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Coexistence of stable states in a biparametric family of bimodal maps.

Tesis que presenta

Francisco Alejandro Jiménez Valdivia

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Director de Tesis:

Dr. Eric Campos Cantón

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Dr. Eric Campos Cantón
(Director de la tesis)

Dr. Cesar Octavio Maldonado Ahumada
(Sinodal de la tesis)

Dr. Haret Codratian Rosu
(Sinodal de la tesis)

Dr. Juan Gonzalo Barajas Ramírez
(Sinodal de la tesis)

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Copia del acta del examen

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Resumen.

Demostramos la existencia de la monoestabilidad y la biestabilidad en el sentido de Lyapunov en dinámicas no lineales discretas y discutimos las propiedades asociadas con este comportamiento. Específicamente, introducimos las condiciones necesarias para asegurar la ocurrencia de biestabilidad dentro de una familia paramétrica de mapeos bimodales, basados en el mapeo diferencia. El mapeo bimodal está definido dentro de una partición regular que consiste en dos subintervalos. En esta investigación, presentamos tres casos de estudio: el primer caso corresponde a mantener fijo la primera moda, mientras que la segunda moda cambia según un parámetro de bifurcación. En el segundo caso, la primera moda cambia según otro parámetro de bifurcación mientras que la segunda moda permanece fija. En el tercer caso, ambos puntos críticos asignados a las modas cambian según un parámetro de bifurcación. Se muestran diagramas de bifurcación para los tres casos de estudios y nos permiten identificar conjuntos invariantes o regiones de atrapamiento en cada subintervalo. Posteriormente, se definen dos conjuntos invariantes para habilitar biestabilidad. Por lo tanto, la monoestabilidad y biestabilidad aparecen en familias paramétricas de acuerdo con el control de conjuntos invariantes y regiones de atrapamiento al variar un parámetro de bifurcación. Los resultados numéricos se alinean con la teoría desarrollada.

Abstract.

We demonstrate the emergence of monostability and bistability in Lyapunov sense in discrete-time nonlinear dynamics and discuss the properties associated with this behavior. Specifically, we introduce the necessary conditions to ensure the occurrence of the bistability within a parametric family of bimodal maps, based on the difference map. The bimodal map is defined within a regular partition consisting of two subintervals, and we present three case studies: the first case corresponds to keeping the first modal fixed, while the second modal changes according to a parameter. In the second case, the first modal changes according to another parameter while the second modal remains fixed. In the third case, both critical points assigned to the modals change according to a bifurcation parameter. Bifurcation diagrams are shown for three case studies and let us identify invariant sets or trapping regions in each subinterval. Subsequently, two invariant sets are defined to enable the bistability. Therefore, monostability and bistability emerge in parametric families based on the control of invariant sets and trapping regions through variations in a bifurcation parameter. The numerical results are consistent with the developed theory.

Chapter 1

Introduction

In this chapter, we provide a historical overview of multimodal maps, as well as monostability and bistability, including their study and analysis within dynamical systems. We also introduce the concepts of trapping regions and invariant sets, examined in both discrete-time and continuous-time dynamical systems, such as the logistic map and the Lorenz attractor. Additionally, we discuss the motivation behind this work and outline the objective we aim to achieve. The chapter is structured into five sections: the first provides a brief overview of multimodal maps; the second delves into the background of multistability, with examples of monostable and bistable dynamical systems and important applications of multistability, as well as the concepts of trapping regions and invariant sets. The third section explains the motivation for this thesis, and the fourth section concludes with the objective of this work and an outline of the thesis.

1.1 Multimodals maps

A monoparametric multimodal map $f(x, \alpha) : I \to I$, where $I \subset \mathcal{D} \subseteq \mathbb{R}$ is a map that contains more than one maximum critical point in the interval I, which in this context are called modals and are denoted by C_i. These modals represent local maxima or minima, so if there is only one modal denoted by C_0 within I, then the map is called unimodal; if it exhibits two modals C_i in I with $i = 0, 1$, then it is a bimodal map. [1].

Two well-known examples of monoparametric unimodal maps are the logistic map [1]

$$f_L(x,\alpha) = \alpha x(1-x) \quad \text{where} \quad \alpha \in (0,4] \subset \mathbb{R},$$

and the tent map [3]

$$f_T(x,\mu) = \begin{cases} \mu x & \text{for } x < \dfrac{1}{2}, \\ \\ \mu(1-x) & \text{for } x \geq \dfrac{1}{2}, \end{cases} \quad \text{where} \quad \mu \in (0,2] \subset \mathbb{R},$$

since we know that both maps have a single modal in the interval $I = [0,1] \subset \mathbb{R}$, where these modals are located at $(C_0, f_L(C_0,\alpha)) = \left(\frac{1}{2}, \frac{\alpha}{4}\right)$ for the logistic map and at $(C_0, f_T(C_0,\mu)) = \left(\frac{1}{2}, \frac{\mu}{2}\right)$ for the tent map. Figures 1.1 and 1.2 depict the logistic map and the tent map for different values of their respective parameters. Once a brief introduction of what a unimodal map is given, we will address what a multimodal map is. We know that a unimodal map has only one modal in the interval where it is defined; consequently, a multimodal map exhibits multiple modals within the defined interval. Figure 1.3 shows a family of multimodal maps. A family of maps that satisfy the brief definition given of a multimodal map are bimodal maps, which exhibit two modals within the interval where they are defined. An example of a monoparametric bimodal map is the difference map [4]

$$f_D(x,\beta) = \begin{cases} 2\beta x(1-2x) & \text{for } 0 \leq x < \dfrac{1}{2}, \\ \\ 2\beta(x-1)(1-2x) & \text{for } \dfrac{1}{2} \leq x \leq 1. \end{cases}$$

Where $\beta \in (0,4] \subset \mathbb{R}$. The difference map f_D is constructed by taking the difference between the logistic map and the tent map, thus creating a quadratic map that contains two unimodal maps within the interval $I = [0,1] \subset \mathbb{R}$. Further elaboration on the aforementioned is provided in Chapter 2. The difference map serves as inspiration for the study carried out in this research work, turning the difference map into a biparametric map (dependent on parameters β_1 and β_2). Figure 1.4 illustrates the difference map.

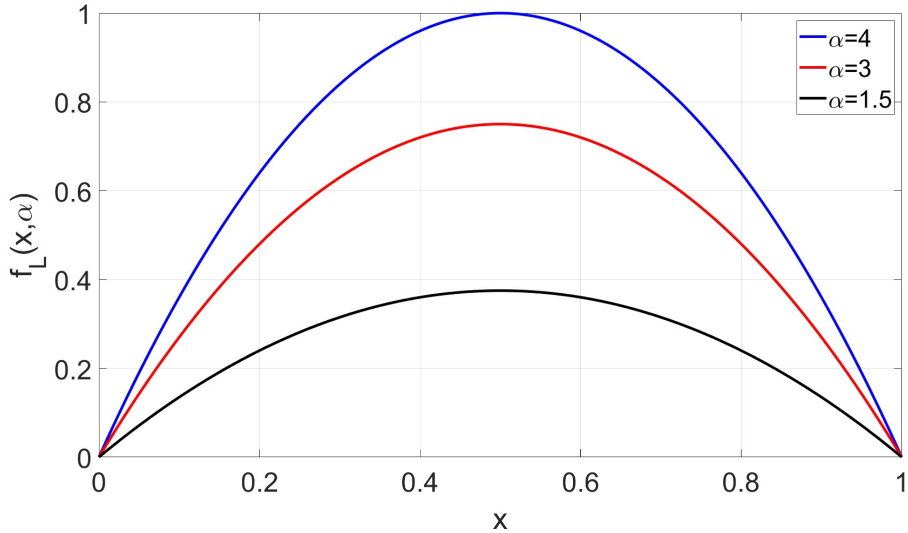


Figure 1.1: The logistic map for different values of the parameter α.

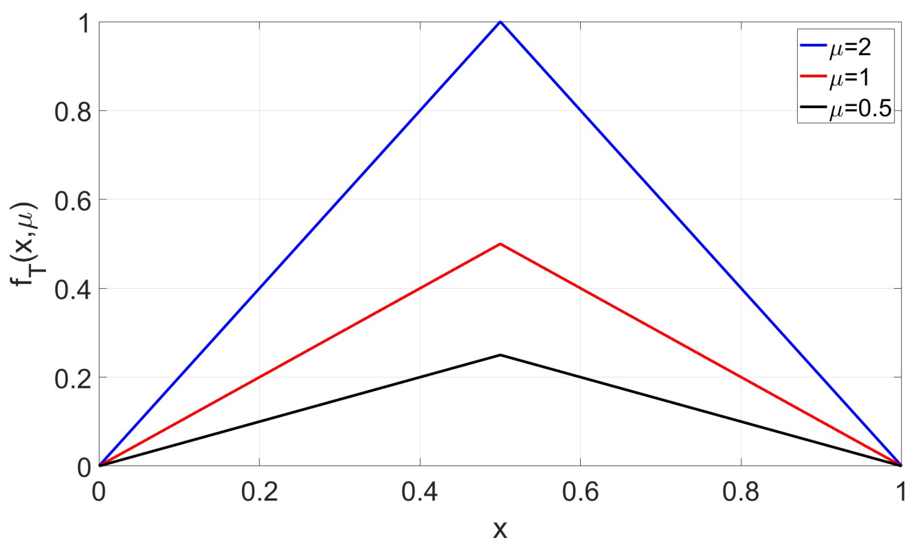


Figure 1.2: The Tent map for different values of the parameter μ.

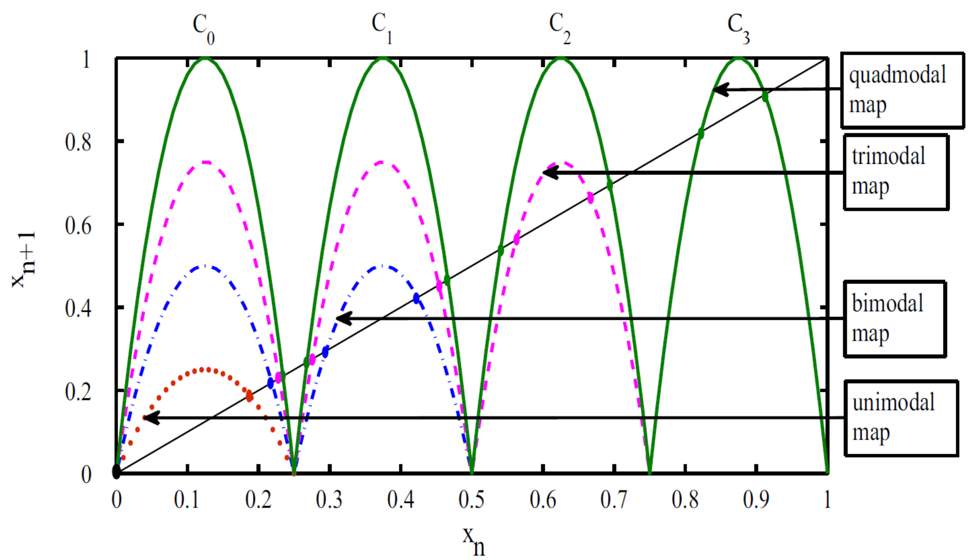


Figure 1.3: Family of multimodal maps based on the logistic map [1].

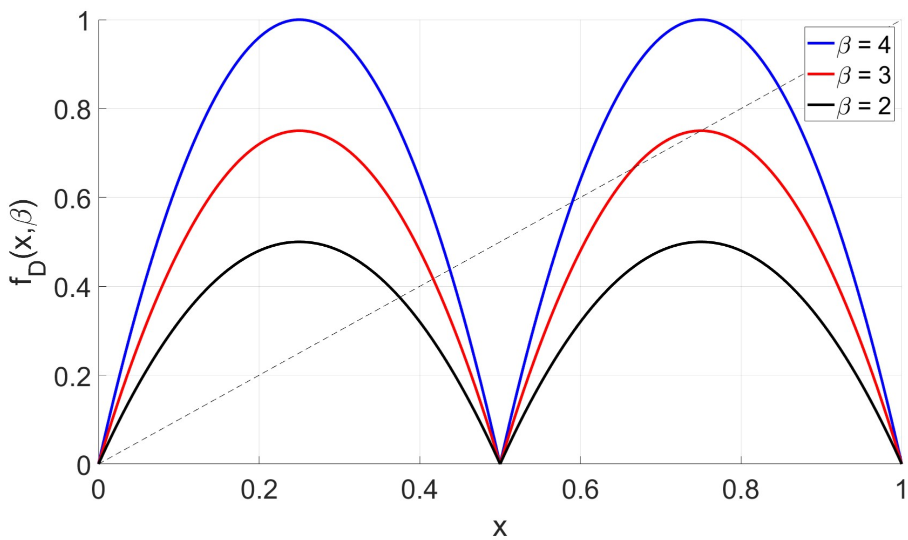


Figure 1.4: Difference map showing different values of the parameter β.

After providing a brief explanation of multimodal maps, we will gain a clearer understanding of what constitutes a family of monoparametric and biparametric bimodal maps. This concept is crucial because, within this family, we will investigate the necessary conditions for achieving monostability and bistability.

1.2 Monostability, bistability, trapping region and invariant set in dynamical systems

Stability can have different meanings and definitions in the field of dynamical systems, but in this chapter, our goal is to provide a clear interpretation. We use the term *stable* (specifically *monostable* in this context) to describe a system that maintains a consistent behavior within a defined set of boundaries over time, showing only one stable state. If the same system exhibits this behavior within two different sets of boundaries, it is termed *bistable*, indicating two stable states. Generally, if a system exhibits this behavior across multiple separate sets of boundaries, it is called *multistable*, meaning it has multiple stable states.

To explain further, a *stable* discrete dynamical system is one where initial conditions lead to orbits—sequences of points that describe the system's evolution as time progresses—that converge towards a fixed point. When orbits converge towards a specific region instead of a single fixed point, we refer to it as *generalized stability*. In monostable systems, all orbits from initial conditions converge towards a single fixed point. In bistable systems, orbits from different initial conditions converge towards one of two possible fixed points. Similarly, in multistable systems, orbits can converge towards multiple distinct fixed points.

The term multistability was first introduced in 1971 in the context of a psychological study on visual perception [2]. This study explored how a single image could be interpreted in fundamentally different ways depending on the viewer's initial visual focus. It demonstrated that the same visual input could lead to various distinct perceptions, each representing a different stable state. This observation gave rise to the concept of multistability, which suggests that an image—or more broadly, a system—can exhibit multiple stable states or attractors, each of which can be perceived or realized depending on initial conditions or contextual factors. In essence, multistability reflects the idea that certain visual stimuli or dynamic systems can possess several equally

Figure 1.5: Image first presented in 1915 with the title '*My Wife and My Mother-in-law*', where the bistability in visual perception is demonstrated [2].

valid and stable interpretations or states, depending on how they are observed or interacted with. In the field of dynamical systems, in brief, a multistable system has the ability to have multiple stable states [5]. If we relate visual multistability to dynamical systems, we say that the initial visual focus is analogous to the initial condition, which, depending on where we place our initial visual focus, will be the perception we have of a multistable image, where each possible perception is a stable state.

In Figure 1.5, we observe a bistable image, a classic example of multistability in visual perception. This type of image illustrates how the same visual stimulus can be interpreted in multiple stable ways. Specifically, if we focus our attention on point c in the image, the figure that appears is that of a young woman. This interpretation is based on the characteristics of the face and profile that are configured when focusing on point c. On the other hand, when we direct our attention to point a, the image reveals the figure of an elderly woman. In this perspective, the facial features are rearranged in such a way that the image is perceived as the face of an older woman.

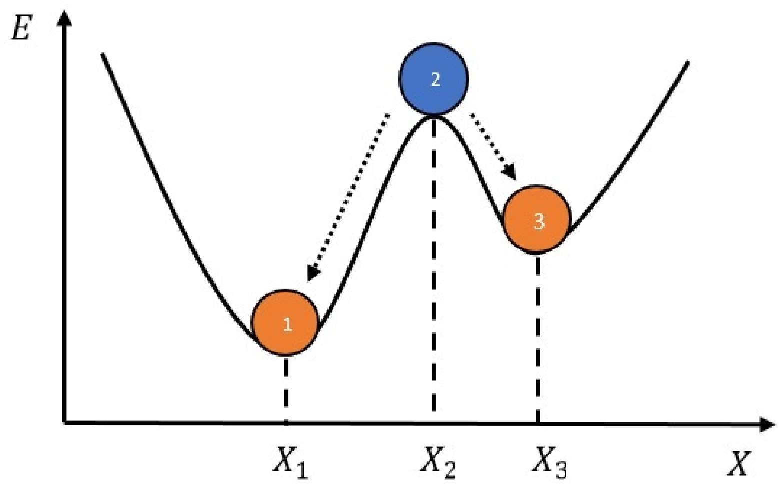


Figure 1.6: Bistability in the energetic description of a system.

In this work, as mentioned earlier, the characteristics of interest are monostability and bistability. In Figure 1.6, we can observe the energy graph of a system that exhibits two stable states (bistability), in which three equilibrium points are denoted by colored spheres and marked with the numbers 1, 2, and 3. Let us observe that equilibrium points 1 and 3 are stable, while equilibrium point 2 is unstable. Note that if we consider only the interval $X_1 \leq X < X_2$, the system is monostable, similarly considering $X_2 < X \leq X_3$. Additionally, note that the equilibrium point X_2 is both the upper and lower bound in these defined intervals. With this, it is evident that under any arbitrary initial condition, regardless of how close it is to the unstable equilibrium point, the trajectory will converge to the respective stable equilibrium point depending on the interval in which the initial condition lies.

The logistic map $f_L(x, \alpha) = \alpha x(1 - x)$ is clearly an example of a monostable system, exhibiting two equilibrium points (fixed points in discrete time) in the interval $I = [0, 1] \subset \mathbb{R}$, which are $x = 0$ and $x = \dfrac{\alpha - 1}{\alpha}$. We say that the logistic map is a monostable system because for $0 < \alpha \leq 4$, the dynamics converge to a fixed point of f_L or remain within the interval I. It is observed that when the parameter $0 < \alpha < 1$, the point $x = 0$ is the only stable point in the interval $I = [0, 1] \subset \mathbb{R}$. However, when $1 < \alpha < 3$, the origin changes

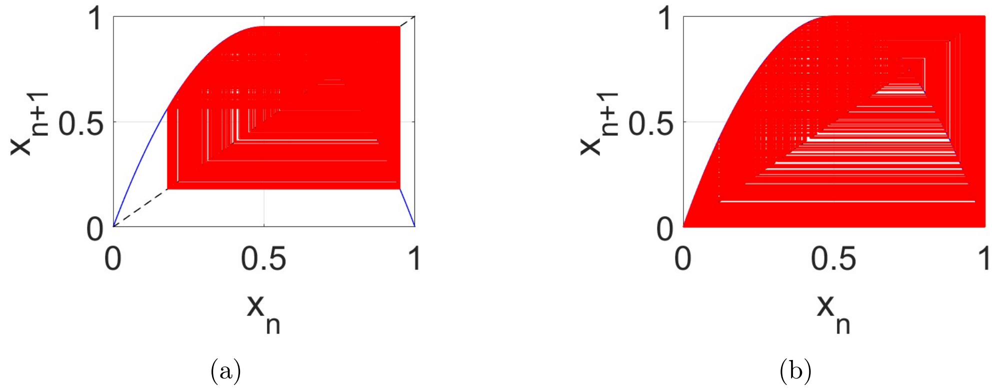


Figure 1.7: Logistic map for different values of $\alpha \in (0, 4] \subset \mathbb{R}$: (a) $\alpha = 3.8$ and (b) $\alpha = 4$.

from stable to unstable, and the point $x = \dfrac{\alpha - 1}{\alpha}$ becomes stable. However, when $3 < \alpha \leq 4$, both equilibrium points are unstable. Nonetheless, if we wish to study when the dynamics diverge, that is, when the dynamics no longer remain within the interval I, we can observe that for $\alpha > 4$, the dynamics escape from the interval I after a certain number of iterations. This is because the modal of f_L is located at $(C_0, f_L(C_0, \alpha)) = \left(\dfrac{1}{2}, \dfrac{\alpha}{4}\right)$, so we observe that for $\alpha > 4$, the modal does not belong to the interval I, since $\dfrac{\alpha}{4} > 1$. Therefore, when $\alpha > 4$ the logistic map is not a monostable system. However, two important concepts arise here, which are trapping region and invariant set.

Let us observe in Figure 1.7a the behavior of the logistic map when the parameter $\alpha = 3.8$, while in Figure 1.7b when $\alpha = 4$, both for initial conditions within I. We can observe from Figure 1.7a that when $\alpha = 3.8$, it holds that $f_L(I) \subset I$ since it does not cover the entire interval I, whereas when $\alpha = 4$, we satisfy $f_L(I) = I$ as we see that the trajectory covers the entire interval I. With this, two fundamental concepts for the study of monostability and bistability in a family of bimodal maps are introduced, namely trapping region and invariant set. Thus, briefly, we can observe that when $0 < \alpha < 4$, there exists a trapping region since $f_L(I) \subset I$, and when $\alpha = 4$, there is an invariant set, as $f_L(I) = I$ is fulfilled. The precise definitions of trapping region and invariant set are provided in Chapter 2.

We will review some applications and studies of the multistability in dynamic systems of science and technology. As mentioned earlier, multistability appears in different dynamic systems within science and technology. For example, in Biology, genetic switches have been constructed in biological systems that exhibit multistability [6]. In physics, the study of multistability in networks of oscillators [7], and within Chemistry, the bi-stability and multi-stability occurring within an ethanol reactor have been studied using numerical tools [8]. Some other studies include the investigation of multistability in symmetric chaotic systems [9], multistability of self-reproducing chaotic systems [5], the study of multistability in a discrete economic model [10], multistability in a memristive chaotic system [11], the multistability in social systems [12], among others [13].

The bistability is essential in neuronal dynamics, as it is present in neuronal interactions as well as in cellular signaling [14]. Within climatological dynamics, coexisting attractors related to bistability have been found through deep oceanic convection [15, 16]. Similarly, in electronics, multivibrators—classified as astable, monostable, and bistable—play a crucial role. An astable multivibrator alternates continuously between two states, generating a square wave without an external input signal, and is commonly used in clock generators and oscillators [17]. A monostable multivibrator has one stable and one unstable state; upon receiving a trigger, it switches to the unstable state for a period before returning to the stable state, making it useful for timing applications such as pulse generators [17]. A bistable multivibrator, or flip-flop, has two stable states and remains in one state until an external signal switches it to the other. This type is fundamental in digital memory construction and data storage registers [18]. These configurations are widely used in digital and analog electronics for various synchronization, storage, and signal generation functions.

1.3 Motivation

The dynamics of a family of multimodal maps with non-uniform intervals have been previously studied [19], where it was found that a bifurcation parameter could induce changes in the dynamics of this family. That work serves as motivation, as multistability was numerically observed; however, the study of multistability was not the primary objective of that research.

Inspired by the multistability observed in this previous work, this investigation focuses specifically on monostability and bistability within a family of parametric bimodal maps with uniform intervals, from both theoretical and numerical perspectives.

This work takes as its starting point the investigation of a family of bimodal maps with uniform intervals, with the purpose of approaching this work from a less complex standpoint, as well as becoming familiar with the study of multimodal maps. Subsequently, once the considerations for the existence of bistability in a family of bimodal maps are established, the conditions for the existence of monostability within this map are provided. Later, to verify the theoretical study, the relevant simulations are carried out to demonstrate a relationship between the numerical and theoretical results.

1.4 Objective

The general objective of this thesis is to study the necessary conditions to guarantee the existence of monostability and bistability in a parametric family of bimodal maps. Specific objectives include constructing a series of propositions to ensure monostability and bistability, followed by conducting numerical simulations based on these propositions to verify and observe bistability. Another specific objective is to characterize trapping regions, as well as invariant sets and points within this family of maps.

1.5 Organization

This thesis is divided into chapters. Chapter 1 provides a brief overview of multimodal maps, including a historical review of multistability. A dynamical system is presented to illustrate monostability, as well as to define invariant sets and trapping regions. Finally, the applications of monostability and bistability in science and technology are discussed. The chapter concludes with an explanation of the motivation behind this work and the general and specific objectives of the thesis.

Chapter 2 introduces the definitions and theorems that form the foundation for demonstrating the obtained results and enriching the theoretical development. Additionally, this chapter defines the parametric bimodal map based on the difference map, which will be used to study bistability. Chapter 3 presents the analytical results, including a series of propositions that ensure bistability in the proposed bimodal map from Chapter 2.

Chapter 4 presents three case studies based on the proposed bimodal map. For each case, the relevant analytical and numerical results are provided using the findings from Chapter 3, aiming to demonstrate the relationship between theoretical and numerical results and to observe monostability and bistability. Chapter 4 concludes with the presentation of bifurcation diagrams for each case study. Finally, Chapter 5 presents the conclusions and outlines directions for future work.

Chapter 2

Mathematical preliminaries

In this chapter, important mathematical foundations that appear throughout the development of this research are presented. The content presented in this chapter consists of definitions and theorems within the field of real number topology, linear algebra, mathematical analysis, and discrete dynamical systems. Additionally, definitions of unimodal and bimodal maps are provided in this chapter, along with examples of these aforementioned maps, concluding with the bimodal map used for the development of this research.

2.1 Topology of the Real Numbers

Below are some definitions found in the field of real number topology. These definitions will help us better understand concepts presented in this chapter, as well as the analytical results obtained. The definitions and theorems within this section include: the existence of a norm, interior point, open set, closed set, bounded set, compact set, and metric space [20].

Definition 1 *Let* $\boldsymbol{x} = (x_1, x_2, \cdots, x_{n-1}, x_n)$ *and* $\boldsymbol{y} = (y_1, y_2, \cdots, y_{n-1}, y_n)$ *be in* $\mathbb{R}^n$*. We define;*

a) Equality:

$$\boldsymbol{x} = \boldsymbol{y} \quad \textit{if, and only if,} \quad x_1 = y_1, x_2 = y_2, \cdots, x_{n-1} = y_{n-1}, x_n = y_n.$$

b) Sum:

$$\boldsymbol{x}+\boldsymbol{y}=(x_1+y_1,x_2+y_2,\cdots,x_{n-1}+y_{n-1},x_n+y_n).$$

c) Multiplication by real numbers (scalars):

$$a\boldsymbol{x}=(ax_1,ax_2,\cdots,ax_{n-1},ax_n)\quad a\in\mathbb{R}.$$

d) Difference:

$$\boldsymbol{x}-\boldsymbol{y}=\boldsymbol{x}+(-1)\boldsymbol{y}.$$

e) Zero vector or origin:

$$\boldsymbol{0}=(0,0,\cdots,0,0).$$

f) Inner product or dot product:

$$\boldsymbol{x}\cdot\boldsymbol{y}=\sum_{k=1}^{n}x_k\,y_k$$

g) Norm o length

$$\|\boldsymbol{x}\|=(\boldsymbol{x}\cdot\boldsymbol{x})^{1/2}=\left(\sum_{k=1}^{n}x_k^2\right)^{1/2}$$

Theorem 1 *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be points in $\mathbb{R}^n$, then*

1. $\|\boldsymbol{x}\|\geq 0$
2. $\|a\boldsymbol{x}\|=|a|\|\boldsymbol{x}\|\quad\forall a\in\mathbb{R}$
3. $\|\boldsymbol{x}\cdot\boldsymbol{y}\|\leq\|\boldsymbol{x}\|\|\boldsymbol{y}\|$ *Cauchy-Schwarz inequality*
4. $\|\boldsymbol{x}+\boldsymbol{y}\|\leq\|\boldsymbol{x}\|+\|\boldsymbol{y}\|$ *Triangle inequality.*

With the existence of a norm in the space $\mathbb{R}^n$, we can provide the definition of an n-ball. This definition is presented in order to enunciate and define in a clearer way the concepts of sets presented later.

Definition 2 *Let $\boldsymbol{a}$ be a point in $\mathbb{R}^n$ and let $r \in \mathbb{R}^+$. The set of all points $\boldsymbol{x}$ in $\mathbb{R}^n$ such that*

$$\|\boldsymbol{x} - \boldsymbol{a}\| < r$$

is called an open n-ball of radius r with center at $\boldsymbol{a}$, usually denoted as $\boldsymbol{B}(\boldsymbol{a}, r)$ or $\boldsymbol{B}(\boldsymbol{a})$ [20].

Definition 3 *Let $\mathcal{S} \subset \mathbb{R}^n$ and $\boldsymbol{a} \in \mathcal{S}$. Then $\boldsymbol{a}$ is called an interior point of $\mathcal{S}$ if there exists a $\boldsymbol{B}(\boldsymbol{a})$ where all its points belong to $\mathcal{S}$ [20].*

From Definition 2, we observe that we have an n-ball whose boundary is not contained since the distance between any point x contained in the ball (different from its center) and its center x must be strictly less than its radius r. In Definition 3, an interior point is defined, where this point is interior if it belongs to a set $\mathcal{S}$ and there exists an $\varepsilon > 0$ such that the n-ball centered at a with radius ε is completely contained in $\mathcal{S}$. In mathematical context, it is expressed as

$$\exists \varepsilon > 0 \text{ such that } \boldsymbol{B}(\boldsymbol{x}, \varepsilon) \subseteq \mathcal{S}.$$

Note that the radius r presented in Definition 2 is replaced by ε, however, the interpretation is the same for both cases. Next, the definition for an open set is presented, this definition is given since following Definition 4, using the definition of an open set, what a closed set is defined.

Definition 4 *A set $\mathcal{S} \in \mathbb{R}^n$ is called open if all its points are interior points [20].*

Definition 5 *Let $\mathcal{S} \subset \mathbb{R}^n$ be a set, $\mathcal{S}$ is said to be closed if its complement is an open set [20].*

If we observe, the previous definition mentions a set operation called complement, let us remember that this operation generates a new set, usually denoted as $\mathcal{S}^c \subset \mathbb{R}^n$ containing the elements that do not belong to the set $\mathcal{S}$, mathematically it is expressed as

$$\mathcal{S}^c = \{x \in \mathbb{R}^n : x \notin \mathcal{S}\}$$

so that if we take the union of these two sets

$$\mathcal{S}^c \cup \mathcal{S} = \mathbb{R}^n$$

and according to the definition of complement, we know that the elements not contained in the set $\mathcal{S}$ necessarily have to be in $\mathcal{S}^c$, therefore, $\mathcal{S}^c$ has to be a closed set in order to generate the entire space $\mathbb{R}^n$ where these sets are defined. Despite the previous definition, an alternative definition of a closed set will be presented later, however, to reach this definition, it is necessary to present the definition of a limit point.

Definition 6 *If $\mathcal{S} \subseteq \mathbb{R}^n$ and $\boldsymbol{x}$ is a point in $\mathbb{R}^n$ where $\boldsymbol{x}$ is not necessarily contained in $\mathcal{S}$. Then $\boldsymbol{x}$ is a limit point of $\mathcal{S}$ if every n-ball $\boldsymbol{B}(\boldsymbol{x})$ contains at least one point of $\mathcal{S}$ [20].*

A limit point is also known as a limit or accumulation point. In simple words, a point contained in $\mathcal{S}$ is considered a limit point if, no matter how small the n-ball is, it will always be possible to find at least another point belonging to $\mathcal{S}$ and within the n-ball. Knowing what a limit point is, we proceed to provide the definition of a closed set.

Definition 7 *A set $\mathcal{S} \in \mathbb{R}^n$ is closed if and only if it contains all its limit points [20].*

Definition 8 *A set $\mathcal{S} \in \mathbb{R}^n$ is said to be bounded if it is completely contained within an n-ball $\boldsymbol{B}(\boldsymbol{a}, r)$ for some $r > 0$ and some $\boldsymbol{a} \in \mathbb{R}^n$ [20].*

Definition 9 *A set $\mathcal{S} \in \mathbb{R}^n$ is compact if it is closed and bounded [21].*

Definition 10 *A set $\mathcal{X}$ is said to be a metric space if to every two points $p, q \in \mathcal{X}$ there is associated a real number $d(p, q)$ called distance from p to q, such that*

1. $d(p, q) > 0$ *if* $p \neq q$; $d(p, p) = 0$;
2. $d(p, q) = d(q, p)$;
3. $d(p, q) \leq d(p, r) + d(r, q)$, *for all* $r \in \mathcal{X}$.

2.2 Linear Algebra

In this section, some concepts used in Linear Algebra are defined. The presented definitions are; field, vector space, and linear transformation. These definitions are presented in order to enunciate what a linear transformation is since when defining the derivative of a function, it is mentioned that the derivative is a linear transformation.

Definition 11 *A field $\mathcal{F}$ is a set with two operations $+$ and $\cdot$ called addition and multiplication respectively, they are defined such that, for each pair of elements $x, y \in \mathcal{F}$ there exist unique elements $x + y$ and $x \cdot y$ in $\mathcal{F}$ for which the following conditions are satisfied. Let $a, b, c \in \mathcal{F}$*

1. *$a + b = b + a$ and $a \cdot b = b \cdot a$ (commutativity in addition and multiplication).*
2. *$(a+b)+c = a+(b+c)$ and $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ (associativity in addition and multiplication).*
3. *There exist distinct elements 0 and 1 in $\mathcal{F}$ such that*

$$0 + a = a \qquad 1 \cdot a = a$$

(existence of identity elements in addition and multiplication).

4. *For each element $a \in \mathcal{F}$ and for each element $b \in \mathcal{F}$ with $b \neq 0$, there exist elements $c, d \in \mathcal{F}$ such that*

$$a + c = 0 \qquad b \cdot d = 1$$

(existence of inverse for addition and multiplication).

5. *$a \cdot (b + c) = a \cdot b + a \cdot c$ (distributivity of multiplication over addition) [22].*

Definition 12 *A vector space $\mathcal{V}$ over a field $\mathcal{F}$ consists of a set in which two operations are defined (called addition and multiplication), such that for any $x, y \in \mathcal{V}$ there exists a unique $x + y$ in $\mathcal{V}$, and for each $a \in \mathcal{F}$ and $x \in \mathcal{V}$ there exists a unique element $ax \in \mathcal{V}$, so that the following conditions are met. Let $x, y, z \in \mathcal{V}$*

1. $x + y = y + x$ *(commutativity in addition).*

2. $(x + y) + z = x + (y + z)$ *(associativity in addition).*

3. *There exists an element in* $\mathcal{V}$ *denoted by* 0 *such that* $x + 0 = x$ *for any element* $x \in \mathcal{V}$ *(additive identity).*

4. *For each element* $x \in \mathcal{V}$ *there exists a unique element* $y \in \mathcal{V}$ *such that* $x + y = 0$ *(additive inverse).*

5. *For each element* $x \in \mathcal{V}$, $1x = x$ *(multiplicative identity).*

6. *For each pair of elements* $a, b \in \mathcal{F}$ *it holds that* $(ab)x = a(bx)$ *(associativity in multiplication).*

7. *For each element* $a \in \mathcal{F}$ *it holds that* $a(x + y) = ax + ay$ *(distributive property).*

8. *For each pair of elements* $a, b \in \mathcal{F}$ *it holds that* $(a + b)x = ax + bx$ *(distributive property) [22].*

Definition 13 *Let* $\mathcal{V}$ *and* $\mathcal{W}$ *be vector spaces over* $\mathcal{F}$*. We call a function* $T : \mathcal{V} \rightarrow \mathcal{W}$ *a linear transformation from* $\mathcal{V}$ *to* $\mathcal{W}$ *if,* $\forall x, y \in \mathcal{V}$ *and* $c \in \mathcal{F}$*, the following conditions are met:*

1. $T(x + y) = T(x) + T(y)$.

2. $T(cx) = cT(x)$.

2.3 Mathematical Analysis

In the development presented in the following Chapter, functions with a set of specific characteristics are worked with, among these are continuity and differentiability. Therefore, the definitions of function are presented, the limit operator of a function and its existence are defined, furthermore, once the limit operator is defined, continuity and derivative of a function are defined.

Definition 14 *A function f is a rule that assigns each element of a set $\mathcal{D}$ to a unique element of a second set $\mathcal{C}$. The sets $\mathcal{D}$ and $\mathcal{C}$ are called the domain and codomain of the function. The set of elements of the codomain $\mathcal{C}$ that have an element of the set $\mathcal{D}$ assigned to them is called the range of the function. The notation $f : \mathcal{D} \to \mathcal{C}$ is used to indicate a function f with domain $\mathcal{D}$ and codomain $\mathcal{C}$. If we use $f : \mathcal{D} \to \mathcal{D}$, it indicates that the domain and codomain are the same set [23].*

Definition 15 *Let $f : \mathcal{D} \to \mathbb{R}$ be a function, $I \subseteq \mathcal{D} \subseteq \mathbb{R}$ an open interval and let $x \in I$ be a point of I. Then $L \in \mathbb{R}$ is the limit of a function f at x denoted as*

$$\lim_{z \to x} f(z) = L$$

if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|z - x| < \delta$, then we have $|f(z) - L| < \varepsilon$ [24, 25].

Definition 16 *Let $\mathcal{D} \subseteq \mathbb{R}$ be an open interval of nonzero length from which at most a finite number of points have been removed, and let $f : \mathcal{D} \to \mathbb{R}$ be a function. Then f is called continuous if and only if*

1. *$f(x)$ is defined, i.e., $x \in \mathcal{D}$, and*
2. *$\lim_{z \to x} f(z)$ exists, and*
3. *$\lim_{z \to x} f(z) = f(x)$, and*
4. *if x is an endpoint of $\mathcal{D}$, use the left or right limits in 2 and 3, as appropriate.*

f is called continuous on $\mathcal{D}$ if and only if f is continuous at each $x \in \mathcal{D}$ [24].

Definition 17 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $x \in \mathbb{R}$ be a point. f is said to be differentiable at x if there exists a linear transformation $Df(x) \in \mathbb{R}$ that satisfies*

$$\lim_{|h| \to 0} \frac{|f(x+h) - f(x) - Df(x)h|}{|h|} = 0$$

The linear transformation $Df(x)$ is called the derivative of f at x [26].

It is worth noting that the previous definition may confuse the reader since, as such, the usual definition of derivative is not provided. However, it should be noticed that for the limit presented in Definition 17 to be zero, it is necessary that the linear transformation $Df(x)$ has the following form

$$Df(x) = \lim_{|h| \to 0} \frac{|f(x+h) - f(x)|}{|h|}$$

which is the definition that is usually presented when starting with the concept of derivative.

Definition 18 *Let $f : \mathcal{D} \to \mathbb{R}$ be a function. Suppose f is differentiable on $\mathcal{D}$. Then $f \in C^1$ if the derivative $Df : \mathcal{D} \to \mathbb{R}$ is continuous on $\mathcal{D}$ [26].*

2.4 Discrete-time Dynamical Systems

In this section, definitions within the area of dynamical systems are presented, which involve the definitions of discrete dynamical system, fixed point, periodic point, and orbit. Subsequently, the conditions for a fixed point to be an attractor or a repeller are presented, as well as the conditions for an orbit to be attractive or repulsive. Finally, the definition of chaos in the sense of Devaney, multimodal map, and some examples of unimodal and bimodal maps are given.

Definition 19 *A discrete dynamical system is a relation of the form*

$$x_{k+1} = f(x_k), \quad k \in 0, 1, 2, .., N \tag{2.1}$$

where $x_k \in \mathbb{R}$ and x_0 is the initial condition [27].

Definition 20 *Let $f : I \to I$ be a function and let $c \in I$ be a point. Then, if it satisfies that $f(c) = c$, then c is a fixed point of f [23].*

Theorem 2 *Let $I = [a, b]$ be a closed interval and let $f : I \to I$ be a continuous function. Then f has a fixed point in I [23].*

Definition 21 *Let f be a function. Then x is a periodic point of f with period k if $f^k(x) = x$. In other words, a point is a periodic point of f with period k if it is a fixed point of f^k. The periodic point x has primary period k_0 if $f^{k_0}(x) = x$ and $f^n(x) \neq x$ when $0 < n < k_0$ [23, 28]. Where*

$$
\begin{aligned}
f^2(x) &= (f \circ f)(x) \\
f^3(x) &= (f \circ f \circ f)(x) = (f \circ f^2)(x) \\
f^4(x) &= (f \circ f \circ f \circ f)(x) = (f \circ f^3)(x) \\
&\vdots \\
f^k(x) &= (\underbrace{f \circ f \cdots f \circ f}_{k-\ times})(x) = (f \circ f^{k-1})(x) \qquad \text{with } k \in \mathbb{N}
\end{aligned}
$$

Definition 22 *Let $f : I \to I$ be a function and $x_0 \in I$ a point, we define the orbit of x_0 under f, denoted as $\mathcal{O}(x_0, f)$, as the sequence of points*

$$\mathcal{O}(x_0, f) = x_0, f(x_0), f^2(x_0), \cdots f^{k-1}(x_0), f^k(x_0)$$

The point x_0 is called the initial condition of the orbit [29].

Definition 23 *Let $x^* \in I$ be the fixed point of the discrete-time dynamical system defined by the equation $x_{k+1} = f(x_k)$ denoted by (2.1) Then;*

- *the fixed point x^* is stable in the Lyapunov sense if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that for any $x \in I$ with $|x - x^*| < \delta$, then $|f^n(x^*) - f^n(x)| < \varepsilon$, for all $n \in \mathbb{N}$ large enough.*
- *The fixed point x^* is unstable if it is not stable.*
- *the fixed point x^* is asymptotically stable in the Lyapunov sense if it is stable in the Lyapunov sense and also if there exists $\delta = \delta(\varepsilon)$ such that for any $x \in I$ with $|x - x^*| < \delta$, then $\lim_{n\to\infty} |f^n(x^*) - f^n(x)| = 0$ [27, 30].*

Definition 24 *Let $f : I \to I$ be a C^1 function and let $x \in I$ be a fixed point of f. Then x is an attracting fixed point if $|Df(x)| < 1$. The point x is a repelling fixed point if $|Df(x)| > 1$. Finally, the fixed point x is a neutral fixed point if $|Df(x)| = 1$ [29].*

Definition 25 *Let $f : I \to I$ be a C^1 function and let $x_1, x_2, \cdots, x_k$ be a periodic orbit of period k, where $x_i \in I$ for all $i \in 1, 2, ..., k$. Then, the periodic orbit is attracting if [31];*

$$|Df(x_1) \cdot Df(x_2) \cdots Df(x_k)| < 1$$

and repelling if

$$|Df(x_1) \cdot Df(x_2) \cdots Df(x_k)| > 1$$

Definition 26 *Suppose $\mathcal{X}$ is a set and $\mathcal{Y} \subset \mathcal{X}$. We say that $\mathcal{Y}$ is dense in $\mathcal{X}$ if, for every point $x \in \mathcal{X}$, there exists a point $y \in \mathcal{Y}$ arbitrarily close to x [29].*

Definition 27 *A dynamical system f is transitive if for every pair of points x, y and any $\varepsilon > 0$ there exists a third point z within ε of x whose orbit is within ε of y [29].*

Definition 28 *A dynamical system f exhibits sensitive dependence on initial conditions if there exists a $\beta > 0$ such that, for any x and any $\varepsilon > 0$, there exists $y \in \mathbb{R}^+$ within ε of x and $k \in \mathbb{N}$ such that the distance between $f^k(x)$ and $f^k(y)$ is at least β [29].*

Definition 29 *A dynamical system f is chaotic if;*

1. *The periodic points of f are dense.*
2. *f is transitive.*
3. *f exhibits sensitive dependence on initial conditions [29].*

Definition 30 *Let $I = [0, 1]$ be a compact set, the mapping $f : I \to I$ is called k-modal if it is continuous on I and moreover, it has k critical points denoted by c_i with $i \in 0, 1, .., k-1$ in I. Furthermore, there exist intervals I_i such that $\bigcup_{i=1}^{k} I_{i-1} = I$ such that $\forall i = 0, 1, .., k-1$ it holds that $c_i \in I_i$ and $f(c_i) > f(x, \gamma, \beta_i)$ $\forall x \in I_i$ and $x_i \neq c_i$, where γ, β_i are parameters. The case when it has only one critical point, i.e., $k = 1$, the mapping is called unimodal, while the case when it has two critical points ($k = 2$) is called bimodal [4].*

The logistic map and the tent map are examples of unimodal maps, as both are continuous on I with a single critical point, moreover, they are increasing for $x \in \left[0, \frac{1}{2}\right)$ and decreasing for $x \in \left(\frac{1}{2}, 1\right]$ [4]. Additionally, in Chapter 1, Section 1.1, we discuss and analyze the monostability of the logistic and the tent map. The definitions of these maps are provided below.

Definition 31 *Let $I = [0, 1] \subset \mathbb{R}$. We refer to $f_L : I \to I$ as the logistic map defined as*

$$f_L(x, \alpha) = \alpha x(1 - x), \tag{2.2}$$

where the parameter $\alpha \in (0, 4] \subset \mathbb{R}$ [4].

Definition 32 *Let $I = [0, 1] \subset \mathbb{R}$. We refer to $f_T : I \to I$ as the tent map, expressed as*

$$f_T(x, \mu) = \begin{cases} \mu x & \text{for } x < \frac{1}{2}, \\ \mu(1 - x) & \text{for } x \geq \frac{1}{2}, \end{cases} \tag{2.3}$$

where $\mu \in (0, 2]$ is a parameter [4].

The bimodal map used for this work is based on the difference map $f_D(x, \beta)$, which is introduced in [4].

Definition 33 *Let f_L and f_T be the logistic map and the tent map shown in (2.2) and (2.3) respectively, taking the maximum value of their parameters $\alpha = 4$ and $\mu = 2$. We define $f_D(x.\beta)$ as the difference between the logistic map and the tent map multiplied by the parameter $\beta \in (0, 4]$, i.e., $f_D(x, \beta) = \beta(f_L(x, 4) - f_T(x, 2))$, as follows:*

$$f_D(x, \beta) = \begin{cases} 2\beta x(1 - 2x) & \text{for } 0 \leq x < \frac{1}{2}, \\ 2\beta(x - 1)(1 - 2x) & \text{for } \frac{1}{2} \leq x \leq 1. \end{cases} \tag{2.4}$$

We can observe that the difference map $f_D(x, \beta)$ is continuous on I and has two critical points located at $c_0 = (x, f_D(x, \beta)) = \left(\frac{1}{4}, \frac{\beta}{4}\right)$ and $c_1 = (x, f_D(x, \beta)) = \left(\frac{3}{4}, \frac{\beta}{4}\right)$ and $f_D(c_0, \beta) = f_D(c_1, \beta) = \frac{\beta}{4}$.

Definition 34 *A subset $\mathcal{A} \subset I$ is said to be an invariant set for a discrete-time dynamical system (2.1) if it satisfies that $f^n(\mathcal{A}) = \mathcal{A}$, $\forall; n \in \mathbb{N}$ [32].*

Definition 35 *A subset $\mathcal{A} \subset I$ is called a trapping region of a discrete-time dynamical system (2.1) if for every orbit of $x_0 \in \mathcal{A}$ will move to the region's interior and remain there as the system evolves $f^n(\mathcal{A}) \subset int(\mathcal{A})$, $\forall; n \in \mathbb{N}$ [32].*

Definition 36 *A family of functions f_β undergoes a saddle-node (or tangent) bifurcation at the parameter value $\beta = \beta_0$ if there exists an open interval I and an $\varepsilon > 0$ such that:*

1. *For $\beta_0 - \varepsilon < \beta < \beta_0$, f_β has no fixed points in the interval I.*
2. *For $\beta = \beta_0$, f_β has one fixed point in I, and this fixed point is neutral.*
3. *For $\beta_0 < \beta < \beta_0 + \varepsilon$, f_β has two fixed points in I, one attracting and one repelling [29].*

Definition 37 *A family of functions f_β undergoes a period-doubling bifurcation at the parameter value $\beta = \beta_0$ if there exists an open interval I and an $\varepsilon > 0$ such that:*

1. *For each β in the interval $[\beta_0 - \varepsilon, \beta_0 + \varepsilon]$, there exists a unique fixed point p_β for f_β in I.*
2. *For $\beta_0 - \varepsilon < \beta \leq \beta_0$, f_β does not have period-2 cycles in I, and p_β is an attractor.*
3. *For $\beta_0 < \beta < \beta_0 + \varepsilon$, exists a unique period-2 cycle q^1_β, q^2_β in I with $f_\beta(q^1_\beta) = q^2_\beta$. This period-2 cycle is attractive, while the fixed point p_β is repelling.*

4. As $\beta \to \beta_0$, we have $q^i_\beta \to p_{\beta_0}$, $\forall i = \{1, 2\}$ [29].

In the next chapter, the parametric family of bimodal maps that will be used is introduced and defined. Additionally, a series of theoretical results are presented that ensure bistability within this family. These results include an analysis of fixed points, saddle-node and period-doubling bifurcations, as well as the characterization of invariant sets and trapping regions that give rise to bistability.

Chapter 3

Bistability in a bimodal map

The family of bimodal maps based on the difference map (2.4) that will be used for this study is defined as follows:

$$f_D(x,\gamma,\beta_1,\beta_2) = \begin{cases} f_{D_1} & \text{for } \zeta_0 \le x < \zeta_1, \\ \\ f_{D_2} & \text{for } \zeta_1 \le x \le \zeta_2. \end{cases} \tag{3.1}$$

Where f_{D_1} and f_{D_2} are unimodal maps defined as:

$$f_{D_1}(x,\gamma,\beta_1) = \gamma\beta_1(x-\zeta_0)(\zeta_1 - x), \tag{3.2}$$

$$f_{D_2}(x,\gamma,\beta_2) = \gamma\beta_2(x-\zeta_1)(\zeta_2 - x), \tag{3.3}$$

and the parameters $\gamma, \beta_i \in (0,4] \subset \mathbb{R}$ must satisfy that:

$$0 < \gamma\beta_i \le \frac{4}{(\zeta_i - \zeta_{i-1})^2}, \tag{3.4}$$

for $i = 1,2$ and $\zeta_k \in [0,1] \subset \mathbb{R}$ for $k = 0,1,2$ where it holds that $\zeta_0 < \zeta_1 < \zeta_2$.

The bistability arises when a dynamic system presents two invariant sets, for the same parameter values, i.e. for a dynamical system $f : I \to I$ there are two invariant sets $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $f(\mathcal{A}_1) \subset \mathcal{A}_1$ and $f(\mathcal{A}_2) \subset \mathcal{A}_2$, with $\mathcal{A}_1 \subset I$ and $\mathcal{A}_2 \subset I$. In this Chapter, we show that the dynamical system (3.1) presents two invariant sets in the interval $I = [0,1]$ for a set of parameter values and the system oscillates in one of them depending of the initial condition x_0.

The analysis of fixed points and their stability are of great importance to understand the bifurcations that a dynamical system presents. Therefore we will begin to study the fixed points of the system (3.1).

Proposition 1 *The bimodal map defined in (3.1) presents fixed points in the interval* $\zeta_1 \leq x \leq \zeta_2$ *if:*

$$\frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1}+\sqrt{\zeta_2}}{\zeta_2-\zeta_1}\right)^2 < \beta_2 \leq \frac{4}{\gamma(\zeta_2-\zeta_1)^2}. \tag{3.5}$$

Proof. For find the upper bound of the interval where the parameter β_2 take it values with the purpose that the map f_D has fixed points in the interval $[\zeta_1, \zeta_2] \subset (0, 1] \subset \mathbb{R}$ it is suffices to find the maximum point that the second modal can take, which, it is located at the point $(x, f(x)) = \left(\frac{\zeta_1+\zeta_2}{2}, 1\right)$, knowing the maximum point that f_D can take, we can substitute into the equation $f_{D_2}(x, \gamma, \beta_1)$ showed in (3.3), thus obtaining

$$f_{D_2}\left(\frac{\zeta_1+\zeta_2}{2}, \gamma, \beta_1\right) \leq 1,$$

analogously, we have the following expression

$$\gamma\beta_2\left(\frac{\zeta_1+\zeta_2}{2}-\zeta_1\right)\left(\zeta_2-\frac{\zeta_1+\zeta_2}{2}\right) \leq 1,$$

by simplifying, we obtain

$$\gamma\beta_2\left(\frac{\zeta_1+\zeta_2-2\zeta_1}{2}\right)\left(\frac{2\zeta_2-\zeta_1-\zeta_2}{2}\right) = \gamma\beta_2\left(\frac{\zeta_2-\zeta_1}{2}\right)\left(\frac{\zeta_2-\zeta_1}{2}\right) \leq 1,$$

again, by simplifying and solving the preceding inequality for the parameter β_2, we obtain the maximum bound

$$\beta_2 \leq \frac{4}{\gamma(\zeta_2-\zeta_1)^2}, \tag{3.6}$$

now, for find the lower bound of the interval $\zeta_1 \leq x \leq \zeta_2$, we proceed to find the fixed points for the second modal showed by (3.5), thus, to located the fixed points, we perform $f_{D_2}(x, \gamma, \beta_2) = x$, thus

$$\gamma\beta_2(x-\zeta_1)(\zeta_2-x) = x,$$

by expanding the preceding equation, thus obtaining

$$\gamma\beta_2 x^2 - \gamma\beta_2(\zeta_1+\zeta_2)x + \gamma\beta_2\zeta_1\zeta_2 = x,$$

by simplifying and setting equal to zero, thus

$$\gamma\beta_2 x^2 - [\gamma\beta_2(\zeta_1+\zeta_2) - 1]x + \gamma\beta_2\zeta_1\zeta_2 = 0, \tag{3.7}$$

by solving (3.7), the expression for the fixed point is obtained

$$x = \frac{\gamma\beta_2(\zeta_1+\zeta_2) - 1 \pm \sqrt{[\gamma\beta_2(\zeta_1+\zeta_2)-1]^2 - 4\gamma^2\beta_2^2\zeta_1\zeta_2}}{2\gamma\beta_2}, \tag{3.8}$$

the preceding equation involves solutions in the complex numbers and the real numbers field, however, the difference map described in (3.1) that maps into the real numbers field, hence the discriminant Δ of (3.8) must satisfy that $\Delta > 0$ since we desire that f_{D_2} has two fixed points, thus

$$\Delta = [\gamma\beta_2(\zeta_1+\zeta_2)-1]^2 - 4\gamma^2\beta_2^2\zeta_1\zeta_2,$$

with the above, the inequality holds

$$[\gamma\beta_2(\zeta_1+\zeta_2)-1]^2 - 4\gamma^2\beta_2^2\zeta_1\zeta_2 > 0, \tag{3.9}$$

by expanding the preceding inequality, and use the parameter β_2 as variable, is obtained

$$\gamma^2\beta_2^2(\zeta_1+\zeta_2)^2 - 2\gamma\beta_2(\zeta_1+\zeta_2) + 1 - 4\gamma^2\beta_2^2\zeta_1\zeta_2 > 0,$$

grouping terms, we obtain

$$[\gamma^2(\zeta_1+\zeta_2)^2 - 4\gamma^2\zeta_1\zeta_2]\beta_2^2 - 2\gamma(\zeta_1+\zeta_2)\beta_2 + 1 > 0, \tag{3.10}$$

by solving the preceding expression by the general formula, is obtained

$$\beta_2 = \frac{\gamma(\zeta_1+\zeta_2) \pm \sqrt{[\gamma(\zeta_1+\zeta_2)]^2 - [\gamma^2(\zeta_1+\zeta_2)^2 - 4\gamma^2\zeta_1\zeta_2]}}{\gamma^2(\zeta_2-\zeta_1)^2}, \tag{3.11}$$

by simplifying the terms within the square root

$$\beta_2 = \frac{\gamma(\zeta_1+\zeta_2) \pm \sqrt{4\gamma^2\zeta_1\zeta_2}}{\gamma^2(\zeta_2-\zeta_1)^2},$$

by simplifying the preceding equation, we obtain

$$\beta_2 = \frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1} \pm \sqrt{\zeta_2}}{\zeta_2 - \zeta_1}\right)^2,$$

finally, by the preceding expression, we obtain the following solution sets that satisfied the inequality (3.10)

$$\beta_2 > \frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1} + \sqrt{\zeta_2}}{\zeta_2 - \zeta_1}\right)^2 \quad \text{and} \quad \beta_2 < \frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1} - \sqrt{\zeta_2}}{\zeta_2 - \zeta_1}\right)^2,$$

thus, the solution set that adequately satisfies the condition of the problems is

$$\beta_2 > \frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1} + \sqrt{\zeta_2}}{\zeta_2 - \zeta_1}\right)^2, \tag{3.12}$$

thus obtaining the lower bound for the parameter β_2 in which, f_D in the interval $(\zeta_1, \zeta_2] \subset [0, 1] \subset \mathbb{R}$ has fixed points. Finally, if we combine the inequalities (3.6) and (3.12) we obtain the interval that we desired showing.

$$\frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1} + \sqrt{\zeta_2}}{\zeta_2 - \zeta_1}\right)^2 < \beta_2 \leq \frac{4}{\gamma(\zeta_2 - \zeta_1)^2}. \tag{3.13}$$

Proposition 1 defined two fixed points, denoted by $x^*_{2,1}$ and $x^*_{2,2}$ in the interval $\zeta_1 \leq x \leq \zeta_2$ when β_2 fulfills (3.13) as follows

$$x^*_{2,1} = \frac{\gamma\beta_2(\zeta_1 + \zeta_2) - 1 + \sqrt{[\gamma\beta_2(\zeta_1 + \zeta_2) - 1]^2 - 4\gamma^2\beta_2^2\zeta_1\zeta_2}}{2\gamma\beta_2}, \tag{3.14}$$

and

$$x^*_{2,2} = \frac{\gamma\beta_2(\zeta_1 + \zeta_2) - 1 - \sqrt{[\gamma\beta_2(\zeta_1 + \zeta_2) - 1]^2 - 4\gamma^2\beta_2^2\zeta_1\zeta_2}}{2\gamma\beta_2}. \tag{3.15}$$

■

Proposition 2 *The fixed point $x^*_{2,2}$ of the bimodal map f_D (3.1) in the interval $\zeta_1 \leq x \leq \zeta_2$ given by* (3.15) *is always repulsive if*

$$\frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1} + \sqrt{\zeta_2}}{\zeta_2 - \zeta_1}\right)^2 < \beta_2 \leq \frac{4}{\gamma(\zeta_2 - \zeta_1)^2}. \tag{3.16}$$

Proof. For this proof, it is enough verifying that the following condition $\left|Df_{D_2}(x^*_{2,2}, \gamma, \beta_2)\right| > 1$ is fulfilled. If we find the condition for β_2 such that the absolute value of the derivative of f_{D_2} evaluated at the fixed point $x^*_{2,2}$ is greater than one then this fixed point is always repulsive. Thus, we proceed to find the derivative of the (3.3) we obtain

$$Df_{D_2}(x, \gamma, \beta_2) = -\gamma\beta_2[2x - (\zeta_1 + \zeta_2)], \tag{3.17}$$

by substituting the fixed point (3.15) into (3.17) and applied the condition $\left|Df_{D_2}(x^*_{2,2}, \gamma, \beta_2)\right| > 1$ is obtained

$$\left|-\gamma\beta_2\left[2x^*_{2,2} - (\zeta_1 + \zeta_2)\right]\right| > 1, \tag{3.18}$$

by simplifying the last expression, results

$$\left|\gamma\beta_2\left[\frac{\gamma\beta_2(\zeta_1 + \zeta_2) - 1 - \sqrt{\Delta} - \gamma\beta_2(\zeta_1 + \zeta_2)}{\gamma\beta_2}\right]\right| > 1,$$

where

$$\Delta = [\gamma\beta_2(\zeta_1 + \zeta_2) - 1]^2 - 4\gamma^2\beta_2^2\zeta_1\zeta_2,$$

by simplifying, we have

$$\left|1 + \sqrt{\Delta}\right| = \left|1 + \sqrt{[\gamma\beta_2(\zeta_1 + \zeta_2) - 1]^2 - 4\gamma^2\beta_2^2\zeta_1\zeta_2}\right| > 1,$$

by properties of absolute value, the preceding expression can expressed into two inequalities that which can be described by (3.19) and (3.20)

$$1 + \sqrt{[\gamma\beta_2(\zeta_1 + \zeta_2) - 1]^2 - 4\gamma^2\beta_2^2\zeta_1\zeta_2} < -1, \tag{3.19}$$

$$1 + \sqrt{[\gamma\beta_2(\zeta_1 + \zeta_2) - 1]^2 - 4\gamma^2\beta_2^2\zeta_1\zeta_2} > 1, \tag{3.20}$$

in the inequality (3.19) we can check that if we substrates one to both sides, is obtained

$$\sqrt{[\gamma\beta_2(\zeta_1 + \zeta_2) - 1]^2 - 4\gamma^2\beta_2^2\zeta_1\zeta_2} < -2,$$

the preceding inequality has not solution in the real numbers field, however, if doing the same processing in the inequality (3.20) we can see that the solutions of this inequality are included in the real numbers field, thus, we obtain

$$\sqrt{[\gamma\beta_2(\zeta_1+\zeta_2)-1]^2-4\gamma^2\beta_2^2\zeta_1\zeta_2} > 0,$$

squaring both sides, results

$$[\gamma\beta_2(\zeta_1+\zeta_2)-1]^2-4\gamma^2\beta_2^2\zeta_1\zeta_2 > 0,$$

by expanded and group terms, using the parameter β_2 as variable, the inequality (3.10) are obtained. Thus, by solving it using the general formula, we concluded that the solution set for β_2 that satisfying the conditions in our problem was the next

$$\beta_2 > \frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1}+\sqrt{\zeta_2}}{\zeta_2-\zeta_1}\right)^2, \tag{3.21}$$

by the last result, we concluded that the fixed point $x^*_{2,2}$ is always repulsive if β_2 satisfies (3.21).

For prove the upper bound of the interval (3.13), we know by the proposition one that β_2 has upper bound described by

$$\beta_2 \leq \frac{4}{\gamma(\zeta_2-\zeta_1)^2},$$

thus, it is enough verifying that take the upper bound of the parameter β_2, the fixed point $x^*_{2,2}$ of the $\zeta_1 \leq x \leq \zeta_2$ substituted into the f_{D_2} expressed in (3.3) is repulsive. Thus, took the upper bound

$$\beta_{2_M} = \frac{4}{\gamma(\zeta_2-\zeta_1)^2} \tag{3.22}$$

by substituting the preceding equation into (3.3), we obtained

$$f_{D_2}(x,\gamma,\beta_{2_M}) = \gamma\beta_{2_M}(x-\zeta_1)(\zeta_2-x), \tag{3.23}$$

when differentiating the preceding equation, results

$$Df_{D_2}(x,\gamma,\beta_{2_M}) = -\gamma\beta_{2_M}[2x-(\zeta_1+\zeta_2)]. \tag{3.24}$$

Now, by substituting (3.15) into of the derivative (3.24) we obtained

$$Df_{D_2}(x^*_{2,2},\gamma,\beta_{2_M}) = -2\gamma\beta_{2_M}\Lambda,$$

where

$$\Lambda = \frac{\gamma\beta_{2_M}(\zeta_1+\zeta_2)-1-\sqrt{[\gamma\beta_{2_M}(\zeta_1+\zeta_2)-1]^2-4\gamma^2\beta_{2_M}^2\zeta_1\zeta_2}}{2\gamma\beta_{2_M}},$$

by simplifying the previous expression

$$Df_{D_2}(x^*_{2,2}) = 1+\sqrt{\left[\frac{4}{(\zeta_2-\zeta_1)^2}(\zeta_1+\zeta_2)-1\right]^2-\frac{64}{(\zeta_2-\zeta_1)^4}\zeta_1\zeta_2}. \tag{3.25}$$

Now, let us analyze the discriminant to verifying that $\Delta > 0$, we know that

$$\Delta = \left[\frac{4}{(\zeta_2-\zeta_1)^2}(\zeta_1+\zeta_2)-1\right]^2-\frac{64}{(\zeta_2-\zeta_1)^4}\zeta_1\zeta_2,$$

by expanding

$$\Delta = \frac{16}{(\zeta_2-\zeta_1)^4}(\zeta_1+\zeta_2)^2-\frac{8}{(\zeta_2-\zeta_1)^2}(\zeta_1+\zeta_2)+1-\frac{64}{(\zeta_2-\zeta_1)^2}\zeta_1\zeta_2,$$

by grouping terms, we obtain

$$\Delta = \frac{16}{(\zeta_2-\zeta_1)^4}\left[(\zeta_1+\zeta_2)^2-4\zeta_1\zeta_2\right]-\frac{8}{(\zeta_2-\zeta_1)^2}(\zeta_1+\zeta_2)+1,$$

$$\Delta = \frac{16}{(\zeta_2-\zeta_1)^4}(\zeta_2-\zeta_1)^2-\frac{8}{(\zeta_2-\zeta_1)^2}(\zeta_1+\zeta_2)+1,$$

by simplifying

$$\Delta = \frac{16}{(\zeta_2-\zeta_1)^2}-\frac{8}{(\zeta_2-\zeta_1)^2}(\zeta_1+\zeta_2)+1.$$

Finally, we obtain that

$$\Delta = \frac{8}{(\zeta_2-\zeta_1)^2}\left[2-(\zeta_1+\zeta_2)\right]+1,$$

and since $\zeta_1 < \zeta_2$ and $\zeta_2 \leq 1$ then $\zeta_1+\zeta_2 < 2$ therefore

$$\Delta = \frac{8}{(\zeta_2-\zeta_1)^2}\left[2-(\zeta_1+\zeta_2)\right]+1 > 0.$$

Based on the previous analysis, we conclude that

$$\Delta = \left[\frac{4}{(\zeta_2-\zeta_1)^2}(\zeta_1+\zeta_2)-1\right]^2 - \frac{64}{(\zeta_2-\zeta_1)^4}\zeta_1\zeta_2 > 0,$$

and therefore, (3.25) is always upper than one. Consequently, if the parameter β_2 took the upper bound expressed by

$$\beta_2 = \frac{4}{\gamma(\zeta_2-\zeta_1)^2},$$

then, the fixed point

$$x^*_{2,2} = \frac{\gamma\beta_2(\zeta_1+\zeta_2)-1-\sqrt{[\gamma\beta_2(\zeta_1+\zeta_2)-1]^2-4\gamma^2\beta_2^2\zeta_1\zeta_2}}{2\gamma\beta_2},$$

satisfies that $Df_{D_2}(x^*_{2,2}) > 1$. Therefore, the fixed point is repulsive if (3.16) is satisfied. ■

Proposition 3 *Consider the bimodal map given by (3.1). The fixed point* $x^*_{2,1}$ *in the interval* $\zeta_1 \leq x \leq \zeta_2$ *expressed by (3.14) is attractive if:*

$$\frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1}+\sqrt{\zeta_2}}{\zeta_2-\zeta_1}\right)^2 < \beta_2 < \frac{\zeta_1+\zeta_2+2\sqrt{(\zeta_1+\zeta_2)^2-3\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2}, \tag{3.26}$$

and repulsive if:

$$\frac{\zeta_1+\zeta_2+2\sqrt{(\zeta_1+\zeta_2)^2-3\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2} < \beta_2 \leq \frac{4}{\gamma(\zeta_2-\zeta_1)^2}. \tag{3.27}$$

Proof. For find the interval where the parameter β_2 makes that the fixed point $x^*_{2,1}$ is repulsive in the interval $[\zeta_1, \zeta_2]$ is repulsive, we use the condition that $\left|Df_{D_2}(x^*_{2,1}, \gamma, \beta_2)\right| < 1$, thus, the derivative of (3.3) results

$$Df_{D_2}(x, \gamma, \beta_2) = -\gamma\beta_2[2x - (\zeta_1+\zeta_2)].$$

By substituting the fixed point $x^*_{2,1}$ shown in (3.14) into the preceding expression, by simplifying we obtain

$$Df_{D_2}(x^*_{2,1}, \gamma, \beta_2) = \left|1 - \sqrt{[\gamma\beta_2(\zeta_1+\zeta_2)-1]^2 - 4\gamma^2\beta_2^2\zeta_1\zeta_2}\right|.$$

Now, we choose a $\varepsilon > 0$ and the lower bound of the interval for the parameter β_2 that which is expressed by (3.26). The lower bound will be denoted by

$$\beta_m = \frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1}+\sqrt{\zeta_2}}{\zeta_2-\zeta_1}\right)^2. \tag{3.28}$$

An analogously way to write the parameter β_m can be seen below

$$\beta_m = \frac{\zeta_1+\zeta_2+2\sqrt{\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2}. \tag{3.29}$$

Previously, we mention an ε, this is mentioned with the purpose that the parameter β_2 can not took the upper and lower bounded, since the interval shown in (3.26) is open and consequently, this interval not include the boundaries. Now, the purpose is found the lower and upper value that ε can take it, for this, we know, by the interval expressed in (3.26) the upper bound of β_2, that which will be denoted by β_M and is expressed by

$$\beta_M = \frac{\zeta_1+\zeta_2+2\sqrt{(\zeta_1+\zeta_2)^2-3\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2}. \tag{3.30}$$

Thus, we can write (3.30) of the following way

$$\beta_M = \frac{\zeta_1+\zeta_2+2\sqrt{(\zeta_1-\zeta_2)^2+\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2}.$$

Now, let β_0 the parameter shown in (3.29) but adding ε within of the radical, results

$$\beta_0 = \frac{\zeta_1+\zeta_2+2\sqrt{\zeta_1\zeta_2+\varepsilon}}{\gamma(\zeta_2-\zeta_1)^2} \tag{3.31}$$

such that if we want to find the interval where ε can be chosen arbitrarily, it is enough to determine the interval of values that ε must take on β_0 in order to satisfy

$$\beta_m < \beta_0 < \beta_M. \tag{3.32}$$

By substituting the previously denoted values, we obtain

$$\frac{\zeta_1+\zeta_2+2\sqrt{\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2} < \frac{\zeta_1+\zeta_2+2\sqrt{\zeta_1\zeta_2+\varepsilon}}{\gamma(\zeta_2-\zeta_1)^2} < \frac{\zeta_1+\zeta_2+2\sqrt{(\zeta_1-\zeta_2)^2+\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2}. \tag{3.33}$$

By simplifying the preceding inequality, it results

$$\sqrt{\zeta_1\zeta_2} < \sqrt{\zeta_1\zeta_2+\varepsilon} < \sqrt{(\zeta_1-\zeta_2)^2+\zeta_1\zeta_2},$$

analogously, we obtain

$$\zeta_1\zeta_2 < \zeta_1\zeta_2 + \varepsilon < (\zeta_1 - \zeta_2)^2 + \zeta_1\zeta_2.$$

Finally, results

$$0 < \varepsilon < (\zeta_1 - \zeta_2)^2. \tag{3.34}$$

The preceding interval is the interval where ε can be chosen arbitrarily. now, we proceed to verify if the following condition $\left|Df_{D_2}(x^*_{2,1}, \gamma, \beta_2)\right| < 1$ is satisfying. Previously was obtain the following expression

$$Df_{D_2}(x^*_{2,1}, \gamma, \beta_2) = \left|1 - \sqrt{\gamma^2(\zeta_2 - \zeta_1)^2\beta_2^2 - 2\gamma(\zeta_1 + \zeta_2)\beta_2 + 1}\right|. \tag{3.35}$$

The preceding expression is the result of applied the condition

$$\left|Df_{D_2}(x^*_{2,1}, \gamma, \beta_2)\right| < 1.$$

By simplifying and substituting β_0 into β_2, it is obtained

$$Df_{D_2}(x^*_{2,1}, \gamma, \beta_0) = \left|1 - \sqrt{\Theta}\right|,$$

where

$$\Theta = \gamma^2(\zeta_2 - \zeta_1)^2\beta_0^2 - 2\gamma(\zeta_1 + \zeta_2)\beta_0 + 1,$$

and β_0 is defined in (3.31). By substituting β_0 in Θ and simplifying we have

$$\Theta = \frac{(\zeta_2 - \zeta_1)^2 + 4\zeta_1\zeta_2 + 4\varepsilon - (\zeta_1 + \zeta_2)^2}{(\zeta_2 - \zeta_1)^2} = \frac{4\varepsilon}{(\zeta_2 - \zeta_1)^2}.$$

Thus, the expression $Df_{D_2}(x^*_{2,1}, \gamma, \beta_0) = \left|1 - \sqrt{\Theta}\right|$ results

$$Df_{D_2}(x^*_{2,1}, \gamma, \beta_0) = \left|1 - \sqrt{\frac{4\varepsilon}{(\zeta_2 - \zeta_1)^2}}\right|.$$

Finally, it results

$$Df_{D_2}(x^*_{2,1}, \gamma, \beta_0) = \left|1 - \frac{2\sqrt{\varepsilon}}{(\zeta_2 - \zeta_1)}\right| < 1,$$

from the last expression, we can see that for each ε taken from the interval $0 < \varepsilon < (\zeta_1 - \zeta_2)^2$ it holds that $\left|Df_{D_2}(x^*_{2,1}, \gamma, \beta_2)\right| < 1$ and how the preceding interval was found using the upper and lower bounds of the parameter

β_2 expressed in (3.26), this shows that for each value of the parameter β_2 contained in

$$\frac{(\sqrt{\zeta_1}+\sqrt{\zeta_2})^2}{\gamma(\zeta_2-\zeta_1)^2} < \beta_2 < \frac{\zeta_1+\zeta_2+2\sqrt{(\zeta_1+\zeta_2)^2-3\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2}. \tag{3.36}$$

The fixed point $x^*_{1,2}$ is always attractive. Now, to demonstrate the interval for the parameter β_2 where the fixed point $x^*_{2,1}$, which is included in the interval $\zeta_1 \le x \le \zeta_2$, is always repulsive, we use the following condition:

$$\left|Df_{D_2}(x^*_{2,1},\gamma,\beta_2)\right| > 1. \tag{3.37}$$

Thus, by substituting the fixed point denoted by (3.14) into the derivative of f_{D_2} as expressed by (3.17) and applying the condition (3.37), it results

$$\left|1-\sqrt{[\gamma\beta_2(\zeta_1+\zeta_2)-1]^2-4\gamma^2\beta_2^2\zeta_1\zeta_2}\right| > 1. \tag{3.38}$$

The preceding inequality is fulfilled if

$$\sqrt{[\gamma\beta_2(\zeta_1+\zeta_2)-1]^2-4\gamma^2\beta_2^2\zeta_1\zeta_2} > 2. \tag{3.39}$$

By squaring both sides of the previous inequality, is obtained

$$[\gamma\beta_2(\zeta_1+\zeta_2)-1]^2-4\gamma^2\beta_2^2\zeta_1\zeta_2 > 4.$$

By expanding, results

$$\gamma^2(\zeta_2-\zeta_1)^2\beta_2^2-2\gamma(\zeta_1+\zeta_2)\beta_2+1 > 4. \tag{3.40}$$

By subtracting four from both sides, it results in

$$\gamma^2(\zeta_2-\zeta_1)^2\beta_2^2-2\gamma(\zeta_1+\zeta_2)\beta_2-3 > 0. \tag{3.41}$$

By solving the previous inequality using the general, produces

$$\beta_2 = \frac{\gamma(\zeta_1+\zeta_2)\pm\sqrt{\gamma^2(\zeta_1+\zeta_2)^2-(-3)(\gamma^2(\zeta_2-\zeta_1)^2)}}{\gamma^2(\zeta_2-\zeta_1)^2}. \tag{3.42}$$

By simplifying the previous expression

$$\beta_2 = \frac{(\zeta_1+\zeta_2)\pm\sqrt{(\zeta_1+\zeta_2)^2+3(\zeta_2-\zeta_1)^2}}{\gamma(\zeta_2-\zeta_1)^2}.$$

Analogously, we have

$$\beta_2 = \frac{(\zeta_1+\zeta_2) \pm 2\sqrt{\zeta_1^2 - \zeta_1\zeta_2 + \zeta_2^2}}{\gamma(\zeta_2-\zeta_1)^2},$$

when factoring the term inside the square root, it results in

$$\beta_2 = \frac{(\zeta_1+\zeta_2) \pm 2\sqrt{(\zeta_1+\zeta_2)^2 - 3\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2}.$$

Thus, the solution set are given by

$$\beta_2 > \frac{(\zeta_1+\zeta_2) + 2\sqrt{(\zeta_1+\zeta_2)^2 - 3\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2}, \tag{3.43}$$

$$\beta_2 < \frac{(\zeta_1+\zeta_2) - 2\sqrt{(\zeta_1+\zeta_2)^2 - 3\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2}. \tag{3.44}$$

However, the solution set that satisfies our conditions is

$$\beta_2 > \frac{(\zeta_1+\zeta_2) + 2\sqrt{(\zeta_1+\zeta_2)^2 - 3\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2}, \tag{3.45}$$

that which is the lower bound of the interval shown in (3.27). Now, to demonstrate the upper bound it is enough notice the inequality (3.41) and we defined by $\mathcal{G}$ the left term, thus

$$\mathcal{G}(\beta_2) = \gamma^2(\zeta_2-\zeta_1)^2\beta_2^2 - 2\gamma(\zeta_1+\zeta_2)\beta_2 - 3.$$

Notice that $\mathcal{G}(\beta_2)$ is a continuous function, therefore, it was previously shown that when β_2 takes the lower bound, it makes that $\mathcal{G}(\beta_2) > 1$ and how the upper bound of the parameter β_2 shown in the interval defined by (3.27) satisfies that

$$\beta_2 = \frac{4}{\gamma(\zeta_2-\zeta_1)^2}. \tag{3.46}$$

Therefore, due to continuity of $\mathcal{G}(\beta_2) > 1$, when β_2 reaches the upper bound, it ensures that $\mathcal{G}(\beta_2) > 1$, and it can be concluded that the fixed point $x_{2,1}^*$ included in $\zeta_1 \leq x \leq \zeta_2$ is always repulsive if β_2 takes values within the following interval

$$\frac{\zeta_1+\zeta_2 + 2\sqrt{(\zeta_1+\zeta_2)^2 - 3\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2} < \beta_2 \leq \frac{4}{\gamma(\zeta_2-\zeta_1)^2} \tag{3.47}$$

■

Proposition 4 *Consider the bimodal map* (3.1). *A saddle-node bifurcation occurs in the subdomain* $\zeta_1 \leq x \leq \zeta_2$ *when*

$$\beta_0 = \frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1}+\sqrt{\zeta_2}}{\zeta_2-\zeta_1}\right)^2. \tag{3.48}$$

Proof. To demonstrate this Proposition, we need to show that every point stated in Definition 36 holds true. Regarding the first point of the aforementioned definition, it is necessary to demonstrate that prior to β_0, the bimodal map f_D, specifically the unimodal map f_{D_2}, does not possess fixed points. To prove this, let's observe that the discriminant Δ of the fixed points $x^*_{2,1}$ and $x^*_{2,2}$, defined by (3.14) and (3.15) respectively, where:

$$\Delta = [\gamma\beta_2(\zeta_1+\zeta_2)-1]^2 - 4\gamma^2\beta_2^2\zeta_1\zeta_2.$$

Notice, from Proposition 1, that discriminant satisfies $\Delta > 0$ for any

$$\frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1}+\sqrt{\zeta_2}}{\zeta_2-\zeta_1}\right)^2 < \beta_2 \leq \frac{4}{\gamma(\zeta_2-\zeta_1)^2}.$$

Moreover, notice that $\Delta = 0$ when $\beta_2 = \beta_0$. Therefore, for any $\beta_2 = \beta_0 - \varepsilon$ with $\varepsilon > 0$, it satisfies that $\Delta < 0$, therefore, f_{D_1} has not fixed points, fulfilling issue 1 of Definition 36.

For demonstrates the issue 2 of Definition 36, it is enough to notice that when $\beta_2 = \beta_0$ the discriminant $\Delta = 0$, which implies that the fixed points $x^*_{2,1}$ and $x^*_{2,2}$ satisfy $x^*_{2,2} = x^*_{2,1}$. Therefore, f_{D_2} has only one fixed point, fulfilling issue 2 of Definition 36.

Finally, to demonstrate issue 3 of the Definition 36, it is sufficient to consider Propositions 2 and 3,to state when β_2 satisfies the condition

$$\frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1}+\sqrt{\zeta_2}}{\zeta_2-\zeta_1}\right)^2 < \beta_2 < \frac{\zeta_1+\zeta_2+2\sqrt{(\zeta_1+\zeta_2)^2-3\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2},$$

there exist two fixed points $x^*_{2,1}$ and $x^*_{2,2}$, where the $x^*_{2,2}$ is a repulsive fixed point and $x^*_{2,1}$ is an invariant point. ■

Proposition 5 *Consider the bimodal map (3.1). A period-doubling bifurcation occurs in the interval* $\zeta_1 \leq x \leq \zeta_2$ *when:*

$$\beta_0 = \frac{(\zeta_1+\zeta_2) + 2\sqrt{(\zeta_1+\zeta_2)^2 - 3\zeta_1\zeta_2}}{\gamma(\zeta_1-\zeta_2)^2}. \tag{3.49}$$

Proof. In the proof of proposition 3, we demonstrated issues 1 and 2 of Definition 37. When it was concluded that for some $\beta_2 < \beta_0$, the fixed point $x^*_{2,1}$ in the interval $\zeta_1 \leq x \leq \zeta_2$, defined in (3.14), was attractive, and for some $\beta_2 > \beta_0$, this fixed point was repulsive. Now, we must demonstrate issues 3 and 4. For point 3, we need to show the existence of a 2-cycle. To demonstrate this, we use

$$f_{D_2}(f_{D_2}(x,\gamma,\beta_2)) = f^2_{D_2}(x,\gamma,\beta_2) = x. \tag{3.50}$$

We know that

$$f_{D_2}(x,\gamma,\beta_2) = -\gamma\beta_2[x^2 - (\zeta_1+\zeta_2)x + \zeta_1\zeta_2], \tag{3.51}$$

of the previous equation, we define

$$\gamma\beta_2 = r, \quad \zeta_1+\zeta_2 = a \quad \text{and} \quad \zeta_1\zeta_2 = b.$$

By substituting the previous defined parameters into (3.51)

$$f_{D_2}(x,r,a,b) = -rx^2 + rax - rb. \tag{3.52}$$

Thus, if we applied $f_{D_2}(f_{D_2}(x,r,a,b)) = f^2_{D_2}(x,r,a,b) = x$ it results

$$f^2_{D_2}(x,r,a,b) = -r(-rx^2+rax-rb)^2 + ra(-rx^2+rax-rb) - rb = x. \tag{3.53}$$

By expanding and simplifying

$$f^2_{D_2}(x,r,a,b) - x =$$
$$-r^3x^4 + 2r^3ax^3 - r^2[r(2b+a^2)+a)]x^2 + (2r^3ab + r^2a^2 - 1)x - rb(1+ra+r^2b).$$

From the previous equation, we know that the fixed points denoted by (3.14) and (3.15) are solution of $f^2_{D_2}(x,r,a,b) - x{=}0$, and we also know that these fixed points are determined by $f_{D_2}(x,r,a,b) - x$, so, if we make

$$\frac{f^2_{D_2}(x,r,a,b) - x}{f_{D_2}(x,r,a,b) - x} = -\frac{f^2_{D_2}(x,r,a,b) - x}{rx^2 + (1-ra)x + rb}, \tag{3.54}$$

by dividing, it results

$$-\frac{f_{D_2}^2(x,r,a,b)-x}{rx^2+(1-ra)x+rb} = rx^2 - r(ra+1)x + (r^2b+ra+1). \tag{3.55}$$

By setting to zero and returning to the previous parameters, we have

$$\gamma\beta_2 x^2 - \gamma\beta_2[\gamma\beta_2(\zeta_1+\zeta_2)+1]x + [\gamma^2\beta_2^2\zeta_1\zeta_2 + \gamma\beta_2(\zeta_1+\zeta_2)+1] = 0. \tag{3.56}$$

Thus, the solutions of the preceding equation are the periodic points with period 2 of $f_{D_2}(x,\gamma,\beta_2)$. By solving using the general formula, we have

$$x_1 = \frac{\gamma\beta_2(\zeta_1+\zeta_2)+1+\sqrt{\gamma^2\beta_2^2(\zeta_2-\zeta_1)^2-2\gamma\beta_2(\zeta_1+\zeta_2)-3}}{2\gamma\beta_2}, \tag{3.57}$$

$$x_2 = \frac{\gamma\beta_2(\zeta_1+\zeta_2)+1-\sqrt{\gamma^2\beta_2^2(\zeta_2-\zeta_1)^2-2\gamma\beta_2(\zeta_1+\zeta_2)-3}}{2\gamma\beta_2}. \tag{3.58}$$

We can notice that the expressions (3.57) and (3.58) depend on β_2; therefore, if the discriminant Δ satisfies that $\Delta > 0$, then x_1 and x_2 belong to $\mathbb{R}$, where

$$\Delta = \gamma^2\beta_2^2(\zeta_2-\zeta_1)^2 - 2\gamma\beta_2(\zeta_1+\zeta_2) - 3. \tag{3.59}$$

Thus, we have

$$\gamma^2(\zeta_2-\zeta_1)^2\beta_2^2 - 2\gamma(\zeta_1+\zeta_2)\beta_2 - 3 > 0. \tag{3.60}$$

By solving the preceding inequality using the general formula, we have

$$\beta_2 = \frac{\gamma(\zeta_1+\zeta_2)\pm\sqrt{\gamma^2(\zeta_1+\zeta_2)^2+3\gamma^2(\zeta_2-\zeta_1)^2}}{\gamma^2(\zeta_2-\zeta_1)^2}.$$

By simplifying, we have

$$\beta_2 = \frac{\zeta_1+\zeta_2\pm\sqrt{(\zeta_1+\zeta_2)^2+3(\zeta_2-\zeta_1)^2}}{\gamma(\zeta_2-\zeta_1)^2},$$

$$\beta_2 = \frac{\zeta_1+\zeta_2\pm 2\sqrt{\zeta_1^2-\zeta_1\zeta_2+\zeta_2^2}}{\gamma(\zeta_2-\zeta_1)^2}.$$

Finally, it results

$$\beta_2 = \frac{\zeta_1+\zeta_2\pm 2\sqrt{(\zeta_1+\zeta_2)^2-3\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2}.$$

Therefore, the solution sets are

$$\beta_2 > \frac{\zeta_1 + \zeta_2 + 2\sqrt{(\zeta_1 + \zeta_2)^2 - 3\zeta_1\zeta_2}}{\gamma(\zeta_2 - \zeta_1)^2}, \quad \beta_2 < \frac{\zeta_1 + \zeta_2 - 2\sqrt{(\zeta_1 + \zeta_2)^2 - 3\zeta_1\zeta_2}}{\gamma(\zeta_2 - \zeta_1)^2}.$$

However, the solution set that satisfies the conditions of β_2 is

$$\beta_2 > \frac{\zeta_1 + \zeta_2 + 2\sqrt{(\zeta_1 + \zeta_2)^2 - 3\zeta_1\zeta_2}}{\gamma(\zeta_2 - \zeta_1)^2}. \tag{3.61}$$

Consequently, if β_2 satisfies the preceding condition then there is a 2-cycle and by the proof of the proposition 3, the fixed point $x^*_{2,1}$ is a repulsive fixed point when

$$\beta_2 > \frac{\zeta_1 + \zeta_2 + 2\sqrt{(\zeta_1 + \zeta_2)^2 - 3\zeta_1\zeta_2}}{\gamma(\zeta_2 - \zeta_1)^2}. \tag{3.62}$$

Now, we must demonstrate that

$$|Df_{D_2}(x_1)Df_{D_2}(x_2)| < 1.$$

By substituting (3.57) and (3.58) into (3.17), and by simplifying we have

$$\left|(1 + \sqrt{\Delta})(1 - \sqrt{\Delta})\right|, \tag{3.63}$$

where

$$\Delta = \gamma^2\beta_2^2(\zeta_2 - \zeta_1)^2 - 2\gamma\beta_2(\zeta_1 + \zeta_2) - 3.$$

Now, by expanding (3.63)

$$|1 - \Delta|.$$

We notice that $\Delta > 0$ when β_2 satisfies (3.47), which includes the condition (3.62), then, we concluded that

$$|1 - \Delta| < 1.$$

However, by absolute value properties, we have

$$-1 < 1 - \Delta < 1,$$

analogously

$$0 < \Delta < 2,$$

from $\Delta > 0$ it results

$$\beta_2 > \frac{\zeta_1 + \zeta_2 + 2\sqrt{(\zeta_1 + \zeta_2)^2 - 3\zeta_1\zeta_2}}{\gamma(\zeta_2 - \zeta_1)^2}. \tag{3.64}$$

And from $\Delta < 2$ we have

$$0 < \beta_2 < \frac{\zeta_1 + \zeta_2 + \sqrt{6(\zeta_1 + \zeta_2)^2 - 20\zeta_1\zeta_2}}{\gamma(\zeta_2 - \zeta_1)^2}. \tag{3.65}$$

Consequently, from (3.64) and (3.65) results

$$\frac{\zeta_1 + \zeta_2 + 2\sqrt{(\zeta_1 + \zeta_2)^2 - 3\zeta_1\zeta_2}}{\gamma(\zeta_2 - \zeta_1)^2} < \beta_2 < \frac{\zeta_1 + \zeta_2 + \sqrt{6(\zeta_1 + \zeta_2)^2 - 20\zeta_1\zeta_2}}{\gamma(\zeta_2 - \zeta_1)^2}. \tag{3.66}$$

Since $\zeta_1 < \zeta_2$ and $\zeta_2 \leq 1$, we know that $\zeta_1 + \zeta_2 < 2$ and $\zeta_1\zeta_2 < 1$ then, from the preceding we have

$$\beta_2 < \frac{\zeta_1 + \zeta_2 + \sqrt{6(\zeta_1 + \zeta_2)^2 - 20\zeta_1\zeta_2}}{\gamma(\zeta_2 - \zeta_1)^2} < \frac{4}{\gamma(\zeta_2 - \zeta_1)^2}.$$

Therefore, we can notice that the following condition $|Df_{D_2}(x_1)Df_{D_2}(x_2)| < 1$ is satisfied if the parameter β_2 takes values within of

$$\frac{\zeta_1 + \zeta_2 + 2\sqrt{(\zeta_1 + \zeta_2)^2 - 3\zeta_1\zeta_2}}{\gamma(\zeta_2 - \zeta_1)^2} < \beta_2 < \frac{4}{\gamma(\zeta_2 - \zeta_1)^2}. \tag{3.67}$$

Finally, issue 3 of Definition 37 has been demonstrated. Now, to demonstrate issue 4 of 37, it is enough to show that

$$\lim_{\beta_2 \to \beta_0} x_i = x^*_{2,1}, \tag{3.68}$$

that is

$$\lim_{\beta_2 \to \beta_0} \frac{\gamma\beta_2(\zeta_1 + \zeta_2) + 1 \pm \sqrt{\gamma^2\beta_2^2(\zeta_2 - \zeta_1)^2 - \gamma\beta_2(\zeta_1 + \zeta_2) - 3}}{2\gamma\beta_2} = x^*_{2,1,\beta_0}, \tag{3.69}$$

where $x^*_{2,1,\beta_0}$ is the fixed point of the interval $\zeta_1 \leq x \leq \zeta_2$, as shown in (3.14). By substituting β_0 into β_2

$$x^*_{2,1,\beta_0} = \frac{\gamma\beta_0(\zeta_1 + \zeta_2) - 1 + \sqrt{(\gamma\beta_0(\zeta_1 + \zeta_2) - 1)^2 - 4\gamma^2\beta_0^2\zeta_1\zeta_2}}{2\gamma\beta_0}, \tag{3.70}$$

where β_0 is defined in (3.49), we can notice that

$$\lim_{\beta_2 \to \beta_0} \sqrt{\gamma^2 \beta_2^2 (\zeta_2 - \zeta_1)^2 - \gamma \beta_2 (\zeta_1 + \zeta_2) - 3} \to 0.$$

Thus, by simplifying (3.69) we have

$$\lim_{\beta_2 \to \beta_0} \frac{\gamma \beta_2 (\zeta_1 + \zeta_2) + 1}{2\gamma \beta_2} = x^*_{2,1,\beta_0}. \tag{3.71}$$

From (3.70) we have

$$x^*_{2,1,\beta_0} = \frac{\gamma \left(\frac{\zeta_1 + \zeta_2 + 2\sqrt{\alpha}}{\gamma(\zeta_2 - \zeta_1)^2} \right) (\zeta_1 + \zeta_2) - 1 + \sqrt{\Pi}}{2\gamma \left(\frac{\zeta_1 + \zeta_2 + 2\sqrt{\alpha}}{\gamma(\zeta_2 - \zeta_1)^2} \right)}, \tag{3.72}$$

where

$$\alpha = (\zeta_1 + \zeta_2)^2 - 3\zeta_1 \zeta_2, \tag{3.73}$$

and

$$\Pi = \gamma^2 (\zeta_2 - \zeta_1)^2 \left(\frac{\zeta_1 + \zeta_2 + 2\sqrt{\alpha}}{\gamma(\zeta_2 - \zeta_1)^2} \right)^2 - 2\gamma(\zeta_1 + \zeta_2) \left(\frac{\zeta_1 + \zeta_2 + 2\sqrt{\alpha}}{\gamma(\zeta_2 - \zeta_1)^2} \right) + 1. \tag{3.74}$$

By expanding the previous equation

$$\Pi = \frac{(\zeta_1 + \zeta_2)^2 + 4(\zeta_1 + \zeta_2)\sqrt{\alpha} + 4\alpha - 2(\zeta_1 + \zeta_1)^2 - 4(\zeta_1 + \zeta_2)\sqrt{\alpha} + (\zeta_2 - \zeta_1)^2}{(\zeta_2 - \zeta_1)^2}.$$

Finally, by simplifying

$$\Pi = \frac{-(\zeta_1 + \zeta_2)^2 + 4\alpha + (\zeta_2 - \zeta_1)^2}{(\zeta_2 - \zeta_1)^2} = \frac{3(\zeta_1 + \zeta_2)^2 + (\zeta_2 - \zeta_1)^2 - 12\zeta_1 \zeta_2}{(\zeta_2 - \zeta_1)^2} = 4.$$

By substituting Π into (3.72)

$$x^*_{2,1,\beta_0} = \frac{\gamma \left(\frac{\zeta_1 + \zeta_2 + 2\sqrt{\alpha}}{\gamma(\zeta_2 - \zeta_1)^2} \right) (\zeta_1 + \zeta_2) - 1 + \sqrt{4}}{2\gamma \left(\frac{\zeta_1 + \zeta_2 + 2\sqrt{\alpha}}{\gamma(\zeta_2 - \zeta_1)^2} \right)}. \tag{3.75}$$

Finally after simplification, we obtain the following solution

$$x^*_{2,1,\beta_0} = \frac{\gamma\left(\dfrac{\zeta_1+\zeta_2+2\sqrt{\alpha}}{\gamma(\zeta_2-\zeta_1)^2}\right)(\zeta_1+\zeta_2)+1}{2\gamma\left(\dfrac{\zeta_1+\zeta_2+2\sqrt{\alpha}}{\gamma(\zeta_2-\zeta_1)^2}\right)} = \frac{\gamma\beta_0(\zeta_1+\zeta_2)+1}{2\gamma\beta_0}. \tag{3.76}$$

Now, from the expression (3.71) we have

$$\lim_{\beta_2\to\beta_0} \frac{\gamma\beta_2(\zeta_1+\zeta_2)+1}{2\gamma\beta_2} = \frac{\gamma\beta_0(\zeta_1+\zeta_2)+1}{2\gamma\beta_0} = x^*_{2,1,\beta_0}. \tag{3.77}$$

Therefore, it is demonstrated that when $\beta_2 \to \beta_0$ then $x_i \to x^*_{2,1,\beta_0}$ for $i = 1, 2$, and based on the above, we conclude the proof of the four issues of Definition 37, and we can assert the existence of a period-doubling bifurcation. ■

Proposition 6 *The difference map shown in (3.1) exhibits a trapping region or an invariant set* $\mathcal{A}_2 \subset (\zeta_1, \zeta_2] \subset (0, 1]$ *if*

$$\frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1}+\sqrt{\zeta_2}}{\zeta_2-\zeta_1}\right)^2 \le \beta_2 \le \frac{(\zeta_1+\zeta_2)+\sqrt{9(\zeta_1+\zeta_2)^2-32\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2}, \tag{3.78}$$

where

$$\mathcal{A}_2 = \left[x^*_{2,2}, \frac{\gamma\beta_2(\zeta_1+\zeta_2)+1+\sqrt{(\gamma\beta_2\zeta_1+\zeta_2)-1)^2-4\gamma^2\beta_2^2\zeta_1\zeta_2}}{\gamma\beta_2}\right], \tag{3.79}$$

and $x^*_{2,2}$ *denotes the fixed point of the interval* $\zeta_1 \le x \le \zeta_2$ *as defined by (3.15).*

Proof. To demonstrate this proposition, it is enough to find the upper value of the parameter β_2 such that the second iteration of f_{D_2} with x_{m_2} as the initial condition x_0 is higher than the fixed point $x^*_{2,2}$, where x_{m_2} denotes the midpoint of the interval $\zeta_1 \le x \le \zeta_2$ and is defined by

$$x_{m_2} = \frac{\zeta_1+\zeta_2}{2}. \tag{3.80}$$

The previous explanation can be resumed by finding a certain value for β_2 such that

$$f_{D_2}^2(x_{m_2}, \gamma, \beta_2) \geq x_{2,2}^*. \tag{3.81}$$

This proposition will be demonstrated; thus, we have

$$f_{D_2}(x, \gamma, \beta_2) = -\gamma\beta_2[x^2 - (\zeta_1 + \zeta_2)x + \zeta_1\zeta_2]. \tag{3.82}$$

We use

$$\gamma\beta_2 = r, \quad \zeta_1 + \zeta_2 = a \quad \text{and} \quad \zeta_1\zeta_2 = b. \tag{3.83}$$

By expanding (3.81) and by substituting the terms defined in (3.83), it results

$$f_{D_2}(x, r, a, b) = -rx^2 + rax - rb. \tag{3.84}$$

By substituting x_{m_2} into (3.84) and by simplifying, we have

$$f_{D_2}(x_{m_2}, r, a, b) = r\frac{(a^2 - 4b)}{4}. \tag{3.85}$$

When performing the second iteration $f_{D_2}^2(x_{m_2}, r, a, b)$ we have

$$f_{D_2}^2(x_{m_2}, r, a, b) = -r\left(r\frac{(a^2 - 4b)}{4}\right)^2 + ra\left(r\frac{(a^2 - 4b)}{4}\right) - rb. \tag{3.86}$$

If we defined

$$c = a^2 - 4b, \tag{3.87}$$

then, by expanding (3.86) and by substituting (3.87), it results

$$f_{D_2}^2(x_{m_2}, r, a, b) = \frac{-r^3c^2 + 4r^2ac - 16rb}{16}. \tag{3.88}$$

Now, we know that the fixed point $x_{2,2}^*$ as defined in (3.15) is

$$x_{2,2}^* = \frac{\gamma\beta_2(\zeta_1 + \zeta_2) - 1 - \sqrt{(\gamma\beta_2(\zeta_1 + \zeta_2) - 1)^2 - 4\gamma^2\beta_2^2\zeta_1\zeta_2}}{2\gamma\beta_2}.$$

The previous expression by substituting the parameters defined in (3.83) becomes

$$x_{2,2}^* = \frac{ra - 1 - \sqrt{(ra - 1)^2 - 4r^2b}}{2r}. \tag{3.89}$$

Applying the condition shown in (3.81), it results

$$\frac{-r^3c^2+4r^2ac-16rb}{16} \geq \frac{ra-1-\sqrt{(ra-1)^2-4r^2b}}{2r}. \tag{3.90}$$

By simplifying

$$-r^4c^2+4r^3ac-16r^2b \geq 8(ra-1-\sqrt{(ra-1)^2-4r^2b}).$$

Finally, we have

$$r^4c^2-4r^3ac+16r^2b+8ra-8 \leq 8\sqrt{(ra-1)^2-4r^2b}.$$

By squaring both sides, it results

$$(r^4c^2-4r^3ac+16r^2b+8ra-8)^2 \leq (8\sqrt{(ra-1)^2-4r^2b})^2.$$

By expanding and simplifying both sides, we have

$$c^4r^8-8ac^3r^7+16c^2(2b+a^2)r^6+16ac(c-8b)r^5- \\ 16(c^2+4ac^2-16b^2)r^4+64a(c+4b)r^3 \leq 0.$$

By dividing the previous expression by r^3, becomes

$$c^4r^5-8ac^3r^4+16c^2(2b+a^2)r^3+16ac(c-8b)r^2- \\ 16(c^2+4ac^2-16b^2)r+64a(c+4b) \leq 0, \tag{3.91}$$

when factoring the previous expression, it results in

$$(cr^2-2ar-8)(c^3r^3-6ac^2r^2+4c[8b+a^2+2c]r-8a[2c+8b-a^2]) \leq 0. \tag{3.92}$$

In the previous expression, we take r as the variable, which is the term containing β_2. The third-degree factor, when solved in terms of r, yields two complex roots and one real root; however, none of them satisfy the requirements for this study. Now, for the second-degree factor, we obtain the inequality

$$(cr^2-2ar-8) \leq 0. \tag{3.93}$$

By solving using the general formula

$$r = \frac{a \pm \sqrt{a^2+8c}}{c}. \tag{3.94}$$

By substituting $c = (a^2 - 4b)$ we have

$$r = \frac{a \pm \sqrt{a^2 + 8(a^2 - 4b)}}{a^2 - 4b}.$$

By simplifying

$$r = \frac{a \pm \sqrt{9a^2 - 32b}}{a^2 - 4b}.$$

We know that $a = \zeta_1 + \zeta_2$, $b = \zeta_1\zeta_2$ and $r = \gamma\beta_2$, thus, the preceding expression becomes

$$\gamma\beta_2 = \frac{(\zeta_1 + \zeta_2) \pm \sqrt{9(\zeta_1 + \zeta_2)^2 - 32\zeta_1\zeta_2}}{(\zeta_1 + \zeta_2)^2 - 4\zeta_1\zeta_2}.$$

Solving for β_2 and simplifying

$$\beta_2 = \frac{(\zeta_1 + \zeta_2) \pm \sqrt{9(\zeta_1 + \zeta_2)^2 - 32\zeta_1\zeta_2}}{\gamma(\zeta_1 - \zeta_2)^2}.$$

Thus, we obtained the interval for β_2 that satisfies $f^2_{D_2}(x_{m_2}, \gamma, \beta_2) \geq x^*_{2,2}$; this interval is

$$\frac{(\zeta_1 + \zeta_2) - \sqrt{9(\zeta_1 + \zeta_2)^2 - 32\zeta_1\zeta_2}}{\gamma(\zeta_1 - \zeta_2)^2} \leq \beta_2 \leq \frac{(\zeta_1 + \zeta_2) + \sqrt{9(\zeta_1 + \zeta_2)^2 - 32\zeta_1\zeta_2}}{\gamma(\zeta_1 - \zeta_2)^2}. \quad (3.95)$$

However, we can notice from (3.94) that the lower bound of the interval (3.95)

$$\frac{(\zeta_1 + \zeta_2) - \sqrt{9(\zeta_1 + \zeta_2)^2 - 32\zeta_1\zeta_2}}{\gamma(\zeta_1 - \zeta_2)^2} < 0.$$

Furthermore, from Proposition 1, we know the minimum bound of the parameter β_2 for which fixed points exist in the interval $\zeta_1 \leq x \leq \zeta_2$, and this minimum bound satisfies

$$\frac{(\zeta_1 + \zeta_2) - \sqrt{9(\zeta_1 + \zeta_2)^2 - 32\zeta_1\zeta_2}}{\gamma(\zeta_1 - \zeta_2)^2} < \frac{(\sqrt{\zeta_1} + \sqrt{\zeta_2})^2}{\gamma(\zeta_2 - \zeta_1)^2}.$$

Thus, by the continuity of f_{D_2}, it is shown that the interval for β_2 will be defined by

$$\frac{(\sqrt{\zeta_1} + \sqrt{\zeta_2})^2}{\gamma(\zeta_2 - \zeta_1)^2} \leq \beta_2 \leq \frac{(\zeta_1 + \zeta_2) + \sqrt{9(\zeta_1 + \zeta_2)^2 - 32\zeta_1\zeta_2}}{\gamma(\zeta_2 - \zeta_1)^2}. \quad (3.96)$$

So, the interval of β_2 defined in (3.96) makes the set $\mathcal{A}_2 \subset (\zeta_1, \zeta_2] \subset (0, 1]$ previously defined as

$$\mathcal{A}_2 = \left[x^*_{2,2}, \frac{\gamma\beta_2(\zeta_1 + \zeta_2) + 1 + \sqrt{(\gamma\beta_2\zeta_1 + \zeta_2) - 1)^2 - 4\gamma^2\beta_2^2\zeta_1\zeta_2}}{\gamma\beta_2} \right], \quad (3.97)$$

let be a trapping region or an invariant set. ■

Proposition 7 *The difference map shown in (3.1) exhibits a trapping region or an invariant set* $\mathcal{A}_1 = [\zeta_0, \zeta_1) = [0, 1/2)$ *if*

$$\frac{1}{\gamma\zeta_1} \leq \beta_1 \leq \frac{4}{\gamma\zeta_1}. \quad (3.98)$$

Proof. To demonstrate this proposition, it is enough to find the upper value of the parameter β_1 such that the second iteration of f_{D_1} with x_{m_1} as the initial condition x_0 is higher than the lower bound of $\mathcal{A}_1$. That is $\zeta_0 = 0$, where x_{m_1} denotes the midpoint of the interval $\zeta_0 \leq x < \zeta_1$ and is defined by

$$x_{m_1} = \frac{\zeta_0 + \zeta_1}{2} = \frac{\zeta_1}{2}. \quad (3.99)$$

The previous explanation can be resumed by finding a certain value for β_1 such that

$$f^2_{D_1}(x_{m_1}, \gamma, \beta_1) \geq 0, \quad (3.100)$$

and this proposition will be demonstrated; thus, we have

$$f_{D_1}(x, \gamma, \beta_1) = -\gamma\beta_1(x^2 - \zeta_1 x). \quad (3.101)$$

By expanding (3.101) it results

$$f_{D_1}(x, \gamma, \beta_1) = -\gamma\beta_1 x^2 + \gamma\beta_1\zeta_1 x. \quad (3.102)$$

By substituting x_{m_1} into (3.103) and by simplifying, we have

$$f_{D_1}(x_{m_1}, \gamma, \beta_1) = \frac{\gamma\beta_1\zeta_1^2}{4}. \quad (3.103)$$

When performing the second iteration $f^2_{D_2}(x_{m_2},\gamma,\beta_1)$ we have

$$f^2_{D_1}(x_{m_1},\gamma,\beta_1) = -\gamma\beta_1\left(\frac{\gamma\beta_1\zeta_1^2}{4}\right)^2 + \gamma\beta_1\zeta_1\left(\frac{\gamma\beta_1\zeta_1^2}{4}\right). \tag{3.104}$$

Then, by expanding (3.104), it results

$$f^2_{D_2}(x_{m_2},\gamma,\beta_1) = \frac{-\gamma^3\beta_1^3\zeta_1^4 + 4\gamma^2\beta_1^2\zeta_1^3}{16}. \tag{3.105}$$

By dividing by $\gamma^2\beta_1^2\zeta_1^3$ we obtain

$$f^2_{D_2}(x_{m_2},\gamma,\beta_1) = \frac{-\gamma\zeta_1\beta_1 + 4}{16}.$$

Applying the condition shown in (3.100), it result

$$\frac{-\gamma\zeta_1\beta_1 + 4}{16} \geq \zeta_0.$$

Analogously

$$\frac{-\gamma\zeta_1\beta_1 + 4}{16} \geq 0.$$

By simplifying

$$-\gamma\zeta_1\beta_1 + 4 \geq 0.$$

Finally, by solving for β_1 we have

$$\beta_1 \leq \frac{4}{\gamma\zeta_1}. \tag{3.106}$$

That is the upper bound of the interval of β_1 expressed in (3.98). To demonstrate the lower bound, we proceed to find the fixed points of $f_{D_1}(x,\gamma,\beta_1)$, this is

$$f_{D_1}(x,\gamma,\beta_1) = -\gamma\beta_1 x^2 + \gamma\beta_1\zeta_1 x = x. \tag{3.107}$$

By simplifying we have

$$f_{D_1}(x,\gamma\beta_1) = -\gamma\beta_1 x^2 + (\gamma\beta_1\zeta_1 - 1)x = 0. \tag{3.108}$$

By solving using the general formula, we have

$$x^*_{1,1} = \frac{\gamma\beta_1\zeta_1 - 1 + \sqrt{(\gamma\beta_1\zeta_1 - 1)^2}}{2\gamma\beta_1}, \quad x^*_{1,2} = \frac{\gamma\beta_1\zeta_1 - 1 - \sqrt{(\gamma\beta_1\zeta_1 - 1)^2}}{2\gamma\beta_1}.$$

Finally, we have the fixed points

$$x^*_{1,1} = \frac{\gamma\beta_1\zeta_1 - 1}{\gamma\beta_1}, \tag{3.109}$$

$$x^*_{1,2} = 0. \tag{3.110}$$

However, the fixed point must satisfy $x^*_{1,1} \geq 0$, thus

$$\frac{\gamma\beta_1\zeta_1 - 1}{\gamma\beta_1} \geq 0. \tag{3.111}$$

When solving for β_1, the results is

$$\beta_1 \geq \frac{1}{\gamma\zeta_1}. \tag{3.112}$$

The preceding result is the lower bound of the interval for β_1 defined in (3.98). Therefore, it is shown that $\mathcal{A}_1$ is a trapping region or an invariant set when β_2 takes values within the interval

$$\frac{1}{\gamma\zeta_1} \leq \beta_1 \leq \frac{4}{\gamma\zeta_1}. \tag{3.113}$$

■

Proposition 8 *The bimodal map f_D given by* (3.1) *exhibits monostability in the interval $\zeta_1 \leq x \leq \zeta_2$ if*

$$\frac{4x^*_{2,2}}{\gamma(\zeta_1 - \zeta_0)^2} < \beta_1 \leq \frac{4}{\gamma(\zeta_1 - \zeta_0)^2}, \tag{3.114}$$

and

$$\frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1} + \sqrt{\zeta_2}}{\zeta_2 - \zeta_1}\right)^2 \leq \beta_2 \leq \frac{(\zeta_1 + \zeta_2) + \sqrt{9(\zeta_1 + \zeta_2)^2 - 32\zeta_1\zeta_2}}{\gamma(\zeta_2 - \zeta_1)^2}. \tag{3.115}$$

Proof. For this proof, it suffices to find the condition for β_1 such that for any initial condition $x_0 \in \mathcal{A}_2^c$ after a certain number of iterations, the orbit goes into the interval $\mathcal{A}_2$. Where $\mathcal{A}_2^c = I \backslash \mathcal{A}_2$ denotes the complement of $\mathcal{A}_2$. To conduct the proof, we take the midpoint of the interval $\zeta_0 \leq x < \zeta_1$

as denoted in (3.99) and substitute it into (3.2), where the goal is for this operation to be greater than the fixed point $x^*_{2,2}$ denoted in (3.15), ensuring the existence of monostability. The above can be expressed as:

$$f_{D_1}(x_{m_1}, \gamma, \beta_1) > x^*_{2,2}.$$

Performing the substitution, we have

$$-\gamma\beta_1 \left[\frac{(\zeta_0 + \zeta_1)^2}{4} - \frac{(\zeta_0 + \zeta_1)^2}{2} + \zeta_1\zeta_0 \right] > x^*_{2,2}.$$

By simplifying the previous inequality

$$\gamma\beta_1 \left[\frac{(\zeta_1 - \zeta_0)^2}{4} \right] > x^*_{2,2}.$$

Finally, by solving for β_1 in the previous expression, we find that

$$\beta_1 > \frac{4x^*_{2,2}}{\gamma(\zeta_1 - \zeta_0)^2}, \tag{3.116}$$

which is the lower bound of the interval (3.114). To prove the upper bound, let us note that the maximum point of the unimodal map f_{D_1} is located at $(x, f(x)) = (x_{m_1}, 1)$. Thus, by substituting this maximum point into f_{D_1}, we have that

$$f_{D_1}(x_{m_1}, \gamma, \beta_1) \leq 1.$$

By simplifying, we have

$$\gamma\beta_1 \left[\frac{(\zeta_1 - \zeta_0)^2}{4} \right] \leq 1.$$

Where, upon solving for β_1, we obtain

$$\beta_1 \leq \frac{4}{\gamma(\zeta_1 - \zeta_0)^2}, \tag{3.117}$$

which is the upper bound of the interval (3.114). Finally, by combining the expressions (3.116) and (3.117), we obtain the interval

$$\frac{4x^*_{2,2}}{\gamma(\zeta_1 - \zeta_0)^2} < \beta_1 \leq \frac{4}{\gamma(\zeta_1 - \zeta_0)^2}. \tag{3.118}$$

Notice that the interval for β_2 arises from Proposition 6, where if β_2 satisfies (3.115), then there exists a trapping region or an invariant set $\mathcal{A}_2$ contained within $[\zeta_1, \zeta_2]$, denoted by (3.79). ■

The following proposition is a result of the combination of Propositions 1, 6, and 7. It can be observed that the interval (3.119) is the complement of the intervals stated in Propositions 1 and 5. This is done to prevent, after an arbitrarily finite number of iterations, the dynamics from remaining in the second mode. The interval (3.120) corresponds to the one presented in Proposition 7, aiming to maintain the dynamics in the first mode. This is achieved by characterizing an invariant set, as demonstrated in Proposition 7.

Proposition 9 *The bimodal map f_D given by* (3.1) *exhibits monostability in the interval $\zeta_0 \leq x < \zeta_1$ if*

$$\beta_2 \in \left[0, \frac{(\sqrt{\zeta_1} + \sqrt{\zeta_2})^2}{\gamma(\zeta_2 - \zeta_1)^2}\right) \bigcup \left(\frac{(\zeta_1 + \zeta_2) + \sqrt{9(\zeta_1 + \zeta_2)^2 - 32\zeta_1\zeta_2}}{\gamma(\zeta_2 - \zeta_1)^2}, \frac{4}{\gamma(\zeta_2 - \zeta_1)^2}\right] \tag{3.119}$$

and

$$\frac{1}{\gamma\zeta_1} \leq \beta_1 \leq \frac{4}{\gamma\zeta_1}. \tag{3.120}$$

Proof. This proposition arises from Propositions 1,6 and 7. From Proposition 7 states that if

$$\frac{1}{\gamma\zeta_1} \leq \beta_1 \leq \frac{4}{\gamma\zeta_1},$$

then there exists a trapping region or invariant set denoted as $\mathcal{A}_1$. From Proposition 1 we know that there exists fixed points in the subdomain $\zeta_1 \leq x \leq \zeta_2$ when

$$\frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1} + \sqrt{\zeta_2}}{\zeta_2 - \zeta_1}\right)^2 < \beta_2 \leq \frac{4}{\gamma(\zeta_2 - \zeta_1)^2}.$$

By the previous, notice that if

$$0 < \beta_2 < \frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1} + \sqrt{\zeta_2}}{\zeta_2 - \zeta_1}\right)^2,$$

then, there not exists fixed points in the subdomain $\zeta_1 \leq x \leq \zeta_2$. Therefore, any initial condition $x_0 \in [0, 1]$, the orbit converges to invariant set or trapping region $\mathcal{A}_1$ denoted in the Proposition 7. By the Proposition 6, we know that there exists a trapping region or an invariant set when

$$\frac{1}{\gamma}\left(\frac{\sqrt{\zeta_1}+\sqrt{\zeta_2}}{\zeta_2-\zeta_1}\right)^2 \leq \beta_2 \leq \frac{(\zeta_1+\zeta_2)+\sqrt{9(\zeta_1+\zeta_2)^2-32\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2}.$$

By the previous, if β_2 takes values within

$$\frac{(\zeta_1+\zeta_2)+\sqrt{9(\zeta_1+\zeta_2)^2-32\zeta_1\zeta_2}}{\gamma(\zeta_2-\zeta_1)^2} < \beta_2 \leq \frac{4}{\gamma(\zeta_2-\zeta_1)^2},$$

then, there not exists an invariant set or a trapping region in $\zeta_1 \leq x \leq \zeta_2$, this means that any initial condition $x_0 \in [0, 1]$ converges to unique invariant set or trapping region $\mathcal{A}_1$ in the interval $I = [0, 1]$. ■

In the next chapter, numerical results based on the findings from this chapter are presented. The numerical results cover three cases. In the first case, the parameters γ and β_1 are kept fixed, while $\beta_2 \in (0, 4]$ varies. In the second case, the parameters γ and β_2 remain fixed, while $\beta_1 \in (0, 4]$ varies. Finally, β_1 and β_2 are kept fixed and equal, while $\gamma \in (0, 4]$ varies. In each case, monostability and bistability are studied as the specified parameters change. The results presented include cobweb diagrams, as well as bifurcation diagrams for each case study.

Chapter 4

Families of bimodal maps

In this Chapter 4, we study three parametric families of bimodal maps by applying the propositions given in the previous Chapter 3. The parametric families of bimodal maps present the transition from monostability to bistability and vice versa. Additionally, in each case study, the corresponding bifurcation diagram is presented for the studied family.

4.1 The monoparametric family of maps for the parameter β_2

Below are the assumptions that must be taken into account to generate a monoparametric family of bimodal maps based on the system (3.1) for this case study:

- A regular partition of the interval $I = [0, 1] \subset \mathbb{R}$ given by the set of points $\left\{\zeta_0 = 0, \zeta_1 = \dfrac{1}{2}, \zeta_2 = 1\right\}$.
- The parameters $\gamma = 4$ and $\beta_1 = 2$.
- The parameter $\beta_2 \in (0, 4] \in \mathbb{R}$.

Under the previous assumptions, a monoparametric family of bimodal maps (3.1) is defined as

$$f_D(x,\beta_2) = 4\begin{cases} 2x\left(\frac{1}{2}-x\right) & \text{for } 0 \le x < \frac{1}{2}, \\ \\ \beta_2\left(x-\frac{1}{2}\right)(1-x) & \text{for } \frac{1}{2} \le x \le 1. \end{cases} \tag{4.1}$$

According to equation (3.98) of the Proposition 7, the system (4.1) presents an invariant set $\mathcal{A}_1$ when $\beta_1 = \dfrac{4}{\gamma\zeta_1} = 2$. Based on the previous result, the following proposition emerges.

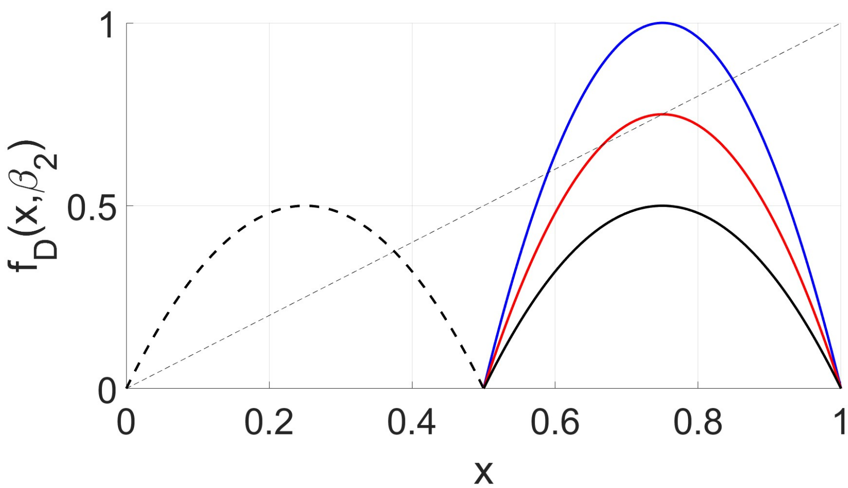


Figure 4.1: Bimodal maps of the monoparametric family f_D (4.1) for different values of β_2. The black line corresponds to $\beta_2 = 2$, the red line $\beta_2 = 3$, and finally, the blue one $\beta_2 = 4$.

Proposition 10 *The set* $\mathcal{A}_1 = \left[0,\ \frac{1}{2}\right) \subset I \subset \mathbb{R}$ *is the invariant set for* $f_D(x,\beta_2)$ *given by* (4.1) *when* $\beta_1 = 2$.

Proof. For this proof is enough to demonstrate that when the parameter $\beta_1 = 2$ satisfies the invariant set definition, showed in Definition 34. We notice that f_{D_1} has the following form

$$f_{D_1}(x,\gamma=4,\beta_1=2) = 8x\left(\frac{1}{2}-x\right).$$

By deriving the previous expression, we have

$$Df_{D_1}(x) = 4(1 - 4x)$$

If we set $Df_{D_1}(x) = 0$, we obtain the critical value. Then, following the previous procedure, we find that the critical value x_{c_1} is located at $x_{c_1} = \frac{1}{4}$. Notice that $x_{c_1} = x_{m_1}$. Also, notice that the function f_{D_1} is monotonically increasing for $0 \leq x < x_{c_1}$ and monotonically decreasing for $x_{c_1} < x < \frac{1}{2}$. Therefore, there exists a maximum critical point at $f_{D_1}(x_{c_1}) = f_{D_1}\left(\frac{1}{4}\right) = \frac{1}{2}$. Also, we have that $f_{D_1}\left(\frac{1}{2}\right) = f_{D_1}(0) = 0$, indicating that $f_{D_1}(0) < f_{D_1}\left(\frac{1}{4}\right)$ and $f_{D_1}\left(\frac{1}{2}\right) < f_{D_1}\left(\frac{1}{4}\right)$. Therefore, by continuity of the function $f_{D_1}(x)$ and also, by the result that when $f_{D_1}(x_{c_1}) = f_{D_1}\left(\frac{1}{4}\right) = \frac{1}{2}$, which is the upper bound of $\mathcal{A}_1$, satisfying $f_{D_1}(\mathcal{A}_1) = \mathcal{A}_1$, we can conclude that $\mathcal{A}_1 = \left[0, \frac{1}{2}\right)$ is an invariant set when $\beta_1 = 2$. ■

According to equation (3.5) of the Proposition 1, we know that the bimodal maps of the monoparametric family given by (4.1) have fixed points in the interval $\frac{1}{2} \leq x \leq 1$, if the parameter $\beta_2 \in \left(\frac{3}{2} + \sqrt{2}, 4\right] \subset \mathbb{R}$. Then the maps of the monoparametric family (4.1) have not fixed points in the interval $\frac{1}{2} \leq x \leq 1$, if the parameter $\beta_2 \in \left(0, \frac{3}{2} + \sqrt{2}\right) \subset \mathbb{R}$.

Proposition 11 *The monoparametric family of maps given by* (4.1) *presents monostability when* $\beta_2 \in \left(0, \frac{3}{2} + \sqrt{2}\right)$.

Proof. Graphically, notice that the graph of f_{D_2} is below the identity function when $\beta_2 \in \left(0, \frac{3}{2} + \sqrt{2}\right)$, analytically, if $\beta_2 \in \left(0, \frac{3}{2} + \sqrt{2}\right)$ then the discriminant Δ of the expression for the fixed points showed in (3.8) satisfies $\Delta < 0$ and therefore, have not fixed points in the real numbers field.

According to previous aforementioned comments, for every initial condition $x_0 \in \left[\frac{1}{2}, 1\right]$, the orbit $\mathcal{O}(x_0, f_D) = \{x_0, x_1, x_2, \ldots\}$ enters the interval $0 \leq x < \frac{1}{2}$ for a few iterations. Since there is only one invariant set $\mathcal{A}_1 = \left[0,\, \frac{1}{2}\right)$ then the only possibility for orbits with initial condition $x_0 \in I$ is to converge or remain in $\mathcal{A}_1$. ■

For $\beta_2 = \frac{3}{2} + \sqrt{2}$, the bimodal map has only one fixed point in the interval $\frac{1}{2} \leq x \leq 1$ at $x_2^* = \frac{\sqrt{2}}{2}$, given that when $\beta_2 = \frac{3}{2} + \sqrt{2}$, the discriminant Δ of the expression for the fixed points for f_{D_2} expressed in (3.8) is $\Delta = 0$.

Figure 4.1 shows maps of the monoparametric family (4.1) for different values of $\beta_2 \in \left(\frac{3}{2} + \sqrt{2}, 4\right]$. These maps have fixed points at the intersection between the unimodal map f_{D_2} and the identity function $f(x) = x$. In this case, the Propositions 6 let us compute a second trapping region or an invariant set $\mathcal{A}_2 \subset \left[\frac{1}{2},\ 1\right]$ of the monoparametric family (4.1), the result is:

$$\mathcal{A}_2(\beta_2) = \left[\frac{6\beta_2 - 1 - \sqrt{(6\beta_2 - 1)^2 - 32\beta_2^2}}{8\beta_2}, \frac{6\beta_2 + 1 + \sqrt{(6\beta_2 - 1)^2 - 32\beta_2^2}}{8\beta_2}\right]. \tag{4.2}$$

A trapping region or an invariant set $\mathcal{A}_2(\beta_2)$ exists if $\frac{3}{2} + \sqrt{2} < \beta_2 \leq \frac{3 + \sqrt{17}}{2}$. Note that $\mathcal{A}_2$ depends on the parameter β_2; thus, we begin conducting the relevant simulations to observe how the bistability occurs.

Proposition 12 *The trapping region* $\mathcal{A}_2(3) = \left[\frac{2}{3}, \frac{5}{6}\right]$ *contains an invariant point at* $x_{2,1}^* = \frac{3}{4}$ *of the bimodal map of monoparametric family* (4.1) *for* $\beta_2 = 3$.

Proof. For this proof is enough to demonstrate that within the trapping region $\mathcal{A}_2(3)$, there exist two fixed point $x_{2,1}^*$ and $x_{2,2}^*$ denoted by (3.14) and (3.15) respectively, where the fixed point $x_{2,1}^*$ is an attractive fixed point and therefore, any initial condition $x_0 \in \mathcal{A}_2(3)$ converges to the attractive fixed point $x^*_{2,1}$. From Proposition 1, we know that the bimodal map (4.1) has two fixed points located at $x^*_{2,2} = \dfrac{2}{3}$ and $x^*_{2,1} = \dfrac{3}{4}$ for $\beta_2 = 3$, given that this value of β_2 belongs to the interval showed in (3.5). The equation for the unimodal map in the interval $\dfrac{1}{2} \leq x \leq 1$ is

$$f_{D_2}(x, \gamma = 4, \beta_2 = 3) = 12\left(x - \frac{1}{2}\right)(1 - x),$$

therefore, by deriving the previous expression

$$Df_{D_2}(x) = -6(4x - 3), \tag{4.3}$$

applying the Definition 24 for the fixed point $x^*_{2,2}$ substituting into (4.3) we have that

$$\left|Df_{D_2}(x^*_{2,2})\right| = \left|-6\left(4\left(\frac{2}{3}\right) - 3\right)\right| = |2| > 1,$$

therefore, the fixed point $x^*_{2,2}$ is a repulsive fixed point. We doing the same procedure for the fixed point $x^*_{2,1}$ and it results

$$\left|Df_{D_2}(x^*_{2,1})\right| = \left|-6\left(4\left(\frac{3}{4}\right) - 3\right)\right| = 0 < 1.$$

Therefore, the fixed point $x^*_{2,1}$ is an attractive fixed point. By the previous results, we can state that the former is a repulsive fixed point and the second is an attracting fixed point $x^*_{2,1} = \dfrac{3}{4}$ that fulfills the conditions of Proposition 3. We know that the map (4.1) is a unimodal map in the interval $x \in \left[\dfrac{2}{3}, \dfrac{5}{6}\right]$ and has a critical value at $x_{c_2} = \dfrac{3}{4}$ (notice that $x_{c_2} = x_{m_2}$) given that if we set $Df_{D_2}(x) = 0$ we find that the solution that satisfies the $Df_{D_2}(x) = 0$ is $x_{c_2} = \dfrac{3}{4}$. Also, the map is monotonically increasing at $\dfrac{2}{3} \leq x < x_{c_2}$ and monotonically decreasing at $x_{c_2} < x \leq \dfrac{5}{6}$ and there exists a maximum critical point at $f_{D_2}(x_c) = f_{D_2}\left(\dfrac{3}{4}\right) = \dfrac{3}{4}$. Analogously, by the previous, we have that $f_D\left(\dfrac{3}{4}\right) = \dfrac{3}{4}$, and $f_D\left(\dfrac{2}{3}\right) = f_D\left(\dfrac{5}{6}\right) = \dfrac{2}{3}$, such that

$f_D\left(\frac{2}{3}\right) < f_D\left(\frac{3}{4}\right)$ and $f_D\left(\frac{5}{6}\right) < f_D\left(\frac{3}{4}\right)$, then for continuity the function $f_D(x_i) : \left[\frac{2}{3}, \frac{5}{6}\right] \to \left[\frac{2}{3}, \frac{5}{6}\right]$. As $x^*_{2,1} = \frac{3}{4}$ is an attracting point then any initial condition $x_0 \in \left(\frac{2}{3}, \frac{5}{6}\right)$ converges to $x^*_{2,1} = \frac{3}{4}$. ■

Figures 4.2 and 4.3 show the orbit for $x_0 = 0.68$ and we can see how the orbit converges to the fixed point $x^*_{2,1} = 0.75$, while $x^*_{2,2}$ is a repulsive fixed point according to Proposition 2. Figure 4.4 shows the orbit of an initial condition $x_0 \notin \mathcal{A}_2(3)$, after a certain number of iterations, the dynamics enters to the invariant set $\mathcal{A}_1$. Now, we consider a new value of $\beta_2 = 3.3 \in \left[\frac{3}{2} + \sqrt{2}, \frac{3+\sqrt{17}}{2}\right] \subset \mathbb{R}$ and its corresponding trapping region is $\mathcal{A}_2(3.3) = \left[\frac{47-\sqrt{31}}{66}, \frac{52+\sqrt{31}}{66}\right] \subset \mathbb{R}$. Figures 4.5 and 4.6 shows the dynamics of f_D in which we notice that the orbit remains within $\mathcal{A}_2(3.3)$ indicating bistability. Furthermore, the period-two orbit is exhibited.

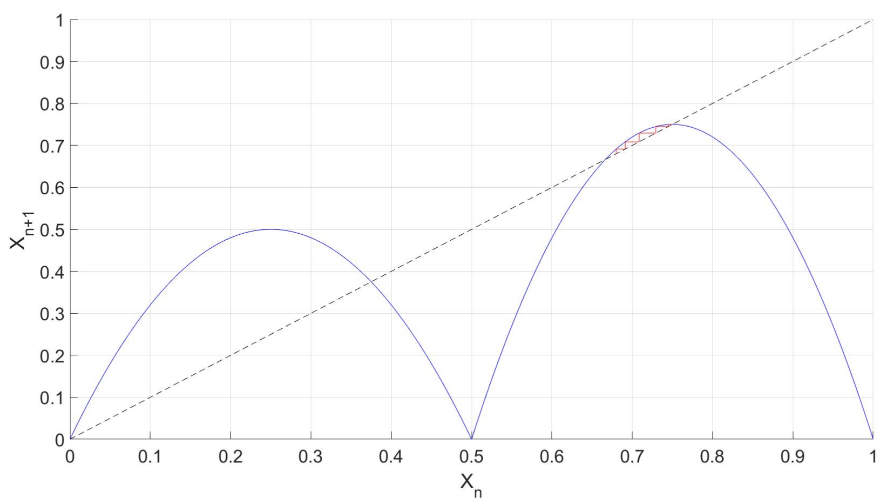


Figure 4.2: Cobweb diagram for the bimodal map f_D with $\beta_2 = 3$ using the initial condition $x_0 = 0.68 \in \mathcal{A}_2(3)$.

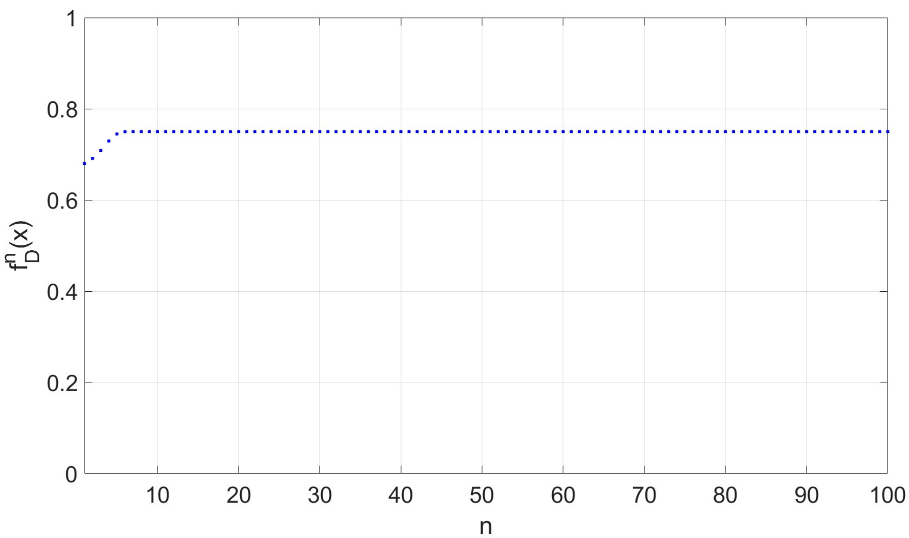


Figure 4.3: Orbit of the bimodal map f_D with $\beta_2 = 3$ using the initial condition $x_0 = 0.68 \in \mathcal{A}_2(3)$, where it's possible to observe the convergence of the trajectory towards the attractor fixed point $x^*_{2,1} = \dfrac{3}{4}$.

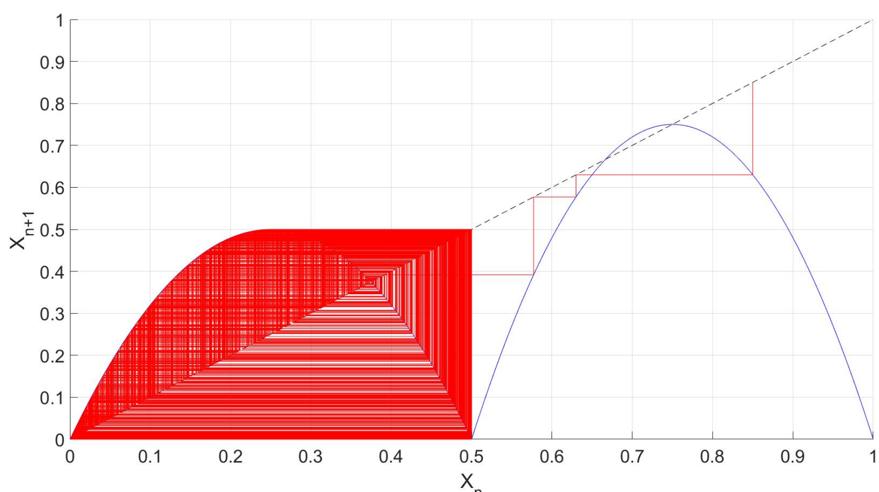


Figure 4.4: Cobweb diagram for the bimodal map f_D with $\beta_2 = 3$ with initial condition $x_0 = 0.85$ such that $x_0 \notin \mathcal{A}_2(3)$.

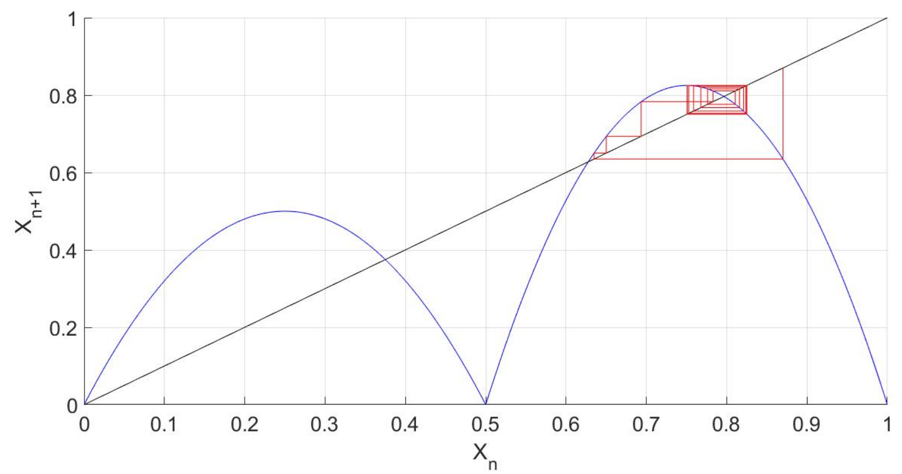


Figure 4.5: Cobweb diagram for the bimodal map f_D with $\beta_2 = 3.3$ using different initial conditions $x_0 \in \mathcal{A}_2(3.3)$; $x_0 = 0.87$.

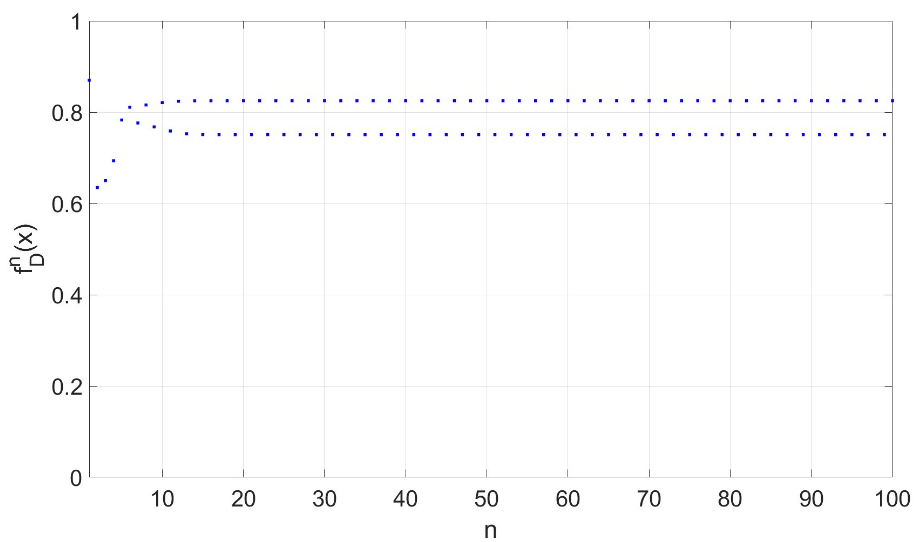


Figure 4.6: Orbit of the bimodal map f_D with $\beta_2 = 3.3$ using the initial condition $x_0 \in \mathcal{A}_2(3.3)$; $x_0 = 0.87$, where it's possible to observe a period-two orbit.

Proposition 13 *The trapping region* $\mathcal{A}_2(3.3) = \left[\dfrac{47-\sqrt{31}}{66}, \dfrac{52+\sqrt{31}}{66}\right] \subset \mathbb{R}$ *contains attracting periodic orbit* $x_1 = \dfrac{52+\sqrt{6}}{66}$ *and* $x_2 = \dfrac{52-\sqrt{6}}{66}$ *of the bimodal map of monoparametric family* (4.1) *for* $\beta_2 = 3.3$.

Proof. For this proof, we utilize the result obtained from Proposition 5. We know that the period-doubling bifurcation occurs when β_2 satisfies (3.49), specifically when $\beta_2 = \dfrac{3}{2}+\sqrt{3}$. Furthermore, the period-two points defined in (3.57) and (3.58) satisfy $x_1 > x_2$ when $\beta_2 > \dfrac{3}{2}+\sqrt{3}$. For the trapping region $\mathcal{A}_2(3.3)$ defined above, the value of the parameter β_2 is set to $\beta_2 = 3.3$, which satisfies $3.3 > \dfrac{3}{2}+\sqrt{3}$. Therefore, based on Proposition 5, we conclude that the period-two points are $x_1 = \dfrac{52+\sqrt{6}}{66}$ and $x_2 = \dfrac{52-\sqrt{6}}{66}$. To demonstrate whether this periodic orbit is attracting, it is sufficient to show that the following condition is fulfilled:

$$|Df_{D_2}(x_1)||Df_{D_2}(x_2)| < 1.$$

Here

$$f_{D_2}(x, \gamma = 4, \beta_2 = 3.3) = 13.2\left(x - \frac{1}{2}\right)(1-x),$$

and therefore, their derivative

$$Df_{D_2}(x) = -26.4x + 19.8.$$

Applying the previous condition, we have:

$$|Df_{D_2}(x_1)||Df_{D_2}(x_2)| = \left|\frac{1}{25}\right| < 1.$$

Hence, the periodic orbit is attracting. ■

Taking a new value of $\beta_2 = \dfrac{3+\sqrt{17}}{2}$, its respective invariant set is $\mathcal{A}_2\left(\dfrac{3+\sqrt{17}}{2}\right) = \left[\dfrac{9-\sqrt{17}}{8}, \dfrac{3+\sqrt{17}}{8}\right] \subset \mathbb{R}$. In Figure 4.7, we can observe the dynamics of f_D for initial conditions $x_0 = 0.61 \in \mathcal{A}_2\left(\dfrac{3+\sqrt{17}}{2}\right)$, where we notice that the orbit remains within the invariant set $\mathcal{A}_2\left(\dfrac{3+\sqrt{17}}{2}\right)$, giving rise to bistability.

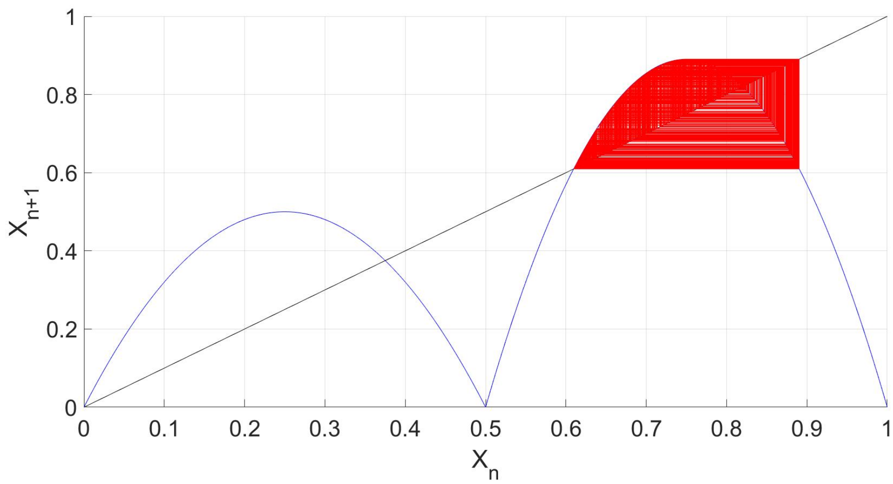


Figure 4.7: Cobweb diagram for the bimodal map f_D with $\beta_2 = \dfrac{3+\sqrt{17}}{2}$ using the initial condition $x_0 = 0.61 \in \mathcal{A}_2\left(\dfrac{3+\sqrt{17}}{2}\right)$.

Proposition 14 $\mathcal{A}_2\left(\dfrac{3+\sqrt{17}}{2}\right) = \left[\dfrac{9-\sqrt{17}}{8}, \dfrac{3+\sqrt{17}}{8}\right] \subset \mathbb{R}$ *is an invariant set of the bimodal map of monoparametric family* (4.1) *for* $\beta_2 = \dfrac{3+\sqrt{17}}{2}$.

Proposition 15 *There exist two invariant sets* $\mathcal{A}_1 = \left[0, \dfrac{1}{2}\right)$ *and* $\mathcal{A}_2\left(\dfrac{3+\sqrt{17}}{2}\right) = \left[\dfrac{9-\sqrt{17}}{8}, \dfrac{3+\sqrt{17}}{8}\right]$ *of the bimodal map of monoparametric family* (4.1) *for* $\beta_2 = \dfrac{3+\sqrt{17}}{2}$ *then the system presents bistability.*

Finally, the last case is when $\beta_2 = 3.57 \in \left(\frac{3+\sqrt{17}}{2}, 4\right]$, for this realization its corresponding set $\mathcal{A}_2(3.57) = \left[\frac{1021-\sqrt{22849}}{1428}, \frac{1121+\sqrt{22849}}{1428}\right] \subset \mathbb{R}$ is not an invariant set neither a trapping region, then bistability disappears. Every orbit $x_0 \in \mathcal{A}_2(3.57)$ converges to the invariant set $\mathcal{A}_1$. In Figure 4.8, we can observe the dynamics of the bimodal map f_D with $x_0 \in \mathcal{A}_2(3.57)$.

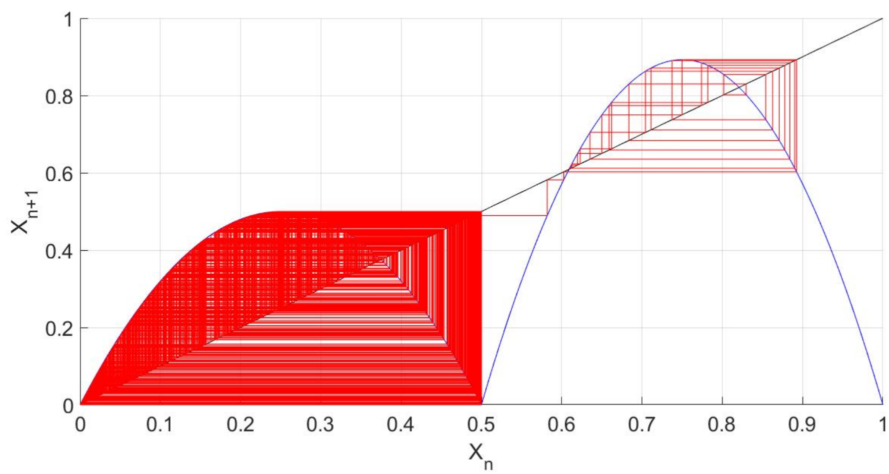


Figure 4.8: Cobweb diagram for the bimodal map f_D with $\beta_2 = 3.57$ using the initial condition $x_0 = 0.61 \in \mathcal{A}_2(3.57)$.

With the previous results, it can be verified that indeed, the conditions provided in the propositions presented in Chapter 3 work to generate bistability in the bimodal map f_{D_2} as presented in equation (3.1). Particularly, we have studied bistability in the monoparametric family of bimodal maps (4.1) when $\beta_2 \in \left[\frac{3}{2}+\sqrt{2}, \frac{3+\sqrt{17}}{2}\right]$, and other case, the system (4.1) presents monostability. The bifurcation diagram for the parameter $\beta_2 \in [0,4]$ is shown in Figure 4.9. Below we briefly describe the different behaviors observe in the monoparametric family of multimodal maps given by (4.1).

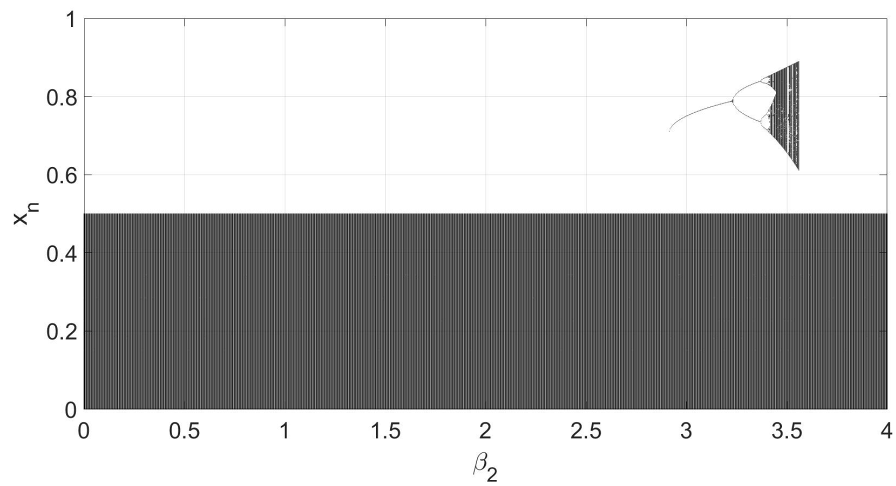


Figure 4.9: Bifurcation diagram of the bimodal map f_D given by (4.1) with respect to the parameters β_1 and β_2, where $\beta_1 = 2$ and $\beta_2 \in (0, 4] \subset \mathbb{R}$.

- For $\beta_2 \in \left[0, \dfrac{3}{2} + \sqrt{2}\right)$, the maps presents two fixed points in the interval $0 \leq x < \dfrac{1}{2}$ and none in the interval $\dfrac{1}{2} \leq x \leq 1$. Every orbit $x_0 \in [0, 1]$ finally oscillates in the invariant set $\mathcal{A}_1 = \left[0, \dfrac{1}{2}\right)$, then the family of maps exhibits a monostable behavior.

- For $\beta_2 = \dfrac{3}{2} + \sqrt{2}$, the family presents a tangent bifurcation and the system presents two fixed points in the interval $0 \leq x < \dfrac{1}{2}$ and one in the interval $\dfrac{1}{2} \leq x \leq 1$. The map exhibits a monostable behavior and every orbit $x_0 \in [0, 1]$ converges to the invariant set $\mathcal{A}_1 = \left[0, \dfrac{1}{2}\right)$.

- For $\beta_2 \in \left(\dfrac{3}{2} + \sqrt{2}, \dfrac{3}{2} + \sqrt{3}\right)$, the family of maps presents two fixed points in the interval $0 \leq x < \dfrac{1}{2}$ and two in the interval $\dfrac{1}{2} \leq x \leq 1$. The map exhibits a bistable behavior and every orbit $x_0 \in [0, 1]$ converges to the invariant set $\mathcal{A}_1 = \left[0, \frac{1}{2}\right)$ or belongs to the trapping region $\mathcal{A}_2(\beta_2)$ that contains a attracting fixed point.

- For $\beta_2 = \frac{3}{2} + \sqrt{3}$, the family presents a period-doubling bifurcation.

- For $\beta_2 \in \left(\frac{3}{2} + \sqrt{3}, \frac{3+\sqrt{17}}{2}\right]$, the family presents four repelling fixed points and at the beginning the bistability is generated by the trapping region $\mathcal{A}_2(\beta_2)$ and the invariant set $\mathcal{A}_1$. Finally, the bistability is generated by two invariant sets $\mathcal{A}_1$ and $\mathcal{A}_2(\beta_2)$ when $\beta_2 = \frac{3+\sqrt{17}}{2}$.

- For $\beta_2 \in \left(\frac{3+\sqrt{17}}{2}, 4\right]$, the family presents monostability again in the interval $0 \leq x < \frac{1}{2}$.

From the previous numerical results of this study cases, we can observe that any initial condition $x_0 \in [\zeta_1, \zeta_2] \backslash \mathcal{A}_2(\beta_2)$ converges to the invariant set $\mathcal{A}_1$ within the interval $\zeta_0 \leq x < \zeta_1$. Therefore, there exists a basin of attraction in $x \in [\zeta_1, \zeta_2] \backslash \mathcal{A}_2(\beta_2) \subset [0, 1]$ and an attractor in $\mathcal{A}_1$ for any $\beta_2 \in (0, 4] \subset \mathbb{R}$.

4.2 The monoparametric family of maps for the parameter β_1

The assumptions taken into account to generate a monoparametric family of bimodal maps based on the system (3.1) are as follows:

- A regular partition of the interval $I = [0, 1]$ given by the set of points $\{\zeta_0 = 0, \zeta_1 = \frac{1}{2}, \zeta_2 = 1\}$.

- The parameters $\gamma = 4$ and $\beta_2 = 3.56$ is chosen to ensure the existence of the trapping region $\mathcal{A}_2(\beta_2)$ in the subdomain $\zeta_1 \leq x \leq \zeta_2$, thus ensuring monostability and bistability..

- The parameter $\beta_1 \in (0, 4] \subset \mathbb{R}$.

Under the previous assumptions, a monoparametric family of bimodal maps (3.1) is defined as

$$f_D(x, \beta_1) = 4 \begin{cases} \beta_1 x \left(\frac{1}{2} - x\right) & \text{for } 0 \leq x < \frac{1}{2}, \\ \\ 3.56 \left(x - \frac{1}{2}\right)(1 - x) & \text{for } \frac{1}{2} \leq x \leq 1. \end{cases} \tag{4.4}$$

Figure 4.15 displays the graph of system (4.4) for different values of β_1: The red line represents $\beta_1 = 1$, the black line $\beta_1 = 2$, and finally, the pink one $\beta_1 = 4$. Considering the map (4.4), we proceed to study bistability and monostability.

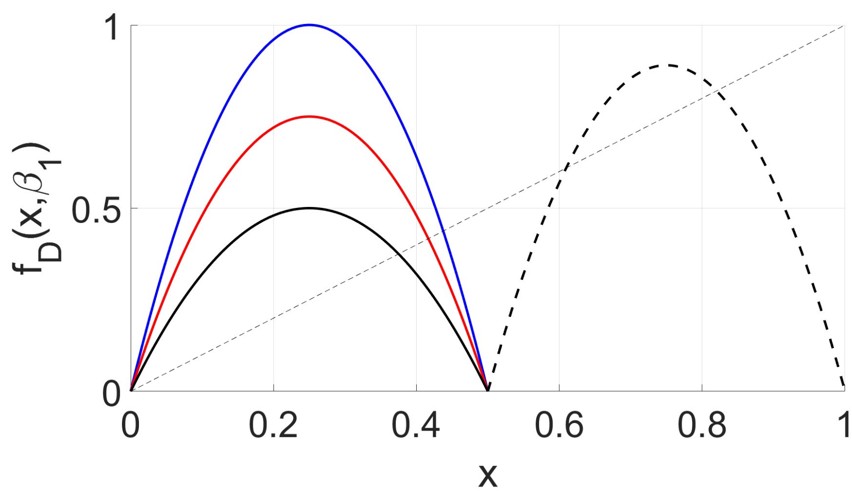


Figure 4.10: Bimodal map f_D (4.4) with $\gamma = 4$, $\beta_2 = 3.56$ and different values of β_1. The black line represents $\beta_1 = 2$, the red line $\beta_1 = 3$, and finally, the blue one $\beta_1 = 4$.

Proposition 16 *The system* (4.4) *has a trapping region* $\mathcal{A}_1 = \left[0, \frac{1}{2}\right) \subset \mathbb{R}$ *where all trajectories* $x_0 \in \mathcal{A}_1$ *converges to* $x^*_{1,1} = \frac{2\beta_1 - 1}{4\beta_1}$ *or* $x^*_{1,2} = 0$ *if*

$\beta_1 \in \left(0, \dfrac{3}{2}\right).$

Proof. For this proof, it's necessary to show that when $\beta_1 \in \left(0, \dfrac{3}{2}\right)$ there exists a trapping region $\mathcal{A}_1 = \left[0, \dfrac{1}{2}\right) \subset \mathbb{R}$ and additionally, we need proof that the fixed points $x^*_{1,1}$ and $x^*_{1,2}$ are attractive when β_1 takes the values previously mentioned. We proceed to calculate the critical point $x_{c_1} \in \mathcal{A}_1$ for the first modal. For this, we calculate the derivative of f_{D_1}, getting

$$Df_{D_1}(x) = 2\beta_1(1 - 4x).$$

By setting $Df_{D_1} = 0$ we have

$$2\beta_1(1 - 4x) = 0.$$

Since $\beta_1 > 0$, we have

$$1 - 4x = 0,$$

by solving the previous equation, we obtain that the critical point of the first modal located at $x_{c_1} = \dfrac{1}{4} \in \mathcal{A}_1$. Now, we defined the interval $I_0 = [0, x_c]$, $I_1 = \left[x_c, \dfrac{1}{2}\right)$, such that $\mathcal{A}_1 = I_0 \cup I_1$. Therefore, notice that

$$f_{D_1}(I_0, \beta_1) = f_{D_1}(I_1, \beta_1) = \left[0, \frac{3}{8}\right) \quad \forall \beta_1 \in \left(0, \frac{3}{2}\right).$$

By the previous, notice that we can conclude that

$$f_{D_1}(I_0 \cup I_1, \beta_1) = f_{D_1}(\mathcal{A}_1, \beta_1) = \left[0, \frac{3}{8}\right) \subset \mathcal{A}_1 \quad \forall \beta_1 \in \left(0, \frac{3}{2}\right).$$

Therefore, the orbit of each initial condition $x_0 \in \mathcal{A}_1$ remains in $\mathcal{A}_1$, i.e., $f(\mathcal{A}_1) \subset \mathcal{A}_1$ that is the definition of trapping region. Now, we proceed to show that one of the fixed points $x^*_{1,1}$ or $x^*_{1,2}$ is attractive when $0 < \beta_1 < \dfrac{3}{2}$. We know that the equation for the first modal is $f_{D_1}(x, \beta_1) = 4\beta_1 x \left(\dfrac{1}{2} - x\right)$

and their respective derivative is $Df_{D_1}(x, \beta_1) = 2\beta_1(1 - 4x)$. Now, we find the condition on β_1 such that

$$\left|Df_{D_1}(x^*_{1,n})\right| < 1 \quad \text{for } n = \{1, 2\}.$$

For the fixed point $x^*_{1,1}$, we have

$$\left|Df_{D_1}(x^*_{1,1})\right| = |2(\beta_1 - 1)| < 1,$$

analogously we have

$$-1 < 2(\beta_1 - 1) < 1.$$

By solving the previous inequality for β_1, we conclude that when $\frac{1}{2} < \beta_1 < \frac{3}{2}$, the fixed point $x^*_{1,1}$ is attractive. Performing the same procedure for the fixed point $x^*_{1,2}$, we have

$$\left|Df_{D_1}(x^*_{1,2})\right| = |2\beta_1| < 1.$$

Again, by solving the previous expression for β_1, we conclude that when $0 < \beta_1 < \frac{1}{2}$, the fixed point $x^*_{1,2}$ is attractive. Combining the two intervals for β_1 found previously, we conclude that if $0 < \beta_1 < \frac{3}{2}$, then all trajectories $x_0 \in \mathcal{A}_1$ converge to $x^*_{1,1}$ or $x^*_{1,2}$. ■

Proposition 17 *The system* (4.4) *has an invariant set* $\mathcal{A}_1 = \left[0, \frac{1}{2}\right) \subset \mathbb{R}$ *if* $\beta_1 = 2$.

Proof. This proof is similar to the previous one. For this proof it's necessary to show that when $\beta_1 = 2$ there exists an invariant set $\mathcal{A}_1 = \left[0, \frac{1}{2}\right) \subset \mathbb{R}$. We know from the proof of the Proposition 16 that the point $x_{c_1} \in \mathcal{A}_1$ is the critical point for the first modal. We proceed to state the following intervals which were mentioned previously: $I_0 = [0, x_c]$, $I_1 = \left[x_c, \frac{1}{2}\right)$, and we remember that these intervals satisfy $\mathcal{A}_1 = I_0 \cup I_1$. Therefore, notice that

$$f_{D_1}(I_0, \beta_1 = 2) = f_{D_1}(I_1, \beta_1 = 2) = \left[0, \frac{1}{2}\right).$$

Based on the above, we can conclude that

$$f_{D_1}(I_0 \cup I_1, \beta_1 = 2) = f_{D_1}(\mathcal{A}_1, \beta_1 = 2) = \left[0, \frac{1}{2}\right) = \mathcal{A}_1.$$

Notice that the previous result satisfies the definition of invariant set. Therefore, for any orbit $x_0 \in \mathcal{A}_1$ it remains in $\mathcal{A}_1$ and moreover, completely covers $\mathcal{A}_1$. ∎

Notice that when $\beta_1 = 2$, the system (4.4) has an invariant set $\mathcal{A}_1$ in the subdomain $\zeta_0 \leq x < \zeta_1$ and a trapping region $\mathcal{A}_2(\beta_2 = 3.56)$ within $\zeta_1 \leq x \leq \zeta_2$. Only the invariant set $\mathcal{A}_1$ has a basin of attraction given by $I \backslash \mathcal{A}_1(\beta = 2) \cup \mathcal{A}_2(\beta_1 = 3.56)$. Figures 4.11 and 4.12 show the cobweb diagram and their respective trajectories diagram for the bimodal map f_D with $\beta_1 = 2$ and $\beta_2 = 3.56$ using initial conditions, where one of them belongs to the invariant set $\mathcal{A}_1$ and the other belongs to trapping region $\mathcal{A}_2(3.56)$.

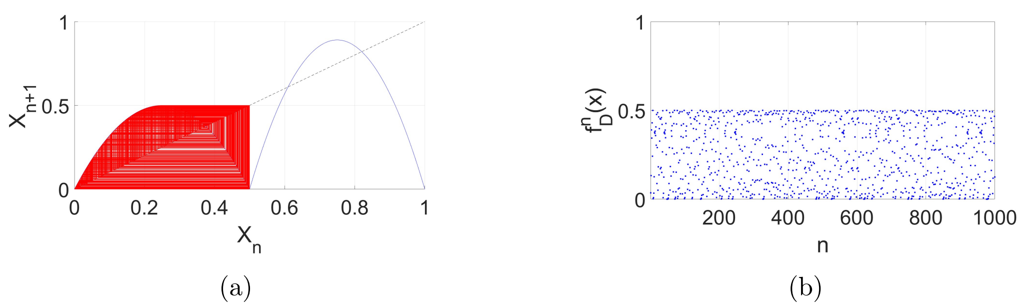
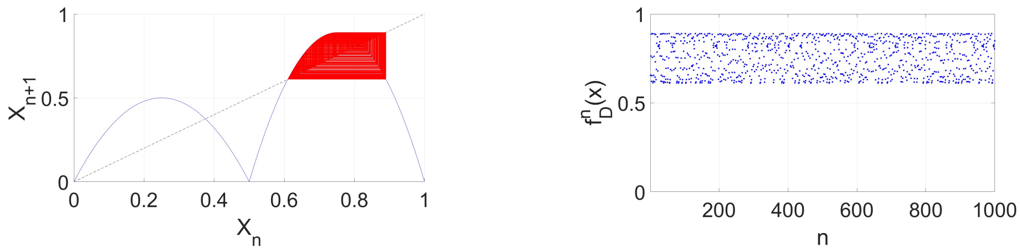


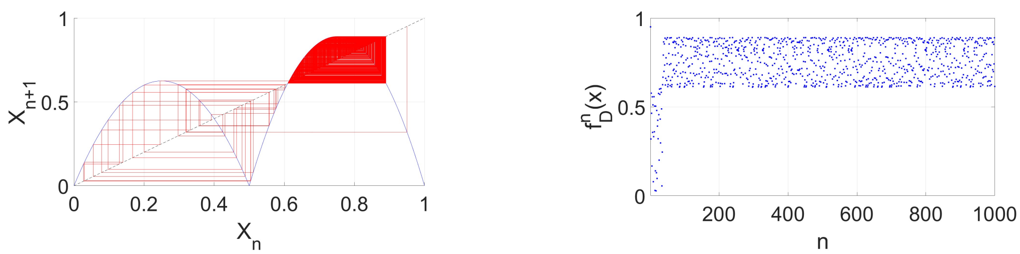
Figure 4.11: (a) Cobweb diagram and (b) orbit of the bimodal map f_D with $\beta_1 = 2$, $\beta_2 = 3.56$ and $x_0 = 0.0357 \in \mathcal{A}_1$.

If we consider Proposition 8, we know that when $\frac{3}{2} + \sqrt{3} \leq \beta_2 \leq \frac{3+\sqrt{17}}{2}$ and $4x^*_{2,2} < \beta_1 \leq 4$, monostability exists in the interval $\zeta_1 \leq x \leq \zeta_2$. Thus, taking $\beta_1 = 2.5$, we note that regardless of the initial condition, after a finite and small number of iterations, the orbit will eventually converge to the invariant set $\mathcal{A}_2(3.56)$. For this case, the invariant set $\mathcal{A}_2$ has a basin of attraction given by $I \backslash \mathcal{A}_2(3.56)$. Figure 4.13 shows the cobweb diagram for the bimodal map f_D with $\beta_1 = 2.5$ using an initial condition in the basin of attraction $x_0 = 0.9502 \notin \mathcal{A}_2(3.56)$.



(a) (b)

Figure 4.12: (a) Cobweb diagram and (b) orbit of the bimodal map f_D with $\beta_1 = 2$, $\beta_2 = 3.56$ and $x_0 = 0.6557 \in \mathcal{A}_2(3.56)$.



(a) (b)

Figure 4.13: (a) Cobweb diagram and (b) orbit of the the bimodal map f_D with $\beta_1 = 2.5$ and $\beta_2 = 3.56$ using the initial condition $x_0 = 0.9502 \notin \mathcal{A}_2(3.56)$.

The bifurcation diagram based on the behavior presented in this case study was constructed and is shown in Figure 4.14, in which we can observe that the behaviors in the previously presented cobweb diagram are related to this bifurcation diagram. Below we briefly describe the different behaviors observe in the monoparametric family of multimodals maps given by (4.4).

- For $\beta_1 \in \left(0, \dfrac{1}{2}\right)$ and $\beta_1 \in \left(\dfrac{1}{2}, 4\right]$ the fixed point $x^*_{1,2} = 0$ is attractive and repulsive, respectively.
- For $\beta_1 \in \left(\dfrac{1}{2}, \dfrac{3}{2}\right)$ and $\beta_1 \in \left(\dfrac{3}{2}, 4\right]$ the fixed point $x^*_{1,1} = \dfrac{2\beta_1 - 1}{4\beta_1}$ is attractive and repulsive, respectively.

- For $\beta_1 = \frac{3}{2}$, the family presents a period-doubling bifurcation.
- For $\beta_1 \in (0, 2)$ the family of bimodal maps (4.4) presents two trapping regions $\mathcal{A}_1 = \left[0, \frac{1}{2}\right)$ and $\mathcal{A}_2(3.56) \subset \left[\frac{1}{2}, 1\right]$, therefore the maps exhibit a bistable behavior and every orbit $x_0 \notin \mathcal{A}_2(3.56)$ converges or belongs to the trapping region $\mathcal{A}_1 = \left[0, \frac{1}{2}\right]$.
- For $\beta_1 = 2$, the map (4.4) exhibits a bistable behavior via an invariant set $\mathcal{A}_1 = [0, 0.5)$ and a trapping region $\mathcal{A}_2(3.56)$.
- The family of bimodal maps (4.4) presents monostability in $\zeta_1 \leq x \leq \zeta_2$ when

$$4x^*_{2,2} < \beta_1 \leq 4,$$

and

$$\frac{3}{2} + \sqrt{3} \leq \beta_2 \leq \frac{3 + \sqrt{17}}{2},$$

where $x^*_{2,2}$ is the fixed point of the subdomain $[\zeta_1, \zeta_2]$, denoted by (3.15).

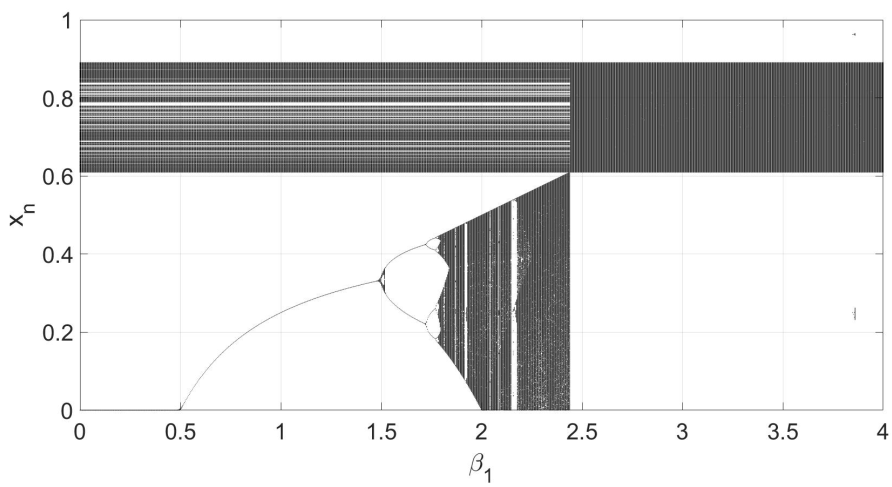


Figure 4.14: Bifurcation diagram of the bimodal map f_D given by (4.4) with respect to the parameters β_1 and β_2, where $\beta_2 = 3.56$ and $\beta_1 \in (0, 4]$.

4.3 The monoparametric family of maps for the parameter γ

The assumptions taken into account to generate a monoparametric family of bimodal maps based on the system (3.1) are as follows:

- A regular partition of the interval $[0, 1]$ given by the set of points $\{\zeta_0 = 0, \zeta_1 = \frac{1}{2}, \zeta_2 = 1\}$.
- The parameters $\beta_1 = \beta_2 = 4$.
- The parameter $\gamma \in (0, 4] \subset \mathbb{R}$.

Under the previous assumptions, a monoparametric family of bimodal maps (3.1) is defined as

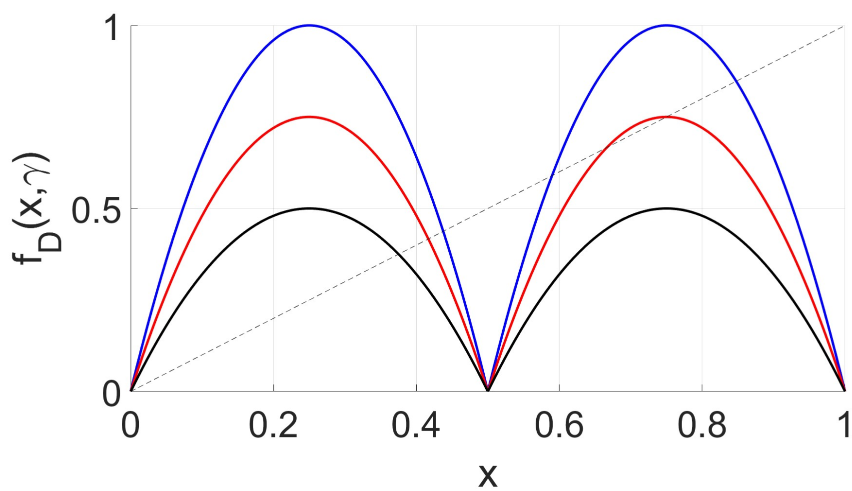


Figure 4.15: Bimodal map f_D (4.5) with $\gamma = 4$, $\beta_1 = \beta_2 = 4$ and different values of $\gamma \in (0, 4]$. The black line represents $\gamma = 2$, the red line $\gamma = 3$, and finally, the blue one $\gamma = 4$.

$$f_D(x,\gamma) = \gamma \begin{cases} 4x\left(\frac{1}{2} - x\right) & \text{for } 0 \le x < \frac{1}{2}, \\ 4\left(x - \frac{1}{2}\right)(1-x) & \text{for } \frac{1}{2} \le x \le 1. \end{cases} \tag{4.5}$$

We can state briefly the behavior for the fixed points when the parameter γ adopts specifically certain values.

- when $\gamma \in \left(0, \frac{1}{2}\right)$ the fixed point $x^*_{1,2}$ denoted by (3.110), is an invariant point, moreover, this fixed points is unique within I.
- When $\gamma \in \left(\frac{1}{2}, \frac{3}{2}\right)$ the fixed point $x^*_{1,2}$ is repulsive, and a new invariant appears, this invariant is the fixed point $x^*_{1,1}$ defined by (3.109).
- When $\gamma = \frac{3}{2}$ a period-doubling bifurcations occurs.

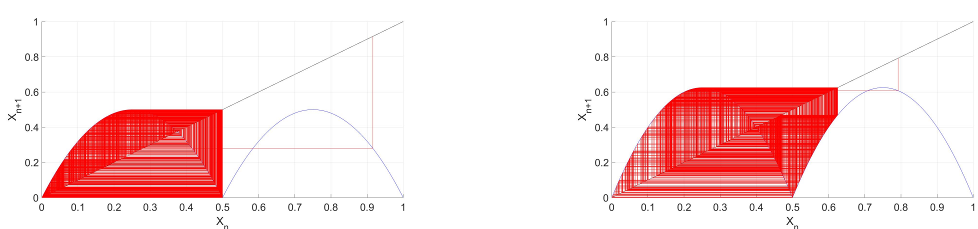
Proposition 18 *The bimodal map* (4.5) *exhibits monostability and an invariant set* $\mathcal{A} = \left[0, \frac{\gamma}{4}\right] \subset I$ *when* $\frac{1}{2} \le \frac{\gamma}{4} < \frac{3+2\sqrt{2}}{8}$.

Proof. For this proof, it's necessary to show that within of the interval $I = [0,1] \subset \mathbb{R}$ there exists an invariant set when the parameter $\gamma \in \left[2, \frac{3+2\sqrt{2}}{2}\right)$, also, we need proof that f_D exhibits monostability. For the first part of this proof, we show that exist an invariant set, for this, let us consider the following three intervals $I_0 = [0, x_{c_1}]$, $I_1 = \left[x_{c_1}, \frac{1}{2}\right]$ and $I_2 = \left[\frac{1}{2}, \frac{\gamma}{4}\right]$, where x_{c_1} is the critical point belongs to subdomain $0 \le x < \frac{1}{2}$, where $x_{c_1} = \frac{1}{4}$. Notice that the set $\mathcal{A} = I_0 \cup I_1 \cup I_2$.

If we take any $x \in \left[0, \frac{1}{2}\right)$ then the expression for the respective modal is $f_{D_1}(x) = 4\gamma x\left(\frac{1}{2} - x\right)$. By evaluating I_0 and I_1 in f_{D_1} we have the following results: $f_{D_1}(I_0) = \left[0, \frac{\gamma}{4}\right]$, $f_{D_1}(I_1) = \left[0, \frac{\gamma}{4}\right]$. Now, if we take any $x \in \left[\frac{1}{2}, 1\right]$ then their respective modal is given by $f_{D_2}(x) = 4\gamma\left(x - \frac{1}{2}\right)(1-x)$. Notice that interval I_2 belongs to the subdomain defined by $\frac{1}{2} \leq x \leq 1$ therefore, by evaluating I_2 in f_{D_2} we have $f_{D_2}(I_2) = \left[0, \frac{\gamma}{4}(\gamma-2)(4-\gamma)\right]$. We take the upper bound of the parameter γ in this case $\gamma = \frac{3+2\sqrt{2}}{2}$. We proceed to substituting in $f_{D_2}(I_2)$, where we obtain

$$\frac{\gamma}{4}(\gamma-2)(4-\gamma) = \left(3\sqrt{2} - \frac{13}{4}\right)\left(\frac{\gamma}{4}\right) < \frac{\gamma}{4}.$$

And as the upper bound of the set $\mathcal{A}$ is not reached by the second modal when $\gamma = \frac{3+2\sqrt{2}}{2}$, then, we conclude that $f(\mathcal{A}) = \mathcal{A}$. Therefore, $\mathcal{A}$ is an invariant set. To show monostability, notice that f_{D_2} has not fixed point. Therefore, for any initial condition $x_0 \in I\backslash\mathcal{A}$ the orbit converges to $\mathcal{A}$. Therefore, the bimodal map (4.5) presents monostability in the subdomain $0 \leq x \leq \frac{\gamma}{4}$ when $2 \leq \gamma < \frac{3+2\sqrt{2}}{2}$. ∎



(a) (b)

Figure 4.16: Cobweb diagram for the bimodal map f_D for: a) $\gamma = 2$ using the initial condition $x_0 = 0.9157$, and b) $\gamma = 2.5$ using the initial condition $x_0 = 0.7922$.

Figures 4.16 shows the Cobweb diagram for the bimodal map f_D for: a) $\gamma = 2$ using the initial conditions $x_0 = 0.9157$, and b) $\gamma = 2.5$ using the initial conditions $x_0 = 0.7922$. It is possible to see that the bimodal map (4.5) exhibits monostability in the invariant set $\mathcal{A} = \left[0, \frac{\gamma}{4}\right]$ for all $x_0 \in I$.

- If $\frac{3+2\sqrt{2}}{2} \leq \gamma < \frac{3+2\sqrt{3}}{2}$, there exists an invariant point denoted $x^*_{2,1}$ defined by (3.14), and at $\gamma = \frac{3+2\sqrt{3}}{2}$ a period-doubling bifurcations occurs.

Proposition 19 *If* $\gamma \in \left(\frac{3+2\sqrt{2}}{2}, \frac{3+\sqrt{17}}{2}\right)$ *then the system* (4.5) *has a trapping region* $\mathcal{A}(\gamma) \subset \left[\frac{1}{2}, 1\right] \subset \mathbb{R}$ *defined as*

$$\mathcal{A}(\gamma) = \left[\frac{6\gamma - 1 - \sqrt{(6\gamma-1)^2 - 32\gamma^2}}{8\gamma}, \frac{6\gamma + 1 + \sqrt{(6\gamma-1)^2 - 32\gamma^2}}{8\gamma}\right].$$

Proof. To demonstrate this proposition, we will find the value for the parameter γ such that $f^2_{D_2}(x_{c_2}, \gamma) > x^*_{2,2}$, with $x_{c_2} = \frac{\zeta_1 + \zeta_2}{2} \in [\zeta_1, \zeta_2]$ as the critical point of the subdomain $\frac{1}{2} \leq x \leq 1$. The previous condition arises from the definition of the invariant set. Now, we defined the following $r, a, b,$ and c such that $r = \gamma\beta_2$, $a = \zeta_1 + \zeta_2$, $b = \zeta_1\zeta_2$ and $c = a^4 - 4b$. Therefore, by the previous definitions, notice that x_{c_1} and $x^*_{2,2}$ results

$$x_{c_1} = \frac{a}{2}, \quad x^*_{2,2} = \frac{ra - 1 - \sqrt{(ra-1)^2 - 4r^2 b}}{2r}$$

Now, we proceed to solving

$$f^2_{D_2}(x_{c_2}, \gamma) > x^*_{2,2}.$$

We know that

$$f_{D_2}(x, \gamma, \beta_2) = -\gamma\beta_2[x^2 - (\zeta_1 + \zeta_2)x + \zeta_1\zeta_2] = -rx^2 + rax - rb,$$

by substituting x_{c_2} into the last expression and by simplifying, we obtain

$$f_{D_2}(x_{c_2}, r, a, b) = r\frac{(a^2 - 4b)}{4},$$

when performing $f_{D_2}(f_{D_2}(x_{c_2}, r, a, b)) = f^2_{D_2}(x_{c_2}, r, a, b)$, we have

$$f^2_{D_2}(x_{m_2}, r, a, b) = -r\left(r\frac{(a^2 - 4b)}{4}\right)^2 + ra\left(r\frac{(a^2 - 4b)}{4}\right) - rb,$$

since $c = a^4 - 4b$ then it results

$$f^2_{D_2}(x_{c_2}, r, a, b) = \frac{-r^3c^2 + 4r^2ac - 16rb}{16}.$$

Therefore, if we make $f^2_{D_2}(x_{c_2}, r, a, b) > x^*_{2,2}$ we obtain the following result

$$\frac{-r^3c^2 + 4r^2ac - 16rb}{16} < \frac{ra - 1 - \sqrt{(ra-1)^2 - 4r^2b}}{2r}.$$

By simplifying and factoring the previous inequality, the result is:

$$(cr^2 - 2ar - 8)(c^3r^3 - 6ac^2r^2 + 4c(8b + a^2 + 2c)r - 8a(2c + 8b - a^2)) < 0.$$

We take the quadratic factor $(cr^2 - 2ar - 8) < 0$ and solving for r, where we obtain the following:

$$r = \frac{2a \pm \sqrt{4a^2 + 32c}}{2c} = \frac{a \pm \sqrt{a^2 + 8c}}{c}.$$

By substituting $a = (\zeta_1 + \zeta_2)$, $b = \zeta_1\zeta_2$, $c = a^4 - 4b$ and $r = \gamma\beta_2$ and by simplifying, we have

$$\gamma\beta_2 = \frac{(\zeta_1 + \zeta_2) + \sqrt{9(\zeta_1 + \zeta_2)^2 - 32\zeta_1\zeta_2}}{(\zeta_2 - \zeta_1)^2}.$$

Analogously

$$\gamma = \frac{(\zeta_1 + \zeta_2) \pm \sqrt{9(\zeta_1 + \zeta_2)^2 - 32\zeta_1\zeta_2}}{\beta_2(\zeta_2 - \zeta_1)^2}.$$

Then, the solution for the inequality $(cr^2 - 2ar - 8) < 0$ is

$$\frac{(\zeta_1 + \zeta_2) - \sqrt{9(\zeta_1 + \zeta_2)^2 - 32\zeta_1\zeta_2}}{\beta_2(\zeta_2 - \zeta_1)^2} < \gamma < \frac{(\zeta_1 + \zeta_2) + \sqrt{9(\zeta_1 + \zeta_2)^2 - 32\zeta_1\zeta_2}}{\beta_2(\zeta_2 - \zeta_1)^2}.$$

However, notice that the lower bound satisfies that

$$\gamma = \frac{(\zeta_1+\zeta_2) - \sqrt{9(\zeta_1+\zeta_2)^2 - 32\zeta_1\zeta_2}}{\beta_2(\zeta_2-\zeta_1)^2} < 0,$$

also, from Proposition 1 we know that there exists fixed points in f_{D_2}, where

$$\gamma > \frac{1}{\beta_2}\left(\frac{\sqrt{\zeta_1}+\sqrt{\zeta_2}}{\zeta_2-\zeta_1}\right)^2.$$

Then, notice that

$$\gamma = \frac{(\zeta_1+\zeta_2) - \sqrt{9(\zeta_1+\zeta_2)^2 - 32\zeta_1\zeta_2}}{\beta_2(\zeta_2-\zeta_1)^2} < 0 < \frac{1}{\beta_2}\left(\frac{\sqrt{\zeta_1}+\sqrt{\zeta_2}}{\zeta_2-\zeta_1}\right)^2.$$

Therefore, the solution set for the inequality $cr^2 - 2ar - 8 < 0$ is

$$\frac{1}{\beta_2}\left(\frac{\sqrt{\zeta_1}+\sqrt{\zeta_2}}{\zeta_2-\zeta_1}\right)^2 < \gamma < \frac{(\zeta_1+\zeta_2) + \sqrt{9(\zeta_1+\zeta_2)^2 - 32\zeta_1\zeta_2}}{\beta_2(\zeta_2-\zeta_1)^2},$$

and by substituting $\beta_2 = 4$, $\zeta_1 = \dfrac{1}{2}$ and $\zeta_2 = 1$, we obtain the following interval

$$\frac{3+2\sqrt{2}}{2} < \gamma < \frac{3+\sqrt{17}}{2},$$

that which is the solution set that the parameter γ must take for $\mathcal{A}(\gamma)$ to be an invariant set. ■

Proposition 20 *If* $\gamma = \dfrac{3+\sqrt{17}}{2}$ *then the system* (4.5) *has an invariant set* $\mathcal{A}(\gamma) \subset \left[\dfrac{1}{2}, 1\right] \subset \mathbb{R}$ *defined as*

$$\mathcal{A}(\gamma) = \left[\frac{6\gamma - 1 - \sqrt{(6\gamma-1)^2 - 32\gamma^2}}{8\gamma}, \frac{6\gamma + 1 + \sqrt{(6\gamma-1)^2 - 32\gamma^2}}{8\gamma}\right].$$

Proof. This proof is similar to the previous one. To demonstrate this proposition, we will find the value for the parameter γ such that $f^2_{D_2}(x_{c_2}, \gamma) = x^*_{2,2}$, with $x_{c_2} = \dfrac{\zeta_1 + \zeta_2}{2} \in [\zeta_1, \zeta_2]$ as the critical point of the subdomain $\dfrac{1}{2} \leq x \leq 1$. The previous condition arises from the definition of the invariant set. Now, we defined the following parameters: $r = \gamma\beta_2$, $a = \zeta_1 + \zeta_2$, $b = \zeta_1\zeta_2$ and $c = a^4 - 4b$. Therefore, by the previous definitions, notice that x_{c_1} and $x^*_{2,2}$ results

$$x_{c_1} = \frac{a}{2} \quad \text{and} \quad x^*_{2,2} = \frac{ra - 1 - \sqrt{(ra-1)^2 - 4r^2 b}}{2r}.$$

Now, we proceed to solving

$$f^2_{D_2}(x_{c_2}, \gamma) = x^*_{2,2}.$$

We know that

$$f_{D_2}(x, \gamma, \beta_2) = -\gamma\beta_2[x^2 - (\zeta_1 + \zeta_2)x + \zeta_1\zeta_2] = -rx^2 + rax - rb.$$

By substituting x_{c_2} into the last expression and by simplifying, we have

$$f_{D_2}(x_{c_2}, r, a, b) = r\frac{(a^2 - 4b)}{4}.$$

When performing $f_{D_2}(f_{D_2}(x_{c_2}, r, a, b)) = f^2_{D_2}(x_{c_2}, r, a, b)$. We have

$$f^2_{D_2}(x_{m_2}, r, a, b) = -r\left(r\frac{(a^2 - 4b)}{4}\right)^2 + ra\left(r\frac{(a^2 - 4b)}{4}\right) - rb.$$

Since $c = a^4 - 4b$ then it results

$$f^2_{D_2}(x_{c_2}, r, a, b) = \frac{-r^3c^2 + 4r^2ac - 16rb}{16}.$$

Therefore, if we make $f^2_{D_2}(x_{c_2}, r, a, b) = x^*_{2,2}$ we obtain the following result

$$\frac{-r^3c^2 + 4r^2ac - 16rb}{16} = \frac{ra - 1 - \sqrt{(ra-1)^2 - 4r^2 b}}{2r}.$$

By simplifying and factoring the previous inequality, the result is:

$$(cr^2 - 2ar - 8)(c^3r^3 - 6ac^2r^2 + 4c(8b + a^2 + 2c)r - 8a(2c + 8b - a^2)) = 0.$$

We take the quadratic factor $(cr^2 - 2ar - 8) = 0$ and solving for r, we obtain:

$$r = \frac{2a \pm \sqrt{4a^2 + 32c}}{2c} = \frac{a \pm \sqrt{a^2 + 8c}}{c}.$$

By substituting $a = \zeta_1 + \zeta_2$, $b = \zeta_1\zeta_2$, $c = a^4 - 4b$ and $r = \gamma\beta_2$ and by simplifying, we have

$$\gamma\beta_2 = \frac{(\zeta_1 + \zeta_2) + \sqrt{9(\zeta_1 + \zeta_2)^2 - 32\zeta_1\zeta_2}}{(\zeta_2 - \zeta_1)^2}.$$

Analogously

$$\gamma = \frac{(\zeta_1 + \zeta_2) \pm \sqrt{9(\zeta_1 + \zeta_2)^2 - 32\zeta_1\zeta_2}}{\beta_2(\zeta_2 - \zeta_1)^2}.$$

Notice that the solution

$$\gamma = \frac{(\zeta_1 + \zeta_2) - \sqrt{9(\zeta_1 + \zeta_2)^2 - 32\zeta_1\zeta_2}}{\beta_2(\zeta_2 - \zeta_1)^2} < 0.$$

Therefore, do not satisfies that $\gamma \in (0, 4] \in \mathbb{R}$. Therefore, we take

$$\gamma = \frac{(\zeta_1 + \zeta_2) + \sqrt{9(\zeta_1 + \zeta_2)^2 - 32\zeta_1\zeta_2}}{\beta_2(\zeta_2 - \zeta_1)^2},$$

and by substituting $\beta_2 = 4$, $\zeta_1 = \dfrac{1}{2}$ and $\zeta_2 = 1$, we obtain $\gamma = \dfrac{3 + \sqrt{17}}{2}$, which is the value that γ must take for $\mathcal{A}(\gamma)$ to be an invariant set. ■

Proposition 21 . *The bimodal map* (4.5) *exhibits monostabilty in the subdomain* $\zeta_1 \leq x \leq \zeta_2$ *when* $\gamma \in \left(\dfrac{3}{2} + \sqrt{2}, \dfrac{3 + \sqrt{17}}{2} \right]$.

Proof. For this proof, we will proceed to take results from the previous Propositions. From Proposition 18 we know that the bimodal map do not have monostability in $0 \leq x \leq \dfrac{\gamma}{4}$ when $\gamma \geq \dfrac{3 + 2\sqrt{2}}{2}$. Moreover, we know from the Proposition 19 that there exists a trapping region $\mathcal{A}(\gamma)$ belongs to

$\zeta_1 \leq x \leq \zeta_2$ when $\dfrac{3+2\sqrt{2}}{2} < \gamma < \dfrac{3+\sqrt{17}}{2}$ and finally, from Proposition 20 we know that when $\gamma = \dfrac{3+\sqrt{17}}{2}$ the bimodal map has an invariant set $\mathcal{A}(\gamma)$. Therefore, since $\gamma \in \left(\dfrac{3}{2}+\sqrt{2}, \dfrac{3+\sqrt{17}}{2}\right]$ then, there exists a trapping region or an invariant set $\mathcal{A}(\gamma)$ allowing to the bimodal map has monostability in $\zeta_1 \leq x \leq \zeta_2$. ■

Since the map f_{D_1} no longer satisfies Proposition 18 but does satisfy Proposition 21, for any initial condition $x_0 \in [0,1]$, the orbit either converges to or belongs to the trapping region $\mathcal{A}(\gamma)$, as defined by (4.2), within the subdomain $\zeta_1 \leq x \leq \zeta_2$, resulting in monostability. From Figure 4.17, we can see that when $\gamma = \dfrac{3+\sqrt{17}}{2}$, there exists monostability in $I = [0,1]$ satisfying Proposition 20 and, consequently, Proposition 21.

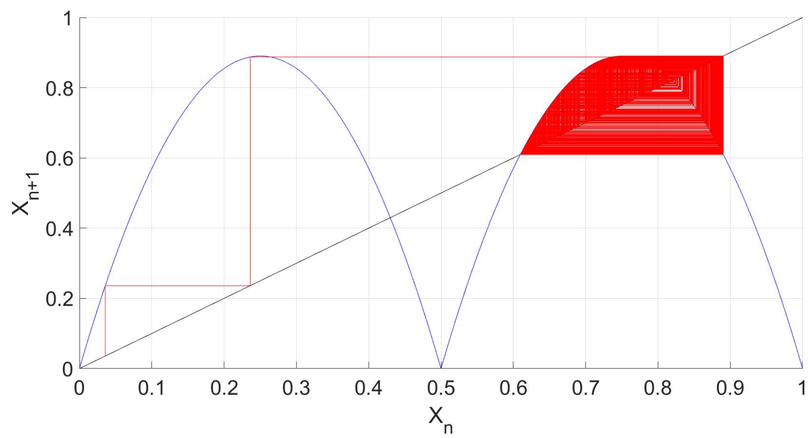


Figure 4.17: Cobweb diagram for the bimodal map f_D with $\gamma = \dfrac{3+\sqrt{17}}{2}$ using the initial condition $x_0 = 0.0357 \in [0,1]$.

Proposition 22 *If* $\gamma \in (0,4) \in \mathbb{R}$ *then the system* (4.5) *has a trapping region* $\mathcal{A} = I = [0,1] \subset \mathbb{R}$.

Proof. To demonstrate this proposition, we need to show that there exists a trapping region if $\gamma \in (0,4) \in \mathbb{R}$. For reach this goal, we consider the following four intervals $I_0 = [0, x_{c_1}]$, $I_1 = \left[x_{c_1}, \frac{1}{2}\right]$, $I_2 = \left[\frac{1}{2}, x_{c_2}\right]$ and $I_3 = [x_{c_2}, 1]$, such that $\mathcal{A} = \bigcup_{i=0}^{3} I_i$, where $x_{c_1} = \frac{1}{4}$ and $x_{c_2} = \frac{3}{4}$ are the critical points of the subdomains $0 \leq x < \frac{1}{2}$ and $\frac{1}{2} \leq x \leq 1$, respectively. For any $x \in \left[0, \frac{1}{2}\right)$ the equation of this modal is

$$f_{D_1}(x, \gamma) = 4\gamma x \left(\frac{1}{2} - x\right).$$

Therefore, if we evaluate each interval I_i for $i = \{0, 1\}$ in f_{D_1} we obtain that $f_{D_1}([0, x_{c_1}], \gamma) = \left[0, \frac{\gamma}{4}\right]$ and $f_{D_1}\left(\left[x_{c_1}, \frac{1}{2}\right], \gamma\right) = \left[0, \frac{\gamma}{4}\right]$. Now, for any $x \in \left[\frac{1}{2}, 1\right]$ the equation for this modal is

$$f_{D_2}(x, \gamma) = 4\gamma \left(x - \frac{1}{2}\right)(1 - x),$$

if we evaluate we obtain the following results $f_{D_2}\left(\left[\frac{1}{2}, x_{c_2}, \gamma\right]\right) = \left[0, \frac{\gamma}{4}\right]$ and $f_{D_2}\left(\left[x_{c_2}, 1\right], \gamma\right) = \left[0, \frac{\gamma}{4}\right]$ and since $\gamma \in (0,4)$ specifically $\gamma < 4$, then,

$$f_{D_1}(I_0, \gamma) = f_{D_1}(I_1, \gamma) = f_{D_2}(I_2, \gamma) = f_{D_2}(I_3, \gamma) = [0, 1) \quad \text{for} \quad \gamma \in (0, 4).$$

Analogously

$$f_{D_1}(I_0 \cup I_1, \gamma) = f_{D_2}(I_2 \cup I_3, \gamma) = f_D\left(\bigcup_{i=0}^{3} I_i, \gamma\right) = f_D(\mathcal{A}, \gamma) = [0, 1),$$

and since $[0, 1) \subset [0, 1]$. Therefore, we have that $f_D(\mathcal{A}, \gamma) \subset \mathcal{A}$. ■

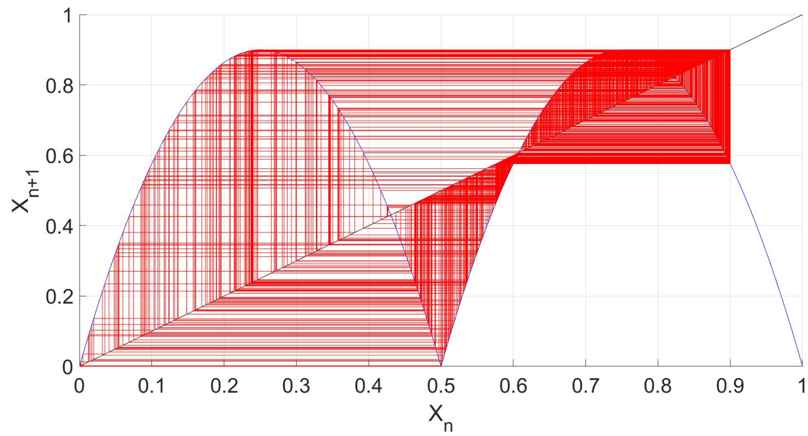


Figure 4.18: Cobweb diagram for the bimodal map f_D with $\gamma = 3.6$ using the initial condition $x_0 = 0.8491 \in [0, 1]$.

Figure 4.18 depicts the cobweb diagram for the parameter value $\gamma = 3.6$, according to Proposition 22. The bimodal map (4.5) exhibits monostability within $I = [0, 1] \subset \mathbb{R}$.

Proposition 23 *The bimodal map* (4.5) *has an invariant set* $\mathcal{A} = I = [0, 1] \subset \mathbb{R}$ *when* $\gamma = 4$.

Proof. This proof is similar to the previous one, again, we need to show that there exists an invariant set if $\gamma = 4$. For this, we consider the following four intervals $I_0 = [0, x_{c_1}]$, $I_1 = \left[x_{c_1}, \frac{1}{2}\right]$, $I_2 = \left[\frac{1}{2}, x_{c_2}\right]$ and $I_3 = [x_{c_2}, 1]$, such that $\mathcal{A} = \bigcup_{i=0}^{3} I_i$, where $x_{c_1} = \frac{1}{4}$ and $x_{c_2} = \frac{3}{4}$ are the critical points of the subdomains $0 \leq x < \frac{1}{2}$ and $\frac{1}{2} \leq x \leq 1$, respectively. For any $x \in \left[0, \frac{1}{2}\right)$ the equation of this modal is

$$f_{D_1}(x, \gamma = 4) = 16x\left(\frac{1}{2} - x\right).$$

Therefore, if we evaluate each interval I_i for $i = \{0, 1\}$ in f_{D_1} we obtain that $f_{D_1}([0, x_{c_1}], \gamma = 4) = [0, 1]$ and $f_{D_1}\left(\left[x_{c_1}, \frac{1}{2}\right], \gamma = 4\right) = [0, 1]$. Now, for any

$x \in \left[\frac{1}{2}, 1\right]$ the equation for this modal is

$$f_{D_2}(x, \gamma = 4) = 16\left(x - \frac{1}{2}\right)(1 - x),$$

if we evaluate we obtain the following results $f_{D_2}\left(\left[\frac{1}{2}, x_{c_2}\right], \gamma = 4\right) = [0, 1]$ and $f_{D_2}([x_{c_2}, 1], \gamma = 4) = [0, 1]$ and since $\gamma \in (0, 4)$. By the previous results, we obtain

$$f_{D_1}(I_0, \gamma = 4) = f_{D_1}(I_1, \gamma = 4) = f_{D_2}(I_2, \gamma = 4) = f_{D_2}(I_3, \gamma = 4) = [0, 1].$$

Analogously

$$\begin{aligned} f_{D_1}(I_0 \cup I_1, \gamma = 4) &= f_{D_2}(I_2 \cup I_3, \gamma = 4) \\ &= f_D\left(\bigcup_{i=0}^{3} I_i, \gamma = 4\right) = f_D(\mathcal{A}, \gamma = 4) = [0, 1]. \end{aligned}$$

Therefore, we have that $f_D(\mathcal{A}, \gamma = 4) = \mathcal{A}$. ■

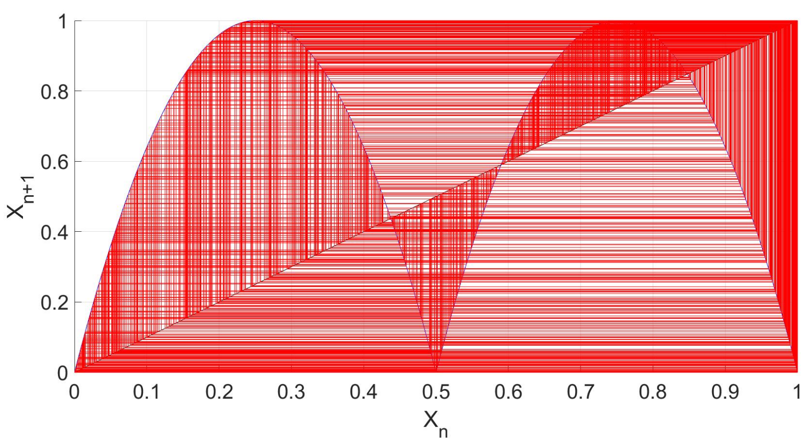


Figure 4.19: Cobweb diagram for the bimodal map f_D with $\gamma = 4$ using the initial condition $x_0 = 0.9340 \in [0, 1]$.

Figure 4.19 depicts the cobweb diagram for the parameter value $\gamma = 4$, according to the Proposition 23. The bimodal map (4.5) exhibits monostability within $I = [0, 1] \subset \mathbb{R}$.

Proposition 24 *The bimodal map* (4.5) *shows monostability in* $I = [0, 1]$ *if* $\gamma \in (0, 4] \subset \mathbb{R}$.

Proof. Notice that when $0 < \gamma < \dfrac{1}{2}$, then there exists an invariant point $x^*_{1,2}$ inside of I, more precisely, in the subdomain $0 \leq x < \dfrac{1}{2}$. When $\dfrac{1}{2} < \gamma < \dfrac{3}{2}$, now the invariant point is $x^*_{1,1}$ and belongs to $0 \leq x < \dfrac{1}{2}$. By the Proposition 18, if $\gamma \in \left[2, \dfrac{3+2\sqrt{2}}{2}\right)$, there is only one invariant set $\mathcal{A} \in \left[0, \dfrac{\gamma}{4}\right] \subset I$. From the Propositions 19 and 20, if $\dfrac{3+2\sqrt{2}}{2} < \gamma \leq \dfrac{3+\sqrt{17}}{2}$ now the only trapping region or invariant set is in $\dfrac{1}{2} \leq x \leq 1$. Finally, notice that when $\gamma \in \left(\dfrac{3+\sqrt{17}}{2}, 4\right]$, for any $x_0 \in I$ the orbit remains in I, given rise to a trapping region or invariant set. Therefore, observe that if $0 < \gamma \leq 4$ the bimodal map (4.5) always shows monostability in I. ■

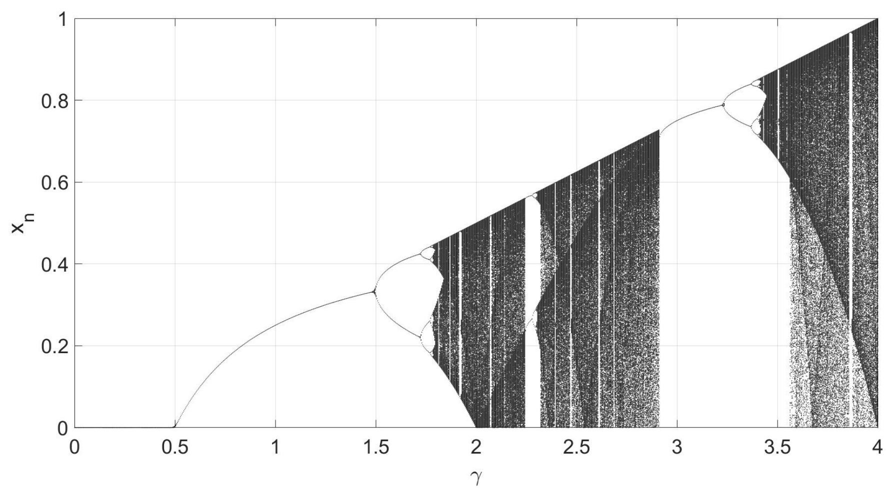


Figure 4.20: Bifurcation diagram of the bimodal map f_D given by (4.5) with respect to the parameter $\gamma \in (0, 4] \in \mathbb{R}$.

Figure 4.20 shows the corresponding bifurcation diagram for this case study, highlighting a clear relationship between the behaviors found in the cobweb diagram for each case. Below we briefly describe the different behaviors observe in the monoparametric family of multimodals maps given by (4.5).

- For $\gamma \in \left(0, \frac{1}{2}\right)$ the bimodal map (4.5) exhibits an invariant point at $x^*_{1,2} = 0$.
- For $\gamma \in \left[2, \frac{3}{2} + \sqrt{2}\right)$ the bimodal map (4.5) has an invariant set $\mathcal{A} \subset I$, where $\mathcal{A} = \left[0, \frac{\gamma}{4}\right]$ and $\frac{1}{2} \leq \frac{\gamma}{4} < \frac{3+2\sqrt{2}}{8}$. Moreover, the bimodal map (4.5) exhibits monostability in the interval $0 \leq x \leq \frac{\gamma}{4}$ when $\gamma \in \left[2, \frac{3}{2} + \sqrt{2}\right)$.
- When $\gamma \in \left(\frac{3}{2} + \sqrt{2}, \frac{3+\sqrt{17}}{2}\right)$ the bimodal map (4.5) has a trapping region $\mathcal{A}(\gamma)$ in the interval $\frac{1}{2} \leq x \leq 1$, and it exhibits monostability.
- When $\gamma = \frac{3+\sqrt{17}}{2}$ the bimodal map (4.5) has an invariant set $\mathcal{A}(\gamma)$ in the interval $\frac{1}{2} \leq x \leq 1$.
- For $\gamma \in (0, 4)$ the bimodal map (4.5) has a trapping region $\mathcal{A} = [0, 1] \subset \mathbb{R}$.
- For $\gamma = 4$ the bimodal map (4.5) has a invariant set $\mathcal{A} = [0, 1] \subset \mathbb{R}$.
- For $\gamma \in (0, 4] \subset \mathbb{R}$ the bimodal map (4.5) always shows monostability in $I = [0, 1] \in \mathbb{R}$.

Chapter 5

Conclusion

In the present work, the necessary conditions for ensuring the existence of monostability and bistability in a bimodal map were provided through propositions. These propositions were analytically demonstrated using tools commonly employed in the study of discrete dynamical systems. Subsequently, numerical simulations were conducted to observe the existence of monostability and bistability.

Some results presented were introduced for regular and irregular partitions, therefore this fact allows us to present the statements in terms of the set of points $\{\zeta_0, \zeta_1, \zeta_2\}$. We found that the location and stability of the fixed points of the bimodal map (3.1) in the interval $[\zeta_0, \zeta_1)$ are controlled by the parameter β_1 when it takes values in an interval determined by ζ_0 and ζ_1. In a similar way, it is possible to determine the location and stability of the fixed points of the bimodal map (3.1) in the interval $[\zeta_1, \zeta_2]$ by β_2 when it takes values in an interval determined by ζ_1 and ζ_2.

A trapping region or an invariant set in the first modal appears if the modal value $f_{D_1}(x_{c_1})$ is less than the value of the fixed point $x^*_{2,2}$ of the second modal. And in the second modal a trapping region or an invariant set appears if the second iteration of the critical point of the second modal $f^2_{D_2}(x_{c_2})$ is equal to or greater than the value of the fixed point $x^*_{2,2}$. Therefore, the fixed point $x^*_{2,2}$ is crucial to exhibits monostability or bistability in the bimodal map f_D given by (3.1).

We introduced three parametric families of the bimodal maps by considering the parameters β_1, β_2 and γ as $f_D(x, \beta_1)$, $f_D(x, \beta_2)$ and $f_D(x, \gamma)$, respectively. In the first two families $f_D(x, \beta_1)$ and $f_D(x, \beta_2)$, the value of first modal and the value of the second modal change, respectively, while the value of the other modal remains fixed, then by controlling the values of the first and second modal allow these families to exhibit monostability or bistability in the interval I. In the last family $f_D(x, \gamma)$, the parameter γ does not allow different values for the first and second modal and this restricts the system to only exhibits monostability in the interval I. Within the development, the respective Cobweb diagrams were shown for each case, the bifurcation diagrams were also developed, with the objective of observing the behavior of the different families of the difference map when their parameters take different values.

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