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Global stabilization of the PVTOL aircraft with lateral coupling and bounded inputs

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López Araujo Daniela Juanita

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Director de la Tesis:

Dr. Arturo Zavala Río

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To my family.

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Resumen

Esta tesis se enfoca en la estabilización global de un avión planar de despegue y aterrizaje verticales (PVTOL, por sus siglas en inglés). Este sistema tiene una dinámica compleja que hace difícil el diseño de controladores orientados a resolver el problema de estabilización global. Esto se debe principalmente a su naturaleza subactuada. Adicionalmente, otras limitaciones son consideradas en este estudio: el carácter positivo (unidireccional) de la propulsión y la naturaleza acotada de las entradas. Un primer enfoque que resuelve el problema de estabilización global, tomando en cuenta las restricciones mencionadas, fue propuesto por A. Zavala-Río, I. Fantoni y R. Lozano en 2003; no obstante el parámetro de acoplamiento lateral ε fue despreciado debido al pequeño valor que usualmente tiene en la práctica, simplificando así la dinámica del sistema. Sin embargo los resultados obtenidos en simulación para corroborar la eficiencia del esquema propuesto, muestran que el objetivo de control es logrado aún tomando valores positivos de ε. Por otro lado, algunos trabajos recientes han propuesto soluciones al problema de estabilización global considerando la dinámica completa del sistema, pero dependen del valor exacto de ε. Así pues, en esta tesis se demuestra analíticamente que el esquema de control inicialmente propuesto por A. Zavala-Río, I. Fantoni y R. Lozano en 2003 bajo las restricciones de entrada arriba mencionadas, considerando $\varepsilon = 0$, logra el objetivo de estabilización global aún cuando $\varepsilon > 0$, para valores suficientemente pequeños de ε pero sin la necesidad de conocer su valor exacto.

Abstract

This thesis focuses on the global stabilization of the Planar Vertical Take-Off and Landing (PVTOL) aircraft. Such a system has a complex dynamics that renders difficult the design of controllers oriented to solve the global stabilization problem. This is mainly due to its under-actuated nature. In addition, other limitations are considered in this study: the positive (unidirectional) character of the thrust and the bounded nature of the inputs. A first approach that solves the global stabilization problem, taking the mentioned restrictions into account, was proposed by A. Zavala-Río, I. Fantoni, and R. Lozano in 2003; however the lateral coupling parameter ε was neglected due to the small value that it usually has in practice, simplifying the system dynamics. Nevertheless the simulation results that were performed to corroborate the efficiency of the proposed scheme, show that the control objective is achieved even taking positive values of ε. On the other hand, recent works have proposed solutions to the global stabilization problem considering the whole system dynamics, but they depend on the exact knowledge of ε. Thus, in this thesis, it is analytically proved that the control scheme initially proposed by A. Zavala-Río, I. Fantoni, and R. Lozano in 2003 under the above mentioned input restrictions, considering $\varepsilon = 0$, achieves the global stabilization objective even when $\varepsilon > 0$, provided that ε is small enough but without the need to know its exact value.

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Chapter 1

Introduction

Since its introduction in the literature of control systems and design, the Planar Vertical Take-Off and Landing (PVTOL) aircraft system has been a subject of great interest in the control community. Due to its special properties (in particular, to its under-actuated nature), it represents a challenge in nonlinear control design. There exist two main research interests in literature: stabilization and trajectory tracking. Many different techniques have been used to investigate these problems. (Hauser, *et al.*, 1992) proposed an input-output linearization resulting in bounded tracking and stabilization for the V/STOL aircraft. (Martin, *et al.*, 1996) extended this result using a flat output for the system; the control scheme achieved output tracking of the PVTOL non-minimum phase flat system taking into account the coupling between the rolling moment and the lateral acceleration. (Sepulchre, *et al.*, 1997) applied a linear high gain approximation of backstepping to the model excluding the lateral coupling. (Lin, *et al.*, 1999) studied robust hovering control of the PVTOL using an optimal control approach. (Olfati-Saber, 2002) used smooth static state feedback to address global configuration stabilization for the VTOL aircraft in an unbounded input context. (Saeki and Sakaue, 2001) transformed the model into an equivalent one where the coupling terms were not present, and then designed a controller by applying a linear high gain approximation of backstepping to the model. (Marconi, *et al.*, 2002) designed an error feedback dynamic regulator that is robust with respect to uncertainties on the model parameters, and achieved global convergence for the autonomous VTOL vehicle landing on a platform oscillating in the vertical direction. (Setlur, *et al.*, 2001) presented a nonlinear controller for the VTOL aircraft guaranteeing trajectory tracking to a reference signal and forcing the tracking error trajectories to converge into an arbitrarily small neighborhood around the origin.

Some authors have supported their algorithms through experimental PVTOL setups. For instance, (Lozano, *et al.*, 2004) presented a simple nonlinear controller for a PVTOL aircraft tested in a real-time application, and (Palomino, *et al.*, 2003) stabilized the PVTOL aircraft with the aid of a vision system. Some others have designed observers when the full state of the PVTOL is not completely measurable. For instance, (Do, *et al.*, 2003) developed a nonlinear output feedback controller for the VTOL aircraft without velocity measurements, while (Sanchez, *et al.*, 2004) designed a nonlinear observer to estimate the angular position of the PVTOL aircraft which constitutes one of the main difficulties in real experiments.

More recently, (Wood and Cazzolato, 2007) proposed a nonlinear control scheme using a feedback law that casts the system into a cascade structure and proved its global stability. Global stabilization was also achieved by (Ye, *et al.*, 2007) through a saturated control technique by previously transforming the PVTOL dynamics into a chain of integrators with nonlinear perturbations. Further, based on partial feedback linearization, a prediction-based nonlinear controller was proposed in (Chemori and Marchand, 2008); stabilization is achieved by forcing the linearized system to track optimal trajectories.

In the previously cited works the lateral coupling has been neglected or the exact knowledge of this term has been considered to design the controllers. On the other hand, from all the previously cited works, (Zavala-Río, *et al.*, 2003) was the first to simultaneously consider the bounded nature of both inputs and the positive character of the thrust to develop a globally stabilizing scheme. Nevertheless, robustness of the previously proposed algorithms has hardly been addressed. For instance (Lin, *et al.*, 1999) has developed a robust control scheme for the PVTOL aircraft with respect to uncertainties of the coupling parameter. However a nominal value of ε was needed. (Teel, 1992) proposed a control law based on the exact knowledge of ε and showed robustness of his approach when the initial conditions are close enough to the origin. The supposition of the exact knowledge of the lateral coupling can be defended due to its dependence on the physical parameters of the aircraft. Nevertheless in real experiments it can be difficult to estimate or measure.

The main objective of this work is to demonstrate that using the control scheme proposed in (Zavala-Río, *et al.*, 2003), where $\varepsilon = 0$ was supposed, global stabilization is achieved even in the presence of the lateral coupling, provided that ε keeps a small enough value, but without the need of its exact value. It was not required to modify the original control algorithm to accomplish the proof of the presented analysis.

1.1 Obtaining the PVTOL aircraft dynamics

Let us consider an aircraft whose motion is restricted to evolve on a vertical plane, see Figure 1.1. To get its model, two reference frames will be considered. Let $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ represent the orthonormal basis of a fixed inertial frame expressed in Cartesian coordinates and $\hat{\imath}_a$, $\hat{\jmath}_a$, and $\hat{k}_a$ account for a similar basis of a moving frame attached to the body of the aircraft, as considered in (Fliess, 1999). The angle between the moving and the inertial frames will be denoted θ. The rotational and translational motion dynamics can be separated by using the centroid of the aircraft as a reference point. The forces acting on the system are

$$\vec{F}_1 = F_1 \sin\alpha \ \hat{\imath}_a + F_1 \cos\alpha \ \hat{\jmath}_a$$

$$\vec{F}_2 = -F_2 \sin\alpha \ \hat{\imath}_a + F_2 \cos\alpha \ \hat{\jmath}_a$$

$$\vec{W} = -mg \ \hat{\jmath}_a$$

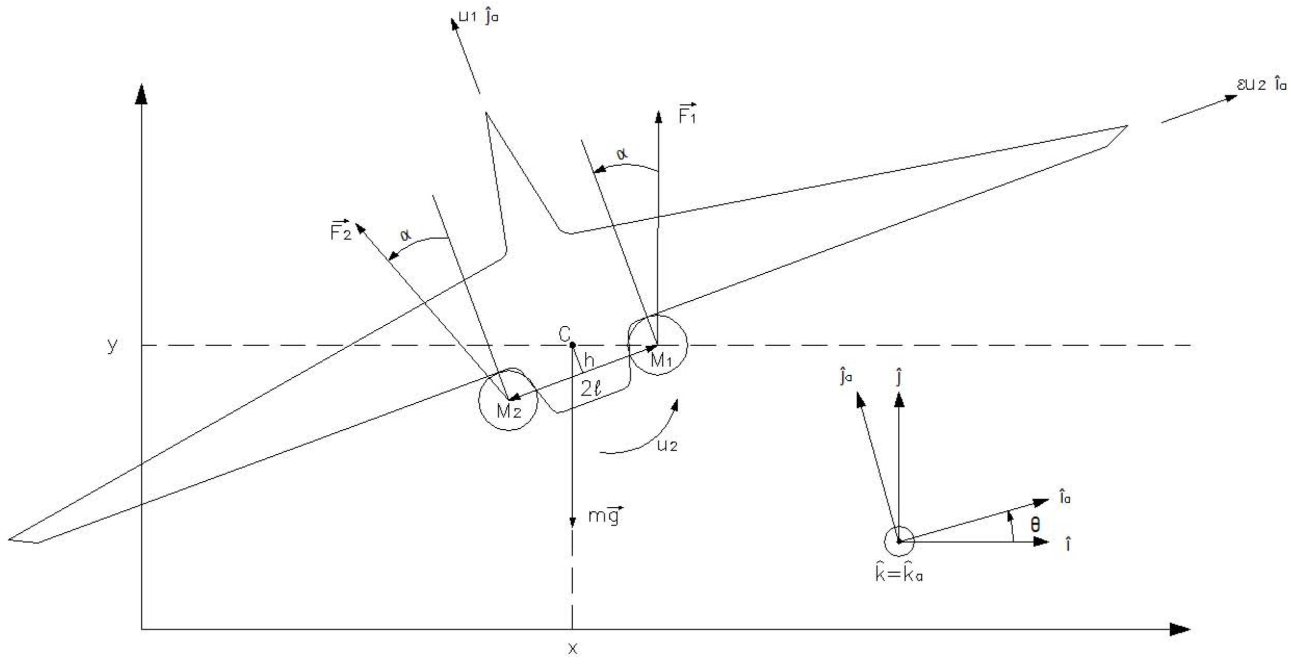

Figure 1.1: Forces acting on the PVTOL aircraft

where α represents a fixed angle formed by the thrust direction of action and the vertical axis of the moving frame. The weight $\vec{W}$ is applied to the center of mass C; the thrusts $\vec{F_1}$ and $\vec{F_2}$ are applied to the points M_1 and M_2 respectively.

First we will develop the equations for the rotational dynamics. The distance from the center of mass to the points M_1 and M_2 is given by

$$\vec{r_1} = l\ \hat{\imath}_a - h\ \hat{\jmath}_a$$

$$\vec{r_2} = -l\ \hat{\imath}_a - h\ \hat{\jmath}_a$$

respectively. The net torque $\sum \vec{\tau}$ applied about the center of mass is equal to the sum of the torques produced by $\vec{F_1}$ and $\vec{F_2}$, *i.e.*

$$\begin{aligned}\sum \vec{\tau} &= \vec{\tau}_{F_1} + \vec{\tau}_{F_2} \\ &= \vec{r_1} \times \vec{F_1} + \vec{r_2} \times \vec{F_2} \\ &= (F_1 - F_2)(h \sin\alpha + l\cos\alpha)\ \hat{k}_a\end{aligned}$$

and from Newton's second law, we have

$$J\ddot{\theta} = (F_1 - F_2)(h\sin\alpha + l\cos\alpha) \tag{1.3}$$

Now we will obtain the equations for the translational motion. Addition of forces about the center of mass results in

$$\sum \vec{F} = m\vec{a}_c = \vec{F_1} + \vec{F_2} + m\vec{g}$$

The vectors $\vec{a}_c$ and $\sum \vec{F}$ can be expressed in Cartesian coordinates as

$$\vec{a}_c = a_x \ \hat{\imath} + a_y \ \hat{\jmath} + a_z \ \hat{k}$$
$$\sum \vec{F} = \sum F_x \ \hat{\imath} + \sum F_y \ \hat{\jmath} + \sum F_z \ \hat{k}$$

Thus, in component form, we have

$$\sum F_x = ma_x = m\ddot{x}_c = -(F_1 + F_2)\cos\alpha\sin\theta + (F_1 - F_2)\sin\alpha\cos\theta \tag{1.4a}$$
$$\sum F_y = ma_y = m\ddot{y}_c = (F_1 + F_2)\cos\alpha\cos\theta + (F_1 - F_2)\sin\alpha\sin\theta - mg \tag{1.4b}$$

Observe that Equations (1.3) and (1.4) can be rewritten as

$$\frac{\ddot{x}_c}{g} = \frac{1}{mg}\left[-(F_1 + F_2)\cos\alpha\sin\theta + (F_1 - F_2)\sin\alpha\cos\theta\right]$$
$$\frac{\ddot{y}_c}{g} = \frac{1}{mg}\left[(F_1 + F_2)\cos\alpha\cos\theta + (F_1 - F_2)\sin\alpha\sin\theta - mg\right]$$
$$\ddot{\theta} = \frac{1}{J}(F_1 - F_2)(h\sin\alpha + l\cos\alpha)$$

Finally, setting

$$u_1 = \frac{F_1 + F_2}{mg}\cos\alpha$$
$$u_2 = \frac{F_1 - F_2}{J}(h\sin\alpha + l\cos\alpha)$$
$$\varepsilon = \frac{J}{mg} \cdot \frac{\sin\alpha}{h\sin\alpha + l\cos\alpha}$$
$$x = \frac{x_c}{g}$$
$$y = \frac{y_c}{g}$$

the equations of motion become

$$\ddot{x} = -u_1\sin\theta + \varepsilon u_2\cos\theta \tag{1.7a}$$
$$\ddot{y} = u_1\cos\theta + \varepsilon u_2\sin\theta - 1 \tag{1.7b}$$
$$\ddot{\theta} = u_2 \tag{1.7c}$$

Remark 1.1 *As pointed out in (Hauser, et al., 1992), the PVTOL aircraft can be seen as a simplified prototype that has a minimum number of states and inputs but retains many of the features that must be considered when designing control laws for a real* *aircraft. This prototype is, for instance, the natural restriction to so-called jet-borne operation (e.g. hover) of the V/STOL (Vertical/Short Take-Off and Landing)*[1] *aircraft in a vertical-lateral plane.*

Remark 1.2 *As pointed out in (Hauser, et al., 1992), the parameter ε is generally small (for instance in V/STOL aircrafts). However, the normalized dynamics in equations* (1.7) *is valid without restrictions on the PVTOL parameter values.*

1.2 Notation

We denote $I\!R$ and $I\!R_+$ the set of real and nonnegative real numbers, respectively. $I\!R^n$ represents the set of n-dimensional vectors whose elements are real numbers. We denote 0_n the origin of $I\!R^n$. For any $x \in I\!R^n$, x_i represents its i^{th} element. Let $A \in I\!R^{n\times n}$ be a symmetric matrix, *i.e.* $A^T = A$. The maximum and minimum eigenvalues of A will be respectively denoted $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$. I_n will denote the $n \times n$ identity matrix.

In Chapter 2, $\|\cdot\|$ will denote any norm, while in the subsequent chapters, it will represent the standard Euclidean vector norm and induced matrix norm, *i.e.*

$$\|x\| \triangleq \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}}$$

for any $x \in I\!R^n$, and

$$\|B\| = \left[\lambda_{\max}(B^T B)\right]^{\frac{1}{2}}$$

for any $B \in I\!R^{m\times n}$. Other type of norms will be explicitly expressed. For instance, the infinite induced matrix norm will be denoted $\|B\|_\infty$, *i.e.*

$$\|B\|_\infty \triangleq \max_i \sum_{j=1}^{n} |b_{ij}|$$

where b_{ij} represents the element in row i and column j of matrix B.

Let $\mathcal{A}$ and $\mathcal{E}$ be subsets (each of them with nonempty interior) of some vector spaces $\mathbb{A}$ and $\mathbb{E}$ respectively. We denote $\mathcal{C}^m(\mathcal{A};\, \mathcal{E})$ the set of m-times continuously differentiable functions from $\mathcal{A}$ to $\mathcal{E}$.[2] In particular, $\mathcal{C}_L^m(\mathcal{A};\, \mathcal{E})$ will stand for the set of m-times continuously differentiable functions from $\mathcal{A}$ to $\mathcal{E}$ whose m^{th} derivative is Lipschitz-continuous. Consider a continuous-time function $h_1 \in \mathcal{C}^2(I\!R_+; I\!R)$ and a scalar function $h_2 \in \mathcal{C}_L^2(I\!R; I\!R)$. The first and second time-derivatives of h_1 are respectively represented as $\dot{h}_1$ and $\ddot{h}_1$, *i.e.* $\dot{h}_1 : t \to \frac{d}{dt}h_1$ and $\ddot{h}_1 : t \to \frac{d^2}{dt^2}h_1$. As for h_2, the following notation will be used: $h_2' : s \to \frac{d}{ds}h_2$ and $h_2'' : s \to \frac{d^2}{ds^2}h_2$, while $h_2''' : s \to D^+h_2''$, where D^+ denotes the upper right-hand (Dini) derivative whose definition, taken from (Khalil, 2002, Appendix C2), is recalled here:

[1] Such as the YAV-8B Harrier; see for instance (Chemori and Marchand, 2008, Figure 1).

[2] Differentiability at any point on the boundary of $\mathcal{A}$ (when such a point is included in $\mathcal{A}$) is considered as the limit from the interior of $\mathcal{A}$.

Definition 1.1 *The* ***upper right-hand derivative*** $D^+v(s)$ *is defined by*

$$D^+v(s) = \limsup_{h \to 0^+} \frac{v(s+h) - v(s)}{h}$$

where $\limsup_{n\to\infty}$ *(the limit superior) of a sequence of real numbers* $\{x_n\}$ *is a real number* y *satisfying*

(i) for every $\epsilon > 0$ *there exists an integer* N *such that* $n > N$ *implies* $x_n < y + \epsilon$*;*

(ii) given $\epsilon > 0$ *and* $m > 0$*, there exists an integer* $n > m$ *such that* $x_n > y - \epsilon$*.*

Let us note that if $v(s)$ is differentiable at s, then $D^+v(s) = \frac{dv}{ds}(s)$. For a Lipschitz-continuous function $v(s)$ that is not differentiable at a finite number of values of s, say $s_1, s_2, ..., s_n$, $D^+v(s)$ is a function with bounded discontinuities but well-defined at such points, $s_1, s_2, ..., s_n$.

Chapter 2

Mathematical Background

The results stated in this chapter are taken from (Khalil, 2002) and will be used to accomplish the proof of the main result.

2.1 Lipschitz continuity

A function satisfying

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\| \tag{2.1}$$

for all (t,x) and (t,y) in some neighborhood of (t_0, x_0), is said to be Lipschitz in x and the positive constant L is called a *Lipschitz constant*. A function $f(x)$ is said to be *locally Lipschitz* on a domain (open and connected set) $D \in \mathbb{R}^n$ if each point of D has a neighborhood D_0 such that f satisfies the Lipschitz condition (2.1) for all point in D_0 with some Lipschitz constant L_0. We say that f is Lipschitz on a set W if it satisfies (2.1) for all points in W with the same Lipschitz constant L. A function $f(x)$ is said to be *globally Lipschitz* if it is Lipschitz on $\mathbb{R}^n$. The same terminology is extended to a function $f(t,x)$, provided the Lipschitz condition holds uniformly in t for all t in a given interval of time.

Lemma 2.1 (Khalil, 2002, Lemma 3.1) *Let $f : [a,b] \times D \to \mathbb{R}^m$ be continuous for some domain $D \in \mathbb{R}^n$. Suppose that $[\partial f/\partial x]$ exists and is continuous on $[a,b] \times D$. If, for a convex subset $W \subset D$, there is a constant $L \geq 0$ such that*

$$\left\| \frac{\partial f}{\partial x}(t,x) \right\| \leq L$$

on $[a,b] \times W$, then

$$\|f(t,x) - f(t,y)\| \leq L\,\|x - y\|$$

for all $t \in [a,b]$, $x \in W$, and $y \in W$.

The Lemma shows how using the knowledge of $[\partial f/\partial x]$ a Lipschitz constant can be calculated.

The Lipschitz property is stronger than continuity but is weaker than continuous differentiability as stated in the following lemmas.

Lemma 2.2 (Khalil, 2002, Lemma 3.2) *If $f(t,x)$ and $[\partial f/\partial x](t,x)$ are continuous on $[a,b]\times D$, for some domain $D \subset \mathbb{R}^n$, then f is locally Lipschitz in x on $[a,b]\times D$.*

Lemma 2.3 (Khalil, 2002, Lemma 3.3) *If $f(t,x)$ and $[\partial f/\partial x](t,x)$ are continuous on $[a,b]\times \mathbb{R}^n$, then f is globally Lipschitz in x on $[a,b]\times \mathbb{R}^n$ if and only if $[\partial f/\partial x]$ is uniformly bounded on $[a,b]\times \mathbb{R}^n$.*

2.2 Lyapunov stability

Throughout this section we will consider autonomous systems of the form

$$\dot{x} = f(x) \tag{2.2}$$

where $f : D \to \mathbb{R}^n$ is a locally Lipschitz map from a domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Without loss of generality it will always be assumed that $f(x)$ satisfies $f(0_n) = 0_n$. We will study the stability of the origin $x = 0_n$ in the sense of Lyapunov (Vidyasagar, 1993).

Definition 2.1 *The origin of (2.2) is*

- ***stable*** *if, for every $\epsilon > 0$ there exists δ such that*

$$\|x(0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon \qquad \forall t \geq 0$$

- ***unstable*** *if it is not stable;*
- ***attractive*** *if there is $\eta > 0$ such that*

$$\|x(0)\| < \eta \quad \Rightarrow \quad x(t) \to 0_n \quad \text{as} \quad t \to \infty$$

- ***asymptotically stable*** *if it is stable and attractive;*
- ***globally attractive*** *if for each pair of positive numbers, M and ϵ, with M arbitrarily large and ϵ arbitrarily small, there exists a finite number $T = T(M,\epsilon)$ such that*

$$\|x(0)\| < M \quad \Rightarrow \quad \|x(t)\| < \epsilon \qquad \forall t \geq T(M,\epsilon)$$

- ***globally asymptotically stable*** *if its stable and globally attractive.*

2.2.1 Linear systems

We first consider the linear time-invariant system

$$\dot{x} = Ax \tag{2.3}$$

As stated for instance in Theorem 4.5 of (Khalil, 2002), the origin of (2.3) is asymptotically stable if and only if all the eigenvalues of A satisfy $\mathrm{Re}\lambda_i < 0$. When this is the case, A is called *Hurwitz matrix* or *stability matrix.*

The next theorem characterizes asymptotic stability of the origin of (2.3) in terms of the solution of the Lyapunov equation

$$PA + A^T P = -Q \tag{2.4}$$

where Q is a symmetric, positive-definite matrix.

Theorem 2.1 (Khalil, 2002, Theorem 4.6) *A matrix A is Hurwitz, that is, $Re\lambda_i < 0$ for all eigenvalues of A, if and only if for any given positive definite symmetric matrix Q there exists a positive definite symmetric matrix P that satisfies the Lyapunov equation* (2.4). *Moreover, if A is Hurwitz, then P is the unique solution of* (2.4).

2.2.2 First Lyapunov method

Consider now the nonlinear system

$$\dot{x} = f(x) \tag{2.5}$$

where $f : D \to I\!R^n$ with $D \subset I\!R^n$, is a continuously differentiable map. Suppose that the origin $x = 0_n$ is in D (open and connected set) and that it is an equilibrium point for the system. In a neighborhood of the origin, system (2.5) can be approximated by its linearization about the origin, $\dot{x} = Ax$, where $A = \frac{\partial f}{\partial x}(0_n)$. The next theorem, known as Lyapunov's first (or indirect) method, gives conditions under which conclusions can be drawn about the stability of the origin as an equilibrium point for the nonlinear system by inspecting its stability as an equilibrium point for the linear system.

Theorem 2.2 (Khalil, 2002, Theorem 4.7) *Let $x = 0_n$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(x)$$

where $f : D \to I\!R^n$ is continuously differentiable and D is a neighborhood of the origin. Let

$$A = \left.\frac{\partial f}{\partial x}(x)\right|_{x=0_n}$$

Then,

(i) The origin is asymptotically stable if $Re\lambda_i < 0$ for all eigenvalues of A.

(ii) The origin is unstable if $Re\lambda_i > 0$ for one or more of the eigenvalues of A.

2.3 LaSalle's invariance principle

When Lyapunov's method fails to asses the asymptotic character of a stable equilibrium point, an important theorem known as LaSalle's invariance principle may be useful to conclude on its attractivity. Before we state that theorem, some definitions are given.

Definition 2.2 *A set M is said to be*

- *an **invariant set** with respect to* (2.2) *if*

$$x(0) \in M \Rightarrow x(t) \in M, \qquad \forall t \in I\!R.$$

- *a* ***positively invariant set*** *if*

$$x(0) \in M \Rightarrow x(t) \in M, \qquad \forall t \geq 0.$$

We now state LaSalle's invariance principle.

Theorem 2.3 (Khalil, 2002, Theorem 4.4) *Let $\Omega \subset D$ be a compact set that is positively invariant with respect to (2.2). Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ for x in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.*

If we want to show that $x(t) \to 0_n$ as $t \to \infty$, we need to establish that the largest invariant set in E is the origin.

2.4 Comparison functions

Definition 2.3 *A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.*

Definition 2.4 *A continuous function $\beta : [0, a) \to [0, \infty) \times [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.*

2.5 Boundedness and ultimate boundedness

Boundedness of solutions can be shown using Lyapunov analysis even when the origin is not an equilibrium point. Let us consider the system

$$\dot{x} = f(t, x) \tag{2.6}$$

where $f : [0, \infty) \times D \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, and $D \subset \mathbb{R}^n$ is a domain that contains the origin.

Definition 2.5 *The solutions of* (2.6) *are*

- ***uniformly bounded*** *if there exists a positive constant c, independent of $t_0 \geq 0$, and for every $a \in (0, c)$ there is $\beta = \beta(a) > 0$, independent of t_0, such that*

$$\|x(t_0)\| \leq a \quad \Rightarrow \quad \|x(t)\| \leq \beta, \quad \forall t \geq t_0 \tag{2.7}$$

- ***globally uniformly bounded*** *if* (2.7) *holds for arbitrarily large a.*

- ***uniformly ultimately bounded*** *with ultimate bound b if there exists positive constants b and c, independent of t_0, and for every $a \in (0, c)$, there is $T = T(a, b) \geq 0$, independent of t_0, such that*

$$\|x(t_0)\| \leq a \quad \Rightarrow \quad \|x(t)\| \leq b, \quad \forall t \geq t_0 + T \tag{2.8}$$

- ***globally uniformly ultimately bounded*** *if* (2.8) *holds for arbitrarily large a.*

Theorem 2.4 (Khalil, 2002, Theorem 4.18) *Let $D \subset I\!R^n$ be a domain that contains the origin and $V : [0, \infty) \times D \to I\!R$ be a continuously differentiable function such that*

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \leq -W_3(x), \quad \forall \|x\| \geq \mu > 0$$

$\forall t \geq 0$ and $\forall x \in D$, where α_1 and α_2 are class $\mathcal{K}$ functions and $W_3(x)$ is a continuous positive definite function. Take $r > 0$ such that $B_r \subset D$ and suppose that

$$\mu < \alpha_2^{-1}(\alpha_1(r))$$

Then, there exists a class $\mathcal{KL}$ function β and for every initial state $x(t_0)$, satisfying $\|x(t_0)\| \leq \alpha_2^{-1}(\alpha_1(r))$, there is $T \geq 0$ (dependent on $x(t_0)$ and μ) such that the solution of (2.6) *satisfies*

$$\|x(t)\| \leq \beta\left(\|x(t_0)\|, t - t_0\right), \quad \forall t_0 \leq t \leq t_0 + T \tag{2.9}$$
$$\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\mu)), \quad \forall t \geq t_0 + T \tag{2.10}$$

Moreover, if $D = I\!R^n$ and α_1 belongs to class $\mathcal{K}_\infty$, then (2.9) *and* (2.10) *hold for any initial state $x(t_0)$, with no restriction on how large μ is.*

Chapter 3

Globally Stabilizing Scheme

3.1 Recalling the PVTOL aircraft dynamics

The equations that model the dynamics of the PVTOL aircraft (see Figure 3.1), whose derivation was thoroughly developed in Section 1.1, are given by

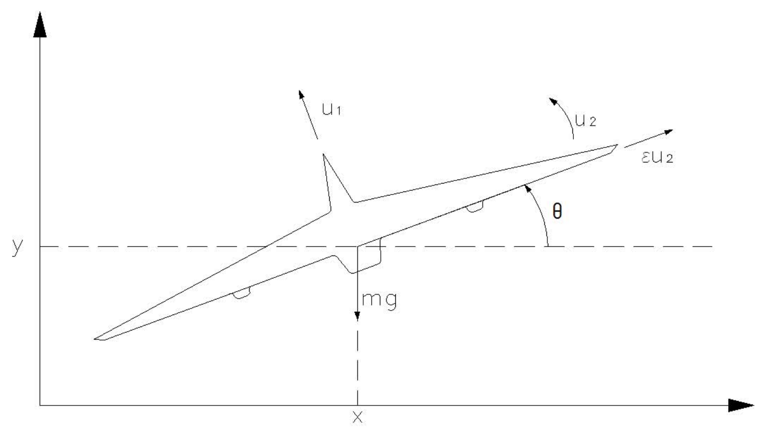

Figure 3.1: The PVTOL aircraft

$$\ddot{x} = -u_1 \sin\theta + \varepsilon u_2 \cos\theta \tag{3.1a}$$

$$\ddot{y} = u_1 \cos\theta + \varepsilon u_2 \sin\theta - 1 \tag{3.1b}$$

$$\ddot{\theta} = u_2 \tag{3.1c}$$

where x and y denote the horizontal and vertical position of the center of gravity and θ is the aircraft's roll angle with respect to the horizon. The control inputs are the thrust and the rolling moment given by the variables u_1 and u_2 respectively. The constant '-1' is the normalized gravitational acceleration and ε is a coefficient which characterizes the coupling between the rolling moment u_2 and the lateral acceleration of the aircraft.

3.2 Globally stabilizing controller

In view of the small value that ε usually takes (Hauser, *et al.*, 1992), a control scheme for a PVTOL aircraft was proposed in (Zavala-Río, *et al.*, 2003) by considering $\varepsilon = 0$ in (3.1), *i.e.* modeling the system dynamics as

$$\ddot{x} = -u_1 \sin\theta \tag{3.2a}$$

$$\ddot{y} = u_1 \cos\theta - 1 \tag{3.2b}$$

$$\ddot{\theta} = u_2 \tag{3.2c}$$

Under this consideration, the **control objective** achieved in (Zavala-Río, *et al.*, 2003) was the global asymptotic stability of the closed-loop system trivial solution $(x, y, \theta)(t) \equiv (0, 0, 0)$ avoiding input saturation, *i.e.* with $0 \leq u_1(t) \leq U_1$ and $|u_2(t)| \leq U_2$, $\forall t \geq 0$, for some constants $U_1 > 1$ and $U_2 > 0$.

Remark 3.1 *Notice, from* (3.2b)*, that* $U_1 > 1$ *is a necessary condition for the PVTOL to be stabilizable at any desired position. Indeed, any steady-state condition implies that the aircraft weight be compensated.*

Remark 3.2 *Achieving global asymptotic stability of the trivial solution implies that the system configuration variables may be globally stabilized to any point on* $\mathbb{R}^2 \times \{0\}$ *through a simple modification on the control law.*

The approach developed in (Zavala-Río, *et al.*, 2003) is based on the use of *linear saturation functions*, as defined in (Teel, 1992), and a special type of them stated in (Zavala-Río, *et al.*, 2003) as *2-level linear saturation functions*, whose definitions are recalled here.

Definition 3.1 *Given positive constants* L *and* M*, with* $L \leq M$*, a function* $\sigma : \mathbb{R} \to \mathbb{R}$ *is said to be a* ***linear saturation*** *for* (L, M) *if it is a nondecreasing Lipschitz-continuous function satisfying*

(a) $\sigma(s) = s$ *when* $|s| \leq L$

(b) $|\sigma(s)| \leq M$ *for all* $s \in \mathbb{R}$

Definition 3.2 *Given positive constants* L^+, M^+, N^+, L^-, M^-*, and* N^-*, with* $L^\pm \leq \min\{M^\pm, N^\pm\}$*, a function* $\sigma : \mathbb{R} \to \mathbb{R}$ *is said to be a* ***2-level linear saturation*** *for* $(L^+, M^+, N^+, L^-, M^-, N^-)$ *if it is a nondecreasing Lipschitz-continuous function satisfying*

(a) $\sigma(s) = s$ *for all* $s \in [-L^-, L^+]$

(b) $-M^- < \sigma(s) < M^+$ *for all* $s \in (-N^-, N^+)$

(c) $\sigma(s) = -M^-$ *for all* $s \leq -N^-$

(d) $\sigma(s) = M^+$ *for all* $s \geq N^+$

Observe that a 2-level linear saturation for $(L^+, M^+N^+, L^-, M^-, N^-)$ is a linear saturation for $(\min\{L^-, L^+\}, \max\{M^-, M^+\})$. The standard saturation function, *i.e.* $\text{sat}(\varsigma) = \text{sign}(\varsigma)\min\{|\varsigma|, 1\}$, is an example of a linear saturation function for $L = M = 1$. In Chapter 4 two examples of 2-level linear saturation functions are given.

We recall the control scheme proposed in (Zavala-Río, *et al.*, 2003), where the thrust u_1 and the rolling moment u_2 were defined as

$$u_1 = \sqrt{r_1^2 + (1 + r_2)^2} \tag{3.3}$$

$$u_2 = \sigma_{41}(\alpha_d) - \sigma_{32}\big(\dot{\theta} - \sigma_{42}(\omega_d) + \sigma_{31}(\dot{\theta} - \sigma_{43}(\omega_d) + \theta - \theta_d)\big) \tag{3.4}$$

where[1]

$$r_1 = -k\sigma_{12}(\dot{x} + \sigma_{11}(kx + \dot{x})) \tag{3.5}$$

$$r_2 = -\sigma_{22}(\dot{y} + \sigma_{21}(y + \dot{y})) \tag{3.6}$$

$$\theta_d = \arctan(-r_1, 1 + r_2) \tag{3.7}$$

$\arctan(a, b)$ represents the (unique) angle α such that $\sin\alpha = a/\sqrt{a^2 + b^2}$ and $\cos\alpha = b/\sqrt{a^2 + b^2}$; k in (3.5) is a positive constant smaller than unity, *i.e.*

$$0 < k < 1 \tag{3.8a}$$

$\sigma_{ij}(\cdot)$ in (3.5) and (3.6) are functions on $\mathcal{C}_L^2(I\!R; I\!R)$ satisfying Definition 3.2, for given $(L_{ij}^+,\ M_{ij}^+,\ N_{ij}^+,\ L_{ij}^-,\ M_{ij}^-,\ N_{ij}^-)$ such that

$$(kM_{12})^2 + \big(1 + M_{22}^-\big)^2 < U_1^2 \tag{3.8b}$$

$$M_{22}^+ < 1 \tag{3.8c}$$

$$M_{i1} < \frac{L_{i2}}{2}, \quad \forall i = 1, 2 \tag{3.8d}$$

with $M_{ij} \triangleq \max\{M_{ij}^-, M_{ij}^+\}$ and $L_{ij} \triangleq \min\{L_{ij}^-, L_{ij}^+\}$, $i = 1, 2,\ j = 1, 2$; the functions $\sigma_{mn}(\cdot)$ in (3.4) are linear saturations for given (L_{mn}, M_{mn}) such that

$$M_{41} + M_{32} < U_2 \tag{3.9a}$$

$$M_{41} + 2M_{42} + 2M_{31} < L_{32} \tag{3.9b}$$

$$M_{41} + M_{42} + 2M_{43} + 2B_{\theta_d} < L_{31} \tag{3.9c}$$

with

$$B_{\theta_d} \triangleq \arctan\big(kM_{12}, 1 - M_{22}^+\big) \tag{3.10}$$

[1] As for the subindices of the linear saturation functions σ_{ij} in r_1, r_2, and u_2, the first of them tells the function that the referred saturation belongs to: $i = 1$ for r_1, $i = 2$ for r_2, $i = 3, 4$ for u_2, with 4 applied to θ_d, ω_d, and α_d. For $i = 1, 2, 3$, the second subindex indicates the nesting level of the referred saturation function in the expression, $j = 1$ for the internal saturation and $j = 2$ for the external one, while for $i = 4$, it differentiates the referred saturation function.

and

$$\omega_d \triangleq \left.\frac{d\theta_d}{dt}\right|_{\varepsilon=0} \quad \text{and} \quad \alpha_d \triangleq \left.\frac{d^2\theta_d}{dt^2}\right|_{\varepsilon=0}$$

whose expressions, calculated considering equations (3.2) as the system dynamics, are given by

$$\omega_d = k\bar{\omega}_d \tag{3.11a}$$

with

$$\bar{\omega}_d = \frac{\overline{r}_1\rho_2 - (1 + r_2)\rho_1}{u_1^2} \tag{3.11b}$$

and

$$\alpha_d = k\bar{\alpha}_d \tag{3.12a}$$

with

$$\bar{\alpha}_d = \frac{\overline{r}_1\varphi_2 - (1 + r_2)\varphi_1}{u_1^2} - \frac{2\mu_1\bar{\omega}_d}{u_1} \tag{3.12b}$$

where

$$\bar{r}_1 \triangleq \frac{r_1}{k} = -\sigma_{12}(\dot{x} + \sigma_{11}(kx + \dot{x})) \tag{3.13a}$$

$$\rho_1 \triangleq \left.\frac{d\bar{r}_1}{dt}\right|_{\varepsilon=0} = -\sigma'_{12}(\dot{x} + \sigma_{11}(kx + \dot{x}))[-u_1 \sin\theta + \sigma'_{11}(kx + \dot{x})(k\dot{x} - u_1 \sin\theta)] \tag{3.13b}$$

$$\rho_2 \triangleq \left.\frac{dr_2}{dt}\right|_{\varepsilon=0} = -\sigma'_{22}(\dot{y} + \sigma_{21}(y + \dot{y})[u_1 \cos\theta - 1 + \sigma'_{21}(y + \dot{y})(\dot{y} + u_1 \cos\theta - 1)] \tag{3.13c}$$

$$\begin{aligned}\varphi_1 \triangleq \left.\frac{d^2\bar{r}_1}{dt^2}\right|_{\varepsilon=0} =& - \sigma''_{12}(\dot{x} + \sigma_{11}(kx + \dot{x}))[-u_1 \sin\theta + \sigma'_{11}(kx + \dot{x})(k\dot{x} - u_1 \sin\theta)]^2 \\ & - \sigma'_{12}(\dot{x} + \sigma_{11}(kx + \dot{x}))(-u_1\dot{\theta}\cos\theta - \dot{u}_1 \sin\theta + \sigma''_{11}(kx + \dot{x})(k\dot{x} - u_1 \sin\theta)^2 \\ & + \sigma'_{11}(kx + \dot{x})(-ku_1 \sin\theta - u_1\dot{\theta}\cos\theta - \dot{u}_1 \sin\theta))\end{aligned} \tag{3.13d}$$

$$\begin{aligned}\varphi_2 \triangleq \left.\frac{d^2r_2}{dt^2}\right|_{\varepsilon=0} =& - \sigma''_{22}(\dot{y} + \sigma_{21}(y + \dot{y}))[u_1 \cos\theta - 1 + \sigma'_{21}(y + \dot{y})(\dot{y} + u_1 \cos\theta - 1)]^2 \\ & - \sigma'_{22}(\dot{y} + \sigma_{21}(y + \dot{y}))[-u_1\dot{\theta}\sin\theta + \dot{u}_1 \cos\theta + \sigma''_{21}(y + \dot{y})(\dot{y} + u_1 \cos\theta - 1)^2 \\ & + \sigma'_{21}(y + \dot{y})(u_1 \cos\theta - 1 + -u_1\dot{\theta}\sin\theta + \dot{u}_1 \cos\theta)]\end{aligned} \tag{3.13e}$$

$$\mu_1 \triangleq \left.\frac{du_1}{dt}\right|_{\varepsilon=0} = \frac{k^2\bar{r}_1\rho_1 + (1 + r_2)\rho_2}{u_1} \tag{3.13f}$$

Remark 3.3 *One can easily verify, from the above stated equations, that if $x = y = \theta = \dot{x} = \dot{y} = \dot{\theta} = 0$, then $r_1 = r_2 = \theta_d = 0$, $u_1 = 1$, $\omega_d = \alpha_d = u_2 = 0$, and consequently $\ddot{x} = \ddot{y} = \ddot{\theta} = 0$.*

3.3 Main Result

Theorem 3.1 *Consider the PVTOL aircraft dynamics (3.1) with input saturation bounds $U_1 > 1$ and $U_2 > 0$. Let the input thrust u_1 be defined as in (3.3),(3.5),(3.6), with constant k and parameters (L^+_{ij}, M^+_{ij}, N^+_{ij}, L^-_{ij}, M^-_{ij}, N^-_{ij}) of the twice differentiable 2-level linear saturation functions $\sigma_{ij}(\cdot)$ in (3.5) and (3.6) satisfying inequalities (3.8), and the input rolling moment u_2 as in (3.4),(3.7),(3.11),(3.12), with parameters (L_{mn}, M_{mn}) of the linear saturation functions $\sigma_{mn}(\cdot)$ in (3.4) satisfying inequalities (3.9). Then, provided that k and ε are sufficiently small,*

(i) global asymptotic stability of the closed-loop system trivial solution $(x, y, \theta)(t) \equiv (0, 0, 0)$ is achieved, with

(ii) $0 < 1 - M^+_{22} \le u_1(t) \le \sqrt{(kM_{12})^2 + \left(1 + M^-_{22}\right)^2} < U_1$ and $|u_2(t)| \le M_{41} + M_{32} < U_2$, $\forall t \ge 0$.

Proof. Item (ii) of the statement results from the definition of u_1, u_2, r_1, and r_2. Its proof is consequently straightforward. We, thus, focus on the proof of item (i). Let us consider the state vector

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} \triangleq \begin{pmatrix} x \\ \dot{x} \\ y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{pmatrix} \tag{3.14}$$

evolving within the normed state space $(\mathbb{R}^6, \|\cdot\|)$. The closed-loop system dynamics can be represented as

$$\dot{z} = f(z) \triangleq \begin{pmatrix} z_2 \\ -u_1 \sin z_5 + \varepsilon u_2 \cos z_5 \\ z_4 \\ u_1 \cos z_5 + \varepsilon u_2 \sin z_5 - 1 \\ z_6 \\ u_2 \end{pmatrix}$$

where u_1 and u_2 take the form defined in equations (3.3) and (3.4). One can easily verify that $f(0_6) = 0_6$; see Remark 3.3. The present stability analysis is carried out showing that, under such a state space representation, provided that ε and k are small enough, the origin is, on the one hand, asymptotically stable and, on the other, globally attractive (see Definition 2.1 in Section 2.2).

The asymptotic stability of the origin is proved by the linearization method (*i.e.* indirect or first Lyapunov method, see Subsection 2.2.2), considering that, provided that k is small enough, within a sufficiently small neighborhood around the origin, we have that the values of all the saturation functions in (3.4) to (3.6) are equal to their respective arguments (see Appendix A), *i.e.*

$$\begin{aligned} r_1 &= -2kz_2 - k^2 z_1 \\ r_2 &= -2z_4 - z_3 \\ u_2 &= \alpha_d - 2(z_6 - \omega_d) - (z_5 - \theta_d) \end{aligned}$$

Under this consideration, the Jacobian matrix of $f(z)$ evaluated at the origin, $A \triangleq [\partial f / \partial z]_{z=0_6}$, is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \varepsilon k^2 & 2\varepsilon k(k+1) & 0 & 0 & -\varepsilon[k(k+4)+1] & -2\varepsilon(k+1) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ k^2 & 2k(k+1) & 0 & 0 & -k(k+4)-1 & -2(k+1) \end{pmatrix}$$

Further, its characteristic polynomial, $P(\lambda) \triangleq |\lambda I - A|$, is given by

$$P(\lambda) = (\lambda+1)^2 \left[\lambda^4 + 2(k+1)(1-\varepsilon k)\lambda^3 + (k^2+4k+1-\varepsilon k^2)\lambda^2 + 2k(k+1)\lambda + k^2\right]$$

Applying the Routh-Hurwitz criterion to $P(\lambda)$ (see Appendix B), one can verify that if $\varepsilon k < 0.8$, the origin of the closed-loop system is indeed asymptotically stable.

The proof of the global attractivity of the origin is divided in 6 parts. The first part shows that θ_d, ω_d, and α_d are bounded signals whose bounds are directly influenced by the parameter k. The second part shows that for any initial condition vector $z(0) \in I\!R^6$, provided that k is small enough, there exists a finite time $t_2 \geq 0$ after which the trajectories of the rotational motion dynamics evolve within a positively invariant set $\mathcal{S}_0 \subset I\!R^2$ where the value of every linear saturation function $\sigma_{mn}(\cdot)$ in (3.4) is equal to its argument. By defining $\dot{\theta}_d = \frac{d\theta_d}{dt}\big|_{\varepsilon \geq 0}$ and the error variable vector $e = (e_1 \quad e_2)^T \triangleq (z_5 - \theta_d \quad z_6 - \dot{\theta}_d)^T$, the third part shows that, for any $z(t_2) \in I\!R^4 \times \mathcal{S}_0$, there exists a finite time $t_3 \geq t_2$ such that $\|e(t)\| \leq \varepsilon k B_{\bar{e}}$, $\forall t \geq t_3$, for some $B_{\bar{e}} > 0$, or equivalently $e(t) \in \mathcal{B}_1 \triangleq \{e \in I\!R^2 : \|e\| \leq \varepsilon k B_{\bar{e}}\}$, $\forall t \geq t_3$. By defining $z_T \triangleq (z_1 \quad z_2 \quad z_3 \quad z_4)^T$ and $\zeta = (z_T^T \quad e^T)^T$, the fourth part shows that for any $\zeta(t_3) \in I\!R^4 \times \mathcal{B}_1$, provided that εk is small enough, there exists a finite time $t' \geq t_3$ after which the trajectories of the translational motion closed-loop dynamics, $z_T(t)$, evolve within a positively invariant set $\mathcal{S}_{12} \subset I\!R^4$ where every linear saturation function $\sigma_{ij}(\cdot)$ in (3.5) and (3.6) is equal to its argument. The fifth part shows that, for any $\zeta(t') \in \mathcal{S}_{12} \times \mathcal{B}_1$, there exists a finite time $t_8 \geq t'$ such that $\|\zeta(t)\| \leq \varepsilon k B_{\bar{\zeta}}$, for some $B_{\bar{\zeta}} > 0$, or equivalently $\zeta(t) \in \mathcal{B}_2 \triangleq \{\zeta \in I\!R^6 : \|\zeta\| \leq \varepsilon k B_{\bar{\zeta}}\}$, $\forall t \geq t_8$. The sixth part proves that for any $\zeta(t_8) \in \mathcal{B}_2$, provided that ε is small enough, $\zeta(t) \to 0_6$ as $t \to \infty$. Since $\zeta = 0_6 \Leftrightarrow z = 0_6$, and in view of the intermediate results obtained in the precedent parts, global attractivity of the origin of the closed-loop system is finally concluded.

First part. From the strictly increasing nature of the arctan function and the definition of r_1 and r_2 in (3.5) and (3.6), it can be seen that $|\theta_d(t)| \leq B_{\theta_d}$ (see (3.10)), $\forall t \geq 0$. Furthermore, note that

$$\frac{\partial B_{\theta_d}}{\partial k} = \frac{M_{12}(1-M_{22}^+)}{(kM_{12})^2 + (1-M_{22}^+)^2} \leq \frac{M_{12}}{1-M_{22}^+} \qquad \forall k > 0$$

whence we have

$$B_{\theta_d} \leq \frac{M_{12}}{1-M_{22}^+} \cdot k \qquad \forall k > 0$$

which bears out the direct influence of k on B_{θ_d}. Now, twice differentiability of $\sigma_{ij}(s)$ ($i = 1,2; j = 1,2$) on $I\!R$ guarantees boundedness of $\sigma'_{ij}(s)$ and $\sigma''_{ij}(s)$ on $[-N_{ij}^-, N_{ij}^+]$ (see Theorem 4.17 in (Apostol, 1974)), *i.e.* there exist positive constants A_{ij} and B_{ij} such that $|\sigma'_{ij}(s)| \leq A_{ij}$ and $|\sigma''_{ij}(s)| \leq B_{ij}$, $\forall s \in [-N_{ij}^-, N_{ij}^+]$. On the other hand, $\sigma'_{ij}(s) = \sigma''_{ij}(s) = 0$ when $|s| \geq N_{ij}^{\pm}$. Therefore, for any scalar $p > 0$, $|s^p \sigma'_{ij}(s)| \leq N_{ij}^p A_{ij}$ and $|s^p \sigma''_{ij}(s)| \leq N_{ij}^p B_{ij}$, $\forall s \in I\!R$, $\forall i,j = 1,2$, with $N_{ij} \triangleq \max\{N_{ij}^-, N_{ij}^+\}$. Hence (see equations (3.13))

$$\begin{aligned}
|\rho_1(t)| &\le A_{12}[B_{u_1} + A_{11}C_1] && \triangleq B_{\rho_1} \\
|\rho_2(t)| &\le A_{22}[B_{u_1} + A_{21}C_2 + 1] && \triangleq B_{\rho_2} \\
|\mu_1(t)| &\le \frac{M_{12}B_{\rho_1}}{1 - M_{22}^{+}} + B_{\rho_2} && \triangleq B_{\mu_1}
\end{aligned}$$

$\forall t \ge 0$, with $B_{u_1} \triangleq \sqrt{M_{12}^2 + (1 + M_{22}^{+})^2}$, $C_1 \triangleq N_{12} + M_{11} + B_{u_1}$, and $C_2 \triangleq N_{22} + M_{21} + B_{u_1} + 1$. Therefore,

$$|\omega_d(t)| \le kB_{\bar{\omega}_d} \quad \forall t \ge 0$$

with

$$B_{\bar{\omega}_d} \triangleq \frac{M_{12}B_{\rho_2}}{(1 - M_{22}^{+})^2} + \frac{B_{\rho_1}}{(1 - M_{22}^{+})}$$

(see Equations (3.11)), showing the boundedness of ω_d and the direct influence of k on its bound. Furthermore, assuming the existence of a finite time $t_1 \ge 0$ such that $|\theta(t)| \le D$, $\forall t \ge t_1$, for some initial-condition-independent positive constant D,[2] we have (see Equations (3.13))

$$\begin{aligned}
|\varphi_1(t)| &\le B_{12}\left(\frac{B_{\rho_1}}{A_{12}}\right)^2 + A_{12}[C_3 + B_{11}C_1^2 + A_{11}C_4] && \triangleq B_{\varphi_1} \\
|\varphi_2(t)| &\le B_{22}\left(\frac{B_{\rho_2}}{A_{22}}\right)^2 + A_{22}[C_3 + B_{21}C_2^2 + A_{21}(C_4 + 1)] && \triangleq B_{\varphi_2}
\end{aligned}$$

$\forall t \ge t_1$, with $C_3 \triangleq \sqrt{(B_{u_1}D)^2 + B_{\mu_1}^2}$ and $C_4 \triangleq \sqrt{(B_{u_1}D)^2 + (B_{u_1} + B_{\mu_1})^2}$. As a result

$$|\alpha_d(t)| \le kB_{\bar{\alpha}_d} \quad \forall t \ge t_1 \tag{3.15}$$

with

$$B_{\bar{\alpha}_d} \triangleq \frac{M_{12}B_{\varphi_2}}{(1 - M_{22}^{+})^2} + \frac{B_{\varphi_1} + 2B_{\mu_1}B_{\bar{\omega}_d}}{(1 - M_{22}^{+})}$$

(see Equations (3.12)), which shows that the ultimate bound of α_d is also directly influenced by k.

Second part. Let us consider the state space representation, defined through (3.14), of the rotational motion closed-loop dynamics (3.1c) and (3.4):

$$\dot{z}_5 = z_6 \tag{3.16a}$$

$$\dot{z}_6 = \sigma_{41}(\alpha_d) - \sigma_{32}\big(z_6 - \sigma_{42}(\omega_d) + \sigma_{31}(z_6 - \sigma_{43}(\omega_d) + z_5 - \theta_d)\big) \tag{3.16b}$$

[2]Such an assumption will be proved to be satisfied with $D = M_{41} + M_{42} + M_{31}$ in the second part of the proof.

We define the positive scalar function $V_1 \triangleq z_6^2$. Its derivative along the trajectories of subsystem (3.16) is given by

$$\dot{V}_1 = 2z_6\dot{z}_6 = 2z_6[\sigma_{41}(\alpha_d) - \sigma_{32}(s_{32})] \tag{3.17}$$

where

$$s_{32} \triangleq z_6 - \sigma_{42}(\omega_d) + \sigma_{31}(z_6 - \sigma_{43}(\omega_d) + z_5 - \theta_d)$$

Suppose that $z_6 > M_{41} + M_{42} + M_{31} > 0$. Under such an assumption, we have

$$\begin{aligned} s_{32} &= z_6 - \sigma_{42}(\omega_d) + \sigma_{31}(\cdot) \\ &> z_6 - M_{42} - M_{31} \\ &> M_{41} > 0 \end{aligned}$$

Then, according to Definition 3.1, either $\sigma_{32}(\cdot) \in (0, L_{32}]$, implying

$$\begin{aligned} \dot{z}_6 &= \sigma_{41}(\cdot) - z_6 + \sigma_{42}(\cdot) - \sigma_{31}(\cdot) \\ &< M_{41} + M_{42} + M_{31} - z_6 \\ &< 0 \end{aligned}$$

or $\sigma_{32}(\cdot) \in (L_{32}, M_{32}]$ entailing

$$\begin{aligned} \dot{z}_6 &= \sigma_{41}(\cdot) - \sigma_{32}(\cdot) \\ &< M_{41} - L_{32} \\ &< M_{41} + 2M_{42} + 2M_{31} - L_{32} \\ &< 0 \end{aligned}$$

(see (3.9b)), *i.e.*

$$z_6 > M_{41} + M_{42} + M_{31} > 0 \Longrightarrow \dot{z}_6 < 0 \tag{3.18}$$

Similarly if $z_6 < -M_{41} - M_{42} - M_{31} < 0$ then

$$\begin{aligned} s_{32} &= z_6 - \sigma_{42}(\omega_d) + \sigma_{31}(\cdot) \\ &< z_6 + M_{42} + M_{31} \\ &< -M_{41} < 0 \end{aligned}$$

Hence, either $\sigma_{32}(\cdot) \in [-L_{32}, 0)$, entailing

$$\begin{aligned} \dot{z}_6 &= \sigma_{41}(\cdot) - z_6 + \sigma_{42}(\cdot) - \sigma_{31}(\cdot) \\ &> -M_{41} - M_{42} - M_{31} - z_6 \\ &> 0 \end{aligned}$$

or $\sigma_{32}(\cdot) \in [-M_{32}, -L_{32})$ implying

$$\begin{aligned}\dot{z}_6 &= \sigma_{41}(\cdot) - \sigma_{32}(\cdot) \\ &> -M_{41} + L_{32} \\ &> -M_{41} - 2M_{42} - 2M_{31} + L_{32} \\ &> 0\end{aligned}$$

i.e.

$$z_6 < -M_{41} - M_{42} - M_{31} < 0 \Longrightarrow \dot{z}_6 > 0 \tag{3.19}$$

Hence, from (3.18) and (3.19), one sees that

$$|z_6| > M_{41} + M_{42} + M_{31} \Longrightarrow \text{sign}(z_6) = -\text{sign}(\dot{z}_6) \Longrightarrow \dot{V}_1 < 0$$

This proves that, for any initial state vector $z(0) \in I\!R^6$, there exists a finite time $t_1 \geq 0$ such that

$$|z_6(t)| \leq M_{41} + M_{42} + M_{31} \triangleq D$$

$\forall t \geq t_1$.[3] Then, for all $t \geq t_1$, we have

$$\begin{aligned}|s_{32}| &\leq |z_6| + M_{42} + M_{31} \\ &\leq M_{41} + 2M_{42} + 2M_{31} \\ &< L_{32}\end{aligned}$$

(see (3.9b)). Therefore, according to Definition 3.1, $\sigma_{32}(s_{32}) = s_{32}$ and (3.16b) becomes

$$\dot{z}_6 = \sigma_{41}(\alpha_d) - z_6 + \sigma_{42}(\omega_d) - \sigma_{31}(z_6 - \sigma_{43}(\omega_d) + z_5 - \theta_d) \tag{3.20}$$

from t_1 on. At this stage, let $q \triangleq z_5 + z_6$ and $V_2 \triangleq q^2$. The derivative of V_2 along the trajectories of subsystem (3.16a) and (3.20) is given by

$$\dot{V}_2 = 2q\dot{q} = 2q[\sigma_{41}(\alpha_d) + \sigma_{42}(\omega_d) - \sigma_{31}(s_{31})]$$

where

$$s_{31} \triangleq q - \sigma_{43}(\omega_d) - \theta_d$$

Following a similar reasoning that the one developed for the analysis of (3.17), suppose that $q > M_{41} + M_{42} + M_{43} + B_{\theta_d}$. Then

$$\begin{aligned}s_{31} &= q - \sigma_{43}(\cdot) - \theta_d \\ &> q - M_{43} - B_{\theta_d} \\ &> M_{41} + M_{42} > 0\end{aligned}$$

[3] Recall that this was assumed in the first part of the proof. Thus, it is demonstrated that such an assumption is actually a fact.

Hence, either $\sigma_{31}(\cdot) \in (0, L_{31}]$, implying

$$\begin{aligned}\dot{q} &= \sigma_{41}(\cdot) + \sigma_{42}(\cdot) - s_{31} \\ &= \sigma_{41}(\cdot) + \sigma_{42}(\cdot) - q + \sigma_{43}(\cdot) + \theta_d \\ &< M_{41} + M_{42} + M_{43} + B_{\theta_d} - q \\ &< 0\end{aligned}$$

or $\sigma_{31}(\cdot) \in (L_{31}, M_{31}]$ entailing

$$\begin{aligned}\dot{q} &= \sigma_{41}(\cdot) + \sigma_{42}(\cdot) - \sigma_{31}(\cdot) \\ &< M_{41} + M_{42} - L_{31} \\ &< M_{41} + M_{42} + 2M_{43} + 2B_{\theta_d} - L_{31} \\ &< 0\end{aligned}$$

(see (3.9c)), *i.e.*

$$q > M_{41} + M_{42} + M_{43} + B_{\theta_d} > 0 \Longrightarrow \dot{q} < 0 \tag{3.21}$$

Similarly if $q < -M_{41} - M_{42} - M_{43} - B_{\theta_d} < 0$, we get

$$\begin{aligned}s_{31} &= q - \sigma_{43}(\cdot) - \theta_d \\ &< q + M_{43} + B_{\theta_d} \\ &< -M_{41} - M_{42} < 0\end{aligned}$$

Therefore, either $\sigma_{31}(\cdot) \in [-L_{31}, 0)$, implying

$$\begin{aligned}\dot{q} =& \sigma_{41}(\cdot) + \sigma_{42}(\cdot) - s_{31} \\ =& \sigma_{41}(\cdot) + \sigma_{42}(\cdot) - q + \sigma_{43}(\cdot) + \theta_d \\ >& - M_{41} - M_{42} - M_{43} - B_{\theta_d} - q \\ >& 0\end{aligned}$$

or $\sigma_{31}(\cdot) \in [-M_{31}, -L_{31})$ entailing

$$\begin{aligned}\dot{q} &= \sigma_{41}(\cdot) + \sigma_{42}(\cdot) - \sigma_{31}(\cdot) \\ &> -M_{41} - M_{42} + L_{31} \\ &> -M_{41} - M_{42} - 2M_{43} - 2B_{\theta_d} + L_{31} \\ &> 0\end{aligned}$$

i.e.

$$q < -M_{41} - M_{42} - M_{43} - B_{\theta_d} < 0 \Longrightarrow \dot{q} > 0 \tag{3.22}$$

Thus form (3.21) and (3.22), one sees that

$$|q| > M_{41} + M_{42} + M_{43} + B_{\theta_d} \Longrightarrow \mathrm{sign}(q) = -\mathrm{sign}(\dot{q}) \Longrightarrow \dot{V}_2 < 0$$

proving that, for any $z(0) \in I\!R^6$, there exists a finite time $t_2 \geq t_1$ such that

$$|q(t)| \leq M_{41} + M_{42} + M_{43} + B_{\theta_d}$$

$\forall t \geq t_2$. Hence, for all $t \geq t_2$, we have

$$\begin{aligned} |s_{31}| &\leq |q| + M_{43} + B_{\theta_d} \\ &< M_{41} + M_{42} + 2M_{43} + 2B_{\theta_d} \\ &< L_{31} \end{aligned}$$

(see (3.9c)). Thus, according to Definition 3.1, $\sigma_{31}(s_{31}) = s_{31}$ and (3.20) becomes

$$\dot{z}_6 = \sigma_{41}(\alpha_d) - (z_6 - \sigma_{42}(\omega_d)) - (z_6 - \sigma_{43}(\omega_d)) - (z_5 - \theta_d) \tag{3.23}$$

from t_2 on. Now, from the first part of the proof, one sees that a sufficiently small k can be chosen such that $|\omega_d(t)| < \min\{L_{42}, L_{43}\}$ and $|\alpha_d(t)| < L_{41}$, $\forall t \geq t_1$. Therefore, provided that such a choice of k is made, the linear saturation functions in (3.23) are equal to their (respective) arguments (according to Definition 3.1) from t_1 on. Hence, $\forall t \geq t_2$, the rotational motion closed-loop dynamics becomes

$$\begin{aligned} \dot{z}_5 &= z_6 \\ \dot{z}_6 &= \alpha_d - 2(z_6 - \omega_d) - (z_5 - \theta_d) \end{aligned}$$

or equivalently

$$\ddot{\theta} = \alpha_d - 2(\dot{\theta} - \omega_d) - (\theta - \theta_d) = u_2 \tag{3.25}$$

Observe that this part of the proof shows that for any $z(0) \in I\!R^6$, provided that k is small enough,

$$\begin{aligned} (\theta(t), \dot{\theta}(t)) \in \mathcal{S}_0 \triangleq \big\{(\theta, \dot{\theta}) \in I\!R^2 :\ & |\dot{\theta}| \leq M_{41} + M_{42} + M_{31}, \\ & |\theta + \dot{\theta}| \leq M_{41} + M_{42} + M_{43} + B_{\theta_d}\big\} \end{aligned}$$

$\forall t \geq t_2 \geq 0$, where every linear saturation in u_2 (see (3.4)) is equal to its argument.

Third part. Let

$$\dot{\theta}_d \triangleq \left.\frac{d\theta_d}{dt}\right|_{\varepsilon \geq 0} \quad \text{and} \quad \ddot{\theta}_d \triangleq \left.\frac{d^2\theta_d}{dt^2}\right|_{\varepsilon \geq 0}$$

From the definition of θ_d in equation (3.7), the system dynamics in (3.1), and the proposed scheme, we get, from t_2 on (consequently taking u_2 as in(3.25)):

$$\dot{\theta}_d = \omega_d + \varepsilon k \Delta_1 \tag{3.26}$$

with

$$\Delta_1 = \frac{u_2}{u_1^2}\Big[\,(1+r_2)\,\sigma_{12}'\,(\dot{x}+\sigma_{11}\,(kx+\dot{x}))\,[1+\sigma_{11}'\,(kx+\dot{x})]\cos\theta \\ -\,\overline{r}_1\sigma_{22}'\,(\dot{y}+\sigma_{21}\,(y+\dot{y}))\,[1+\sigma_{21}'\,(y+\dot{y})]\sin\theta\Big] \tag{3.27}$$

and

$$\ddot{\theta}_d = \alpha_d + \varepsilon k \Delta_2 \tag{3.28}$$

with

$$\begin{aligned}
\Delta_2 = -\frac{u_2}{u_1^2}&\left[\left[\overline{r}_1\Delta_{\dot{\rho}_2} + \Delta_{\dot{\overline{r}}_1}\rho_2 - \Delta_{\dot{r}_2}\rho_1 - (1+r_2)\,\Delta_{\dot{\rho}_1}\right] - \frac{2\Delta_{\dot{u}_1}}{u_1}\overline{\omega}_d\right] \\
&- \left[\frac{\dot{u}_2}{u_1^2} - 2\frac{\dot{u}_1 u_2}{u_1^3}\right]\left[\overline{r}_1\sigma_{22}'(s_{22})\,[1+\sigma_{21}'(s_{21})]\sin\theta - (1+r_2)\sigma_{12}'(s_{12})\,[1+\sigma_{11}'(s_{11})]\cos\theta\right] \\
&- \frac{u_2}{u_1^2}\Big[\rho_1\left[\sigma_{22}'(s_{22})\,[1+\sigma_{21}'(s_{21})]\sin\theta\right] - \rho_2\left[\sigma_{12}'(s_{12})\,[1+\sigma_{11}'(s_{11})]\cos\theta\right] \\
&\qquad + \overline{r}_1\Big[\sigma_{22}''(s_{22})\big[+\sigma_{21}'(s_{21})\,(\dot{y}+u_1\cos\theta-1)+u_1\cos\theta-1\big]\,[1+\sigma_{21}'(s_{21})] \\
&\qquad\qquad + \sigma_{22}'(s_{22})\sigma_{21}''(s_{21})\,(\dot{y}+u_1\cos\theta-1)\Big]\sin\theta \\
&\qquad + \dot{\theta}\Big[\overline{r}_1\sigma_{22}'(s_{22})\,[1+\sigma_{21}'(s_{21})]\cos\theta + (1+r_2)\sigma_{12}'(s_{12})\,[1+\sigma_{11}'(s_{11})]\sin\theta\Big] \\
&\qquad - (1+r_2)\Big[\sigma_{12}''(s_{12})\big[\sigma_{11}'(s_{11})\,(k\dot{x}-u_1\sin\theta)-u_1\sin\theta\big]\,[1+\sigma_{11}'(s_{11})] \\
&\qquad\qquad + \sigma_{12}'(s_{12})\sigma_{11}''(s_{11})\,(k\dot{x}-u_1\sin\theta)\Big]\cos\theta \\
&\qquad + \varepsilon u_2\overline{r}_1\Big[\sigma_{22}''(s_{22})\,[1+\sigma_{21}'(s_{21})]^2 + \sigma_{22}'(s_{22})\sigma_{21}''(s_{21})\Big]\sin^2\theta \\
&\qquad - \varepsilon u_2(1+r_2)\Big[\sigma_{12}''(s_{12})\,[1+\sigma_{11}'(s_{11})]^2 + \sigma_{12}'(s_{12})\sigma_{11}''(s_{11})\Big]\cos^2\theta\Big]
\end{aligned} \tag{3.29}$$

See Appendix C for details on the procedure to get these expressions as well as the definitions of s_{ij}, $i,j=1,2$, Δ_{ρ_2}, $\Delta_{\dot{\overline{r}}_1}$, $\Delta_{\dot{r}_2}$, Δ_{ρ_1}, $\Delta_{\dot{u}_1}$, $\dot{u}_1$, and $\dot{u}_2$.

Remark 3.4 *Observe that every term involved in Δ_1 and Δ_2 is bounded and recall that $u_1 \geq 1 - M_{22}^+ > 0$, wherefrom we conclude that there exist positive constants B_{Δ_1} and B_{Δ_2} such that $|\Delta_1| \leq B_{\Delta_1}$ and $|\Delta_2| \leq B_{\Delta_2}$ for any value of the system states. See Appendix D for an estimation of these bounds.*

Let

$$e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \triangleq \begin{pmatrix} \theta - \theta_d \\ \dot{\theta} - \dot{\theta}_d \end{pmatrix}$$

From equations (3.25), (3.26), and (3.28), we have that

$$\dot{e} = A_0 e + h(t,e) \tag{3.30}$$

from t_2 on, with

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \qquad \text{and} \qquad h(t,e) = \varepsilon k \begin{pmatrix} 0 \\ 2\Delta_1 + \Delta_2 \end{pmatrix}$$

(where the trajectories of the translational motion dynamics, involved in h, are being considered external time-varying functions).

Let us define a quadratic positive definite function $V_3(e) \triangleq e^T P_0 e$, where P_0 is the (unique) solution of the Lyapunov equation $P_0 A_0 + A_0^T P_0 = -I_2$, *i.e*

$$P_0 = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

For such a P_0 we have that

$$\lambda_{\max}(P_0) = \frac{2+\sqrt{2}}{2} \qquad \text{and} \qquad \lambda_{\min}(P_0) = \frac{2-\sqrt{2}}{2} > 0$$

The derivative of $V_3(e)$ along the trajectories of subsystem (3.30) is given by

$$\begin{aligned} \dot{V}_3(e) &= e^T P_0 [A_0 e + h(t,e)] + [A_0 e + h(t,e)]^T P_0 e \\ &= -e^T e + 2e^T P_0 h(t,e) \\ &\leq -\|e\|^2 + 2\lambda_{\max}(P_0)\|e\|\|h\| \\ &\leq -\|e\|^2 + \varepsilon k(2+\sqrt{2})\|e\|(2B_{\Delta_1} + B_{\Delta_2}) \end{aligned}$$

(see Remark 3.4). Defining $B_\Delta \triangleq 2B_{\Delta_1} + B_{\Delta_2}$, we can rewrite the foregoing inequality as

$$\dot{V}_3(e) \leq -(1-\phi_1)\|e\|^2 - \|e\| \left[\phi_1 \|e\| - \varepsilon k(2+\sqrt{2})B_\Delta\right]$$

where ϕ_1 is a strictly positive constant less than unity, *i.e.* $0 < \phi_1 < 1$. Then

$$\dot{V}_3(e) \leq -(1-\phi_1)\|e\|^2, \qquad \forall \quad \|e\| \geq \frac{\varepsilon k(2+\sqrt{2})B_\Delta}{\phi_1}$$

Thus, from Theorem 2.4 in Chapter 2, there exists a finite time $t_3 \geq t_2$ such that

$$\|e(t)\| \leq \frac{\varepsilon k(2+\sqrt{2})B_\Delta}{\phi_1}\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} = \frac{\varepsilon k B_\Delta}{\phi_1}(4+3\sqrt{2}) = \varepsilon k B_{\bar{e}} \tag{3.31}$$

$\forall t \geq t_3$, with

$$B_{\bar{e}} \triangleq \frac{(4+3\sqrt{2})B_\Delta}{\phi_1}$$

In other words, for any $z(t_2) \in I\!R^4 \times \mathcal{S}_0$,

$$e(t) \in \mathcal{B}_1 \triangleq \left\{e \in I\!R^2 : \|e\| \leq \varepsilon k B_{\bar{e}}\right\} \qquad t \geq t_3 \tag{3.32}$$

Fourth part. Let

$$z_T = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \quad \text{and} \quad \zeta = \begin{pmatrix} z_T \\ e \end{pmatrix}$$

Remark 3.5 *One can easily verify from the expressions defining θ_d and $\dot{\theta}_d$ that $\zeta = 0_6 \Leftrightarrow z = 0_6$.*

Observe that, from t_3 on, the translational motion closed-loop dynamics, (3.1a), (3.1b), (3.3)–(3.7), can be expressed as

$$\dot{z}_1 = z_2 \tag{3.33a}$$

$$\dot{z}_2 = -k\sigma_{12}(z_2 + \sigma_{11}(kz_1 + z_2)) + R_1(\zeta) \tag{3.33b}$$

$$\dot{z}_3 = z_4 \tag{3.33c}$$

$$\dot{z}_4 = -\sigma_{22}(z_4 + \sigma_{21}(z_3 + z_4)) + R_2(\zeta) \tag{3.33d}$$

where

$$R_1(\zeta) = -u_1\left[\sin(e_1 + \theta_d) - \sin\theta_d\right] + \varepsilon u_2 \cos(e_1 + \theta_d)$$

and

$$R_2(\zeta) = u_1\left[\cos(e_1 + \theta_d) - \cos\theta_d\right] + \varepsilon u_2 \sin(e_1 + \theta_d)$$

with

$$u_2 = \alpha_d - 2e_2 - e_1 + 2\varepsilon k\Delta_1 \tag{3.34}$$

Let us note that from (3.31), (3.34), and the facts that $|\alpha_d| \le kB_{\bar{\alpha}_d}$ (see (3.15)), $|\sin(e_1 + \theta_d) - \sin\theta_d| \le |e_1|$, $|\cos(e_1 + \theta_d) - \cos\theta_d| \le |e_1|$, $|e_1| \le \|e\|$, and $|2e_2 + e_1| = |(1 \quad 2)e| \le \|(1 \quad 2)\|\|e\| = \sqrt{5}\|e\|$, we have

$$|R_i(\zeta(t))| \le \varepsilon k\left[B_{\bar{\alpha}_d} + 2\varepsilon B_{\Delta_1} + B'B_{\bar{e}}\right] = \varepsilon k B_{\bar{R}_i}$$

$i = 1, 2$, $\forall t \ge t_3$, with

$$B_{\bar{R}_i} \triangleq B_{\bar{\alpha}_d} + 2\varepsilon B_{\Delta_1} + B'B_{\bar{e}}$$

where $B' \triangleq B_{u_1} + \sqrt{5}\varepsilon$ and $B_{u_1} \triangleq \sqrt{(kM_{21})^2 + \left(1 + M_{22}^-\right)^2}$. Further, observe that in view of the uniform boundedness of the terms involved in the translational motion closed-loop dynamics, *i.e.* (3.1a), (3.1b), (3.3), and (3.4), $z_T(t)$ exists and is bounded at any finite time t.

We begin by analyzing the vertical motion closed-loop dynamics, *i.e.* equations (3.33c) and (3.33d). Let us define

$$\gamma \triangleq \min\{L_{21}, L_{22} - 2M_{21}, kL_{11}, k(L_{12} - 2M_{11})\} \tag{3.35}$$

and suppose that the product εk is small enough to satisfy

$$\gamma > \varepsilon k B_{\bar{R}_i}$$

We define the positive scalar function $V_4 = z_4^2$. Its derivative along the system trajectories is given by

$$\dot{V}_4 = 2z_4\dot{z}_4 = 2z_4\left[-\sigma_{22}(z_4 + \sigma_{21}(z_3 + z_4)) + R_2(\zeta)\right] \tag{3.36}$$

Suppose for the moment that $z_4 > M_{21} + \varepsilon k B_{\bar{R}_i} > 0$. Under such an assumption, we have

$$\begin{aligned} s_{22} &= z_4 + \sigma_{21}(\cdot) \\ &> z_4 - M_{21} \\ &> \varepsilon k B_{\bar{R}_i} > 0 \end{aligned}$$

Then, according to Definition 3.2, either $\sigma_{22}(\cdot) \in (0, L_{22}^+]$ implying

$$\begin{aligned} \dot{z}_4 &= -z_4 - \sigma_{21}(\cdot) + R_2(\zeta) \\ &< -z_4 + M_{21} + \varepsilon k B_{\bar{R}_i} \\ &< 0 \end{aligned}$$

or $\sigma_{22}(\cdot) \in (L_{22}^+, M_{22}^+]$ entailing

$$\begin{aligned} \dot{z}_4 &= -\sigma_{22}(\cdot) + R_2(\zeta) \\ &< -L_{22}^+ + \varepsilon k B_{\bar{R}_i} \\ &< -L_{22}^+ + \gamma \\ &< 0 \end{aligned}$$

since $\gamma \leq L_{21} \leq M_{21} < \frac{L_{22}}{2} < L_{22} \leq L_{22}^+$ (see (3.35)). Hence,

$$z_4 > M_{21} + \varepsilon k B_{\bar{R}_i} > 0 \implies \dot{z}_4 < 0 \tag{3.37}$$

Similarly, if $z_4 < -M_{21} - \varepsilon k B_{\bar{R}_i} < 0$, which implies

$$\begin{aligned} s_{22} &= z_4 + \sigma_{21}(\cdot) \\ &< z_4 + M_{21} \\ &< -\varepsilon k B_{\bar{R}_i} < 0 \end{aligned}$$

then either $\sigma_{22}(\cdot) \in [-L_{22}^-, 0)$ entailing

$$\begin{aligned} \dot{z}_4 &= -z_4 - \sigma_{21}(\cdot) + R_2(\zeta) \\ &> -z_4 - M_{21} - \varepsilon k B_{\bar{R}_i} \\ &> 0 \end{aligned}$$

or $\sigma_{22}(\cdot) \in [-M_{22}^-, -L_{22}^-)$ implying

$$\begin{aligned}\dot{z}_4 &= -\sigma_{22}(\cdot) + R_2(\zeta) \\ &> L_{22}^- - \varepsilon k B_{\bar{R}_i} \\ &> L_{22}^- - \gamma \\ &> 0\end{aligned}$$

since $\gamma \leq L_{21} \leq M_{21} < \frac{L_{22}}{2} < L_{22} \leq L_{22}^-$ (see (3.35)). Thus,

$$z_4 < -M_{21} - \varepsilon k B_{\bar{R}_i} < 0 \implies \dot{z}_4 > 0 \tag{3.38}$$

Therefore, from (3.37) and (3.38), we see that

$$\begin{aligned}|z_4| &> M_{21} + \varepsilon k B_{\bar{R}_i} \\ &\implies \operatorname{sign}(z_4) \neq \operatorname{sign}(\dot{z}_4) \\ &\implies \dot{V}_4 < 0\end{aligned}$$

This proves that, for any $\zeta(t_3) \in I\!R^4 \times \mathcal{B}_1$, there exists a finite time $t_4 \geq t_3$ such that

$$|z_4(t)| \leq M_{21} + \varepsilon k B_{\bar{R}_i}$$

$\forall t \geq t_4$. Then, for all $t \geq t_4$, we have

$$\begin{aligned}|z_4 + \sigma_{21}(\cdot)| &\leq |z_4| + M_{21} \\ &\leq 2M_{21} + \varepsilon k B_{\bar{R}_i} \\ &< 2M_{21} + \gamma \leq L_{22}\end{aligned}$$

since $\gamma \leq L_{22} - 2M_{21}$ (see (3.35)). Consequently, according to item (a) of Definition 3.2,

$$\sigma_{22}(z_4 + \sigma_{21}(\cdot)) = z_4 + \sigma_{21}(\cdot)$$

and (3.33d) becomes

$$\dot{z}_4 = -z_4 - \sigma_{21}(z_3 + z_4) + R_2(\zeta)$$

from t_4 on. Let us now define $q_1 \triangleq z_3 + z_4$ and the scalar positive function $V_5 \triangleq q_1^2$. The derivative of V_5 along the system trajectories is given by

$$\dot{V}_5 = 2q_1\dot{q}_1 = 2q_1\left[-\sigma_{21}(q_1) + R_2(\zeta)\right]$$

Following a similar reasoning that the one developed for the analysis of (3.36), we first suppose $q_1 > \varepsilon k B_{\bar{R}_i} > 0$. Then either $\sigma_{21}(\cdot) \in (0, L_{21}]$, implying

$$\begin{aligned}\dot{q}_1 &= -q_1 + R_2(\zeta) \\ &< -q_1 + \varepsilon k B_{\bar{R}_i} \\ &< 0\end{aligned}$$

or $\sigma_{21}(\cdot) \in (L_{21}, M_{21}]$ entailing

$$\begin{aligned}\dot{q}_1 &= -\sigma_{21}(\cdot) + R_2(\zeta) \\ &< -L_{21} + \varepsilon k B_{\bar{R}_i} \\ &< -L_{21} + \gamma \\ &\leq 0\end{aligned}$$

since $\gamma \leq L_{21}$, (see (3.35)). Hence

$$q_1 > \varepsilon k B_{\bar{R}_i} > 0 \implies \dot{q}_1 < 0 \tag{3.39}$$

Similarly, assuming $q_1 < -\varepsilon k B_{\bar{R}_i} < 0$, either $\sigma_{21}(\cdot) \in [-L_{21}, 0)$, implying

$$\begin{aligned}\dot{q}_1 &= -q_1 + R_2(\zeta) \\ &> -q_1 - \varepsilon k B_{\bar{R}_i} \\ &> 0\end{aligned}$$

or $\sigma_{21}(\cdot) \in [-M_{21}, -L_{21})$, entailing

$$\begin{aligned}\dot{q}_1 &= -\sigma_{21}(\cdot) + R_2(\zeta) \\ &> L_{21} - \varepsilon k B_{\bar{R}_i} \\ &> L_{21} - \gamma \\ &\geq 0\end{aligned}$$

according to (3.35). Therefore

$$q_1 < -\varepsilon k B_{\bar{R}_i} < 0 \implies \dot{q}_1 > 0 \tag{3.40}$$

From (3.39) and (3.40), we conclude that

$$\begin{aligned}|q_1| &> \varepsilon k B_{\bar{R}_i} \\ &\implies \operatorname{sign}(q_1) \neq \operatorname{sign}(\dot{q}_1) \\ &\implies \dot{V}_5 < 0\end{aligned}$$

Hence, for any $\zeta(t_3) \in I\!R^4 \times \mathcal{B}_1$, there exists a finite time $t_5 \geq t_4$ such that

$$|q_1(t)| \leq \varepsilon k B_{\bar{R}_i} < \gamma \leq L_{21}$$

(see(3.35)), $\forall t \geq t_5$. Consequently, according to item (a) of Definition 3.2,

$$\sigma_{21}(z_3 + z_4) = z_3 + z_4$$

and (3.33d) becomes

$$\dot{z}_4 = -z_3 - 2z_4 + R_2(\zeta)$$

from t_5 on. At this point, we have that, for any $\zeta(t_3) \in I\!R^4 \times \mathcal{B}_1$, provided that εk is small enough,

$$(z_3(t), z_4(t)) \in \mathcal{S}_1 \triangleq \Big\{(z_3, z_4) \in I\!R^2 : |z_4| \leq M_{21} + \varepsilon k B_{\bar{R}_i}, \qquad |z_3 + z_4| \leq L_{21}\Big\} \tag{3.41}$$

$\forall t \geq t_5$, where all the 2-level linear saturation functions involved in r_2 (see (3.6)) are equal to their arguments.

Let us now analyze the horizontal motion closed-loop dynamics, *i.e.* equations (3.33a) and (3.33b). We define the positive scalar function $V_6 = z_2^2$. Its derivative along the system trajectories is given by

$$\dot{V}_6 = 2z_2\dot{z}_2 = 2z_2\left[-k\sigma_{12}(z_2 + \sigma_{11}(kz_1 + z_2)) + R_1(\zeta)\right]$$

Following a procedure similar to the one developed above for the analysis of (3.36), consider that $z_2 > M_{11} + \varepsilon B_{\bar{R}_i} > 0$. Under such an assumption, we have

$$\begin{aligned} s_{12} &= z_2 + \sigma_{11}(\cdot) \\ &> z_2 - M_{11} \\ &> \varepsilon B_{\bar{R}_i} > 0 \end{aligned}$$

Hence, either $\sigma_{12}(\cdot) \in (0, L_{12}]$ implying

$$\begin{aligned} \dot{z}_2 &= -kz_2 - k\sigma_{11}(\cdot) + R_1(\zeta) \\ &< -kz_2 + kM_{11} + \varepsilon k B_{\bar{R}_i} \\ &< 0 \end{aligned}$$

or $\sigma_{12}(\cdot) \in (L_{12}, M_{12}]$ entailing

$$\begin{aligned} \dot{z}_2 &= -k\sigma_{12}(\cdot) + R_1(\zeta) \\ &< -kL_{12} + \varepsilon k B_{\bar{R}_i} \\ &< -kL_{12} + \gamma \\ &< 0 \end{aligned}$$

since $\gamma \leq k(L_{12} - 2M_{11}) < kL_{12}$ (see (3.35)). Therefore,

$$z_2 > M_{11} + \varepsilon B_{\bar{R}_i} 0 \implies \dot{z}_2 < 0 \tag{3.42}$$

Similarly, if $z_2 < -M_{11} - \varepsilon B_{\bar{R}_i} < 0$, then

$$\begin{aligned} s_{12} &= z_2 + \sigma_{11}(\cdot) \\ &< z_2 + M_{11} \\ &< -\varepsilon B_{\bar{R}_i} < 0 \end{aligned}$$

Then either $\sigma_{12}(\cdot) \in [-L_{12}, 0)$ entailing

$$\begin{aligned}
\dot{z}_2 &= -kz_2 - k\sigma_{11}(\cdot) + R_1(\zeta) \\
&> -kz_2 - kM_{11} - \varepsilon k B_{\bar{R}_i} \\
&> 0
\end{aligned}$$

or $\sigma_{12}(\cdot) \in [-M_{12}, -L_{12})$ implying

$$\begin{aligned}
\dot{z}_2 &= -k\sigma_{12}(\cdot) + R_1(\zeta) \\
&> kL_{12} - \varepsilon k B_{\bar{R}_i} \\
&> kL_{12} - \gamma \\
&> 0
\end{aligned}$$

since $\gamma \leq k(L_{12} - 2M_{11}) < kL_{12}$ (see (3.35)). Thus,

$$z_2 < -M_{11} - \varepsilon B_{\bar{R}_i} < 0 \implies \dot{z}_2 > 0 \tag{3.43}$$

Therefore, from (3.42) and (3.43), we see that

$$\begin{aligned}
|z_2| &> M_{11} + \varepsilon B_{\bar{R}_i} \\
&\implies \operatorname{sign}(z_2) \neq \operatorname{sign}(\dot{z}_2) \\
&\implies \dot{V}_6 < 0
\end{aligned}$$

This proves that, for any $\zeta(t_3) \in I\!R^4 \times \mathcal{B}_1$, there exists a finite time $t_6 \geq t_3$ such that

$$|z_2(t)| \leq M_{11} + \varepsilon B_{\bar{R}_i}$$

$\forall t \geq t_6$. Then, for all $t \geq t_6$

$$|z_2 + \sigma_{11}(\cdot)| \leq |z_2| + M_{11} \leq 2M_{11} + \varepsilon B_{\bar{R}_i} < 2M_{11} + \gamma \leq L_{12}$$

since $\gamma \leq L_{12} - 2M_{11}$ (see (3.35)). Consequently, according to item (a) of Definition 3.2,

$$\sigma_{12}(z_2 + \sigma_{11}(\cdot)) = z_2 + \sigma_{11}(\cdot)$$

and (3.33b) becomes

$$\dot{z}_2 = -kz_2 - k\sigma_{11}(kz_1 + z_2) + R_1(\zeta) \tag{3.44}$$

from t_6 on. Let us now define $q_2 \triangleq kz_1 + z_2$ and the positive scalar function $V_7 \triangleq q_2^2$. The derivative of V_7 along the system trajectories is given by

$$\dot{V}_7 = 2q_2\dot{q}_2 = 2q_2\left[-k\sigma_{11}(q_2) + R_1(\zeta)\right]$$

With a similar reasoning that the one developed above, let us consider

$$q_2 > \varepsilon B_{\bar{R}_i} > 0$$

Then either $\sigma_{11}(\cdot) \in (0, L_{11}]$, implying

$$\begin{aligned}\dot{q}_2 &= -kq_2 + R_1(\zeta) \\ &< -kq_2 + \varepsilon k B_{\bar{R}_i} \\ &< 0\end{aligned}$$

or $\sigma_{21}(\cdot) \in (L_{11}, M_{11}]$ entailing

$$\begin{aligned}\dot{q}_2 &= -k\sigma_{11}(\cdot) + R_1(\zeta) \\ &< -kL_{11} + \varepsilon k B_{\bar{R}_i} \\ &< -kL_{11} + \gamma \\ &\leq 0\end{aligned}$$

according to (3.35). Therefore

$$q_2 > \varepsilon B_{\bar{R}_i} \implies \dot{q}_2 < 0 \tag{3.45}$$

Similarly, assume that $q_2 < -\varepsilon B_{\bar{R}_i} < 0$. Then either $\sigma_{11}(\cdot) \in [-L_{11}, 0)$, implying

$$\begin{aligned}\dot{q}_2 &= -kq_2 + R_1(\zeta) \\ &> -kq_2 - \varepsilon k B_{\bar{R}_i} \\ &> 0\end{aligned}$$

or $\sigma_{11}(\cdot) \in [-M_{11}, -L_{11})$ entailing

$$\begin{aligned}\dot{q}_2 &= -k\sigma_{11}(\cdot) + R_1(\zeta) \\ &> kL_{11} - \varepsilon k B_{\bar{R}_i} \\ &> kL_{11} - \gamma \\ &\geq 0\end{aligned}$$

according to (3.35). Thus

$$q_2 < -\varepsilon B_{\bar{R}_i} < 0 \implies \dot{q}_2 > 0 \tag{3.46}$$

Then, from (3.45), and (3.46) we conclude that

$$\begin{aligned}|q_2| &> \varepsilon B_{\bar{R}_i} \\ &\implies \operatorname{sign}(q_2) \neq \operatorname{sign}(\dot{q}_2) \\ &\implies \dot{V}_7 < 0\end{aligned}$$

Hence, for any $\zeta(t_3) \in I\!R^4 \times \mathcal{B}_1$, there exists a time $t_7 \geq t_6$ such that

$$|q_2(t)| \leq \varepsilon B_{\bar{R}_i} \leq \frac{\gamma}{k} \leq L_{11}$$

(see (3.35)), $\forall t \geq t_7$. Consequently, according to item (a) of Definition 3.2,

$$\sigma_{11}(kz_1 + z_2) = kz_1 + z_2$$

and (3.44) becomes

$$\dot{z}_2 = -k^2 - 2kz_2 + R_1(\zeta)$$

from t_7 on. Thus, we have that, for any $\zeta(t_3) \in I\!\!R^4 \times \mathcal{B}_1$, provided that εk is small enough,

$$(z_1(t), z_2(t)) \in \mathcal{S}_2 \triangleq \Big\{(z_1, z_2) \in I\!\!R^2 : |z_2| \leq M_{11} + \varepsilon B_{\bar{R}_i}, \qquad |kz_1 + z_2| \leq L_{11}\Big\} \tag{3.47}$$

$\forall t \geq t_7$, where all the 2-level linear saturation functions involved in r_1 (see (3.5)) are equal to their argument. Finally, from (3.41) and (3.47) we see that, for any $\zeta(t_3) \in I\!\!R^4 \times \mathcal{B}_1$, provided that εk is small enough,

$$z_T(t) \in \mathcal{S}_{12} \triangleq \mathcal{S}_1 \times \mathcal{S}_2 \qquad \forall t \geq t' \triangleq \max\{t_5, t_7\}$$

where every 2-level linear saturation in u_1 (see (3.3)) is equal to its argument, or more generally, considering (3.32),

$$\zeta(t) \in \mathcal{S}_{12} \times \mathcal{B}_1 \qquad \forall t \geq t' \tag{3.48}$$

where all the linear saturation in u_1 and u_2 (as defined in (3.3) and (3.4)) are equal to their arguments.

Fifth part. As a consequence of the precedent analysis, the closed-loop system may be expressed, from t' on, as

$$\dot{\zeta} = A_1\zeta + g(\zeta)$$

where

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -k^2 & -2k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -2 \end{pmatrix}$$

and

$$g(\zeta) = \begin{pmatrix} 0 \\ -u_1 \sin(e_1 + \theta_d) + u_1 \sin\theta_d + \varepsilon u_2 \cos(e_1 + \theta_d) \\ 0 \\ u_1 \cos(e_1 + \theta_d) - u_1 \cos\theta_d + \varepsilon u_2 \sin(e_1 + \theta_d) \\ 0 \\ \varepsilon k(2\Delta_1 + \Delta_2) \end{pmatrix}$$

The characteristic polynomial of A_1 is given by $|\lambda I_6 - A_1| = (\lambda + k)^2(\lambda + 1)^4$ wherefrom it is clear that A_1 is Hurwitz. Hence, according to Theorem 2.1 in Chapter 2, there exists a (unique) positive definite symmetric matrix P_1 that solves the Lyapunov equation $P_1 A_1 + A_1^T P_1 = -I_6$. Actually, one can verify that such a P_1 is given by the following matrix

$$P_1 = \begin{pmatrix} \frac{k^2+5}{4k} & \frac{1}{2k^2} & 0 & 0 & 0 & 0 \\ \frac{1}{2k^2} & \frac{k^2+1}{4k^3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

whose maximum and minimum eigenvalues are given by

$$\lambda_{\max}(P_1) = \frac{(k^2+2)(k^4+6k^2+1)}{8k^3} \quad \text{and} \quad \lambda_{\min}(P_1) = 1 - \frac{1}{\sqrt{2}} = \frac{1}{2+\sqrt{2}}$$

Let us, on the other hand, note that, on $\mathcal{S}_{12} \times \mathcal{B}_1$ (see (3.48) and (3.32)):

$$\begin{aligned}
\|g(\zeta)\|^2 = &\ (-u_1 \sin(e_1+\theta_d) + u_1 \sin\theta_d + \varepsilon u_2 \cos(e_1+\theta_d))^2 \\
&+ (u_1 \cos(e_1+\theta_d) - u_1\cos\theta_d + \varepsilon u_2 \sin(e_1+\theta_d))^2 + (\varepsilon k)^2 (2\Delta_1+\Delta_2)^2 \\
= &\ u_1^2 (\sin(e_1+\theta_d) - \sin\theta_d)^2 - 2\varepsilon u_1 u_2 \cos(e_1+\theta_d)(\sin(e_1+\theta_d) - \sin\theta_d) \\
&+ (\varepsilon u_2)^2 \cos^2(e_1+\theta_d) \\
&+ u_1^2(\cos(e_1+\theta_d) - \cos\theta_d)^2 + 2\varepsilon u_1 u_2 \sin(e_1+\theta_d)(\cos(e_1+\theta_d) - \cos\theta_d) \\
&+ (\varepsilon u_2)^2 \sin^2(e_1+\theta_d) + (\varepsilon k)^2(2\Delta_1+\Delta_2)^2 \\
= &\ u_1^2 \left[(\sin(e_1+\theta_d) - \sin\theta_d)^2 + (\cos(e_1+\theta_d) - \cos\theta_d)^2\right] \\
&- 2\varepsilon u_1 u_2 \left[\sin(e_1+\theta_d)\cos\theta_d - \sin\theta_d \cos(e_1+\theta_d)\right] \\
&+ (\varepsilon u_2)^2 \left[\sin^2(e_1+\theta_d) + \cos^2(e_1+\theta_d)\right] + (\varepsilon k)^2(2\Delta_1+\Delta_2)^2 \\
= &\ u_1^2 \left[(\sin(e_1+\theta_d) - \sin\theta_d)^2 + (\cos(e_1+\theta_d) - \cos\theta_d)^2\right] \\
&- 2\varepsilon u_1 u_2 \sin e_1 + (\varepsilon u_2)^2 + (\varepsilon k)^2 (2\Delta_1+\Delta_2)^2 \\
\leq &\ 2B_{u_1}^2 |e_1|^2 + 2\varepsilon k B_{u_1} B_{\bar{u}_2} |e_1| + (\varepsilon k)^2 B_{\bar{u}_2}^2 + (\varepsilon k)^2 (2B_{\Delta_1} + B_{\Delta_2})^2 \\
\leq &\ 2B_{u_1}^2 \|e\|^2 + 2\varepsilon k B_{u_1} B_{\bar{u}_2} \|e\| + (\varepsilon k)^2 B_{\bar{u}_2}^2 + (\varepsilon k)^2 (2B_{\Delta_1} + B_{\Delta_2})^2 \\
= &\ (\varepsilon k)^2 \left[2B_{u_1} B_{\bar{e}} (B_{u_1} B_{\bar{e}} + B_{\bar{u}_2}) + B_{\bar{u}_2}^2 + (2B_{\Delta_1} + B_{\Delta_2})^2\right] \\
= &\ (\varepsilon k)^2 B_{\bar{g}}^2
\end{aligned}$$

i.e.

$$\|g(\zeta)\| \leq \varepsilon k B_{\bar{g}}$$

with

$$B_{\bar{g}} \triangleq \left[2B_{u_1} B_{\bar{e}} (B_{u_1} B_{\bar{e}} + B_{\bar{u}_2}) + B_{\bar{u}_2}^2 + (2B_{\Delta_1} + B_{\Delta_2})^2\right]^{\frac{1}{2}}$$

where $B_{\bar{u}_2} = B_{\bar{\alpha}_d} + \varepsilon\sqrt{5} B_{\bar{e}} + 2\varepsilon B_{\Delta_1}$ and the facts that $\alpha_d \leq k B_{\bar{\alpha}_d}$ (see (3.15)), $|2e_2+e_1| = |(1 \quad 2)e| \leq \|(1 \quad 2)\|\|e\| = \sqrt{5}\|e\|$, $|\sin(e_1+\theta_d) - \sin\theta_d| \leq |e_1|$, $|\cos(e_1+\theta_d) - \cos\theta_d| \leq |e_1|$, $|e_1| \leq \|e\|$, and (3.31), have been considered.

Now, let us define the quadratic Lyapunov candidate function $V_8(\zeta) = \zeta^T P_1 \zeta$. On $\mathcal{S}_{12} \times \mathcal{B}_1$ (see (3.48)), its derivative along the system trajectories is given by

$$\begin{aligned}
\dot{V}_8(\zeta) &= \zeta^T P_1 [A_1 \zeta + g(\zeta)] + [A_1\zeta + g(\zeta)]^T P_1 \zeta \\
&= -\zeta^T \zeta + 2\zeta^T P_1 g(\zeta) \\
&\leq -\|\zeta\|^2 + 2\lambda_{\max}(P_1)\|\zeta\|\|g(\zeta)\| \\
&\leq -(1-\phi_2)\|\zeta\|^2 - \phi_2\|\zeta\|^2 + 2\varepsilon k B_{\bar{g}} \lambda_{\max}(P_1)\|\zeta\| \\
&\leq -(1-\phi_2)\|\zeta\|^2 \qquad , \qquad \forall \|\zeta\| > \frac{2\varepsilon k B_{\bar{g}} \lambda_{\max}(P_1)}{\phi_2}
\end{aligned}$$

where ϕ_2 is a strictly positive constant less than unity, *i.e.* $0 < \phi_2 < 1$. Thus, according to Theorem 2.4 in Chapter 2, there exists a finite time $t_8 \geq t'$ such that

$$\|\zeta(t)\| \leq \frac{2\varepsilon k B_g \lambda_{\max}(P_1)}{\phi_2} \sqrt{(2+\sqrt{2})\lambda_{\max}(P_1)} = \varepsilon k B_{\bar{\zeta}} \tag{3.49}$$

for all $t \geq t_8$, with

$$B_{\bar{\zeta}} \triangleq \frac{2B_g \lambda_{\max}(P_1)}{\phi_2} \sqrt{(2+\sqrt{2})\lambda_{\max}(P_1)}$$

In other words, for any $\zeta(t') \in \mathcal{S}_{12} \times \mathcal{B}_1$,

$$\zeta(t) \in \mathcal{B}_2 \triangleq \left\{ \zeta \in I\!\!R^6 : \|\zeta\| \leq \varepsilon k B_{\bar{\zeta}} \right\}$$

$\forall t \geq t_8$, where, according to the precedent parts of the proof, every linear saturation in u_1 and u_2 is equal to its argument.

Remark 3.6 *Observe that $\mathcal{B}_2$ is a positively invariant compact set containing 0_6.*

Sixth part. From t_8 on, the closed loop system dynamics may be written as

$$\dot{\zeta} = A_2 \zeta + \bar{g}(\zeta)$$

where

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -k^2 & -2k & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -2 \end{pmatrix}$$

and

$$\bar{g}(\zeta) = \tilde{g}(\zeta) + \varepsilon \hat{g}(\zeta) \tag{3.50}$$

with

$$\tilde{g}(\zeta) = \begin{pmatrix} 0 \\ -u_1[\sin(e_1+\theta_d) - u_1 \sin\theta_d] + e_1 \\ 0 \\ u_1[\cos(e_1+\theta_d) - u_1 \cos\theta_d] \\ 0 \\ 0 \end{pmatrix}$$

and

$$\hat{g}(\zeta) = \begin{pmatrix} 0 \\ u_2 \cos(e_1+\theta_d) \\ 0 \\ u_2 \sin(e_1+\theta_d) \\ 0 \\ k(2\Delta_1 + \Delta_2) \end{pmatrix}$$

with ζ evolving in $\mathcal{B}_2$ where $\sigma_{ij}(s_{ij}) = s_{ij}$ in u_1 and $\sigma_{mn}(s_{mn}) = s_{mn}$ in u_2, and consequently $\sigma'_{ij}(\cdot) = 1$, $\sigma''_{ij}(\cdot) = \sigma'''_{ij}(\cdot) = 0$, and $\sigma'_{mn}(\cdot) = 1$, $\sigma''_{mn}(\cdot) = 0$.

Let us note that, after several basic developments, we have

$$\begin{aligned}
\frac{\partial \tilde{g}_2}{\partial z_i} &= (1 - \cos e_1) i k^{3-i} && \forall i = 1, 2 \\
\frac{\partial \tilde{g}_2}{\partial z_j} &= (j - 2) \sin e_1 && \forall j = 3, 4 \\
\frac{\partial \tilde{g}_2}{\partial e_1} &= -u_1[\cos(e_1 + \theta_d) - \cos\theta_d] - r_2 \\
\frac{\partial \tilde{g}_2}{\partial e_2} &= 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \tilde{g}_4}{\partial z_i} &= -i k^{3-i} \sin e_1 && \forall i = 1, 2 \\
\frac{\partial \tilde{g}_4}{\partial z_j} &= (2 - j)(\cos e_1 - 1) && \forall j = 3, 4 \\
\frac{\partial \tilde{g}_4}{\partial e_1} &= -u_1[\sin(e_1 + \theta_d) - \sin\theta_d] + r_1 \\
\frac{\partial \tilde{g}_4}{\partial e_2} &= 0
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{i=1}^{6} \left| \frac{\partial \tilde{g}_2}{\partial \zeta_i} \right| &\leq k(k+2)|1 - \cos e_1| + 3|\sin e_1| + u_1|\cos(e_1 + \theta_d) - \cos\theta_d| + |(1 \quad 2) \cdot (z_3 \quad z_4)^T| \\
&\leq (k^2 + 2k)|e_1| + 3|e_1| + B_{u_1}|e_1| + \sqrt{5}\|(z_3 \quad z_4)\| \\
&\leq (k^2 + 2k + 3 + \sqrt{5} + B_{u_1})\|\zeta\|
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{6} \left| \frac{\partial \tilde{g}_4}{\partial \zeta_i} \right| &\leq k(k+2)|\sin e_1| + 3|\cos e_1 - 1| + u_1|\sin(e_1 + \theta_d) - \sin\theta_d| + |(k^2 \quad 2k) \cdot (z_1 \quad z_2)^T| \\
&\leq (k^2 + 2k)|e_1| + 3|e_1| + B_{u_1}|e_1| + k\sqrt{k^2 + 4}\|(z_1 \quad z_2)\| \\
&\leq (k^2 + 2k + 3 + k\sqrt{k^2 + 4} + B_{u_1})\|\zeta\| \\
&\leq (k^2 + 2k + 3 + \sqrt{5} + B_{u_1})\|\zeta\|
\end{aligned}$$

where the facts that $|\sin e_1| \leq |e_1|$, $|\cos e_1 - 1| \leq |e_1|$, $|e_1| \leq \|\zeta\|$, $|(1 \quad 2)(z_3 \quad z_4)^T| \leq \|(1 \quad 2)\| \cdot \|(z_3 \quad z_4)\| = \sqrt{5}\|(z_3 \quad z_4)\|$, $|(k^2 \quad 2k)(z_1 \quad z_2)^T| \leq \|(k^2 \quad 2k)\| \cdot \|(z_1 \quad z_2)\| \leq$

$k\sqrt{k^2+4}\|(z_1 \quad z_2)\| \leq \sqrt{5}\|(z_1 \quad z_2)\|$, and $\|(z_j \quad z_{j+1})\| \leq \|\zeta\|$ with $j = 1$ or $j = 3$, were considered. Then,

$$\left\|\frac{\partial \tilde{g}}{\partial \zeta}\right\|_\infty \leq (k^2 + 2k + 3 + \sqrt{5} + B_{u_1})\|\zeta\|$$

and consequently

$$\left\|\frac{\partial \tilde{g}}{\partial \zeta}\right\| \leq \sqrt{6}(k^2 + 2k + 3 + \sqrt{5} + B_{u_1})\|\zeta\|$$

since $\left\|\frac{\partial \tilde{g}}{\partial \zeta}\right\| \leq \sqrt{6}\left\|\frac{\partial \tilde{g}}{\partial \zeta}\right\|_\infty$ (see for instance (Khalil, 1996, Exercise 2.2)). Hence

$$\left\|\frac{\partial \tilde{g}}{\partial \zeta}\right\| \leq \varepsilon k B_{\tilde{g}} \qquad \forall \zeta \in \mathcal{B}_2$$

with

$$B_{\tilde{g}} \triangleq \sqrt{6}(k^2 + 2k + 3 + \sqrt{5} + B_{u_1})\frac{2B_g\lambda_{\max}(P)}{\phi_2}\sqrt{(2+\sqrt{2})\lambda_{\max}(P)}$$

where (3.49) has been considered. From this and the easily verifiable fact that $\tilde{g}(0_6) = 0_6$, we have that $\|\tilde{g}(\zeta)\| \leq \varepsilon k B_{\tilde{g}}\|\zeta\|$, $\forall \zeta \in \mathcal{B}_2$, according to Lemma 2.3 in Chapter 2. On the other hand, by analyzing every term involved in $\hat{g}(\zeta)$, one can easily see that $\hat{g}(\zeta)$ is continuously differentiable on $\mathcal{B}_2$. Hence, the Jacobian matrix of $\hat{g}(\zeta)$, $\frac{\partial \hat{g}}{\partial \zeta}$, exists and is continuous on $\mathcal{B}_2$. Moreover, $\forall \zeta \in \mathcal{B}_2$, $\frac{\partial \hat{g}}{\partial \zeta}$ is bounded in view of the compactness of $\mathcal{B}_2$, and consequently $L = \max_{\zeta \in \mathcal{B}_2}\left\|\frac{\partial \hat{g}}{\partial \zeta}\right\|$ exists and is finite. From this and the easily verifiable fact that $\hat{g}(0_6) = 0_6$, we have that $\|\hat{g}(\zeta)\| \leq L\|\zeta\|$, $\forall \zeta \in \mathcal{B}_2$, according to Lemma 2.3 in Chapter 2. Thus, from (3.50), we have that

$$\|\bar{g}(\zeta)\| \leq \|\tilde{g}(\zeta)\| + \varepsilon\|\hat{g}(\zeta)\| \leq \varepsilon\tilde{B}\|\zeta\|$$

$\forall \zeta \in \mathcal{B}_2$, with $\tilde{B} = kB_{\tilde{g}} + L$.

Now, the characteristic polynomial of A_2 is given by $|\lambda I_6 - A_2| = (\lambda + k)^2(\lambda + 1)^4$ whence one sees that A_2 is Hurwitz. Then, according to Theorem 2.1 in Chapter 2, there exists a (unique) symmetric positive definite matrix P_2 that solves the Lyapunov equation $P_2A_2 + A_2^TP_2 = -I_6$. Consider the positive definite scalar function $V_9(\zeta) = \zeta^TP_2\zeta$. Its derivative along the closed-loop system trajectories is given by

$$\begin{aligned}
\dot{V}_9(\zeta) &= \zeta^TP_2[A_2\zeta + \bar{g}(\zeta)] + [A_2\zeta + \bar{g}(\zeta)]^TP_2\zeta \\
&= -\zeta^T\zeta + 2\zeta^TP_2\bar{g}(\zeta) \\
&\leq -\|\zeta\|^2 + 2\lambda_{\max}(P_2)\|\zeta\|\|\bar{g}(\zeta)\| \\
&\leq -\|\zeta\|^2 + 2\varepsilon\tilde{B}\lambda_{\max}(P_2)\|\zeta\|^2 \\
&\leq -\left(1 - 2\varepsilon\tilde{B}\lambda_{\max}(P_2)\right)\|\zeta\|^2
\end{aligned}$$

$\forall \zeta \in \mathcal{B}_2$. Then for a sufficiently small value of ε, such that $\varepsilon < \dfrac{1}{2\tilde{B}\lambda_{\max}(P_2)}$, $\dot{V}_9(\zeta)$ is negative definite on $\mathcal{B}_2$. Moreover, recall that $\mathcal{B}_2$ is compact and positively invariant (see Remark 3.6). Observe, on the other hand, that $E \triangleq \{\zeta \in \mathcal{B}_2 : \dot{V}_9(\zeta) = 0\} = \{0_6\}$. Consequently, the largest invariant set contained in E is E itself. Therefore, from LaSalle's invariance principle (see Theorem 2.3 in Chapter 2), we conclude that $\zeta(t) \to 0_6$ as $t \to \infty$, for any $\zeta(t_8) \in \mathcal{B}_2$. Finally, from the precedent parts of the proof and Remark 3.5, we conclude that $z(t) \to 0_6$ as $t \to \infty$, for any $z(0) \in I\!\!R^6$. Thus, by Definition 2.1, global asymptotic stability of the origin is concluded. □

Remark 3.7 *Let us note that if $\varepsilon = 0$, in which case $\dot{\theta}_d = \omega_d$ and $\ddot{\theta}_d = \alpha_d$, then the third part proves that, for any $z(t_2) \in I\!\!R^4 \times \mathcal{S}_0$, $e(t) \to 0_2$ as $t \to \infty$. Further, through the application of La Salle's invariance principle, the fifth part proves that, for any $\zeta(t') \in \mathcal{S}_{12} \times \mathcal{B}_1$, $\zeta(t) \to 0_6$ as $t \to \infty$. Consequently, in the $\varepsilon = 0$ case, the fifth part ends the proof.*

Remark 3.8 *The global character of the asymptotic stability of the closed loop trivial solution, whose proof has just been developed, holds in the Euclidean space where the coordinates used to express the system dynamics were considered to evolve.*

Chapter 4

Simulation Results

Simulation results using MATLAB© and SIMULINK© are presented in this chapter. First, the twice differentiable 2-level linear saturation function used in (Zavala-Río, *et al.*, 2003) was used to define every linear saturation function involved in the proposed approach; then a three-times differentiable one presented here was used for the same purpose. The former is given by

$$\sigma(s) = \begin{cases} -M^{-} & \text{if} \quad s \leq -N^{-} \\ -R_{-}(-s) & \text{if} \quad s \in (-N^{-}, -L^{-}) \\ s & \text{if} \quad s \in [-L^{-}, L^{+}] \\ R_{+}(s) & \text{if} \quad s \in (L^{+}, N^{+}) \\ M^{+} & \text{if} \quad s \geq N^{+} \end{cases} \tag{4.1}$$

where

$$R_{\pm}(s) = \frac{(s - M^{\pm})^{6}}{48\,(M^{\pm} - L^{\pm})^{5}} - \frac{5\,(s - M^{\pm})^{2}}{16\,(M^{\pm} - L^{\pm})} + \frac{(s - M^{\pm})}{2} + \frac{19M^{\pm} + 5L^{\pm}}{24}$$

and the latter is defined as

$$\sigma(s) = \begin{cases} -M^{-} & \text{if} \quad s \leq -N^{-} \\ -P_{-}(-s) & \text{if} \quad s \in (-N^{-}, -L^{-}) \\ s & \text{if} \quad s \in [-L^{-}, L^{+}] \\ P_{+}(s) & \text{if} \quad s \in (L^{+}, N^{+}) \\ M^{+} & \text{if} \quad s \geq N^{+} \end{cases} \tag{4.2}$$

where

$$P_{\pm}(s) = \frac{(s - M^{\pm})^{4}}{16\,(M^{\pm} - L^{\pm})^{3}} - \frac{3\,(s - M^{\pm})^{2}}{8\,(M^{\pm} - L^{\pm})} + \frac{(s - M^{\pm})}{2} + \frac{13M^{\pm} + 3L^{\pm}}{16}$$

and $N^{\pm} = 2M^{\pm} - L^{\pm}$, with $M^{\pm} > L^{\pm}$, for both saturation functions.

The form of function (4.2) and those of its first and second derivatives with respect to its argument, with $L^{+} = 0.2$, $L^{-} = 0.2$, $M^{+} = 0.5$, $M^{-} = 0.4$, $N^{+} = 0.8$, and $N^{-} = 0.6$, are shown in Figure 4.1. That of function (4.1), with the same constants $L^{+}, M^{+}, N^{+}, L^{-}, M^{-}, N^{-}$, is presented in Figure 4.2.

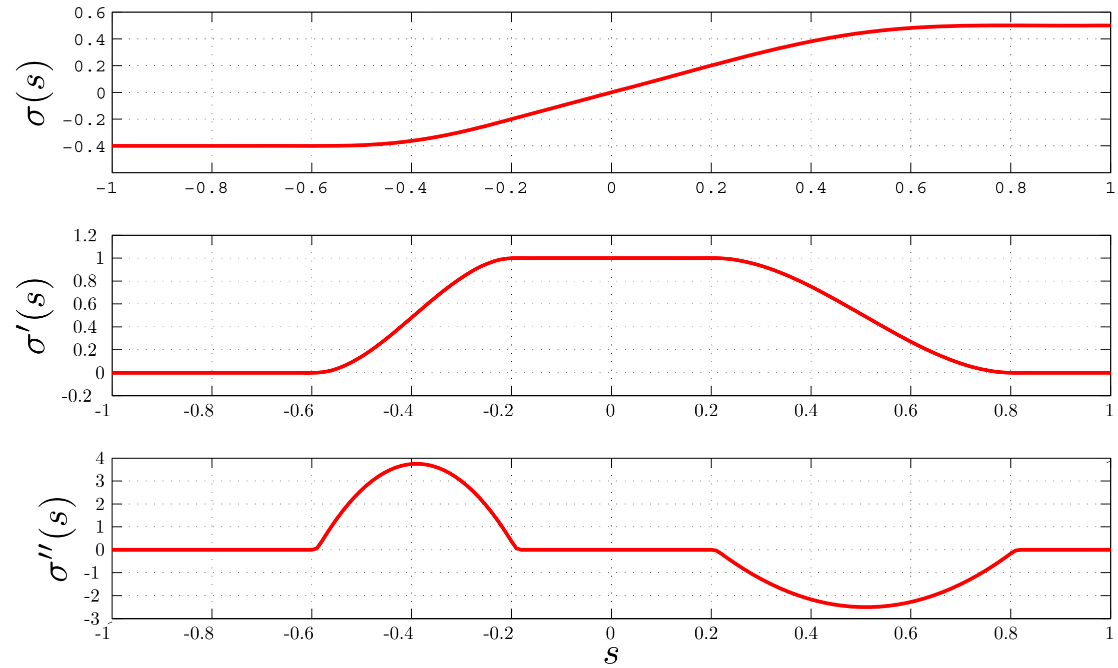

Figure 4.1: Twice differentiable 2-level linear saturation function and its first two derivatives with respect to its argument

The following initial conditions where taken

$$\left(x(0), \dot{x}(0), y(0), \dot{y}(0), \theta(0), \dot{\theta}(0)\right) = \left(50, 0, 50, 0, \frac{3\pi}{5}, 0\right)$$

and

$$\left(x(0), \dot{x}(0), y(0), \dot{y}(0), \theta(0), \dot{\theta}(0)\right) = \left(100, 1, 100, 1, \frac{3\pi}{7}, 1\right) \tag{4.3}$$

The values corresponding to the input bounds, control gain, and saturation function parameters were defined as: $U_1 = 10$, $U_2 = 5$, $k = 0.028$, $M_{22}^{-} = 8$, $L_{22}^{-} = 7$, $M_{21}^{-} = 0.3$, $L_{21}^{-} = 0.2$, $M_{22}^{+} = 0.8$, $L_{22}^{+} = 0.7$, $M_{21}^{+} = 0.3$, $L_{21}^{+} = 0.2$, $M_{12} = 0.8$, $L_{12} = 0.7$, $M_{11} = 0.3$, $L_{11} = 0.2$, $M_{32} = 4$, $L_{32} = 3.2$, $M_{31} = 0.808$, $L_{31} = 0.436$, $M_{41} = 0.032$, $L_{41} = 0.026$, $M_{42} = 0.032$, $L_{42} = 0.026$, $M_{43} = 0.047$, and $L_{43} = 0.037$. These values, which satisfy the conditions required by the proposed algorithm, were taken from (Zavala-Río, *et al.*, 2003). Let us recall that, in the developed framework, we conventionally consider stabilization of the configuration variables (x, y, θ) towards $(0, 0, 0)$.

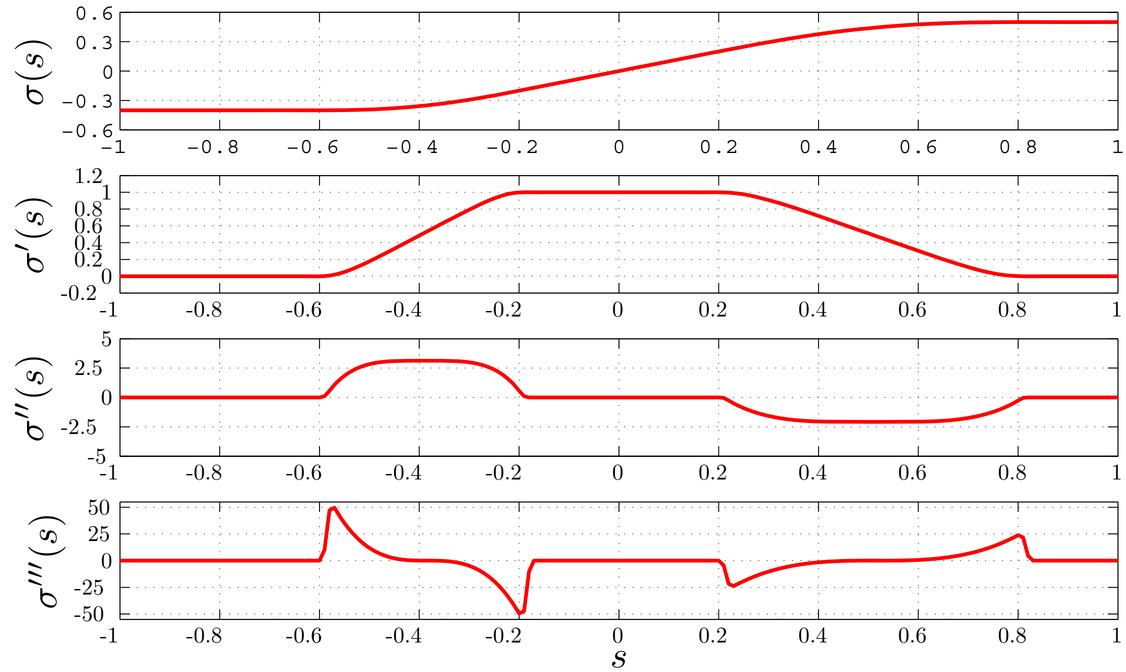

Figure 4.2: Three-times differentiable 2-level linear saturation function and its first two derivatives with respect to its argument

As observed in (Zavala-Río, *et al.*, 2003), convergence to the origin is preserved even when relatively large values of ε are taken. Figure 4.3 shows the differences between the system states when $\varepsilon = 0$, $\varepsilon = 0.5$, and $\varepsilon = 1$. The most significant difference can be appreciated in the horizontal motion response but after 2000s the responses of the system with $\varepsilon = 0.5$ or $\varepsilon = 1$ behaves like the one of the system with $\varepsilon = 0$, also notice that due to the small value of k the time response of x is larger than that of y. Regarding the vertical and rotational motion responses, performance differences among the trajectories for each ε case are almost imperceptible. Figure 4.4 compares the behavior of the control inputs u_1 and u_2 when $\varepsilon = 0$, $\varepsilon = 0.5$, and $\varepsilon = 1$. Notice that regardless of the value that ε takes, the control inputs remain within their saturation bounds.

When the three-times differentiable two-level linear saturation function is used there are no significant differences in the responses of the system states or the system control inputs (see Figures 4.5 and 4.6).

Further, simulations were run with the initial condition vector stated in (4.3) using the saturation function described in (4.2). The results are shown in Figures 4.7 and 4.8. The most considerable difference is the faster rate of convergence due to the smaller value chosen for θ. Convergence is preserved and the control inputs continue in the range of their saturation bounds regardless of the value of ε.

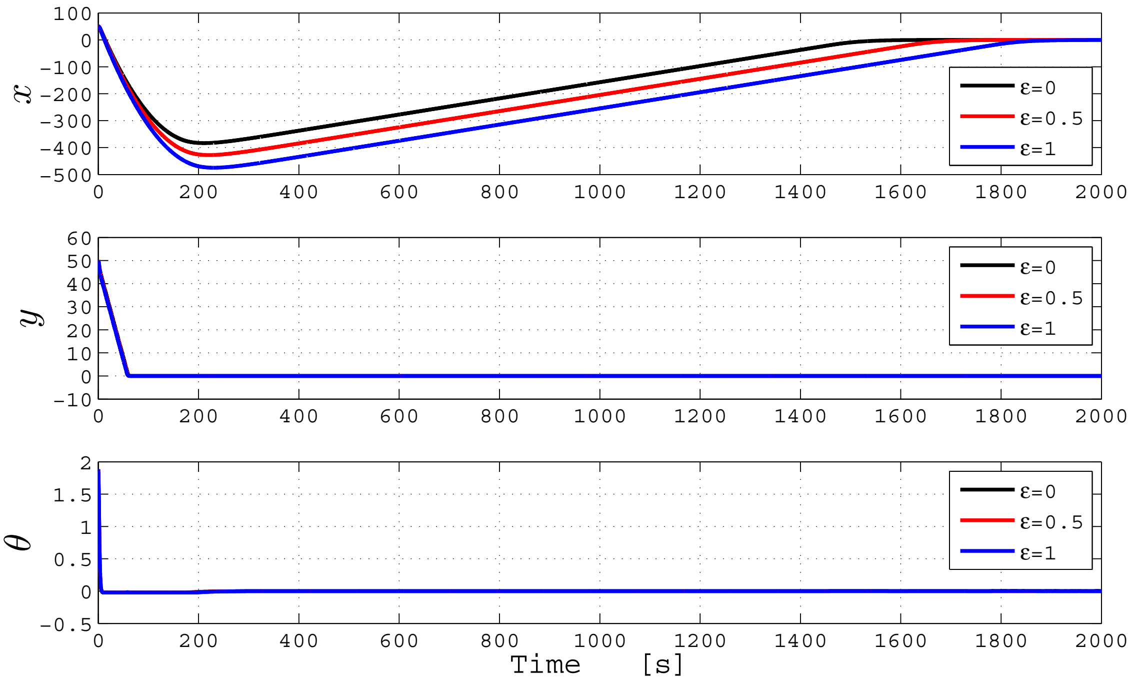

Figure 4.3: Comparison between positions when $\varepsilon = 0$, $\varepsilon = 0.5$, and $\varepsilon = 1$ using the two-times differentiable 2-level saturation function

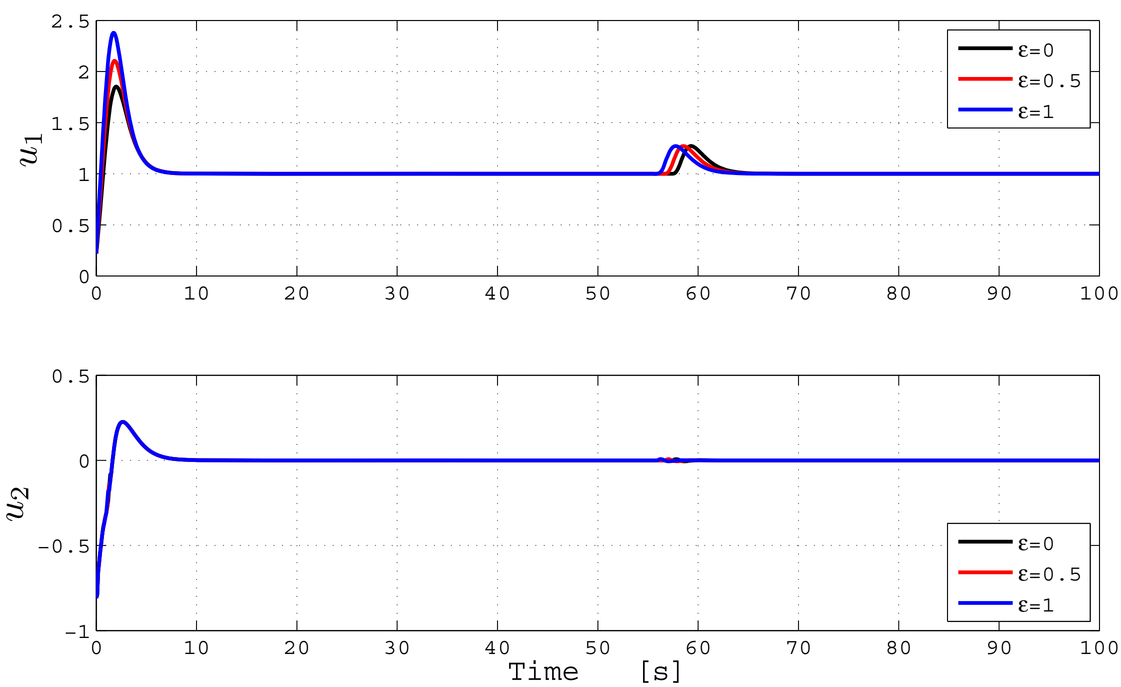

Figure 4.4: Comparison between control inputs when $\varepsilon = 0$, $\varepsilon = 0.5$, and $\varepsilon = 1$ using the two-times differentiable 2-level saturation function

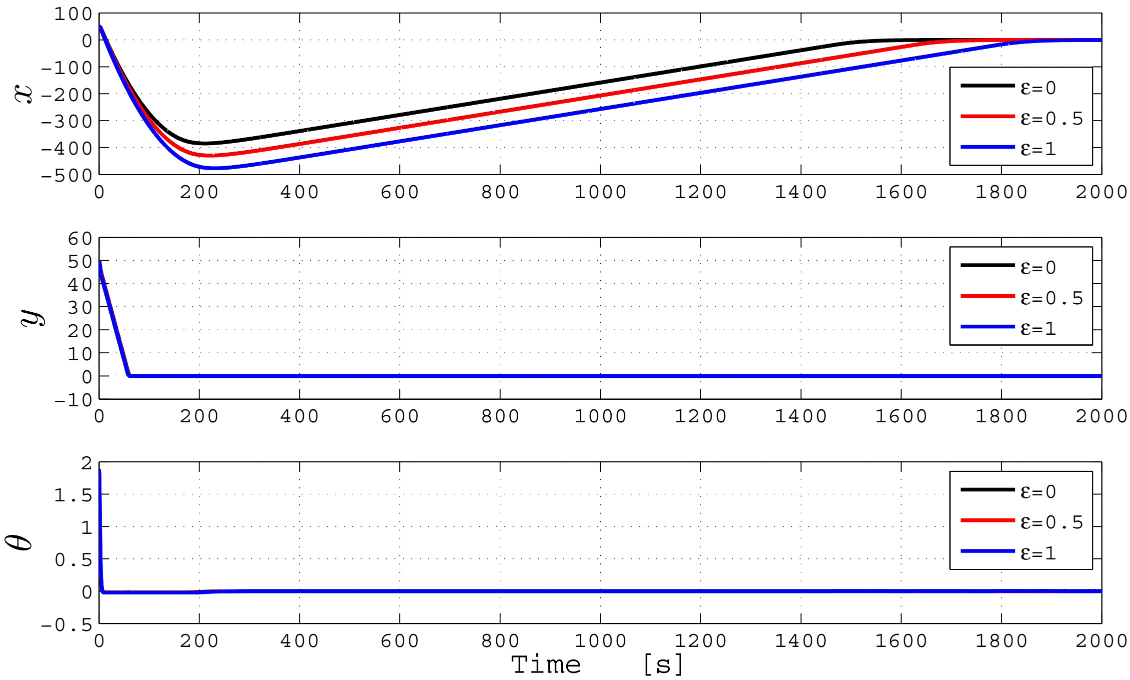

Figure 4.5: Comparison between positions when $\varepsilon = 0$, $\varepsilon = 0.5$, and $\varepsilon = 1$ using the three-times differentiable saturation function

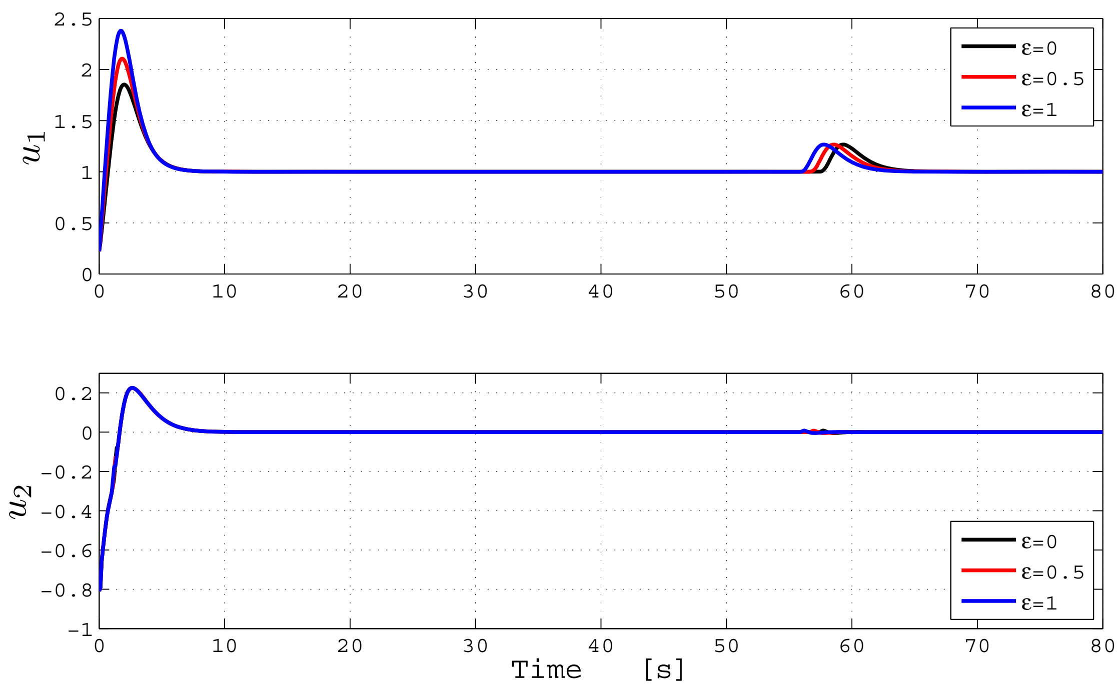

Figure 4.6: Comparison between control inputs when $\varepsilon = 0$, $\varepsilon = 0.5$, and $\varepsilon = 1$ using the three-times differentiable saturation function

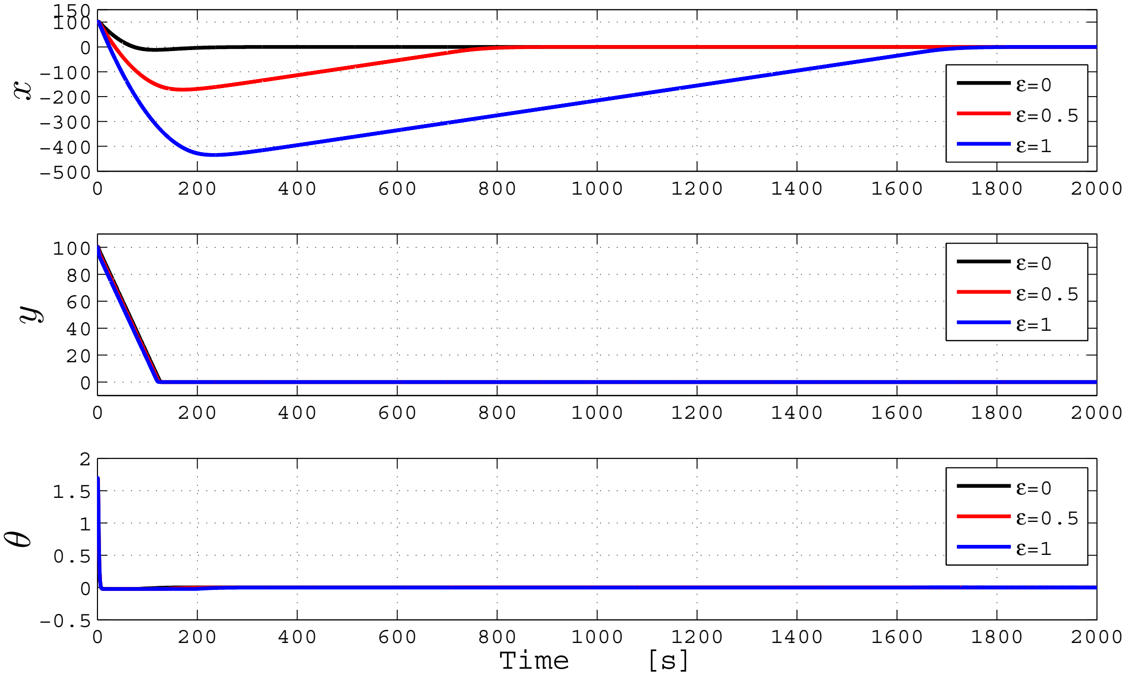

Figure 4.7: Comparison between positions when $\varepsilon = 0$, $\varepsilon = 0.5$, and $\varepsilon = 1$ for different initial conditions.

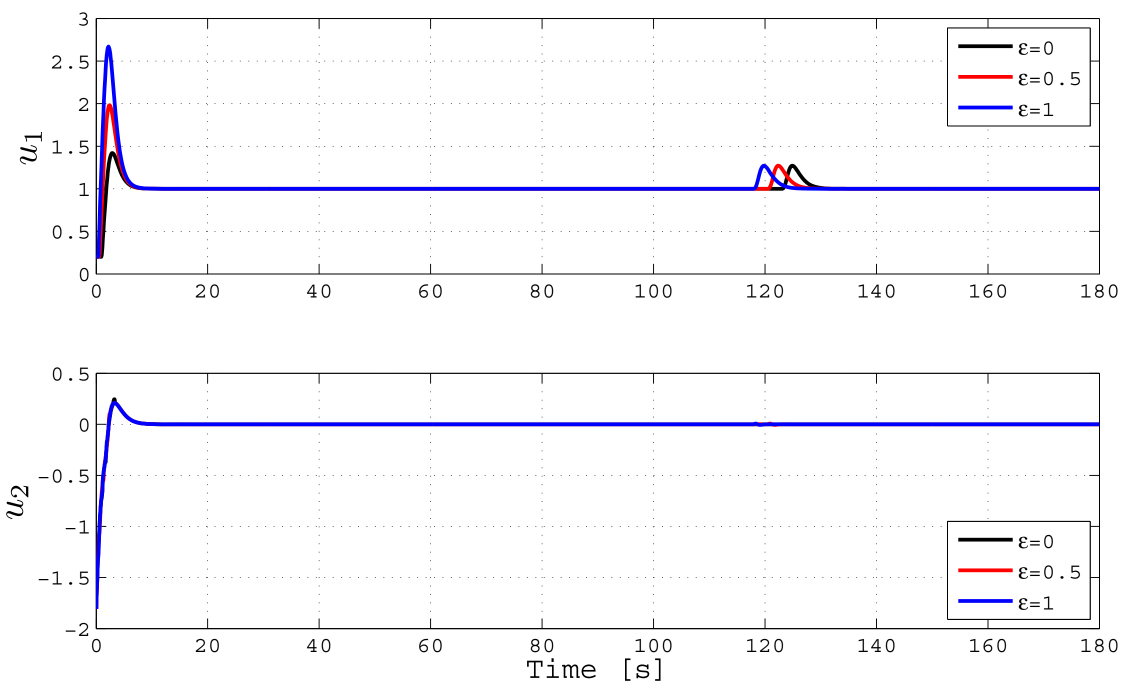

Figure 4.8: Comparison between control inputs when $\varepsilon = 0$, $\varepsilon = 0.5$, and $\varepsilon = 1$ for different initial conditions.

Chapter 5

Conclusions

This work has focused on the global stabilization of the PVTOL aircraft. Such a system has a complex dynamics that renders difficult the design of controllers oriented to solve the global stabilization problem. This is mainly due to its under-actuated nature. In addition, other limitations have been considered in this study: the positive character of the thrust and the bounded nature of the inputs. A first approach that solves the global stabilization problem, taking the mentioned restrictions into account, was proposed in (Zavala-Río, *et al.*, 2003); however the lateral coupling parameter ε was neglected due to the small value that it usually has in practice, simplifying the system dynamics. On the other hand, recent works, like those in (Wood and Cazzolato, 2007) and (Ye, *et al.*, 2007), have proposed solutions to the global stabilization problem considering the whole system dynamics; nevertheless they depend on the exact knowledge of ε.

In this thesis, it has been analytically proved that the control scheme first proposed in (Zavala-Río, *et al.*, 2003) under the above mentioned input restrictions, considering $\varepsilon = 0$, achieves the global stabilization objective even when $\varepsilon > 0$, provided that ε is small enough but without the knowledge of its exact value. A certain degree of robustness of such an approach with respect to ε can be concluded from the result developed in this thesis. The analytical developments have been corroborated through numerical simulation results.

Potential future work may focus on the improvement of the system time response by trying different definitions of r_1 and r_2. On the other hand, it may be worth investigating if the small enough condition of ε can be relaxed, for instance by designing an adaptive control law.

Appendix A

Linear saturations around the origin

For all $z \in B_\rho \triangleq \{z \in I\!R^6 \big| \|z\| < \rho\}$, with $\rho = \min\{L_{11}/2, L_{21}/2, M_{41} + M_{42}, M_{43} + B_{\theta_d}\}$, we have

$$|z_1| < \frac{L_{11}}{2} \quad \text{and} \quad |z_2| < \frac{L_{11}}{2} < L_{11} \leq M_{11} < \frac{L_{12}}{2}$$

$$\implies \begin{cases} |z_2 + \sigma_{11}(\cdot)| \leq |z_2| + |\sigma_{11}(\cdot)| < \frac{L_{12}}{2} + M_{11} < L_{12} \\ \qquad \implies \sigma_{12}(z_2 + \sigma_{11}(\cdot)) = z_2 + \sigma_{11}(\cdot) \\ \text{and} \\ |kz_1 + z_2| \leq k|z_1| + |z_2| < |z_1| + |z_2| < L_{11} \\ \qquad \implies \sigma_{11}(kz_1 + z_2) = kz_1 + z_2 \end{cases}$$

(according to item (a) of Definition 3.1), entailing $r_1 = 2kz_2 + k^2 z_1$, $\forall z \in B_\rho$,

$$|z_3| < \frac{L_{21}}{2} \quad \text{and} \quad |z_4| < \frac{L_{21}}{2} < L_{21} \leq M_{21} < \frac{L_{22}}{2}$$

$$\implies \begin{cases} |z_3 + \sigma_{21}(\cdot)| \leq |z_3| + |\sigma_{21}(\cdot)| < \frac{L_{22}}{2} + M_{21} < L_{22} \\ \qquad \implies \sigma_{22}(z_3 + \sigma_{21}(\cdot)) = z_3 + \sigma_{21}(\cdot) \\ \text{and} \\ |z_3 + z_4| \leq |z_3| + |z_4| < L_{21} \\ \qquad \implies \sigma_{21}(z_3 + z_4) = z_3 + z_4 \end{cases}$$

implying $r_2 = 2z_4 + z_3$, $\forall z \in B_\rho$, and

$$|z_5| < M_{43} + B_{\theta_d} \quad \text{and} \quad |z_6| < M_{41} + M_{42} < M_{41} + M_{42} + M_{31}$$

$$\implies \begin{cases} |z_6 - \sigma_{42}(\cdot) + \sigma_{31}(\cdot)| \leq |z_6| + |\sigma_{42}(\cdot)| + |\sigma_{31}(\cdot)| \\ \qquad\qquad\qquad\qquad\qquad < M_{41} + 2(M_{42} + M_{31}) \\ \qquad\qquad\qquad\qquad\qquad < L_{32} \\ \qquad \implies \sigma_{32}(z_6 - \sigma_{42}(\cdot) + \sigma_{31}(\cdot)) = z_6 - \sigma_{42}(\cdot) + \sigma_{31}(\cdot) \\ \text{and} \\ |z_6 - \sigma_{43}(\cdot) + z_5 - \theta_d| \leq |z_6| + |\sigma_{43}(\cdot)| + |z_5| + |\theta_d| \\ \qquad\qquad\qquad\qquad\qquad < M_{41} + M_{42} + 2M_{43} + 2B_{\theta_d} \\ \qquad\qquad\qquad\qquad\qquad < L_{31} \\ \qquad \implies \sigma_{31}(z_6 - \sigma_{43}(\cdot) + z_5 - \theta_d) = z_6 - \sigma_{43}(\cdot) + z_5 - \theta_d \end{cases}$$

where the satisfaction of inequalities (3.9b) and (3.9c) has been considered, entailing $u_2 = \sigma_{41}(\alpha_d) - (z_6 - \sigma_{42}(\omega_d)) - (z_6 - \sigma_{43}(\omega_d)) - (z_5 - \theta_d)$, $\forall z \in B_\rho$. Moreover, from the first part of the proof, we see that k can be chosen small enough to guarantee that $|\dot{\theta}_d| < \min\{L_{42}, L_{43}\}$ and $|\ddot{\theta}_d| < L_{41}$ (after a sufficiently long time) $\forall z \in B_\rho$, implying $\sigma_{42}(\omega_d) = \sigma_{43}(\omega_d) = \omega_d$, and $\sigma_{41}(\alpha_d) = \alpha_d$, $\forall z \in B_\rho$. Then, provided that such a choice of k is made, we have $u_2 = \alpha_d - 2(z_6 - \omega_d) - (z_5 - \theta_d)$, $\forall z \in B_\rho$.

Appendix B

Stability conditions through the Routh-Hurwitz criterion

Let

$$\tilde{P}(\lambda) = \left[\lambda^4 + 2(k+1)(1-\varepsilon k)\lambda^3 + (k^2+4k+1-\varepsilon k^2)\lambda^2 + 2k(k+1)\lambda + k^2\right]$$

Since $P(\lambda) = (\lambda+1)^2\tilde{P}(\lambda)$, if all the roots of $\tilde{P}(\lambda)$ have negative real part, so does $P(\lambda)$. Thus applying the Routh-Hurwitz criterion to $\tilde{P}(\lambda)$, we have

$$\begin{array}{c|ccc}
\hline
s^4 & 1 & k^2+4k+1-\varepsilon k^2 & k^2 \\
s^3 & a_1 & 2k(k+1) & \\
s^2 & b_1 & k^2 & \\
s^1 & c_1 & & \\
s^0 & d_1 & &
\end{array}$$

where

$$a_1 = 2(k+1)(1-\varepsilon k) \tag{B.1a}$$

$$\begin{aligned}
b_1 &= \frac{2(k+1)(1-\varepsilon k)[k^2+4k+1-\varepsilon k^2] - 2k(k+1)}{2(k+1)(1-\varepsilon k)} \\
&= \frac{(1-\varepsilon k)[k^2+4k+1-\varepsilon k^2] - k}{1-\varepsilon k} \\
&= k^2+4k+1-\varepsilon k^2 - \frac{k}{1-\varepsilon k}
\end{aligned} \tag{B.1b}$$

$$\begin{aligned}
c_1 &= \frac{2k(k+1)b_1 - 2k^2(k+1)(1-\varepsilon k)}{b_1} \\
&= \frac{2k(k+1)[b_1 - k(1-\varepsilon k)]}{b_1}
\end{aligned} \tag{B.1c}$$

$$d_1 = k^2 \tag{B.1d}$$

The Routh-Hurwitz criterion states that the roots of $\tilde{P}(\lambda)$ have all negative real part if $a_1 > 0$, $b_1 > 0$, $c_1 > 0$, and $d_1 > 0$. Observe that for all $k \in (0,1)$, $d_1 > 0$. On the other hand, notice, from the expressions (B.1), that

$$b_1 > k(1-\varepsilon k) > 0 \quad \Longrightarrow \quad a_1 > 0,\ b_1 > 0,\ \text{and}\ c_1 > 0$$

Let us prove that $\varepsilon k < 0.8 \quad \Longrightarrow \quad b_1 > k(1-\varepsilon k) > 0$. Let

$$h(k) \triangleq \frac{(k+1)^2}{(k+1)^2 + k}$$

Then

$$h'(k) \triangleq \frac{(k+1)(k-1)}{[(k+1)^2 + k]^2}$$

Observe that

$$h'(k) < 0 \quad \forall k \in (0,1)$$

Then, $\forall k \in (0,1)$:

$$\begin{aligned}
\varepsilon k < 0.8 = h(1) \leq h(k) = \frac{(k+1)^2}{(k+1)^2 + k} & \\
\Longrightarrow \qquad & 1 - \varepsilon k > 1 - \frac{(k+1)^2}{(k+1)^2 + k} = \frac{k}{(k+1)^2 + k} \\
\Longrightarrow \qquad & (k+1)^2 + k > \frac{k}{1-\varepsilon k} \\
\Longrightarrow \qquad & k^2 + 4k + 1 - k + \varepsilon k^2 - \varepsilon k^2 > \frac{k}{1-\varepsilon k} \\
\Longrightarrow \qquad & k^2 + 4k + 1 - \varepsilon k^2 - \frac{k}{1-\varepsilon k} > k(1-\varepsilon k) > 0
\end{aligned}$$

i.e.

$$\varepsilon k < 0.8 \qquad \Longrightarrow \qquad b_1 > k(1-\varepsilon k) > 0$$

Therefore $\varepsilon k < 0.8 \quad \Longrightarrow \quad a_1 > 0,\ b_1 > 0,\ \text{and}\ c_1 > 0$.

Appendix C

On the derivation of $\dot{\theta}_d$ and $\ddot{\theta}_d$

From the definition of θ_d in (3.7), we have

$$\dot{\theta}_d = \frac{k\bar{r}_1\dot{r}_2 - k(1+r_2)\dot{\bar{r}}_1}{u_1^2} \tag{C.1}$$

where

$$\begin{aligned}
\dot{\bar{r}}_1 \triangleq \left.\frac{d\bar{r}_1}{dt}\right|_{\varepsilon\geq 0} &= -\sigma_{12}'(s_{12})\left[-u_1\sin\theta + \varepsilon u_2\cos\theta + \sigma_{11}'(s_{11})(k\dot{x} - u_1\sin\theta + \varepsilon u_2\cos\theta)\right] \\
&= -\sigma_{12}'(s_{12})\left[-u_1\sin\theta + \sigma_{11}'(s_{11})(k\dot{x} - u_1\sin\theta)\right] + \varepsilon u_2\sigma_{12}'(s_{12})(1+\sigma_{11}'(s_{11}))\cos\theta \\
&= \rho_1 - \varepsilon u_2\sigma_{12}'(s_{12})\left[1+\sigma_{11}'(s_{11})\right]\cos\theta \\
&= \rho_1 - \varepsilon u_2\Delta_{\dot{\bar{r}}_1}
\end{aligned} \tag{C.2}$$

with

$$\Delta_{\dot{\bar{r}}_1} = \sigma_{12}'(s_{12})\left[1+\sigma_{11}'(s_{11})\right]\cos\theta$$

and

$$\begin{aligned}
\dot{r}_2 \triangleq \left.\frac{dr_2}{dt}\right|_{\varepsilon\geq 0} &= -\sigma_{22}'(s_{22})\left[u_1\cos\theta + \varepsilon u_2\sin\theta - 1 + \sigma_{21}'(s_{21})(\dot{y} + u_1\cos\theta + \varepsilon u_2\sin\theta - 1)\right] \\
&= -\sigma_{22}'(s_{22})\left[u_1\cos\theta - 1 + \sigma_{21}'(s_{21})(\dot{y} + u_1\cos\theta - 1)\right] - \varepsilon u_2\sigma_{22}'(s_{22})(1+\sigma_{21}'(s_{21}))\sin\theta \\
&= \rho_2 - \varepsilon u_2\sigma_{22}'(s_{22})\left[1+\sigma_{21}'(s_{21})\right]\sin\theta \\
&= \rho_2 - \varepsilon u_2\Delta_{\dot{r}_2}
\end{aligned} \tag{C.3}$$

with

$$\Delta_{\dot{r}_2} \triangleq \sigma_{22}'(s_{22})\left[1+\sigma_{21}'(s_{21})\right]\sin\theta$$

where

$$\begin{aligned}
s_{12} &\triangleq \dot{x} + \sigma_{11}(s_{11}) \qquad & s_{11} &\triangleq kx + \dot{x} \\
s_{22} &\triangleq \dot{y} + \sigma_{21}(s_{21}) \qquad & s_{21} &\triangleq y + \dot{y}
\end{aligned}$$

By substituting (C.2) and (C.3) into (C.1), one obtains the expressions in (3.26) and (3.27). Now, from (3.27), we get $\ddot{\theta}_d = \dot{\omega}_d + \varepsilon k\dot{\Delta}_1$. This expression is obtained considering the following developments:

$$
\begin{aligned}
\dot{\rho}_1 \triangleq \left.\frac{d\rho_1}{dt}\right|_{\varepsilon\geq 0} \\
= \; & -\sigma''_{12}(s_{12})\,[\,-u_1\sin\theta+\varepsilon u_2\cos\theta \\
& \qquad\qquad +\sigma'_{11}(s_{11})(k\dot{x}-u_1\sin\theta+\varepsilon u_2\cos\theta)]\,[-u_1\sin\theta+\sigma'_{11}(s_{11})(k\dot{x}-u_1\sin\theta)] \\
& -\sigma'_{12}(s_{12})\,[\,-\dot{u}_1\sin\theta-u_1\dot{\theta}\cos\theta+\sigma''_{11}(s_{11})(k\dot{x}-u_1\sin\theta+\varepsilon u_2\cos\theta)(k\dot{x}-u_1\sin\theta) \\
& \qquad\qquad +\sigma'_{11}(s_{11})(-ku_1\sin\theta+\varepsilon ku_2\cos\theta-\dot{u}_1\sin\theta-u_1\dot{\theta}\cos\theta)] \\
= \; & \varphi_1-\varepsilon u_2\,\Big[\sigma''_{12}(s_{12})\,(1+\sigma'_{11}(s_{11}))\,(-u_1\sin\theta+\sigma'_{11}(s_{11})\,(k\dot{x}-u_1\sin\theta)) \\
& \qquad +\sigma'_{12}(s_{12})\,(\sigma''_{11}(s_{11})\,(k\dot{x}-u_1\sin\theta)+k\sigma'_{11}(s_{11}))\Big]\cos\theta \\
= \; & \varphi_1-\varepsilon u_2\Delta_{\dot{\rho}_1}
\end{aligned}
$$

with

$$
\begin{aligned}
\Delta_{\dot{\rho}_1} = \; & \Big[\sigma''_{12}(s_{12})\,(1+\sigma'_{11}(s_{11}))\,(-u_1\sin\theta+\sigma'_{11}(s_{11})\,(k\dot{x}-u_1\sin\theta)) \\
& +\sigma'_{12}(s_{12})\,(\sigma''_{11}(s_{11})\,(k\dot{x}-u_1\sin\theta)+k\sigma'_{11}(s_{11}))\Big]\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
\dot{\rho}_2 \triangleq \left.\frac{d\rho_2}{dt}\right|_{\varepsilon\geq 0} \\
= \; & -\sigma''_{22}(s_{22})\,[u_1\cos\theta+\varepsilon u_2\sin\theta-1 \\
& \qquad\qquad +\sigma'_{21}(s_{21})(\dot{y}+u_1\cos\theta+\varepsilon u_2\sin\theta-1)]\,[u_1\cos\theta-1+\sigma'_{21}(s_{21})(\dot{y}+u_1\cos\theta-1)] \\
& -\sigma'_{22}(s_{22})\,[\dot{u}_1\cos\theta-u_1\dot{\theta}\sin\theta+\sigma''_{21}(s_{21})(\dot{y}+u_1\cos\theta+\varepsilon u_2\sin\theta-1)(\dot{y}+u_1\cos\theta-1) \\
& \qquad\qquad +\sigma'_{21}(s_{21})(u_1\cos\theta+\varepsilon u_2\sin\theta-1+\dot{u}_1\cos\theta-u_1\dot{\theta}\sin\theta)] \\
= \; & \varphi_2-\varepsilon u_2\,\Big[\sigma''_{22}(s_{22})\,(1+\sigma'_{21}(s_{21}))\,[u_1\cos\theta-1+\sigma'_{21}\,(s_{21})\,(\dot{y}+u_1\cos\theta-1)] \\
& \qquad +\sigma'_{22}(s_{22})\,[\sigma''_{21}(s_{21})\,(\dot{y}+u_1\cos\theta-1)+\sigma'_{21}(s_{21})]\Big]\sin\theta \\
= \; & \varphi_2-\varepsilon u_2\Delta_{\dot{\rho}_2}
\end{aligned}
$$

with

$$
\begin{aligned}
\Delta_{\dot{\rho}_2} = \; & \Big[\sigma''_{22}(s_{22})\,(1+\sigma'_{21}(s_{21}))\,[u_1\cos\theta-1+\sigma'_{21}\,(s_{21})\,(\dot{y}+u_1\cos\theta-1)] \\
& +\sigma'_{22}(s_{22})\,[\sigma''_{21}(s_{21})\,(\dot{y}+u_1\cos\theta-1)+\sigma'_{21}(s_{21})]\Big]\sin\theta
\end{aligned}
$$

$$
\begin{aligned}
\dot{u}_1 \triangleq \left.\frac{du_1}{dt}\right|_{\varepsilon\geq 0} = \; & \frac{k^2\bar{r}_1\dot{\bar{r}}_1+(1+r_2)\dot{r}_2}{u_1} \\
= \; & \mu_1-\frac{\varepsilon u_2}{u_1}\left[k^2\bar{r}_1\sigma'_{12}(s_{12})\,[1+\sigma'_{11}(s_{11})]\cos\theta+(1+r_2)\,\sigma'_{22}(s_{22})\,[1+\sigma'_{21}(s_{21})]\sin\theta\right] \\
= \; & \mu_1-\varepsilon u_2\Delta_{\dot{u}_1}
\end{aligned}
$$

with

$$\Delta_{\dot{u}_1} = k^2 \bar{r}_1 \sigma'_{12}(s_{12}) \left[1 + \sigma'_{11}(s_{11})\right] \cos\theta + (1 + r_2)\, \sigma'_{22}(s_{22}) \left[1 + \sigma'_{21}(s_{21})\right] \sin\theta$$

$$\dot{\mu}_1 \triangleq \left.\frac{d\mu_1}{dt}\right|_{\varepsilon \geq 0} = \frac{1}{u_1} \left(k^2 \dot{\bar{r}}_1 \rho_1 + k^2 \bar{r}_1 \dot{\rho}_1 + (1 + r_2)\dot{\rho}_2 + \dot{r}_2 \rho_2\right) - \frac{\dot{u}_1}{u_1^2} \left(k^2 \bar{r}_1 \rho_1 + (1 + r_2)\rho_2\right)$$

$$\begin{aligned}
\dot{\omega}_d \triangleq \left.\frac{d\omega_d}{dt}\right|_{\varepsilon \geq 0} &= \frac{k}{u_1^2} \left[\bar{r}_1 \dot{\rho}_2 + \dot{\bar{r}}_1 \rho_2 - \dot{r}_2 \rho_1 - (1 + r_2)\dot{\rho}_1\right] - \frac{2k\dot{u}_1}{u_1^3} \left[\bar{r}_1 \rho_2 - (1 + r_2)\rho_1\right] \\
&= \frac{k}{u_1^2} \left[\bar{r}_1 \dot{\rho}_2 + \dot{\bar{r}}_1 \rho_2 - \dot{r}_2 \rho_1 - (1 + r_2)\dot{\rho}_1\right] - \frac{2\dot{u}_1}{u_1} \omega_d \\
&= \frac{k}{u_1^2} \left[\bar{r}_1 \varphi_2 - (1 + r_2)\varphi_1\right] + 2\frac{\mu_1 \omega_d}{u_1} \\
&\quad + 2\varepsilon k \frac{u_2 \bar{\omega}_d}{u_1^2} \left[k^2 \bar{r}_1 \sigma'_{12}(s_{12})(1 + \sigma'_{11}(s_{11})) \cos\theta + (1 + r_2)\sigma'_{22}(s_{22})(1 + \sigma'_{21}(s_{21})) \sin\theta\right] \\
&\quad - \varepsilon k \frac{u_2}{u_1^2} \left[\rho_2 \sigma'_{12}(s_{12})(1 + \sigma'_{11}(s_{11})) \cos\theta - \rho_1 \sigma'_{22}(s_{22})(1 + \sigma'_{21}(s_{21})) \sin\theta\right]
\end{aligned}$$

$$\dot{\alpha}_d \triangleq \left.\frac{d\alpha_d}{dt}\right|_{\varepsilon \geq 0} = \frac{k}{u_1^2} \left[\dot{\bar{r}}_1 \varphi_2 + \bar{r}_1 \dot{\varphi}_2 - \dot{r}_2 \varphi_1 - (1 + r_2)\, \dot{\varphi}_1\right] - 2\frac{\dot{u}_1}{u_1} \alpha_d - \frac{2}{u_1^2} \left[\dot{\omega}_d \mu_1 + \omega_d \dot{\mu}_1\right]$$

$$\begin{aligned}
\dot{\varphi}_1 &\triangleq \left.\frac{d\varphi_1}{dt}\right|_{\varepsilon \geq 0} \\
= \; & - \sigma'''_{12}(s_{12}) \left(\ddot{x} + \sigma'''_{11}(s_{11}) \left(k\dot{x} + \ddot{x}\right)\right) \left[-u_1 \sin\theta + \sigma'''_{11}(s_{11}) \left(k\dot{x} - u_1 \sin\theta\right)\right]^2 \\
& - 2\sigma''_{12}(s_{12}) \Big[- \dot{u}_1 \sin\theta - u_1 \dot{\theta} \cos\theta + \sigma''_{11}(s_{11}) \left(k\dot{x} - \ddot{x}\right) \left(k\dot{x} - u_1 \sin\theta\right) \\
& \qquad\qquad + \sigma'_{11}(s_{11})(k\ddot{x} - \dot{u}_1 \sin\theta - u_1 \dot{\theta} \cos\theta)\Big] \Big[- u_1 \sin\theta + \sigma'_{11}(s_{11}) \left(k\dot{x} - u_1 \sin\theta\right) \Big] \\
& - \sigma''_{12}(s_{12}) \left(\ddot{x} + \sigma'_{11}(s_{11}) \left(k\dot{x} - \ddot{x}\right)\right) \Big[- u_1 \dot{\theta} \cos\theta - \mu_1 \sin\theta + \sigma''_{11}(s_{11}) \left(k\dot{x} - u_1 \sin\theta\right)^2 \\
& \qquad\qquad + \sigma'_{11}(s_{11})(-ku_1 \sin\theta - u_1 \dot{\theta} \cos\theta - \mu_1 \sin\theta)\Big] \\
& - \sigma'_{12}(s_{12}) \Big[- \dot{u}_1 \dot{\theta} \cos\theta - u_1 \ddot{\theta} \cos\theta + u_1 \dot{\theta}^2 \sin\theta - \dot{\mu}_1 \sin\theta - \mu_1 \dot{\theta} \cos\theta \\
& \qquad + \sigma'''_{11}(s_{11}) \left(k\dot{x} - \ddot{x}\right) \left(k\dot{x} - u_1 \sin\theta\right)^2 \\
& \qquad + 2\sigma''_{11}(s_{11}) \left(k\dot{x} - u_1 \sin\theta\right) \left(k\ddot{x} - \dot{u}_1 \sin\theta - u_1 \dot{\theta} \cos\theta\right) \\
& \qquad + \sigma''_{11}(s_{11}) \left(k\dot{x} - \ddot{x}\right) \left(-ku_1 \sin\theta - u_1 \dot{\theta} \cos\theta - \mu_1 \sin\theta\right) \\
& \qquad + \sigma'_{11}(s_{11}) \big(- k\dot{u}_1 \sin\theta - ku_1 \dot{\theta} \cos\theta - \dot{u}_1 \dot{\theta} \cos\theta \\
& \qquad\qquad - u_1 \ddot{\theta} \cos\theta + u_1 \dot{\theta}^2 \sin\theta - \dot{\mu}_1 \sin\theta - \mu_1 \dot{\theta} \cos\theta\big)\Big]
\end{aligned}$$

$$
\begin{aligned}
\dot{\varphi}_2 \triangleq \left.\frac{d\varphi_2}{dt}\right|_{\varepsilon\geq 0} \\
= \ & -\sigma_{22}'''(s_{22})\left(\ddot{y}+\sigma_{21}'(s_{21})\left(\dot{y}+\ddot{y}\right)\right)\left[u_1\cos\theta-1+\sigma_{21}'(s_{21})\left(\dot{y}+u_1\cos\theta-1\right)\right]^2 \\
& -2\sigma_{22}'(s_{22})\Big[\dot{u}_1\cos\theta-u_1\dot{\theta}\sin\theta+\sigma_{21}''(s_{21})\left(\dot{y}+\ddot{y}\right)\left(\dot{y}+u_1\cos\theta-1\right) \\
& \qquad +\sigma_{21}'(s_{21})(\ddot{y}+\dot{u}_1\cos\theta-u_1\dot{\theta}\sin\theta)\Big]\Big[u_1\cos\theta-1+\sigma_{21}'(s_{21})\left(\dot{y}+u_1\cos\theta-1\right)\Big] \\
& -\sigma_{22}''(s_{22})\left(\ddot{y}+\sigma_{21}'(s_{22})\left(\dot{y}+\ddot{y}\right)\right)\Big[-u_1\dot{\theta}\sin\theta+\mu_1\cos\theta+\sigma_{21}''(s_{21})\left(\dot{y}+u_1\cos\theta-1\right)^2 \\
& \qquad +\sigma_{21}'(s_{21})(u_1\cos\theta-1-u_1\dot{\theta}\sin\theta+\mu_1\cos\theta)\Big] \\
& -\sigma_{22}'(s_{22})\Big[-\dot{u}_1\dot{\theta}\sin\theta-u_1\ddot{\theta}\sin\theta-u_1\dot{\theta}^2\cos\theta+\dot{\mu}_1\cos\theta-\mu_1\dot{\theta}\sin\theta \\
& \qquad +\sigma_{21}'''(s_{21})\left(\dot{y}+\ddot{y}\right)\left(\dot{y}+u_1\cos\theta-1\right)^2 \\
& \qquad +2\sigma_{21}''(s_{21})\left(\dot{y}+u_1\cos\theta-1\right)\left(\ddot{y}+\dot{u}_1\cos\theta-u_1\dot{\theta}\sin\theta\right) \\
& \qquad +\sigma_{21}''(s_{21})\left(\dot{y}+\ddot{y}\right)\left(u_1\cos\theta-1-u_1\dot{\theta}\sin\theta+\mu_1\cos\theta\right) \\
& \qquad +\sigma_{21}'(s_{21})\Big(\dot{u}_1\cos\theta-u_1\dot{\theta}\sin\theta-\dot{u}_1\dot{\theta}\sin\theta \\
& \qquad\qquad -u_1\ddot{\theta}\sin\theta-u_1\dot{\theta}^2\cos\theta+\dot{\mu}_1\cos\theta-\mu_1\dot{\theta}\sin\theta\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\dot{u}_2 \triangleq \left.\frac{du_2}{dt}\right|_{\varepsilon\geq 0} &= \dot{\alpha}_d-2(\ddot{\theta}-\dot{\omega}_d)-(\dot{\theta}-\dot{\theta}_d) \\
&= \dot{\alpha}_d-2(u_2-\dot{\omega}_d)-(\dot{\theta}-\dot{\theta}_d) \\
&= \dot{\alpha}_d-2[\alpha_d-2(\dot{\theta}-\omega_d)-(\theta-\theta_d)-\dot{\omega}_d]-(\dot{\theta}-\dot{\theta}_d)
\end{aligned}
$$

(recall that all these expressions are being calculated considering $t \geq t_2$),

$$
\begin{aligned}
\dot{\Delta}_1 = \ & \frac{\dot{u}_2}{u_1^2}\left[(1+r_2)\Delta_{\dot{r}_1}-\bar{r}_1\Delta_{\dot{r}_2}\right]-\frac{2u_2\dot{u}_1}{u_1^3}\left[(1+r_2)\Delta_{\dot{r}_1}-\bar{r}_1\Delta_{\dot{r}_2}\right] \\
& +\frac{u_2}{u_1^2}\left[\dot{r}_2\Delta_{\dot{r}_1}-\dot{\bar{r}}_1\Delta_{\dot{r}_2}+(1+r_2)\dot{\Delta}_{\dot{r}_1}-\bar{r}_1\dot{\Delta}_{\dot{r}_2}\right] \\
\dot{\Delta}_{\dot{r}_1} = \ & \sigma_{12}''(s_{12})\dot{s}_{12}[1+\sigma_{11}'(s_{11})]\cos\theta+\sigma_{12}'(s_{12})\sigma_{11}''(s_{11})\dot{s}_{11}\cos\theta \\
& -\sigma_{12}'(s_{12})[1+\sigma_{11}'(s_{11})]\sin\theta\cdot\dot{\theta} \\
\dot{\Delta}_{\dot{r}_2} = \ & \sigma_{22}''(s_{22})\dot{s}_{22}[1+\sigma_{21}'(s_{21})]\sin\theta+\sigma_{22}'(s_{22})\sigma_{21}''(s_{21})\dot{s}_{21}\sin\theta \\
& +\sigma_{22}'(s_{22})[1+\sigma_{21}'(s_{21})]\cos\theta\cdot\dot{\theta}
\end{aligned}
$$

where

$$
\begin{aligned}
\dot{s}_{12} &= -u_1\sin\theta+\sigma_{11}'(s_{11})\dot{s}_{11} & \dot{s}_{11} &= -u_1\sin\theta+k\dot{x} \\
\dot{s}_{22} &= u_1\cos\theta+\sigma_{21}'(s_{21})\dot{s}_{21}-1 & \dot{s}_{21} &= \dot{y}+u_1\cos\theta-1
\end{aligned}
$$

The appropriate substitutions give rise to the expressions in (3.28) and (3.29).

Appendix D

Bounds of Δ_1, Δ_2, $\dot{\theta}_d$, and $\ddot{\theta}_d$

.

Even when the terms containing ε are included in the system dynamics, $\dot{\theta}_d$ and $\ddot{\theta}_d$ remain bounded. Indeed, from the first part of the proof, we know that $|\sigma'_{ij}(s)| \leq A_{ij}$, $|\sigma''_{ij}(s)| \leq B_{ij}$, $\forall s \in [-N^-_{ij}, N^+_{ij}]$, and $|s^p \sigma'_{ij}(s)| \leq N^p_{ij} A_{ij}$, $|s^p \sigma''_{ij}(s)| \leq N^p_{ij} B_{ij}$, $\forall s \in I\!R$ $\forall i, j = 1, 2$. Hence

$$
\begin{aligned}
|\omega_d| &\leq kB_{\bar{\omega}_d} \\
|\Delta_1| &\leq \frac{M_{41} + M_{32}}{1 - M^+_{22}} \sqrt{[M_{12}A_{22}(1 + A_{21})]^2 + \left[\left(1 + M^-_{22}\right) A_{12}(1 + A_{11})\right]^2} \\
&\triangleq B_{\Delta_1}
\end{aligned}
$$

wherefrom we see that

$$
|\dot{\theta}_d| \leq kB_{\bar{\omega}_d} + \varepsilon k B_{\Delta_1}
$$

On the other hand, since $\sigma'''_{ij}(s) = D^+ \sigma''_{ij}(s)$ and $\sigma_{ij}(\cdot) \in \mathcal{C}^2_L(I\!R; I\!R)$, there exist positive (real) constants C_{ij} $(i = 1, 2; j = 1, 2)$ such that $|\sigma'''_{ij}(s)| \leq C_{ij}$, $\forall s \in [-N^-_{ij}, N^+_{ij}]$. Moreover, $\sigma'''_{ij}(s) = 0$ when $|s| \geq N^{\pm}_{ij}$. Then, for any scalar $p > 0$, $|s^p \sigma'''_{ij}(s)| \leq N^p_{ij} C_{ij}$, $\forall s \in I\!R$, $\forall i, j = 1, 2$, with $N_{ij} \triangleq \max[-N^-_{ij}, N^+_{ij}]$. Therefore, from their respective expressions defined throughout the text, we have:

$$
\begin{aligned}
|\dot{\omega}_d| \leq\ & kB_{\bar{\alpha}_d} + k\varepsilon \frac{M_{41} + M_{32}}{\left(1 - M^+_{22}\right)^2} \Bigg[\Bigg(M_{12} B_{\Delta_{\rho_2}} + \left(1 + M^-_{22}\right) B_{\Delta_{\rho_1}} \\
& \qquad + \sqrt{\left[A_{12}\left(1 + A_{11}\right) B_{\rho_2}\right]^2 + \left[A_{22}\left(1 + A_{21}\right) B_{\rho_1}\right]^2} \Bigg) \\
& \qquad + \frac{2B_{\dot{u}_1}}{1 - M^+_{22}} B_{\bar{\omega}_d} \Bigg] \\
\triangleq\ & k\left(B_{\bar{\alpha}_d} + \varepsilon B_{\Delta_{\dot{\omega}d}}\right) \\
\triangleq\ & B_{\dot{\omega}_d}
\end{aligned}
$$

with

$$B_{\Delta_{\dot{\omega} d}} \triangleq \frac{M_{41}+M_{32}}{\left(1-M_{22}^{+}\right)^2}\Bigg[\Bigg(M_{12}B_{\Delta_{\rho_2}}+\left(1+M_{22}^{-}\right)B_{\Delta_{\rho_1}} + \sqrt{\left[A_{12}\left(1+A_{11}\right)B_{\rho_2}\right]^2+\left[A_{22}\left(1+A_{21}\right)B_{\rho_1}\right]^2}\Bigg) + \frac{2B_{\dot{u}_1}}{1-M_{22}^{+}}B_{\bar{\omega}_d}\Bigg]$$

$$\begin{aligned}|\dot{\bar{r}}_1| \leq\ & B_{\rho_1}+\varepsilon(M_{41}+M_{32})A_{12}\left[1+A_{11}\right] \\ \triangleq\ & B_{\dot{\bar{r}}_1} \\ |\dot{r}_2| \leq\ & B_{\rho_2}+\varepsilon(M_{41}+M_{32})A_{22}\left[1+A_{21}\right] \\ \triangleq\ & B_{\dot{r}_2}\end{aligned}$$

$$\begin{aligned}|\dot{\rho}_1| \leq\ & B_{\varphi_1}+\varepsilon(M_{41}+M_{32})\Bigg[B_{12}\left(1+A_{11}\right)\left(B_{u_1}+A_{11}\left(N_{12}+M_{11}+B_{u_1}\right)\right) \\ & + A_{12}\left[B_{11}\left(N_{12}+M_{11}+B_{u_1}\right)+A_{11}\right] \\ & + \frac{A_{12}}{1-M_{22}^{+}}\left(1+A_{11}\right)C_6\Bigg] \\ \triangleq\ & B_{\dot{\rho}_1} \\ |\dot{\rho}_2| \leq\ & B_{\varphi_2}+\varepsilon(M_{41}+M_{32})\Bigg[B_{22}\left[1+A_{21}\right]\left[B_{u_1}+1+A_{21}\left(N_{22}+M_{21}+B_{u_1}+1\right)\right] \\ & + A_{22}\left[B_{21}\left(N_{22}+M_{21}+B_{u_1}+1\right)+A_{21}\right] \\ & + \frac{A_{22}}{1-M_{22}^{+}}\left(1+A_{21}\right)C_6\Bigg] \\ \triangleq\ & B_{\dot{\rho}_2}\end{aligned}$$

$$\begin{aligned}|\dot{u}_1| \leq\ & B_{\mu_1}+\varepsilon\frac{M_{41}+M_{32}}{1-M_{22}^{+}}\sqrt{\left[M_{12}A_{12}\left(1+A_{11}\right)\right]^2+\left[\left(1+M_{22}^{-}\right)A_{22}\left(1+A_{21}\right)\right]^2} \\ \triangleq\ & B_{\dot{u}_1}\end{aligned}$$

$$
\begin{aligned}
|\dot{\varphi}_1| \leq\ & C_{12}\left[C_5 + A_{11}\left(N_{12} + M_{11} + C_5\right)\right]\left[B_{u_1} + A_{11}C_1\right]^2 \\
& + 2B_{12}\left[B_{u_1} + A_{11}C_1\right]\left[\sqrt{B_{\dot{u}_1}^2 + \left(B_{u_1}D\right)^2} + B_{11}\left(N_{12} + M_{11} + C_5\right)C_1\right. \\
& \qquad\qquad\qquad\qquad \left. + A_{11}\left(C_5 + \sqrt{B_{\dot{u}_1}^2 + \left(B_{u_1}D\right)^2}\right)\right] \\
& + B_{12}\left[C_5 + A_{11}\left(N_{12} + M_{11} + C_5\right)\right]\left[C_3 + B_{11}C_1^2 + A_{11}C_4\right] \\
& + A_{12}\left[\sqrt{\left[B_{\dot{u}_1}D + B_{\mu_1}D + B_{u_1}\left(M_{41} + M_{32}\right)\right]^2 + \left[B_{u_1}D^2 + B_{\dot{\mu}_1}\right]^2}\right. \\
& \qquad + C_{11}\left(N_{12} + M_{11} + C_5\right)C_1^2 + 2B_{11}C_1\left[C_5 + \sqrt{B_{\dot{u}_1}^2 + \left(B_{u_1}D\right)^2}\right] \\
& \qquad + B_{11}\left(N_{12} + M_{11} + C_5\right)C_4 \\
& \qquad \left. + A_{11}\sqrt{\left[B_{\dot{u}_1} + B_{u_1}D^2 + B_{\dot{\mu}_1}\right]^2 + \left[B_{u_1}D + B_{\dot{u}_1}D + B_{u_1}\left(M_{41} + M_{32}\right) + B_{\mu_1}D\right]^2}\right] \\
& \triangleq B_{\dot{\varphi}_1}
\end{aligned}
$$

$$
\begin{aligned}
|\dot{\varphi}_2| \leq\ & C_{22}\left[B_{u_1} + A_{21}C_2 + 1\right]^2\left[C_5 + 1 + A_{21}\left(N_{22} + M_{21} + C_5 + 1\right)\right] \\
& + 2B_{22}\left[B_{u_1} + A_{21}C_2 + 1\right]\left[\sqrt{B_{\dot{u}_1}^2 + \left(B_{u_1}D\right)^2} + B_{21}\left(N_{22} + M_{21} + C_5 + 1\right)C_2\right. \\
& \qquad\qquad\qquad\qquad \left. + A_{21}\left(C_5 + 1 + \sqrt{B_{\dot{u}_1}^2 + \left(B_{u_1}D\right)^2}\right)\right] \\
& + B_{22}\left[C_5 + 1 + A_{21}C_2\right]\left[C_3 + B_{21}C_2^2 + A_{21}\left(C_4 + 1\right)\right] \\
& + A_{22}\left[\sqrt{\left[B_{\dot{u}_1}D + B_{\mu_1}D + B_{u_1}\left(M_{41} + M_{32}\right)\right]^2 + \left[B_{u_1}D^2 + B_{\dot{\mu}_1}\right]^2}\right. \\
& \qquad + C_{21}\left(N_{22} + M_{21} + C_5 + 1\right)C_2^2 + 2B_{21}C_2\left(C_5 + 1 + \sqrt{B_{\dot{u}_1}^2 + \left(B_{u_1}D\right)^2}\right) \\
& \qquad + B_{21}\left(N_{22} + M_{21} + C_5 + 1\right)\left(B_{u_1} + C_3 + 1\right) \\
& \qquad \left. + A_{21}\sqrt{\left[B_{\dot{u}_1} + B_{u_1}D^2 + B_{\dot{\mu}_1}\right]^2 + \left[B_{u_1}D + B_{\dot{u}_1}D + B_{u_1}\left(M_{41} + M_{32}\right) + B_{\mu_1}D\right]^2}\right] \\
& \triangleq B_{\dot{\varphi}_2}
\end{aligned}
$$

with C_1, C_2, C_3, C_4 defined as in the first part of the proof, $C_5 \triangleq \sqrt{B_{u_1}^2 + \varepsilon^2\left(M_{41} + M_{32}\right)^2}$ and $C_6 \triangleq \sqrt{\left[M_{12}A_{12}(1 + A_{11})\right]^2 + \left[(1 + M_{22}^-)A_{22}(1 + A_{21})\right]^2}$.

$$
\begin{aligned}
|\dot{u}_2| \leq\ & B_{\dot{\alpha}_d} + 2\left[(M_{41}+M_{32}) + B_{\dot{\omega}_d}\right] + (D + kB_{\bar{\omega}_d} + B_{\Delta_1}) \\
& \triangleq B_{\dot{u}_2} \\
|\dot{\alpha}_d| \leq\ & \frac{k}{\left(1-M_{22}^{+}\right)^2}\left[B_{\dot{\bar{r}}_1}B_{\varphi_2} + M_{12}\dot{\varphi}_2 + B_{\dot{r}_2}B_{\varphi_1} + \left(1+M_{22}^{-}\right)\dot{\varphi}_1\right] \\
& + 2k\frac{B_{\dot{u}_1}}{1-M_{22}^{+}}B_{\bar{\alpha}_d} + \frac{2}{\left(1-M_{22}^{+}\right)^2}\left[B_{\dot{\omega}_d}B_{\mu_1} + kB_{\bar{\omega}_d}\dot{\mu}_1\right] \\
& \triangleq B_{\dot{\alpha}_d}
\end{aligned}
$$

$$
\begin{aligned}
|\dot{\Delta}_1| \leq\ & \frac{1}{\left(1-M_{22}^{+}\right)^2}\left[B_{\dot{u}_2} - 2\frac{B_{\dot{u}_1}(M_{41}+M_{32})}{1-M_{22}^{+}}\right]\left[\sqrt{\left[M_{12}A_{22}(1+A_{21})\right]^2 + \left[\left(1+M_{22}^{-}\right)A_{12}(1+A_{11})\right]^2}\right] \\
& + \frac{M_{41}+M_{32}}{\left(1-M_{22}^{+}\right)^2}\left[\sqrt{\left[B_{\rho_1}A_{22}(1+A_{21})\right]^2 + \left[B_{\rho_2}A_{12}(1+A_{11})\right]^2}\right. \\
& \qquad + M_{12}\Big[B_{22}\left[B_{u_1} + 1 + A_{21}(N_{22}+M_{21}+B_{u_1}+1)\right](1+A_{21}) \\
& \qquad\qquad + A_{22}B_{21}(N_{22}+M_{21}+B_{u_1}+1)\Big] \\
& \qquad + D\sqrt{\left[M_{12}A_{22}(1+A_{21})\right]^2 + \left[\left(1+M_{22}^{-}\right)A_{12}(1+A_{11})\right]^2} \\
& \qquad + \left(1+M_{22}^{-}\right)\Big[B_{12}\left[B_{u_1} + A_{11}(N_{12}+M_{11}+B_{u_1})\right](1+A_{11}) \\
& \qquad\qquad + A_{12}B_{11}(N_{12}+M_{11}+B_{u_1})\Big] \\
& \qquad \left. + M_{12}\left[B_{22}(1+A_{21})^2 + A_{22}B_{21}\right] + \left(1+M_{22}^{-}\right)\left[B_{12}(1+A_{11})^2 + A_{12}B_{11}\right]\right] \\
& \triangleq B_{\dot{\Delta}_1}
\end{aligned}
$$

Hence

$$|\ddot{\theta}_d| \leq kB_{\bar{\alpha}_d} + \varepsilon kB_{\Delta_{\dot{\omega}_d}} + \varepsilon kB_{\dot{\Delta}_1}$$

and defining $B_{\Delta_2} \triangleq B_{\Delta_{\dot{\omega}_d}} + B_{\dot{\Delta}_1}$

$$|\ddot{\theta}_d| \leq kB_{\bar{\alpha}_d} + \varepsilon kB_{\Delta_2}$$

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