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Flocking motion of multi-agent systems: A proximity digraph case ⋆

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Abstract: We investigate the coordinated motion of a multi-agent system with heterogeneous distance-dependent communication constraints. In this setup, the underlying interaction network is dynamic since edges appear or disappear as the agents navigate their workspace. Inspired by the gradient-descent method, we provide a distributed controller which preserves the position-dependent communication network connectivity properties. We use a distributed connectivity measure based on the entries of the first-left eigenvector of the network's associated Laplacian matrix to provide the agents with local knowledge of the overall network topology and reveal its dynamics properties. We illustrate our result with a numerical simulation.

Keywords: Multi-agent systems; Distributed control and estimation; Flocking motion; State-dependent digraphs

1. INTRODUCTION

Over the past few years, multi-agent systems coordination has become a topic of interest for researchers from many fields, such as ecology, biology and robotics. A multi-agent system consists of a large number of components, called agents, interacting with each other through communication or sensing networks. In general, agents are simple, and coordination arises from agents' interactions. In coordination tasks, such as the formation control of multiple mobile robots, agents must exchange information with others. Therefore, the capability to preserve the communication network's connectivity is of great relevance.

Often, in physically embodied agents with limited communication capabilities (such as mobile robots), the inter-agent distances dictate the existence of communication edges. Therefore, the distributed controller design, which drives the multi-agent system to achieve the task at hand, must also preserve the communication edges. This observation led to many studies on connectivity preservation over proximity graphs. The first approaches aim to maintain every edge and allow only their additions (Ávila-Martínez and Barajas-Ramírez, 2021). However, these conservative solutions hindered the network's flexibility. To account for edge deletions a measure of the overall graph's connectivity was introduced, and distributed estimation processes were developed (Sabattini et al., 2013; Fang et al., 2017; Ávila-Martínez, 2023). Nonetheless, in many cases agents with different sensing ranges were not contemplated.

The coordinated motion in multi-agent systems takes inspiration from collective behaviours observed in nature, such as consensus, synchronization and flocking. In flocking motion, every agent follows three simple rules: flock centring, collision avoidance and velocity matching (Reynolds, 1987). Coordination emerges from local interactions, and many studies concentrate on bidirectional communication channels. Olfati-Saber presented a theoretical framework to design distributed controllers for flocking motion employing artificial potential functions (APFs) over bidirectional graphs (Olfati-Saber, 2006). However, a multi-agent system with heterogeneous sensing ranges leads to directed communication channels, a case barely studied because of its complexity.

In this paper, we design a distributed controller for flocking motion in second-order multi-agent systems with heterogeneous communication constraints. The main challenge in this setup is to keep the overall network properties which drive the multi-agent system to achieve flocking motion. When connections are distance-dependent, fragmentation, i.e. when the group splits into two or more components, is a common concern that hiders the group's coordination. Uneven communication constraints imply that the agent's actions are not symmetrical, a crucial feature to show the behaviour's stability in the case of bidirectional communication channels. Since the controller's design cannot rely on symmetric actions, we must provide the agents with a local sense of the overall graph's topology. Therefore, graph connectivity measure and its dynamic properties take great relevance in our setup. Yet, these design concerns provide an opportunity to derive methods to coordinate multi-agent systems with heterogeneous capabilities.

1.1 Related works

Coordination in first-order multi-agent systems with position-dependent, heterogeneously constrained, commu-

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nication (or sensing) edges was studied in (Poonawala and Spong, 2017; Maeda et al., 2017; Yoshimoto et al., 2018; Sano et al., 2023). A target determination process was used in (Maeda et al., 2017; Yoshimoto et al., 2018; Sano et al., 2023), where each agent chooses one of its neighbours such that the initial proximity digraph contains a directed spanning tree. Then, the control actions are such the multi-agent system conserves such a tree. On the other hand, a distributed connectivity measure for strongly connected digraphs is introduced in (Poonawala and Spong, 2017). This measure is used to decide whether a connectivity preserving control action is included in the agent's controller or not, and the sensing ranges are assumed to be increased or decreased by the controller as needed. The connectivity measure is based on the first-left Lapacian's eigenvector entries and is computed through the distributed method described in (Poonawala and Spong, 2015).

1.2 Contributions

In contrast to the above discussed researches, we contemplate second-order dynamic agents, explore the dynamic properties of the connectivity measure introduced in (Poonawala and Spong, 2017), and use it in the gradient-descent method to obtain distributed controllers that steer the system into a flocking motion behaviour. With this design, we allow both edge additions and deletions while preserving the proximity graph's strong connectivity.

2. PRELIMINARIES

Let $\mathbb{R}^d$ and $\mathbb{C}^d$ be the d-dimensional sets of real and complex numbers, and $\mathbb{R}^{d\times d}$ the set of $d \times d$ matrices. The set $\mathbb{R}_{\geq 0}$ is the non-negative real scalars set. Denote as $\mathbf{1}_d$ ($\mathbf{0}_d$) the d-dimensional vector with all its entries equal to one (zero). For a matrix $A \in \mathbb{R}^{d\times d}$ denote as $rank(A)$ its rank. Meanwhile, for a set $\mathcal{S}$, denote $|\mathcal{S}|$ its cardinality.

2.1 Algebraic graph theory

A *directed graph* (in short, a *digraph*) of order N is a pair $\mathscr{D} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \{1, 2, \ldots, N\}$ is a set of *nodes* and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ a set of ordered pairs of nodes, called *edges*. For $i, j \in \mathcal{V}$, the ordered pair $e_{ji} = (j, i) \in \mathcal{E}$ denotes an edge that starts at j and ends at node i. In an edge e_{ij}, respectively i is known as the *parent* and j as the *child*. The in- and out-neighbours of node i respectively are

$$\mathcal{N}_i^{in} := \{j \in \mathcal{V} : e_{ji} \in \mathcal{E}\} \text{ and } \mathcal{N}_i^{out} := \{j \in \mathcal{V} : e_{ij} \in \mathcal{E}\}.$$

In $\mathscr{D}$, a *self-loop* is an edge that starts and ends in the same node. A *simple* digraph has no self-loops and no multiple edges between the same pair of nodes (here we only consider simple digraphs). A *directed path* of length m from node i to j is a sequence of edges $e_{ik_1}, e_{k_1k_2}, \ldots, e_{k_mj}$ with distinct nodes k_l and $l = 1, 2, \ldots, m$. A *cycle* is a simple path that starts and ends at the same node. A digraph is acyclic if it contains no cycles.

A *directed tree*, sometimes called a *rooted tree*, is an acyclic digraph with a node, called the *root*, with no out-neighbours and where every other node has a directed path to it. In a directed tree, a *leaf* is a node with no in-neighbours. A directed tree is called a *spanning tree* of $\mathscr{D}$ if is a spanning subgraph of it. Let $\mathcal{T}$ be a spanning tree in $\mathscr{D}$ and $\mathbb{T}_i$ the collection of all spanning trees in $\mathscr{D}$ with node i as its root, thus $\mathcal{T} \in \mathbb{T}_i$ means $\mathcal{T}$ is a spanning tree with root at the ith node.

Digraph $\mathscr{D}$ is *weakly connected* if has no isolated nodes and *strongly connected* if there exists a directed path connecting every nodes pair.

The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ of a digraph $\mathscr{D}$, is a nonnegative matrix with elements $a_{ij} > 0$ if $e_{ji} \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. When $e_{ji} \in \mathcal{E}$, a_{ij} is known as the edge weight. The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N\times N}$ of $\mathscr{D}$ is a zero-row-sum nonnegative definite matrix with elements $l_{ii} = \sum_{j\in\mathcal{N}_i^{in}} a_{ij}$ and $l_{ij} = -a_{ij}$. The following previous results summarize several properties of the Laplacian matrix.

Lemma 1. (Wu (2007)). For a digraph $\mathscr{D}$ with Laplacian L there exists a positive vector $\gamma \in \mathbb{R}^N$ such that $\gamma^T L = \mathbf{0}_N^T$ if and only if $\mathscr{D}$ is a disjoint union of strongly connected subgraphs.

Lemma 2. (Wu (2007)). Consider a digraph $\mathscr{D}$ which contains a directed spanning tree and with Laplacian L. Let a nonnegative vector $\gamma \in \mathbb{R}^N$ such that $\gamma^T L = \mathbf{0}_N^T$, with $\gamma = [\gamma_1, \ldots, \gamma_N]$. Then $\gamma_i = 0$ for all nodes i that do no have directed paths to all other nodes in $\mathscr{D}$ and $\gamma_i > 0$ otherwise.

Proposition 1. (Guo et al. (2008)). For a strongly connected digraph $\mathscr{D}$ and vector $\gamma \in \mathbb{R}^N$ such that $\gamma^T L = \mathbf{0}_N^T$,

$$\gamma_i = \sum_{\mathscr{T}\in\mathbb{T}_i} \Pi\left(\mathcal{E}_{\mathscr{T}}\right), \qquad i \in \mathcal{V}, \tag{1}$$

where $\mathcal{E}_{\mathscr{T}}$ is the edge set of $\mathscr{T}$ and $\Pi\left(\mathcal{E}_{\mathscr{T}}\right) = \prod_{e_{kj}\in\mathcal{E}_{\mathscr{T}}} a_{jk}$.

Lemma 3. (Li and Duan (2015)). For a digraph $\mathscr{D}$ define its generalized algebraic connectivity as

$$\alpha = \min_{\gamma^T x=0, x\neq\mathbf{0}} \left\{ \frac{x^T\left(\Gamma L + L^T\Gamma\right)x}{2x^T\Gamma x} \right\}, \tag{2}$$

where L is its matrix Laplacian, γ is defined as in Lemma 1 and $\Gamma = diag(\gamma)$. Then $\alpha > 0$.

3. PROBLEM FORMULATION

Consider a multi-agent system consisting of N mobile agents with the dynamics of the ith agent given by:

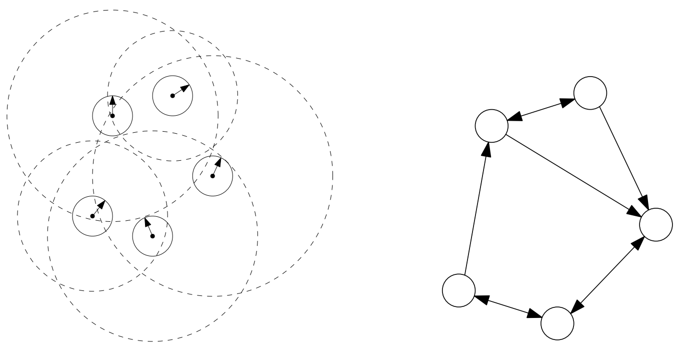
$$\dot{p}_i = v_i, \quad \dot{v}_i = u_i, \qquad i \in \mathcal{V}, \tag{3}$$

where $p_i, v_i, u_i \in \mathbb{R}^n$ are, respectively, its position, velocity, and control input. Each agent has an individual maximum sensing radius $r_i > 0$ within which it gathers information from the nearby agents.

In multi-agent systems, the group's cohesive movement is called flocking motion. In terms of Reynold's boids model, every agent in the group satisfies three heuristic rules: *Flock centering*, *collision avodiance* and *velocity matching*. These rules translate to control objectives as follows.

Definition 1. We say the multi-agent system (3) is on flocking motion over a time interval $[t_0, t_f)$ if the following properties are satisfied:

(1) (Group cohesiveness) There is a constant $\epsilon_p > 0$ such that $\|p_{ij}\| \leq \epsilon_p$;
(2) (Collision avoidance) The inter-agent distances are positive, i.e. $0 < \|p_{ij}\|$;



(a) (b)

Fig. 1. The nearest neighbourhood rule (a) and its resulting proximity digraph (b).

(3) (Bounded velocity mismatch) There exists a sufficiently small constant $\epsilon_v > 0$ such that $\|v_{ij}\| \leq \epsilon_v$;

where $p_{ij} := p_i - p_j$ and $v_{ij} := v_i - v_j$ are, respectively, the inter-agent position and velocity deviations for all $i, j \in \mathcal{V}$.

4. PROXIMITY DIGRAPHS

A *proximity digraph* is a state-dependent digraph [1] in which the relative position between agents regulates the existence of an edge (Mesbahi and Egerstedt, 2010). More precisely, it's a digraph $\mathscr{D}(p) := (\mathcal{V}, \mathcal{E}(p))$ where the multi-agent system's configuration $p = [p_1^T, \dots, p_N^T]^T \in \mathbb{R}^{nN}$ governs the edge set. To keep a short notation, we denote $\mathscr{D} = \mathscr{D}(p)$ and $\mathcal{E} = \mathcal{E}(p)$.

We can model the agents' interactions in system (3) through a proximity digraph (Figure 1 illustrates this observation) where the ith agent's neighbours set is

$$\mathcal{N}_i^{in} := \{j \in \mathcal{V} : \|p_{ij}\| \leq r_i\}. \tag{4}$$

In proximity graphs, the algebraic connectivity reflects its connectivity properties (Sabattini et al., 2013; Fang et al., 2017; Ávila-Martínez, 2023); Nonetheless, computing it is a centralized operation. In (Yang et al., 2010; Fang et al., 2017; Zhang et al., 2022), the authors present distributed computational methods to provide agents with the algebraic connectivity value. However, they are not suitable for digraph setups. In what follows, we provide a distributed connectivity measure for digraphs.

4.1 Digraph connectivity measure

Let $\gamma \in \mathbb{R}^N$ be the first-left eigenvector of the matrix Laplacian associated to proximity digraph $\mathscr{D}$, *i.e.* such that $\gamma^T L = \mathbf{0}_N^T$. Define the *connectivity measure* between the ith agent and its in-neighbour j as a function $\mu : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \mapsto (1, \infty)$ as follows:

$$\mu(\gamma_i, \gamma_j) := \frac{\gamma_i}{\gamma_j} + \frac{\gamma_j}{\gamma_i}. \tag{5}$$

Notice $\mu(\gamma_i, \gamma_j) \to \infty$ if and only if either $\gamma_i \to 0$ or $\gamma_j \to 0$. From now on, we use the shorthand notation $\mu_{ij} = \mu(\gamma_i, \gamma_j)$.

[1] A state-dependent digraph is a mapping between the state space of the networked system and the set of all its possible network configurations. See Mesbahi and Egerstedt (2010) for more details.

The connectivity measure of equation (5) was introduced in (Poonawala and Spong, 2015) and used in (Poonawala and Spong, 2017) to preserve the overall digraph's connectivity in a MAS setup. However, the authors used it to decide whether or not the agents' controller includes a given term. Here, we provide some dynamic properties of the connectivity measure and use it with the gradient-descent method to derive each agent's controller.

4.2 Position-dependent connectivity measure

Let the ijth element of the adjacency matrix A be

$$a_{ij}(\|p_{ij}\|) := \begin{cases} 1 & \text{if } \|p_{ij}\| < \rho_i; \\ \tilde{a}_{ij}(\|p_{ij}\|) & \text{if } \rho_i \leq \|p_{ij}\| \leq r_i; \\ 0 & \text{if } r_i < \|p_{ij}\|; \end{cases} \tag{6}$$

where $\tilde{a}_{ij}(\cdot) : [\rho_i, r_i] \mapsto [0, 1]$ is a differentiable and strictly decreasing bump function. This definition provides a position-dependent nonnegative value ranging from connectivity to non-connectivity between nearby agents, a common phenomenon in sensors and communication devices restricted by distance. Notice also that:

$$\dot{a}_{ij}(\|p_{ij}\|) = v_{ij}^T \nabla_{p_i} a_{ij}(\|p_{ij}\|), \tag{7}$$

where $\nabla_{p_i} a_{ij}(\|p_{ij}\|) = \frac{\partial a_{ij}(\|p_{ij}\|)}{\partial \|p_{ij}\|} \frac{p_{ij}}{\|p_{ij}\|}$ is the gradient with respect to p_{ij} of $a_{ij}(\|p_{ij}\|)$. Finally, we choose $a_{ij}(\|p_{ij}\|)$ such that $\frac{\partial a_{ij}(\|p_{ij}\|)}{\partial \|p_{ij}\|} = b_{ij}(\|p_{ij}\|) a_{ij}(\|p_{ij}\|)$.

In what follows, we provide some properties of the connectivity measure (5) where the Laplacian matrix elements are given by equation (6).

Let $\mathscr{T}$ be a spanning tree in digraph $\mathscr{D}$ and denote as $\mathcal{E}_{\mathscr{T}}$ its edge set. Define as

$$\mathcal{E}_{\mathscr{T}}^i := \left\{ e_{ji} \in \mathcal{E}_{\mathscr{T}} : j \in \mathcal{N}_i^{in} \right\} \tag{8}$$

the set of edges in $\mathcal{E}_{\mathscr{T}}$ ending at the ith node. Now, let $\gamma \in \mathbb{R}^N$ be defined as in Lemma 1. From Proposition 1, we have:

$$\gamma_i = \sum_{\mathscr{T} \in \mathbb{T}_i} \Pi(\mathcal{E}_{\mathscr{T}}^i) \Pi\left(\bar{\mathcal{E}}\right), \qquad \forall i \in \mathcal{V}, \tag{9}$$

where $\bar{\mathcal{E}} = \mathcal{E}(\mathscr{T}) \setminus \mathcal{E}_{\mathscr{T}}^i$. On the other hand, for some $k \in \mathcal{V}$ which is not a leaf of $\mathscr{T}$, we have

$$\gamma_i = \sum_{\mathscr{T} \in \mathbb{T}_i} a_{jk}(\|p_{jk}\|) \Pi(\mathcal{E}_{\mathscr{T}}^k) \Pi\left(\bar{\mathcal{E}}\right), \tag{10}$$

where j is a child of k, and $\bar{\mathcal{E}} = \mathcal{E}(\mathscr{T}) \setminus \left\{ \mathcal{E}_{\mathscr{T}}^i \bigcup e_{jk} \right\}$. This observation allows us to relate the first-eigenvector entry with any other node in the network and evidence that γ_i is a position-dependent value.

Equation (8) allow us to compute the time derivative of γ_i and relate it to both inter-agent position and velocity vectors as follows:

$$\dot{\gamma}_i = \sum_{\mathscr{T} \in \mathbb{T}_i} \left(\Pi(\mathcal{E}_{\mathscr{T}}) \sum_{e_{kj} \in \mathcal{E}_{\mathscr{T}}} b_{jk}(\|p_{jk}\|) \frac{v_{jk}^T p_{jk}}{\|p_{jk}\|} \right). \tag{11}$$

On the other hand, for an agent i and its neighbours $j \in \mathcal{N}_i^{in}$ we are able to compute the gradient respect to the ith agent's position of γ_i and γ_j as:

$$\nabla_{p_i} \gamma_i = \sum_{\mathscr{T} \in \mathbb{T}_i} \left(\Pi(\mathcal{E}_{\mathscr{T}}) \sum_{e_{ji} \in \mathcal{E}_{\mathscr{T}}^i} b_{ij}(\|p_{ij}\|) \frac{p_{ij}}{\|p_{ij}\|} \right) \tag{12}$$

and

$$\nabla_{p_i}\gamma_j = \sum_{\mathscr{T}\in\mathbb{T}_j}\left(\Pi(\mathcal{E}_{\mathscr{T}})\sum_{e_{ki}\in\mathcal{E}^i_{\mathscr{T}}} b_{ik}(\|p_{ik}\|)\frac{p_{ij}}{\|p_{ij}\|}\right) + \sum_{\mathscr{T}\in\mathbb{T}_j}\left(\Pi(\mathcal{E}_{\mathscr{T}})b_{ji}(\|p_{ji}\|)\frac{p_{ji}}{\|p_{ji}\|}\right). \quad (13)$$

The above analysis allows us to reveal the dynamic properties of the connectivity measure in equation (5). That is, μ_{ij} changes as γ_i or γ_j do, and each of the former does as the inter-agent positions change. Hence, the time derivative of μ_{ij} is

$$\dot{\mu}_{ij} = \frac{\partial\mu_{ij}}{\partial\gamma_i}\dot{\gamma}_i + \frac{\partial\mu_{ij}}{\partial\gamma_j}\dot{\gamma}_j \quad (14)$$

and its gradient with respect to the ith agents position is

$$\nabla_{p_i}\mu_{ij} = \frac{\partial\mu_{ij}}{\partial\gamma_i}\nabla_{p_i}\gamma_i + \frac{\partial\mu_{ij}}{\partial\gamma_j}\nabla_{p_i}\gamma_j. \quad (15)$$

So far, we have constructed and discussed the properties of a position-dependent matrix Laplacian first-left eigenvector-based connectivity measure. In the following section, we use it to design a distributed controller to preserve the proximity digraph's strong connectivity while the multi-agent system maintains a flocking behaviour.

5. FLOCKING MOTION OVER PROXIMITY DIGRAPHS

5.1 Controller design

Let $\psi_{ij} : [0, r_i] \times (1, \infty) \mapsto \mathbb{R}_{\geq 0}$ be an Artificial Potential Function (APF) between an agent i and its neighbour j with partial derivatives $\phi_{ij}(s_1, s_2) = \frac{\partial\psi_{ij}(s_1,s_2)}{\partial s_1}$ and $\varphi_{ij}(s_1, s_2) = \frac{\partial\psi_{ij}(s_1,s_2)}{\partial s_2}$, and the following properties:

i) For all $s_1 \in [0, r_i]$ and $s_2 \in (1, \infty)$, $0 \leq \psi_{ij}(s_1, s_2)$;
ii) For all $s_1 \in [0, r_i]$ and any $s_2 \in (1, \infty)$, $\phi_{ij}(s_1, s_2) < 0$;
iii) For any $s_1 \in [0, r_i]$ and all $s_2 \in (1, \infty)$, $\varphi_{ij}(s_1, s_2) > 0$;
iv) $|\phi_{ij}(s_1, s_2)| \leq \bar{\phi}_{ij} < \infty$;
v) $\varphi_{ij}(s_1, s_2) = \frac{\tilde{\varphi}_{ij}(s_1,s_2)}{s_2}$ with $|\tilde{\varphi}_{ij}(s_1, s_2)| \leq \bar{\varphi}_{ij} < \infty$.

Property *i)* states that $\psi_{ij}(s_1, s_2)$ is a non-negative potential function; Properties *ii)* and *iii)* states that $\psi_{ij}(s_1, s_2)$ is a decreasing function of s_1 but increasing for s_2; Property *iv)* and *v)* states that $\phi_{ij}(s_1, s_2)$ and $\varphi_{ij}(s_1, s_2)$ are bounded functions. Let $s_1 = \|p_{ij}\|$ and $s_2 = \mu_{ij}$. Then, the function $\psi_{ij}(\|p_{ij}\|, \mu_{ij})$ increases as the distance between the pair of agents approach zero and decrease as approaches the sensing radius. Also, $\psi_{ij}(\|p_{ij}\|, \mu_{ij})$ increases as the connectivity measure approaches infinity. This definition allows us to obtain an increasing function between pair of agents near collision or moving towards a position which yields a proximity digraph which is not strongly connected while allowing the edge deletions when needed. It is also worth noticing that, for agents which are not point masses, we might displace the value when collisions occur from $\|p_{ij}\| = 0$ to $\|p_{ij}\| = \epsilon_c$, with $\epsilon_c > 0$ which conforms to the physical dimension of the agents, and define the APF's domain for the inter-agent distances as $[\epsilon_c, r_i]$. From now on, when needed, we'll use the shorthand notation $\psi_{ij} = \psi_{ij}(\|p_{ij}\|, \mu_{ij})$, $\phi_{ij} = \phi_{ij}(\|p_{ij}\|, \mu_{ij})$ and $\varphi_{ij} = \varphi_{ij}(\|p_{ij}\|, \mu_{ij})$.

For ψ_{ij}, its time-derivative and gradient respect to the ith agent's position respectively are:

$$\dot{\psi}_{ij} = \phi_{ij}\frac{v_{ij}^T p_{ij}}{\|p_{ij}\|} + \varphi_{ij}\dot{\mu}_{ij} \quad (16)$$

and

$$\nabla_{p_i}\psi_{ij} = \phi_{ij}\frac{p_{ij}}{\|p_{ij}\|} + \varphi_{ij}\nabla_{p_i}\mu_{ij}. \quad (17)$$

Notice that, to compute $\nabla_{p_i}\psi_{ij}$, each agent i must know γ_i, γ_j, for all $j \in \mathcal{N}_i^{in}$, and their gradients with respect to its position. Therefore, we introduce the following assumption.

Assumption 1. Every agent computes γ_i (and shares it with all $k \in \mathcal{N}_i^{out}$), $\nabla_{p_i}\gamma_i$ and $\nabla_{p_i}\gamma_j$, for all $j \in \mathcal{N}_i^{in}$.

Remark 1. Computing γ_i, for all $i \in \mathcal{V}$, in a distributed fashion can be achieved by implementing the algorithm described in (Poonawala and Spong, 2015). On the other hand, the sets $\mathbb{T}_i$ can be obtained easily for a small number of agents.

For all $i \in \mathcal{V}$, consider the following controller:

$$u_i := -\frac{1}{\gamma_i}\sum_{j\in\mathcal{N}_i^{in}}\nabla_{p_i}\psi_{ij}\left(\|p_{ij}\|, \mu_{ij}\right) - c\sum_{j\in\mathcal{N}_i^{in}} a_{ij}(\|p_{ij}\|)v_{ij}, \quad (18)$$

where $c > 0$ is a control gain yet to be design. The key difference between controller (18) and others (Ávila-Martínez and Barajas-Ramírez, 2021; Ávila-Martínez, 2023) is its dependency on the connectivity measure μ_{ij} and γ_i.

5.2 Stability analysis

We establish the stability of the closed-loop system (3)-(18) through its collective energy. Before that, we define a disagreement vector and analyze its dynamic properties.

Denote as $v = [v_1, \ldots, v_N] \in \mathbb{R}^{nN}$ the stack velocity vector of the multi-agent system. Inspired by (Li and Duan, 2015), let $\delta = \left[\left(I_N - \mathbf{1}\gamma^T\right) \otimes I_n\right] v$ denote the velocity *disagreement vector*, where γ is defined as in Lemma 1. Each entry $i \in \mathcal{V}$ of δ and its time-derivative respectively are

$$\delta_i = v_i - \sum_{k=1}^{N}\gamma_k v_k$$

and

$$\dot{\delta}_i = u_i - \sum_{k=1}^{N}\left(\dot{\gamma}_k v_k + \gamma_k u_k\right),$$

where $\dot{\gamma}_k$ is defined by equation (11). The following lemma summarizes some properties of the disagreement vector.

Lemma 4. (Ávila-Martínez, 2022) Consider a strongly connected proximity digraph $\mathscr{D}$ with L as its associated matrix Laplacian. Let $\gamma \in \mathbb{R}^N$ be defined as in Lemma 1 and such that $\gamma^T\mathbf{1} = 1$. Then, the following is satisfied.

i) $\delta = \mathbf{0}$ if and only if $v_1 = \cdots = v_N$;
ii) $\sum_{i=1}^{N}\gamma_i\delta_i = 0$;
iii) $\sum_{i=1}^{N}\dot{\gamma}_i(\delta_i - v_i) = \mathbf{0}$;
iv) If $\|v\| \leq \bar{v}$, then $\|\delta_i\| \leq (1 - \gamma_i)\bar{v}$ for all $i \in \mathcal{V}$.

Define the collective energy of system (3)-(18) as

$$V := \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i^{in}}\psi_{ij} + \frac{1}{2}\sum_{i=1}^{N}\gamma_i\left\|\delta_i\right\|^2. \quad (19)$$

This equation summarizes the artificial potential and kinetic energies of the overall closed-loop system. Its local minimum is on a configuration p^* and a velocity vector v such that, respectively, the APF is locally minima for all $(i,j)\in\mathcal{E}$ and $v_1=\cdots=v_N$. On the other hand, for a strongly connected proximity digraph $\mathscr{D}$ and bounded velocity vector, there exists a finite $0<\bar{V}$ such that $V\le\bar{V}<\infty$. Finally, the time-derivative of (19) is

$$\dot{V} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i^{in}}\dot{\psi}_{ij} + \frac{1}{2}\sum_{i=1}^{N}\dot{\gamma}_i\left\|\delta_i\right\|^2 + \sum_{i=1}^{N}\gamma_i\delta_i^T u_i, \quad (20)$$

where a few terms cancel according to the properties described in Lemma 4.

In proximity digraphs the information network might change over time. Therefore, suppose the topology of $\mathscr{D}$ switches every time instant t_k, with $k=1,2,\ldots,$ and remains fixed over the time interval $[t_{k-1},t_k)$. At each t_k edges are added to or removed from $\mathcal{E}$. The following lemma studies the behaviour of the collective energy dynamics over $[t_{k-1},t_k)$.

Lemma 5. (Ávila-Martínez, 2022) Consider the closed-loop system (3)-(18) in the time interval $[t_{k-1},t_k)$ and denote as V_{k-1} the collective energy at instant t_k. Suppose $\mathscr{D}$ is strongly connected, $\|v\|\le\bar{v}$ at t_{k-1}, and for all $t\in[t_{k-1},t_k)$ the control gain

$$c \ge \frac{\bar{b}}{\check{\gamma}\alpha}\left(4N(\bar{\phi}+\bar{\varphi})(N-1)^2+\bar{v}\right), \quad (21)$$

where $\check{\gamma}=\min_{i\in\mathcal{V}}\{\gamma_i\}$, $\bar{\phi}=\max_{i,j\in\mathcal{V}}\{\bar{\phi}_{ij}\}$, and $\bar{\varphi}=\max_{i,j\in\mathcal{V}}\{\bar{\varphi}_{ij}\}$. Then $V_k\le V_{k-1}$.

Remark 2. In equation (21), the lower bound for the controller gain c changes as the proximity digraph's topology does. Also, the controller's gain grows as both $\check{\gamma}$ and α approach zero. On the one hand, the value $\check{\gamma}$ can be retrieved on each agent using a consensus algorithm. However, a method to compute a positive constant lower bound for α is yet a work in progress.

Theorem 1. Consider the closed-loop system (3)-(18). Suppose at t_0 the proximity digraph $\mathscr{D}$ is strongly connected and the collective energy (19) bounded. Choose c in controller (18) such that fulfils inequality (21) for all $t\ge0$. Then, the following statements simultaneously hold:

i) The proximity digraph $\mathscr{D}$ remains strongly connected for all $t\ge0$;
ii) The multi-agent system is on flocking motion.

Proof. Proof of statement i): Suppose edges are added to or removed from $\mathcal{E}$ at every time instant t_k, with $k=1,2,\ldots,$ and remains fixed over the time interval $[t_{k-1},t_k)$. The following analysis is for the $k=1$, then we extend it to every time interval.

Respectively denote by V_{t_0} and V_t the collective energy from equation (19) at time instants t_0 and $t\in[t_0,t_1)$. Given $\mathscr{D}$ doesn't change at any $t\in[t_0,t_1)$, from Lemma (5), the collective energy doesn't increase, *i.e.* $V_t\le V_{t_0}$ for all $t\in[t_0,t_1)$. Hence, V_t is bounded above by some constant value and there are no APFs in (19) approaching infinity. From the APFs definition, this implies there are no connectivity measure approaching infinity. In consequence, the entries of the first-left Laplacian eigenvector γ are all such that $\gamma_i>0$ for all $i\in\mathcal{V}$. That is, $\mathscr{D}$ remains strongly connected for all $t\in[t_0,t_1)$.

Consider t_1, the time-instant where $\mathscr{D}$ change. Respectively denote as $\mathcal{E}^-$ and $\mathcal{E}^+$ the set of edges removed from and added to $\mathcal{E}$ at t_1. Is clear that the edges deletion reduce the collective energy (19), hence $V_{t_1}\le V_{t_0}$, and the statement of the previous paragraph still holds true. For edge additions this is not the case. The collective energy at t_1 is $V_{t_1}\le V_{t_0}+\sum_{e_{ki}\in\mathcal{E}^+}\psi_{ik}$. However, edge additions just increase some of the vector γ entries, as we can see from equation (1). That is, μ_{ik} are bounded for all $e_{ki}\in\mathcal{E}^+$ and also the APFs ψ_{ik}. This also holds for any other connectivity measures and APFs affected from the edges addition. Hence, V_{t_1} is finite, thus $\mathscr{D}$ is strongly connected.

By a similar reasoning over each time-interval, we conclude the collective energy is finite. That is, there exists $\bar{V}<\infty$ such that $V_t\le\bar{V}$ for all $t\ge0$. Thus, the proximity digraph $\mathscr{D}$ remains strongly connected for all $t\ge0$.

Proof of statement ii): We'll separately prove every property in Definition 1.

(Group cohesiveness) We already show $\mathscr{D}$ is strongly connected for all $t\ge0$. Therefore, in $\mathscr{D}$ the inter-agent distances are bounded above by the sum of their sensing radius. Take $\epsilon_p=\sum_{i=1}^{N}r_i$, hence the group is cohesive.

(Colision avoidance) From the APFs definition we have $\psi_{ij}\to\infty$ as $\|p_{ij}\|\to0$. We already show there exists $\bar{V}<\infty$ such that $V_t\le\bar{V}$ for all $t\ge0$. In consequence, there are no distance $\|p_{ij}\|$ approaching zero for all $i,j\in\mathcal{V}$ and $t\ge0$. Hence, inter-agent collisions are avoided.

(Bounded velocity mismatch) Without lost of generality, suppose $\gamma^T\mathbf{1}=1$. Notice the disagreement vector entries can be rewritten as pondered inter-agent velocity differences as $\delta_i=\sum_{k=1}^{N}\gamma_k v_{ik}$. From statement i), we know there exists $\bar{V}<\infty$ such that $V_t\le\bar{V}$ for all $t\ge0$. From equation (19) this implies $\|\delta_i\|\le\sqrt{\frac{2\bar{V}}{\gamma_i}}$. With these observation, we conclude that, at any time $t\ge0$, the velocity differences $\|v_{ij}\|\le\epsilon_v$ for some $\epsilon_v<\infty$. Thus, inter-agents velocity mismatches are bounded.

6. NUMERICAL SIMULATIONS

Consider a multi-agent system of $N=3$ agents moving on a plane, *i.e.* $n=2$, with dynamics given by equation (3). Each agent has a different sensing radius, namely $r_1=1$, $r_2=2$ and $r_3=5$, resulting in a proximity digraph. We show the initial configuration of the multi-agent system in Figure 2a, where coloured circles are centred at the agents' positions, while the solid arrows indicate the existence of a distance-dependent edge.

Figure 2b shows the final configuration for the closed-loop system (3)-(18) after 20 seconds with the APF

$$\psi_{ij}\left(\|p_{ij}\|,\mu_{ij}\right) = 10\left[(1-\|p_{ij}\|)^2+\left(1-\tanh\left(\frac{1}{\mu_{ij}}\right)\right)\right]$$

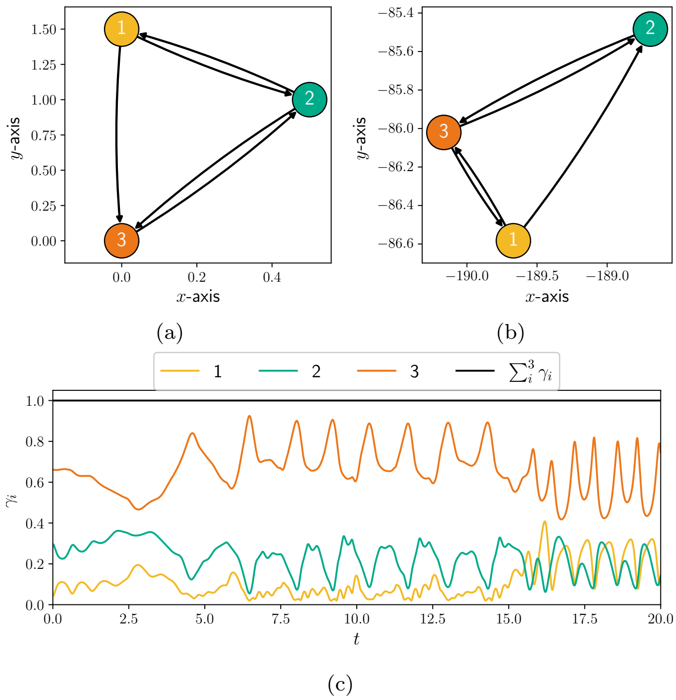


Fig. 2. Closed-loop system (3)-(18) flocking motion.

and $c = 50$. On the other hand, Figure 2c shows the trajectories of each eigenvector entry γ_i. Notice $\gamma_i > 0$ for all $i \in \mathcal{V}$ and $t > 0$, implying the proximity digraph remains strongly connected.

7. FINAL COMMENTS

The distributed controller design for flocking motion in second-order multi-agent systems with heterogeneous communication constraints is a challenging problem. This paper presents a gradient-descent method-based solution. We modelled the agents' interaction with a proximity digraph and implemented a connectivity measure based on the first-left Laplacian matrix eigenvector entries; providing the agents with local knowledge of the overall graph's topology. Then, by defining inter-agent distance-dependent edge weights, we unveil the dynamic properties of the connectivity measure. With a collective energy function and mild assumptions, we proved the closed-loop system's convergence to the desired flocking motion behaviour while preserving the strong connectivity property of the proximity digraph. However, to successfully implement our controller design, the challenge of distributedly computing the sets of the agents' rooted spanning trees remains.

Future works include exploring the model's robustness properties concerning perturbations and designing distributed continuous algorithms to compute the rooted spanning trees and a lower bound for the digraph's generalized algebraic connectivity. We also aim to use our controller design in other multi-agent systems, for example, where the agents have different input constraints.

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