sections III-C and III-D. Under these settings and with additional conditions on $\Delta\mathbf{D}U_i(t)$ as per input-to-state stability (ISS) [32, Lemma 4.6], one should study the robust stability of the system at hand.

B. Robot Consensus Design with Time-Delays

Consider next a network of n heterogeneous differential-wheeled robots, each governed by the dynamics (2). In addition to an inherent delay, a transmission delay τ_0 exists between the robots that exchange information over the network. Therefore, at any time instant, the i-th robot has access to only the delayed states $Z_i(t - \tau_{inh})$ and $\theta(t - \tau_{inh})$ about itself and $Z_j(t - \tau_{inh} - \tau_0)$ about its neighbors, and needs to utilize these measurements to make decisions toward reaching consensus.[4]

Considering that the control mission is to achieve consensus, a typical protocol but affected by inherent and transmission delays reads

$$R_i(t) = \sum_{j \in N_i} \alpha_{ij}[Z_j(t - \tau_{inh} - \tau_0) - Z_i(t - \tau_{inh})], \tag{7}$$

where $\alpha_{ij} > 0$ are constant coupling strengths[5] between the i-th agent and the j-th agent and N_i denotes the set of neighbors of the i-th agent. We say that the system reaches consensus when $\lim_{t\to\infty} \|Z_i(t) - Z_j(t)\| = 0$ for all i, j with $i \neq j$.

In vector form, the consensus protocol (7) reads

$$R(t) = [\mathbf{A} \otimes \mathbf{I}_2]Z(t - \tau_{inh}) + [\mathbf{B} \otimes \mathbf{I}_2]Z(t - \tau_{inh} - \tau_0), \tag{8}$$

where $Z(t) = [Z_{x,1}, Z_{y,1}, Z_{x,2}, Z_{y,2}, \ldots, Z_{x,n}, Z_{y,n}]^\top$, and matrices $\mathbf{A}$ and $\mathbf{B}$ are defined as

$$\mathbf{A}_{ij} = \begin{cases} -\beta_{ii} & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}, \quad \mathbf{B}_{ij} = \begin{cases} 0 & \text{if } j = i \\ \alpha_{ij} & \text{if } j \neq i \end{cases} \tag{9}$$

where $\beta_{ii} = \sum_{l \in N_i} \alpha_{il}$ is the sum of the coupling strengths of the neighbors of the i-th robot. Notice that $\mathbf{A}$ is the negative degree matrix and $\mathbf{B}$ is the adjacency matrix of the weighted graph and hence, $\mathbf{A} + \mathbf{B} = -\mathcal{L}$ is the negative graph Laplacian of the weighted graph. Moreover, since

[4]This setting allows to investigate stability with respect to each delay τ_{inh} versus τ_0, with a unique characteristic, namely induced by self-loops or network transmission. Assuming no major disparities among the robots and the network, such a model may be reasonable however further analysis would be needed to explain the cases where the delays are non-identical, see also Remark 7.

[5]The coupling strengths α_{ij} need not be identical and in general we have $\alpha_{ij} \neq \alpha_{ji}$ indicating the heterogeneous and asymmetric nature of the protocol.

we assume that the graph is connected,[6] $\mathcal{L}$ has only one zero eigenvalue [17], [33], [34].

III. Main Results

In this section, we present the multi-dimensional n robot nonlinear network dynamics combined with predictors distributed over the robots. The network dynamics is next re-written as a combination of linear and nonlinear parts, and since, based on stability theory, the stability of the linear part is essential for achieving asymptotic stability of the nonlinear network dynamics, we address the stability of the linear part in the presence of inherent and transmission delays.

A. Stability of the Nonlinear System

Let $\theta(t) = [\theta_1, \theta_2, \ldots, \theta_n]^\top$ and $\hat{\theta}(t) = \left[\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n\right]^\top$ be defined as the orientation and estimated orientation vectors of all robots, respectively. Then, $e_\theta(t) = \hat{\theta}(t) - \theta(t)$ is the orientation error vector and (2) can be rewritten as

$$\dot{Z}(t) = R(t) + \bar{\mathbf{G}}(t, e_\theta)R(t), \quad \dot{\theta}(t) = [\mathbf{I}_n \otimes \mathbf{D}]U(t), \quad (10)$$

where $U(t) = [u_{l,1}, u_{r,1}, u_{l,2}, u_{r,2}, \ldots, u_{l,n}, u_{r,n}]^\top$ is a vector of robots' wheel speeds, $\bar{\mathbf{G}}(t, e_\theta) = \bar{\mathbf{M}}(e_\theta(t)) - \mathbf{I}_{2n}$ represents the non-linearity of the system, and $\bar{\mathbf{M}}(e_\theta(t)) = \mathrm{diag}\{\mathbf{M}(e_{\theta,1}(t)), \mathbf{M}(e_{\theta,2}(t)), \ldots, \mathbf{M}(e_{\theta,n}(t))\}$ is a block diagonal matrix with blocks formed by (3). Combining (8) with (10), the dynamics of the overall multi-robot system becomes

$$\dot{Z}(t) = \left[\mathbf{I}_{2n} + \bar{\mathbf{G}}(t, e_\theta)\right]\big\{[\mathbf{A} \otimes \mathbf{I}_2]Z(t - \tau_{inh}) + [\mathbf{B} \otimes \mathbf{I}_2]Z(t - \tau_{inh} - \tau_0)\big\}. \quad (11)$$

Following the ideas presented in Section II-A, it can be shown that $\bar{\mathbf{G}}(t, e_\theta)^\top \bar{\mathbf{G}}(t, e_\theta) = \mathrm{diag}_{i=\overline{1,n}}\{4\sin^2(e_{\theta,i}/2)\mathbf{I}_2\}$ and $\left\|\bar{\mathbf{G}}(t, e_\theta)\right\| = \sqrt{\lambda_{\max}^{\bar{\mathbf{G}}(t,e_\theta)^\top \bar{\mathbf{G}}(t,e_\theta)}} \leq \max_{i=\overline{1,n}} |e_{\theta,i}(t)|$ for all t, see [27, Proof of Theorem 1]. Therefore, in order to attenuate the effects of nonlinearities, $\lim_{t\to\infty} \|e_\theta(t)\| = 0$ should hold. To this end, inspired by the results in [26], we use decentralized predictors distributed over the robots such that each robot can utilize its own predictor to obtain its own estimated orientation. In multi-dimensional form, the predictor dynamics read

$$\dot{\hat{\theta}}(t) = [\mathbf{I}_n \otimes \mathbf{D}]U(t) + k_{ob}[\theta(t - \tau_{inh}) - \hat{\theta}(t - \tau_{inh})], \quad (12)$$

Subtracting $\dot{\theta}(t)$ in (10) from (12), we obtain the vector version of the error dynamics in (5), given by

$$\dot{e}_\theta(t) = -k_{ob} e_\theta(t - \tau_{inh}), \quad (13)$$

which is a single-delay linear system.

Remark 2: Stability of (13) is guaranteed by choosing the gain $k_{ob} \in (0, \pi/(2\tau_{inh}))$ [17]. Moreover, for fast decay of $e_\theta(t)$, one should pick $k_{ob} = 1/(e\tau_{inh})$ as proven in [31], [35], where e is the base of the natural logarithm.

Remark 3: As (13) is designed to be asymptotically stable, we have $\|e_\theta(t)\| \to 0$ and hence effects of the nonlinear part $\bar{\mathbf{G}}(t, e_\theta)$ in (11) approach zero. Thus, the linear part of the dynamics will dominate the overall nonlinear dynamics as time grows and hence the stability and performance of the linear part plays an important role on the overall nonlinear robotic network dynamics. For this reason, in the following, we shall focus only on the linear part of (11). For the case when transmission delay does not exist ($\tau_0 = 0$), see [27] for the stability proof based on Lyapunov-Razumikhin theory.

B. Stability of the Linear Delay System

From (11), the linear part of the dynamics is expressed in the general form

$$\dot{x}(t) = \mathbf{A}x(t - \tau_{inh}) + \mathbf{B}x(t - \tau_{inh} - \tau_0), \quad (14)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, which is a partition of the vector Z in (11), $\mathbf{A}$ and $\mathbf{B}$ are defined in (9), and τ_{inh} and τ_0 are constant non-negative time delays. Applying Laplace transform to (14) yields the characteristic equation

$$f(s; \tau_{inh}, \tau_0) = \det\left(s\mathbf{I}_n - \mathbf{A}e^{-s\tau_{inh}} - \mathbf{B}e^{-s(\tau_{inh}+\tau_0)}\right) = 0, \quad (15)$$

where $s \in \mathbb{C}$ is the Laplace variable, and delays $\tau_{inh} \geq 0$ and $\tau_0 \geq 0$, which appear in the exponential terms, are parameters. Stability of (14) can then be investigated based on the distribution of the roots of (15), also called the characteristic roots. On the other hand, given the large-scale nature of the dynamics, it is prohibitive to expand the determinant. Further, due to the presence of exponential terms, (15) bears an infinite number of characteristic roots. That is, system (14) is of infinite dimensional nature even if n is finite. Since it is impractical to expand the determinant and impossible to compute the infinitely many characteristic roots, stability analysis of (14) using (15) is not trivial. We first present several important features of (15) regarding its spectrum and next demonstrate how to analyze stability for any given n without having to expand the determinant:

Definition 1: Given $\mathbf{A}$ and $\mathbf{B}$, system (14) is consensus stable if and only if (15) has no more than one root at the origin and the remaining roots are in $\mathbb{C}^-$.

In the above definition, the single root at $s = 0$ arising as the eigenvalue of $\mathbf{A} + \mathbf{B}$ is excluded since this is an invariant root of (15). That is, as the effects of delays in the exponents in (15) vanish at $s = 0$, this root exists for any delay value, indicating the very nature of the consensus dynamics. Then, with the above definition in mind and given τ_{inh} and τ_0, system (14) is said to be stable "around" the consensus state, or shortly "consensus stable", if and only if

$$f \neq 0 \text{ for } \{s | s \in \bar{\mathbb{C}}^+, s \neq 0\} \text{ and } \left.\frac{df}{ds}\right|_{s=0} \neq 0, \quad (16)$$

[6] If the graph is not connected, one can divide the network into several connected subnetworks and analyze each subnetwork separately.

hold where the derivative condition guarantees $s = 0$ is with multiplicity of one. Further, (14) is said to be τ_0-independent stable at τ_{inh} if for a given $\tau_{inh} \geq 0$ (16) holds for all $\tau_0 \geq 0$.

Property 1: When $\tau_0 = 0$, the linear system with inherent delay, $\Sigma_{\mathbf{A},\mathbf{B}} : \dot{x}(t) = (\mathbf{A} + \mathbf{B})x(t - \tau_{inh})$, is consensus stable for $\tau_{inh} \in [0, \tau^*_{inh})$, where τ^*_{inh} can be computed in view of [36] and moreover the system $\Sigma_{\mathbf{A},\mathbf{B}}$ with some $\tau_{inh} \neq 0$ is consensus stable also for $\tau_0 = 0^+$ as per the continuity of the system characteristic roots with respect to τ_0 [37]. In the remainder of this manuscript, we have that $\tau_{inh} < \tau^*_{inh}$, that is, the system $\Sigma_{\mathbf{A},\mathbf{B}}$ is consensus stable.

In view of Property 1, (14) is guaranteed to be consensus stable up to a certain margin, called the Delay Margin (DM) and denoted by τ^*_0 [38]. In the following, we aim to compute the DM τ^*_0 of this transmission delay for a selection of inherent delays $\tau_{inh} \in [0, \tau^*_{inh})$.

For the system (14) with $\tau_{inh} \in [0, \tau^*_{inh})$, the problem of computing τ^*_0 can be stated as a two-step problem: (i) compute the critical delay values $\tau_0 = \tau_c > 0$ for which

$$f(j\omega; \tau_{inh}, \tau_0) = 0, \text{ for some } \omega \geq 0, \tag{17}$$

holds, where $\omega = 0$ is included as a characteristic root may cross over the origin of the complex plane and destabilize the dynamics for some nonzero τ_0, and (ii) pick the smallest of $\tau_c > 0$ as the DM τ^*_0 of the system. In what follows, the purpose is to find all possible (ω, τ_0) satisfying (17) in order to extract τ^*_0, and if no such pairs exist then we say that (14) is consensus stable at τ_{inh}, independent of any value of $\tau_0 \geq 0$.

Remark 4: Two main approaches taken to study the stability of linear systems with delays are: (a) time domain and (b) frequency domain. In (a), arising computations are handled based on convex optimization with the help of linear matrix inequalities, which bring conservatism to the computation of DM. Where exact DM computation is desired, approach (b) is most preferred. In (b), computation of DM for single-delay problems is well established, see [38]–[40], but computing the DM of a 'large scale' system with more than one delay as is the focus in this manuscript does not simply extend from single-delay studies. One of the first results where one delay is fixed and DM is computed along the other delay axis can be found in [41]. While frequency domain approaches do not impose conservatism in computing DM, they may not be suitable for moderate to large size problems. Especially for problems of size $n > 100$, such approaches may lead to impractical computation times as reported in [42]. Similar challenges arise when utilizing rightmost root solvers, since these solvers do not compute the DM but they either compute the rightmost eigenvalues of (14) or the rightmost roots of (15) for a given delay value. Hence, fast convergence to and accuracy of DM cannot be guaranteed, and these solvers cannot reveal that a system is delay-independent stable, i.e., DM is infinite; see [43] for further discussions.

C. τ_0-independent Consensus Stability

We next study, for a given $\tau_{inh} \neq 0$, the conditions under which system (14) can be made consensus stable independent of $\tau_0 \geq 0$, that is, the delay margin is infinite along the τ_0 axis. Two main technical challenges in revealing these conditions are the large scale of the problem at hand which is complicated to analyze for heterogeneous robots/agents, and since by the nature of consensus, the dynamics has a characteristic root at $s = 0$, there is the risk that for some $\tau_0 > 0$ the system may have double roots at $s = 0$, which would indicate instability.

Since τ_{inh} is given, we separate it from τ_0 as follows

$$g(s; \tau_{inh}, \tau_0) = \det\left(se^{s\tau_{inh}}\mathbf{I}_n - \mathbf{A} - \mathbf{B}e^{-s\tau_0}\right) = 0. \tag{18}$$

We start with the case of heterogeneous robots. That is, $\mathbf{A} = -\operatorname{diag}\{\beta_{11}, \beta_{22}, \ldots, \beta_{nn}\}$ with $\beta_{ii} \in \mathbb{R}^+$, and divide the delay-independent stability analysis into two parts where in the first part we study whether double roots may arise at $s = 0$ and in the second part we study the case of pure imaginary roots, namely, $s = j\omega$ for $\omega > 0$. The case of homogeneous robots will be summarized at the end of the subsection.

1) The case of double roots at $s = 0$:

Given that the robots are non-identical, decomposition of the system characteristic equation to simplify this treatment is impossible and hence it should be performed on the large scale linear system.

Lemma 1: The system in (14) cannot have a double characteristic root at the origin of the complex plane for any finite transmission delay $\tau_0 \geq 0$.

Proof: Define $\mathbf{Q}(s) = se^{s\tau_{inh}}\mathbf{I}_n - \mathbf{A} - \mathbf{B}e^{-s\tau_0}$, then if (14) has a double characteristic root at the origin both $\det \mathbf{Q}(s)$ and $d\big(\det \mathbf{Q}(s)\big)/ds$ must vanish at $s = 0$. According to Jacobi's formula [44], we have that

$$\left.\frac{d\det(\mathbf{Q}(s))}{ds}\right|_{s=0} = \operatorname{tr}\left(\operatorname{adj}(\mathbf{Q}(0)) \left.\frac{d\mathbf{Q}(s)}{ds}\right|_{s=0}\right), \tag{19}$$

where $\operatorname{adj}(\cdot)$ denotes the adjugate of matrix $(\cdot)$, $\operatorname{tr}(\cdot)$ is the trace of matrix $(\cdot)$, and the (i, j) entry of $d\mathbf{Q}(s)/ds$ can be written as

$$\frac{d\mathbf{Q}(s)}{ds} = \begin{cases} (1 + s\tau_{inh})e^{s\tau_{inh}} & \text{if } j = i \\ \alpha_{ij}\tau_0 e^{-s\tau_0} & \text{if } j \neq i \end{cases}. \tag{20}$$

The adjugate of $\mathbf{Q}(0)$ in (19) can be calculated by $\operatorname{adj}(\mathbf{Q}(0)) = \mathbf{P}^\top$ where $\mathbf{P} = \{P_{i,j}\}$ is the cofactor matrix of

$$\mathbf{Q}(0) = -\mathbf{A} - \mathbf{B} = \begin{cases} \beta_{ii} & \text{if } j = i \\ -\alpha_{ij} & \text{if } j \neq i \end{cases}. \tag{21}$$

Let $\mathbf{Q}_{i,j}$ be the submatrix of $\mathbf{Q}(0)$ formed by deleting the i-th row and the j-th column, then $P_{i,j} = (-1)^{i+j}\det(\mathbf{Q}_{i,j})$. The i-th entry on the main diagonal of $\mathbf{P}$ is expressed in terms of the eigenvalues of $\mathbf{Q}_{i,i}$ as $P_{i,i} = \det(\mathbf{Q}_{i,i}) = \prod_{k=1}^{n-1} \lambda_k^{\mathbf{Q}_{i,i}}$. Since $\mathbf{Q}_{i,i}$ is real and strictly diagonal dominant, and all elements on its main diagonal are positive, then, according to Gershgorin circle theorem [45], all of its eigenvalues $\lambda_k^{\mathbf{Q}_{i,i}}$ are on the right-half complex plane

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which guarantees that $P_{i,i} \in \mathbb{R}^+$.Then, for $j = \overline{1, n-1}$, we have

$$P_{i,j} - P_{i,j+1} = (-1)^{i+j}[\det(\mathbf{Q}_{i,j}) + \det(\mathbf{Q}_{i,j+1})].$$

Except for the j-th column, the remaining columns of $\mathbf{Q}_{i,j}$ and $\mathbf{Q}_{i,j+1}$ are the same. Therefore, we have that $\det(\mathbf{Q}_{i,j})+\det(\mathbf{Q}_{i,j+1}) = \det\big(\mathbf{Q}^*_{i,j}\big)$ where $\mathbf{Q}^*_{i,j}$ is a matrix formed by adding the j-th column of $\mathbf{Q}_{i,j+1}$ to the j-th column of $\mathbf{Q}_{i,j}$. Because the sum of each row of matrix $\mathbf{Q}(0)$ is zero, $\mathbf{Q}^*_{i,j}$ also has zero row sums which indicates $\det\big(\mathbf{Q}^*_{i,j}\big) = 0$. Accordingly, $P_{i,j} = P_{i,j+1}$. Combining the above results, we have that all the elements of the i-th row of $\mathbf{P}$ are equal to a positive real value $P_{i,i}$. Thus, for any $\tau_0 \geq 0$, $\mathrm{adj}(\mathbf{Q}(0))$ is a positive real matrix and so is $\mathrm{adj}(\mathbf{Q}(0))\, d\mathbf{Q}(s)/ds|_{s=0}$. Hence, $d\big(\det \mathbf{Q}(s)\big)/ds > 0$ for any $\tau_0 \geq 0$. ■

2) The case of $\omega \neq 0$: For the single-delay case, e.g., $\tau_{inh} = 0$, τ_0-independent stability of (14) can be established from [46] where it is shown that a necessary condition for delay-independent stability is that both $\mathbf{A}$ and $\mathbf{A}+\mathbf{B}$ are Hurwitz. When $\tau_{inh} \neq 0$ however, stronger conditions are needed to establish τ_0-independent 'consensus' stability. Moreover, for the consensus dynamics at hand, here $\mathbf{A}+\mathbf{B}$ is not Hurwitz and hence, the proof of Lemma 1 is essential, allowing us to focus only on $\omega > 0$. With this in mind, below we extend [46] to the case of systems with two delays ($\tau_{inh} \neq 0$).

Lemma 2 (τ_0-independent stability (sufficient condition)): In the presence of heterogeneous robots, the linear part (14) of the nonlinear robotic network dynamics is consensus stable for any transmission delay τ_0 if $0 \leq \tau_{inh} \leq 1/(2p)$, $p = \max_{i=\overline{1,n}} \beta_{ii}$.

Proof: Since $\mathbf{A}$ is Hurwitz, there exists $0 \leq \tau_{inh} < \tau^*_A = \pi/(2\max_{i=\overline{1,n}} \beta_{ii})$, see [17], such that the system $\Sigma_{\mathbf{A}} : \dot{x}(t) = \mathbf{A}x(t-\tau_{inh})$ is stable. Then we have $\det(se^{s\tau_{inh}}\mathbf{I}_n - \mathbf{A}) \neq 0$, $\forall s \in \bar{\mathbb{C}}^+$. Accordingly, $(se^{s\tau_{inh}}\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}$ is analytic on $\bar{\mathbb{C}}^+$ and so is $(se^{s\tau_{inh}}\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}e^{-s\tau_0}, \forall\tau_0 \geq 0$. Define $\hat{\mathbf{B}} = (se^{s\tau_{inh}}\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}$. On $s \in \bar{\mathbb{C}}^+$, we have that

$$\rho((se^{s\tau_{inh}}\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}e^{-s\tau_0}) \leq \rho(\hat{\mathbf{B}})\left|e^{-s\tau_0}\right| \leq \rho(\hat{\mathbf{B}})$$

Since $\mathbf{A}$ is diagonal so is $(se^{s\tau_{inh}}\mathbf{I}_n - \mathbf{A})^{-1}$ and therefore, from (9), we can express $\hat{\mathbf{B}} = \{\hat{\mathbf{B}}_{ij}\}$ as $\hat{\mathbf{B}}_{ij} = \alpha_{ij}/(se^{s\tau_{inh}} + \beta_{ii}), \forall i \neq j$ and $\hat{\mathbf{B}}_{ii} = 0$. Next, we write $\sup_{s\in\bar{\mathbb{C}}^+}\left\{\rho(\hat{\mathbf{B}})\right\} \leq \sup_{s\in\bar{\mathbb{C}}^+}\left\{\max_{i=\overline{1,n}}\sum_{j=1}^n\left|\hat{\mathbf{B}}_{ij}\right|\right\} = \sup_{s\in\bar{\mathbb{C}}^+}\left\{\max_{i=\overline{1,n}}\left|\beta_{ii}/(se^{s\tau_{inh}} + \beta_{ii})\right|\right\}$, where inequality is as per the Gershgorin circle theorem [45] and equality is established since $\beta_{ii} = \sum_{l\in N_i}\alpha_{il}$. Therefore, according to Maximum Modulus Principle [47], the last expression can be studied on the imaginary axis of the complex plane, $\sup_{\omega>0}\left\{\max_{i=\overline{1,n}}\left|\beta_{ii}/(j\omega e^{j\omega\tau_{inh}} + \beta_{ii})\right|\right\}$. For each i, we can further write

$$\sup_{\omega>0}\left\{\frac{\beta_{ii}^2}{\omega^2 + \beta_{ii}^2 - 2\omega\beta_{ii}\sin(\omega\tau_{inh})}\right\}, \tag{22}$$

which is always less than one whenever $\omega^2 - 2\omega\beta_{ii}\sin(\omega\tau_{inh}) > 0$ holds $\forall\omega > 0$. Noticing that this expression is equivalent to $(\omega/\tau_{inh})[\omega\tau_{inh} - 2\beta_{ii}\tau_{inh}\sin(\omega\tau_{inh})]$ for $\tau_{inh} \neq 0$, and noting that, for $a \in [0,1]$, $x - a\sin(x) > 0, \forall x > 0$ holds, it is easy to see that $2\tau_{inh}p \leq 1$ guarantees $\omega^2 - 2\omega\beta_{ii}\sin(\omega\tau_{inh}) > 0$, $\forall i, \forall\omega > 0$. For the case of $\tau_{inh} = 0$, (22) is always upper bounded by one. Consequently, for $0 \leq \tau_{inh} \leq 1/(2p)$, we have $\rho(\hat{\mathbf{B}}) < 1$ holds, and therefore $\rho((se^{s\tau_{inh}}\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}e^{-s\tau_0}) < 1$ holds for $s \in \bar{\mathbb{C}}^+, s \neq 0$, which thus guarantees that $g \neq 0$ in (18), $\forall s \in \bar{\mathbb{C}}^+$, $s \neq 0$, independent of τ_0. Clearly, $1/(2p) < \tau^*_A$. ■

Remark 5: With Lemma 1, we guarantee that the system can only have a single root at $s = 0$ as per the nature of the consensus and additional roots at $s = 0$ cannot arise for any non-negative delay values. With Lemma 2, we show that $0 \leq \tau_{inh} \leq 1/(2p)$ will guarantee that the system characteristic equation is analytic on the imaginary axis and open right half plane (with $s \neq 0$). Combining these results with Property 1, we finally conclude that the consensus dynamics with $0 \leq \tau_{inh} \leq 1/(2p) < \tau^*_{inh}$ is stable no matter how large/small the transmission delay τ_0 is.

3) Special case of homogeneous agents: Assume $\tau_{inh} < \tau^*_{inh}$ as per Property 1. For the special case of homogeneous agents, let $\alpha_{ij} = p/(n-1)$ without loss of generality, $p \in \mathbb{R}^+$, then $\mathbf{A} = -p\mathbf{I}_n$ and $\mathbf{B} = p/(n-1)(\mathbf{1}_n - \mathbf{I}_n)$. In this case we can decompose the system characteristic equation as $f = \Pi_{m=1}^n(se^{s\tau_{inh}} + p - \lambda_m^{\mathbf{B}}e^{-s\tau_o}) = 0$ where $\lambda_m^{\mathbf{B}}$ is the m-th eigenvalue of matrix $\mathbf{B}$ given by $\lambda_m^{\mathbf{B}} \in \{p, -p/(n-1), \ldots, -p/(n-1)\}$. Next, for $s = j\omega$, $\omega \geq 0$, we have the following arguments. When $\omega = 0$, $f = 0$ holds only for $\lambda_m^{\mathbf{B}} = p$, but $df/ds|_{s=0} := \lambda_m^{\mathbf{B}}\tau_0 + 1 \neq 0$ for any $\tau_0 \geq 0$, hence the system cannot have more than one pole at $s = 0$. When $\omega \neq 0$, we have that $j\omega e^{j\omega\tau_{inh}} + p - \lambda_m^{\mathbf{B}}e^{-j\omega\tau_0} = 0$, from which the argument condition leads to $p^2 - \left|\lambda_m^{\mathbf{B}}\right|^2 + \omega^2 - 2p\omega\sin(\omega\tau_{inh}) = 0$. Since $\max_{m=\overline{1,n}}\left|\lambda_m^{\mathbf{B}}\right| = p$, $p^2 - \left|\lambda_m^{\mathbf{B}}\right|^2 \geq 0$ holds. Further, in view of the proof of Lemma 2, we have that $\omega^2 - 2p\omega\sin(\omega\tau_{inh}) > 0$ whenever $2p\tau_{inh} \leq 1$. Hence, for $\tau_{inh} \leq 1/(2p)$, the above argument condition cannot hold. We conclude that the system is consensus stable for any $\tau_0 \geq 0$ if and only if $0 \leq \tau_{inh} \leq 1/(2p)$.

D. τ_0-dependent consensus stability

We now investigate the cases when the system cannot be τ_0-independent consensus stable, i.e., DM τ^*_0 is finite. However, given the large scale problem at hand, computation of the DM is not trivial. In what follows, a computationally efficient algorithm is proposed to obtain an accurate approximation of τ^*_0 for large-scale problems.

1) Factorization of the characteristic equation: For a given $\tau_{inh} < \tau^*_{inh}$, the system at $\tau_0 = \tau^*_0$ has characteristic roots at $s = j\omega, \omega \in \mathbb{R}$. Due to the symmetry of conjugate root pairs, here we only need to discuss the case of $\omega \geq 0$ without loss of generality. Let us start by defining $\phi = \omega\tau_0$. Then, following the ideas in the single-delay treatment [43], we decompose (18) as the product of n independent factors $g\,(j\omega, \tau_{inh}, \phi/\omega) = \prod_{m=1}^n g_m\,(j\omega;\tau_{inh},\phi) = 0$,

where the factors g_m are given by

$$g_m(j\omega;\tau_{inh},\phi) = j\omega e^{j\omega\tau_{inh}} - \lambda_m^{\mathbf{M}} = 0, \quad (23)$$

and $\lambda_m^{\mathbf{M}}$ is the m-th eigenvalue of the matrix

$$\mathbf{M} = \mathbf{A} + \mathbf{B}e^{-j\phi}. \quad (24)$$

With the above factorization, which does not need expanding the determinant in (18), approximating the DM requires finding all possible (ω, ϕ) pairs satisfying (23).

Remark 6: According to the continuity property that eigenvalues of a matrix depend continuously upon matrix entries [45] and because $e^{-j\phi}$ is a periodic function of ϕ with period 2π, it follows that $\lambda_m^{\mathbf{M}}$ is also a continuous function of ϕ with period 2π. Since $\mathbf{A}+\mathbf{B}$ has one zero eigenvalue, we observe that $\phi = 0$, $\forall \tau_0 \neq 0$ could be a solution to (23) corresponding to the characteristic root at $s = 0$ associated with consensus. As multiple roots cannot arise at $s = 0$, per Lemma 1, the remainder of this section considers only $\phi > 0$.

Rewriting next (23) using $j\omega e^{j\omega\tau_{inh}} = \omega e^{j(\omega\tau_{inh}+\pi/2)}$ and taking the magnitude and phase of both sides yield

$$\omega = |\lambda_m^{\mathbf{M}}| = \sqrt{\mathrm{Re}^2(\lambda_m^{\mathbf{M}}) + \mathrm{Im}^2(\lambda_m^{\mathbf{M}})}, \quad (25)$$

$$\frac{\sin(\omega\tau_{inh}+\pi/2)}{\cos(\omega\tau_{inh}+\pi/2)} = \frac{\cos(\omega\tau_{inh})}{-\sin(\omega\tau_{inh})} = \frac{\mathrm{Im}(\lambda_m^{\mathbf{M}})}{\mathrm{Re}(\lambda_m^{\mathbf{M}})}. \quad (26)$$

To find if a feasible solution to (25)-(26) exists, first define

$$F_m(\phi) = \mathrm{Re}(\lambda_m^{\mathbf{M}})\cos(\omega\tau_{inh}) + \mathrm{Im}(\lambda_m^{\mathbf{M}})\sin(\omega\tau_{inh}). \quad (27)$$

Then, if there exists $\phi_{m,k}^*$ satisfying (25) subject to $F_m(\phi) = 0$ then, a DM candidate associated with $\lambda_m^{\mathbf{M}}$ can be defined as

$$\tau_{0,m}^* = \min_{k\in\mathbb{N}^+} \frac{\phi_{m,k}^*}{\omega(\phi_{m,k}^*)}. \quad (28)$$

Proposition 1: System (14) with $\tau_{inh} < \tau_{inh}^*$ has a finite DM $\tau_0^* > 0$ if and only if (27) has at least one root ϕ on $(0, 2\pi)$. The DM is then given by $\tau_0^* = \min_{m=\overline{1,n}} \tau_{0,m}^*$, where the DM candidate $\tau_{0,m}^*$ is defined in (28). Otherwise, the system is τ_0-independent stable: $\tau_{0,m}^* = \infty$ and $\tau_0^* = \infty$.

Proof: According to (25), $\omega \geq 0$ is a continuous function of $\lambda_m^{\mathbf{M}}$ and from Remark 6 we know that $\lambda_m^{\mathbf{M}}$ is a periodic function of ϕ. Consequently, ω and $\phi/\omega(\phi)$ are also continuous functions of ϕ with period 2π. Since $\phi = 0$ indicates only a possible solution at $s = 0$ corresponding to infinite DM already addressed in Section III-C, the nonzero roots of (27) on the interval $\phi \in (0, 2\pi)$ will determine all possible finite DM candidates. ■

At this point, solving the delay margin computation problem is reduced to finding the roots of equation $F_m(\phi) = 0$ on the interval $(0, 2\pi)$. Yet, constructing $F_m(\phi)$ requires computing the eigenvalues $\lambda_1^{\mathbf{M}}, \ldots, \lambda_n^{\mathbf{M}}$ of matrix $\mathbf{M}$ in (24). On the other hand, the fact that the entries of $\mathbf{M}$ are continuous in ϕ implies that its eigenvalues define n continuous functions. It then follows from (27) that $F_m(\phi)$ also defines n continuous functions [48, Theorem 5.2]. Since no analytical formula for $\lambda_1^{\mathbf{M}}, \ldots, \lambda_n^{\mathbf{M}}$ is available in general, solving $F_m(\phi) = 0$ is not a trivial task. However, the task may be addressed using a numerical algorithm as proposed next.

2) Algorithmic implementation: We find τ_0^* by constructing the curves $F_m(\phi)$ in (27). This requires scanning[7] the interval $\phi \in [0, 2\pi]$ in search of nonzero roots of (27). This procedure is referred to as the scanning process. Recall that $F_m(\phi)$ is continuous and hence, a sign change of the function within an interval $[a, b]$ indicates the existence of a root $\phi \in [a, b]$.

Since the eigenvalues of $\mathbf{M}$ are required in $F_m(\phi)$, we construct them by discretizing the interval $[0, 2\pi]$ using N points:

$$\mathcal{I} = \{\phi_p,\ p \in \mathbb{P} = \overline{1, N}\}, \quad (29)$$

where $0 = \phi_1 < \phi_2 < \cdots < \phi_N = 2\pi$. Then, for every $\phi_p \in \mathcal{I}$, we obtain the n-tuple $\lambda^{\mathbf{M}_p} = (\lambda_1^{\mathbf{M}_p}, \ldots, \lambda_n^{\mathbf{M}_p})$, where $\mathbf{M}_p$ follows from (24) as $\mathbf{M}_p = \mathbf{A} + \mathbf{B}e^{-j\phi_p}$.

The problem here is how should any two consecutive $\lambda^{\mathbf{M}_p}$-tuples be connected such that sign changes of $F_m(\phi)$ are preserved. Following [43], consider an initial numbering of the eigenvalues using their real parts sorted increasingly:

$$\mathrm{Re}(\lambda_1^{\mathbf{M}_p}) \leq \mathrm{Re}(\lambda_2^{\mathbf{M}_p}) \leq \cdots \leq \mathrm{Re}(\lambda_n^{\mathbf{M}_p}), \quad (30)$$

and let $\mathcal{E}_m^{\mathbf{M}_p} = (\lambda_m^{\mathbf{M}_p}, \nu_m^{\mathbf{M}_p})$ be an eigenpair consisting of the m-th eigenvalue and m-th eigenvector of $\mathbf{M}_p$. Then, from (30), we have that $\mathcal{E}^{\mathbf{M}_p} = (\mathcal{E}_1^{\mathbf{M}_p}, \ldots, \mathcal{E}_n^{\mathbf{M}_p})$, is an ordered n-tuple of eigenpairs. Consider next the metric

$$d_{ij} = 1 - |\nu_i^{\mathbf{M}_p} \cdot \nu_j^{\mathbf{M}_q}| + |\lambda_i^{\mathbf{M}_p} - \lambda_j^{\mathbf{M}_q}|, \quad (31)$$

measuring the distance between $\mathcal{E}_i^{\mathbf{M}_p}$ and $\mathcal{E}_j^{\mathbf{M}_q}$ with $i, j = \overline{1, n}$ and $p, q \in \mathbb{P}$. Before proceeding further, notice that the distance between $\mathcal{E}_i^{\mathbf{M}_p} \in \mathcal{E}^{\mathbf{M}_p}$ and $\mathcal{E}_j^{\mathbf{M}_q} \in \mathcal{E}^{\mathbf{M}_q}$ is assigned by (30) as d_{ii}. We can compute the total distance of the assignment between two n-tuples as

$$\mathcal{J}(\mathcal{E}^{\mathbf{M}_p}, \mathcal{E}^{\mathbf{M}_q}) = \sum_{i=1}^{n} d_{ii} = d_{11} + \cdots + d_{nn}. \quad (32)$$

However, the initial numbering (30) may not minimize the cost functional (32). Hence, we are interested in the assignment, i.e., the numbering that minimizes (32). Note that the problem of minimizing $\mathcal{J}$ is equivalent to finding a permutation π of $\{1, \ldots, n\}$ minimizing the objective function $\sum_{i=1}^{n} d_{i\pi(i)}$.

In a more formal way, the optimal assignment of eigenpairs is the problem of

$$\min_{\pi\in\mathcal{S}_n} \sum_{i=1}^{n} d_{i\pi(i)}, \quad (33)$$

where $\mathcal{S}_n$ denotes the set of all permutations of n indices. Solving this problem by enumerating all possible assignments requires $n!$ operations, which is computationally

[7] In the previous subsection, in the theoretical developments it was shown that $\phi = 0$ does not need to be considered however in numerical implementation one should still include $\phi = 0$ and $\phi = 2\pi$ in the scanning process in order to avoid missing possible roots close to boundaries of the interval.

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prohibitive in general. To circumvent this issue, we use the Hungarian algorithm (HA)[8], which solves the problem in polynomial time $\mathcal{O}(n^3)$ and guarantees an optimal solution [49], [50].

Finally, using the HA to solve (33) we obtain the eigenvalues of $\mathbf{M}_p$ for all $\phi_p \in \mathcal{I}$ and construct the functions $F_m(\phi_p)$ based on (27) while preserving sign changes on the complete interval $\mathcal{I}$, ultimately, preventing from missing candidate ϕ_p solutions. Further, note that the scanning process may be implemented in parallel by partitioning $\mathcal{I} = \bigcup_{w=1}^{W} \mathcal{I}_w$, where W is the number of available processors. Then, searching solutions for $F_m(\phi_p) = 0$ can be performed in parallel, boosting the capabilities of the procedure [51].

Remark 7 (Multiple delays): With multiple delays, the dynamics (14) would read $\dot{x}(t) = \sum_{\ell=1}^{L} A_\ell x(t - \tau_\ell)$, where each matrix A_ℓ is given, and without loss of generality, one can consider that $\tau_2, ...\tau_L$ are all fixed and seek to calculate the delay margin along the delay axis τ_1. To this end, (15) will be rewritten as $f = |\mathcal{M}(\omega) - A_1 e^{-j\omega\tau_1}| = 0$, where the entries of the matrix $\mathcal{M}(\omega) = j\omega I - \sum_{\ell=2}^{L} A_\ell e^{-j\omega\tau_\ell}$ are numerically known for a given ω. The solution $f = 0$ is as a matter of fact equivalent to solving the generalized eigenvalues $\lambda(\omega) = e^{-j\omega\tau_1}$ of the pair $(\mathcal{M}(\omega), A_1)$ whose magnitude is unity $|\lambda(\omega)| = 1$. Once such eigenvalues are identified, if they exist, then the corresponding delay τ_1 can be easily computed from the phase condition of complex number $\lambda(\omega) = e^{-j\omega\tau_1}$. Furthermore, given the system at hand is classified as a 'retarded' system [52], there always exists an upper-bound on ω above which no solution of $|\lambda(\omega)| - 1 = 0$ can ever exist. This approach differs from that presented in Section III-D, but crossing detection part of the the main algorithm can help study the crossing problem $|\lambda(\omega)| - 1 = 0$.

Remark 8: In our recent work [43], [51], DM is computed for systems with a single delay hence the results therein cannot be utilized to study the stability robustness of the dynamics against transmission delays. To this end, developments in Section III-D.1 are new and essential to be able to run parallel computing to obtain DM for large scale problems. Moreover, results in [43], [51] concern the DM computation only; whereas in the current manuscript, for large-scale problems, we provide conditions under which the system maintains stability independent of transmission delays (Section III-C). Last but not least, here we provide a system's level implementation and control design procedure for a large scale network of robots in which decentralized predictors are utilized to achieve stability despite both inherent and transmission delays. Theoretical and computational contributions are enabled by avoiding expanding any determinants associated with large scale matrices.

Remark 9 (Scalability): One of the critical aspects of nonlinear network systems is the well known concept of scalability [53], [54] describing the property of a network in which the disturbances will not grow without bound even if more agents are added to the network. Since the scope of this manuscript is not focused on issues related to disturbances, we refer interested readers to the cited references. Nevertheless, in the current manuscript, the non-delayed system, i.e., the system with zero delays by setup is always consensus stable regardless of the size of the network. A different definition of scalability, which is closely related to the presented work herein is associated with computational effort required to obtain the delay margin along τ_0 as the network size n grows. Since computation times in the current frequency domain techniques do not scale well with network size (see benchmarks in [43], [51] for single-delay problems), here we addressed this challenge by developing theoretical and computational tools for a two-delay large scale linear system associated with the nonlinear robotic network dynamics.

[8]See [49] for details on the implementation of the algorithm and the eigenshuffle function by John D'Errico for an implementation in Matlab.

IV. Case Studies

We provide several case studies to investigate how network topology affects stability robustness against delays. We also test the effectiveness of the proposed DM calculation algorithm for large scale systems with homogeneous and heterogeneous coupling strengths. The controller is given by (8) and the predictor is combined with the nonlinear dynamics. That is, the overall dynamics follows the nonlinear system in (11). To ensure fast decay of $e_\theta(t) \to 0$, we pick $k_{ob} = 1/(e\tau_{inh})$ as per Remark 2. A discussion on the computational efficiency is provided at the end of the section.

A. Small-scale Robotic Network

We first focus on a small-scale problem with $n = 6$ homogeneous robots. Specifically, we investigate how DM τ_0^* varies for different network topologies, namely, strongly connected, bilateral ring, and directed ring. Matrices $\mathbf{A}$ and $\mathbf{B}$ corresponding to each topology are constructed according to (9). Furthermore, we normalize the Laplacian matrix by the sum of coupling strengths for comparison. In the normalized cases, matrix $\mathbf{A} = -\mathbf{I}_n$ and sum of each row of $\mathbf{B}$ is unity. For the case of directed ring, corresponding matrices are already in normalized form since the coupling strength is $\alpha_{ij} = 1$.

Next, we visit Section III-D to compute τ_0^* for a set of delays $\tau_{inh} < \tau_{inh}^*$. The results are summarized in Fig.2 where colored markers indicate the delay pairs for which the linear system has dominant roots on the imaginary axis. That is, these markers form the stability boundary. Given that this system is consensus stable at the origin of the figure, it is also stable at any point that can be connected to the origin without crossing over the stability boundary.

Comparing the three topologies without normalization (bilateral, directed and strongly connected topology), those with larger number of links cause higher gains, i.e., entries in $\mathbf{A}$ are larger, and with higher gains, as expected, the system can accommodate only smaller delays. With normalization, the couplings are reduced to weak gains, which allow larger stability regions. We can conclude that the strongly connected topology with normalized gains offers the largest stable region, and that the results of Lemma 2 related to τ_0-independent consensus stability are validated also via computations, which cannot find any feasible imaginary roots of the system characteristic equation for $\tau_{inh} < 1/(2p)$.

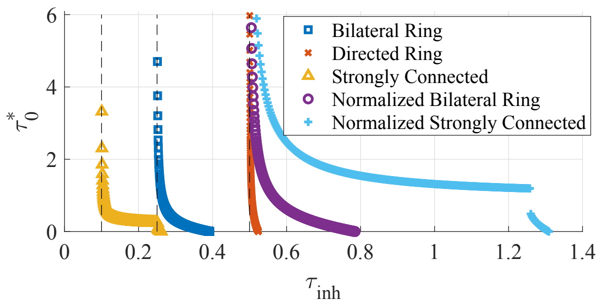


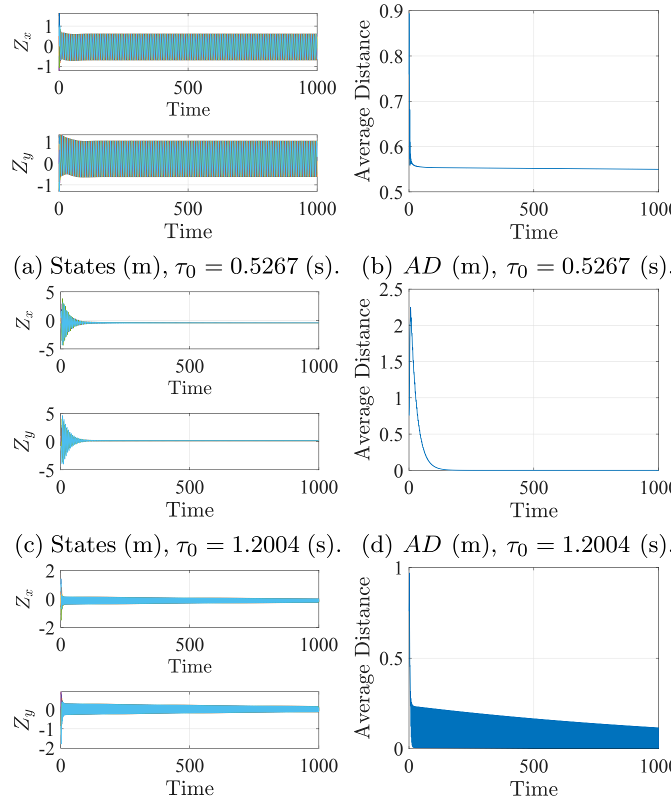
Fig. 2: Robustness with respect to transmission delay τ_0. Calculated DM τ_0^* for different $\tau_{inh} < \tau_{inh}^*$ and for five types of topologies (see legend) in a network of $n = 6$ robots. Black dash lines indicate the boundaries of τ_0-independent consensus stability corresponding to Lemma 2. A small segment of the boundary not captured at $\tau_{inh} \approx 1.25$ can be revealed by increasing resolution along τ_{inh}.

We also present time simulations with the goal to validate that indeed the nonlinear robotic network dynamics reach the border of stability at $\tau_0 = \tau_0^*$. Time-simulations are run with a time step size of 0.0001 (s) for a total simulation time of 1000 (s). The main parameters of the robots are $b = d = 0.052$ (m) and the robots' states are updated based on Euler method. The initial coordinates of the i-th robot's center are $C_{x,i}(0) = \cos[(i-1)\pi/6]$ and $C_{y,i}(0) = \sin[(i-1)\pi/6]$, and the initial orientation is $\theta_i(0) = (i-1)\pi/6 - \pi/i$, $i \in \{1, 2, \ldots, 6\}$. In addition to the states controlled, Z_x and Z_y, we define the average distance (AD) of robots' control points as a scalar metric of consensus: $AD(t) = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} d_{ij}(t)/[n(n-1)]$, where d_{ij} is the distance between the i-th robot's control point and the j-th robot's control point.

Since numerical truncation will not permit to test the precise value of $\tau_0 = \tau_0^*$, for a given τ_{inh}, simulations are conducted for both $\tau_0 = 99\%\tau_0^*$, which should indicate stability and $\tau_0 = 1.01\%\tau_0^*$, which should indicate instability. In all the cases both stability and instability are validated via time-simulations. Due to lack of space, we provide only those cases with normalized coupling strengths at $\tau_0 = 99\%\tau_0^*$ for a given $\tau_{inh} < \tau_{inh}^*$, see Fig. 3.

B. Large-Scale Strongly Connected Robotic Network

We next study a case of $n = 400$ robots with homogeneous coupling strengths and considering the normalized strongly connected robotic network as a benchmark. Similar to the previous section, in the homogeneous case,



(a) States (m), $\tau_0 = 0.5267$ (s). (b) AD (m), $\tau_0 = 0.5267$ (s).
(c) States (m), $\tau_0 = 1.2004$ (s). (d) AD (m), $\tau_0 = 1.2004$ (s).
(e) States (m), $\tau_0 = 0.6301$ (s). (f) AD (m), $\tau_0 = 0.6301$ (s).

Fig. 3: Top row: Directed ring network with $\tau_{inh} = 0.51$ (s) $< \tau_{inh}^* = 0.5235$ (s). The calculated DM is $\tau_0^* = 0.5320$ (s). Center row: Normalized strongly connected network with $\tau_{inh} = 1.2$ (s) $< \tau_{inh}^* = 1.3089$ (s). The calculated DM is $\tau_0^* = 1.2125$ (s). Bottom row: Normalized bilateral ring network with $\tau_{inh} = 0.6$ (s) $< \tau_{inh}^* = 0.7853$ (s). The calculated DM is $\tau_0^* = 0.6365$ (s).

we choose identical coupling strengths $\alpha_{ij} = 1/(n-1)$. According to (9), the corresponding matrices $\mathbf{A}, \mathbf{B}$ will be in normalized form.

Next, we randomize the coupling strengths in two different ways; one in which we scale up and in the other in which we scale down the coupling strengths from the homogeneous case. Coupling strengths are chosen as $\alpha_{ij} = \eta_{ij}/(n-1)$, where scaling factors are selected randomly from a normal distribution with $\eta_{ij} \in [1, 1.1]$ for the scale-up case and with $\eta_{ij} \in [0.9, 1]$ for the scale-down case.

The DM of the homogeneous case is next compared with the two cases of heterogeneous coupling strengths where one is labeled as scale up and the other as scale down, for a range of feasible $\tau_{inh} < \tau_{inh}^*$, see Fig.4. As the coupling strengths are increased corresponding to relatively higher gains, the region of delays that permits consensus stability shrinks. Although not shown, the sufficient condition for τ_0-independent stability can be verified on the figure for $\tau_{inh} < 1/(2p)$ where for the homogeneous case $p = 1$.

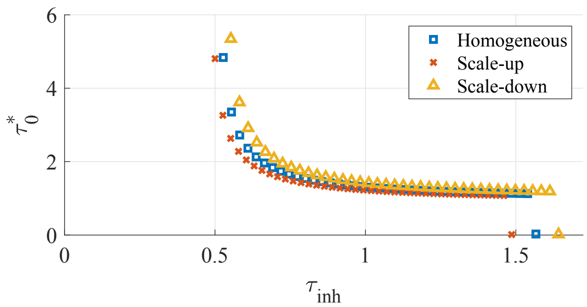


Fig. 4: Robustness with respect to transmission delay τ_0. Calculated DM τ^* for strongly connected case, $n = 400$, with homogeneous couplings, with heterogeneous couplings obtained by scaling up or down the homogeneous couplings (see text for details). A small segment of the boundary not captured at $\tau_{inh} \approx 1.5$ can be revealed by increasing resolution along τ_{inh}.

C. DM Computational Efficiency

In the examples, DM τ_0^* is calculated[9] for a given τ_{inh}, $0 < \tau_{inh} < \tau_{inh}^*$. In Fig.2, DM computations has a mean of 0.13 (s) and standard deviation of 0.026 (s) and in Fig.4, mean and standard deviation of computation time of the homogeneous case was respectively 980.6 (s) and 80.2 (s), while for both the scale-up and scale-down cases, they were respectively 130 (s) and 1.6 (s). The homogeneous case is computationally more expensive since $\theta \in \mathbb{R}^+$, $\mathbf{M} = \mathbf{A} + \mathbf{B}e^{-j\theta}$ has $n-1$ repeated eigenvalues which require more computation time in the eigenvalue assignment process in Section III-D.2.

V. Conclusion

We present theoretical and computational results to reveal stability robustness of large scale linear consensus dynamics against two types of homogeneous time delays. Results combined with input/output linearization and decentralized predictors enable stable consensus of nonlinear robotic network dynamics. Although it is reasonable to assume each robot is aware of its dynamics, further research is needed to study the robustness of the network dynamics against uncertainties. Other developments are needed in optimizing the DM finder presented, especially by reducing computation times when faced with redundant computations associated with repeated eigenvalues and expanding it to handle non-homogeneous delays in light of Remark 7.

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