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Mathematics and Physics of Self-Image Effects

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Juan Pablo Treviño Gutiérrez

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Codirectores de la Tesis:

Dr. Haret Codratian Rosu Barbus y Dr. Hugo Cabrera Ibarra

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Dr. Haret Codratian Rosu Barbus
(Codirector de la tesis)

Dr. Hugo Cabrera Ibarra
(Codirector de la tesis)

Dr. José Luis Morán López
(Sinodal)

Dr. José Salomé Murguía Ibarra
(Sinodal Externo)

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Contents

Resumen

El presente documento es parte del resultado del trabajo de investigación que realicé durante mi maestría. La tesis trata de presentar, de manera global, distintas manifestaciones de fenómenos de auto-reconstrucción. Para ser mas precisos, la auto-reconstrucción se refiere a la capacidad de las estructuras periódicas de reproducirse a si mismas gracias a los efectos de interferencia constructiva y destructiva. La interferencia se presenta por dispersión o difracción de ondas dependiendo del contexto en el que se presenten las estructuras. Lo anterior quiere decir que las estructuras periódicas producen imágenes de si mismas, ya sean ópticas o reconstrucciónes de otra índole. Sobre el documento se desarrollan y explican diversas propiedades matemáticas que poseen las representaciones formales de las imágenes reconstruidas. Estas propiedades están relacionadas con las representaciones corpuscular y ondulatoria de la materia y el principio de complementariedad. Adicionalmente, se presenta un método de análisis de los campos difractados o dispersados a través de una herramienta conocida como Transformada Ondeleta. Esta herramienta y su utilidad se explican con detalle. Es importante señalar que la transfromada ondeleta puede aplicarse de diversas maneras en el presente contexto así como en otras areas de investigación. Finalmente se incluyen comentarios y propuestas experimentales donde es posible e incluso necesaria la aplicación de la teoría aquí presentada.

Abstract

The present document is part of the outcome of the research I have done through my master courses. The thesis develops a global view of the diverse manifestations of self-reconstruction effects. Accurately speaking, self-reconstruction means the capacity of periodic structures to reproduce themselves due to constructive and destructive interference. This interference occurs due to diffraction or dispersion depending on the context in which the effect is studied. In short, periodic structures produce self-images, which can be optical images or other type of reconstructions. Along the document different properties of the mathematical representations of the self-reconstruction effects are discussed. This properties are related to the corpuscular and wave representation of matter and their relationship through complementarity principles. Aditionally, a method to analize diffracted and dispersied fields is presented. This method is based on the Wavelet Transform which will be explained in detail. Worth to point out is the fact that this integral transform can be applied in different ways to study diffracted fields and also in other areas. Finally, some comments are added. Possible experimental setups, where the theory presented here is applicable, are mentioned.

A mi abuelo José, mi amigo y compañero de viaje, mi manager.

A mi querido amigo Brando, donde quiera que se encuentre: compartimos laberintos y rompecabezas... siempre serás mi mejor amigo.

Preface and Organization

This work is concerned with three different contexts in which self-reconstruction of periodic structures appears: optical diffraction, fiber optic dispersion, and quantum revivals. In the first one, the effect is known as Talbot effect, named after its discoverer. The second context is also known by the same name and appears in the class of self-action effects that are found in Fiber Optics. Finally, the self-reconstruction effect in quantum mechanics is called Quantum Revivals and is not developed in detail in this work. Nevertheless, several hints for its generalization are given. Along this document, I will use the terms *Talbot effect*, *Self-Image effect*, and *Self Reconstruction* indistinctively where no ambiguity is possible.

The document is organized into five chapters. The first one is all about historical background and informal explanations. The second chapter is a collection of definitions which are useful to understand the basics of wave theory and different contexts in which it may appear. The core of the theoretical background lies in the third chapter , in which most of the physics is developed and the self-image effects in diffraction and dispersion are developed in detail. In the next two chapters, the relationship between the effects is explained through analogy, and the wave and corpuscular representations are also explained to be mathematically related. Finally in the fifth chapter, the analytic tool known as the Wavelet Transform is explained and applied to the study of irrational multiples of the integer reconstruction distance. Conclusive comments and remarks are added in some of the sections plus a general summary of the document as a whole that can be found at the end.

Chapter 1

Historical Perspective

In any research project, the historical background is a very useful tool to understand the meaning of a particular theory and the resulting technologies. For this reason, in this chapter I will include a short historical background of optics as a whole. Since the main subject of this work is the Talbot effect, I will go on with a historical review of its discovery and development. The three important contexts in which the Talbot effect occurs will be also shortly described. As an additional source of information for the most curious readers, short biographies of important personalities related to these topics can be found in appendix A.

1.1 Light over the Centuries

The interest of the human being in light is as ancient as the observation of day and night, seasons and stars. There is evidence that human beings had used the property of light to travel in a rectilinear path to measure certain time intervals. The amazing structures found in Stonehenge, England, suggest that ancient cultures could measure different time intervals such as seasons by this means. Nevertheless, the true development of Optics was probably initiated very much later. As many other scientific disciplines, Optics began to be developed by ancient greeks. Some of their beliefs were correct and some others were mostly strange to the point view of modern ideas about light and sight. The belief that light rays emanated from the eye and touched the objects is an example of an incorrect idea. The first record of the simplest law in optics, the reflection law, dates back to $\sim$ 300 BC, due to the remarkable mathematician of those times, Euclid, in his *Optika* ($o\pi\tau\iota\kappa\alpha$). Refraction, on the other hand, was probably first described by Claudius Ptolemy near 140 BC.

In more recent investigations, the great personality of Roger Bacon, in the XIII century, comes to attention. This important philosopher believed in the finite speed of light and was the first to glance the relationship between light and sound. Nowadays it is known that these phenomena are physically related through their undulatory nature. Some centuries later, Kepler came up with ideas close to those of Huygens. Kepler suggested that the intensity of light from a point source varies inversely with the square of the distance from the source, that light can be propagated over an unlimited distance, and that the speed of propagation is infinite. The first of the ideas is in fact closely related to the Huygens-Fresnel principle.

Later on, during the Middle Age, there was almost no advance. Arab maintained alive previous knowledge through translation of old books, whereas contributions were only minor deal. This was done only in the general field of geometrical optics. The true theoretical breakthrough had to wait a few more centuries.

Passing to the theory that underlies optical fibers, arguably, it was Willebord Snell who did the first contribution. He discovered the relation between the angles of a ray refracted through an inter-phase. This law, commonly named after him, makes possible the calculation of a critical angle at which total internal reflection occurs. Any ray having an angle greater than the critical, experiences no refraction at all and thus reflects to the same side of the interface from where it came. This is the basic phenomenon that takes place in optical fibers. More recent contributions and developments in fiber optics will be mentioned later in this section.

Around the middle of the $17th$ century two frenchmen, Pierre de Fermat and René Descartes came up independently with different basic principles which would explain Snell's law. Descartes was known for his remarkable works in philosophy which lead to analytic geometry as a consequence. Fermat was a respected mathematician who had correspondence with many other mathematicians and was an enthusiast of challenging problems. In the end, the principle formulated by Fermat turned out to be the correct one. This principle is known with its discoverer's name and also as the least optical path principle. Fermat's principle is also used in optical fibers when the refractive index is continuous (i.e., there is no interphase).

At this point, it is impossible to omit the name of the great Sir Isaac Newton. This awesome personality did a number of discoveries related to optics. He had the idea that light is made of tiny particles and travels in rectilinear paths which he defined as rays. Assuming this, he managed to explain different physical phenomena such as light aberration and chromatic dispersion. Newton knew that waves diffract and rejected the idea that light would do so. For that reason he stuck to the idea of the corpuscular nature of light. Opposed to Newton was Christiaan Huygens who believed that light was an oscillation rather than a motion of corpuscles. According to Huygens, material media would vibrate and create secondary waves from each vibrational point. Unfortunately, Huygens dyed before Newton and had no chance to debate his ideas with him. Also, Huygens was a great scientist and inventor (he built the first mechanical clock) but never as great as Newton, for this reasons his thoughts were not taken as seriously as they ought to.

Nearly two centuries later, Augustine Fresnel was able to describe mathematically the Huygens principle and, therefore, the diffraction principle. He developed a plausible theory and stated the principle of secondary wave production. However, his description of the diffractive effect is restricted to the near field, whereas the german Fraunhofer developed the corresponding theory for the far field. Fresnel's theory was developed more precisely by Kirchhoff who introduced a correcting geometrical factor.

The phenomenon associated to vector fields known as polarization has an interesting story and was also developed by many of the scientists here shortly mentioned. This effect is though slightly tangential to this work and thus not mentioned in detail in this introduction.

Regarding optical fibers it is worth mentioning the pioneering experiment of the french Jaques Babinet and the swiss Daniel Collud in the XIX century. They discovered the possibility to guide light within a water jet [1], however, this experiment did not have very much impact and was forgotten for decades. It was until 1854 when John Tyndal, an english physicist, popularized a new version of the experiment, and about a hundred years later fibers were commonly used to experiment with image transmission. In 1956, Kapany invented the term *Optical Fiber* to describe a filament conformed of a core and a cladding which confined light to the core. This devise was developed by Van Heel and Hopkins in the Netherlands in 1951 [1, 2, 3]. Further information on the development and properties of fiber optics is given in my undergraduate thesis [4].

The reader is encouraged to read and cross references to learn more historical facts of science since they are not only fascinating but useful.

1.2 Brief Overview of the Talbot Effect

The Talbot effect is an optical self-image effect. It has several counterparts in different fields of physics where phenomena can be described through Helmholtz equation. This remarkable effect consists in the reconstruction of the original wave field only by means of constructive and destructive interference. The optical self-image effect was first discovered by Sir William Henry Fox Talbot in 1836. The original experiment was made with a white light source that illuminated a ruler. Different images of different colors appeared periodically onto the optical axis [5].

For many years, the Talbot effect was only taken as a curiosity. Forty five years later, Lord Rayleigh investigated the experiment and gave the first analytic treatment. He was able to calculate the self-image distance as $z_T = a^2/\lambda$, being a the period of the object and λ the wavelength of the light source. Though the Talbot effect can be applied to reconstruct images, it is also useful to transmit information over long distances, and it turns out that it can be used to make complicated calculations such as prime factorization of large numbers. This last application along with other special properties of the Talbot effect could be of great importance to cryptography. The Talbot effect shows up not only in light diffraction, but in any other type of wave diffraction where minimal assumptions work, such as sound. It also appears in light pulse propagation in optical fibers and most impressively in quantum mechanics. Some helpful details of optics and quantum mechanics will be discussed throughout this document.

1.3 A Global View of Talbot-Like Effects

As mentioned before, the Talbot effect appears in every cuasi-monoenergetic wave phenomenon. Yet the most studied fields have been optics and quantum mechanics because of technological and experimental interests and advantages. The three contexts in which this work will be mainly interested are *diffraction, dispersion* and *quantum wave functions*. As it will be shown later, the equations that describe the phenomena have the same mathematical form, though the particular features of the phenomena might not be the same.

Light diffraction is related to the wave nature of light and describes the way a single ray of light deflects from its rectilinear path when obstructed by an opaque object. The reason for this deflection is given by the Huygens-Fresnel principle. On the other hand, fiber dispersion is due mainly to Rayleigh dispersion. This type of dispersion is related specifically to the composition of the propagating media, in this case, the optical fiber. A pulse propagating inside the core of an optical fiber will be broader in time as it propagates. This means that if the pulse is a gaussian pulse, the gaussian width will increase as the pulse propagates. In both effects, interference takes place and makes possible the reconstruction of periodic structures.

Finally, the analogous effect in quantum mechanics is known as Quantum Revivals. The effect was discovered by Eberly et al [6] . This work considered a two level system approached by the well known Jaynes-Cummings model in quantum optics. The wavy nature of matter at the microscopic scale makes possible that a particle spreads and reconstructs itself through time.

Chapter 2

Basic Notions in Wave Physics

There is a great number of oscillatory phenomena in physics. The easiest pedagogical way to introduce oscillations is the example of the spring-mass system without friction. The importance of such example is beyond instructive purposes because of the variety of fields to which it can be extended. Wave representations occur from classical mechanics and its applications through electromagnetism and quantum mechanics. In this chapter, some examples will be explained and a general picture of waves will be given.

2.1 Transversal Oscillations: The Case of Strings

The wave equation is probably the most important partial differential equation. In this section, an easy way to obtain the one dimensional wave equation will be presented. This approach deals with the specific case of an elastic string of uniform density $\rho = \Delta m/\Delta x =$ const. Nevertheless, it is easy to translate the discussion to more complicated cases.

The problem is to describe the vertical or rather transversal position of a point of the string at a distance x from a given end. This quantity is denoted $u(x,t)$ according to the notation of Boyce and DiPrima [7]. Assuming that the longitudinal displacement is negligible, the total force exerted at a given point of the string is the tension times the tangent vector in both directions. It has been asumed that the tension is constant over the entire string. Following the argument given by Williamson [8], the total force is written as

$$F = \tau \mathbf{t}(x+\Delta x,t) - \tau \mathbf{t}(x,t),$$

where τ is the tension and $\mathbf{t}(x,t)$ is the unit vector tangent to the string at the point x of the axis. By Newton's law and recalling the linear mass distribution $m = \rho\Delta x$, the force on each point of the string is:

$$F = m\mathbf{a} = \rho\Delta x\ddot{u}\ . \tag{2.1}$$

In the transversal direction, the tangent vector is the space derivative of u, then the two previous expressions are combined to obtain

$$\rho\frac{\partial^2 u}{\partial t^2} = \tau\left[\frac{\partial_x u(x+\Delta x,t) - \partial_x u(x,t)}{\Delta x}\right]. \tag{2.2}$$

In the limit $\Delta x \to 0$, equation (2.2) turns into

$$\frac{\rho}{\tau}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \tag{2.3}$$

where $\tau/\rho = v^2$ is the square of the velocity at which a maximum vertical displacement travels in the horizontal direction. This equation is the one-dimensional wave equation, and it is usually written in the general form

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \tag{2.4}$$

or in operatorial form the trivial generalization to three dimensions is written as

$$\Box^2 u = 0, \tag{2.5}$$

where the operator $\Box^2 = (1/v^2)\partial_{tt} - \nabla^2$ is the d'Alembertian operator.

For given values of time, the level sets of the d'Alembertian solution $u(x,t)$ are recognized as wavefronts. The formal definition of the wavefront is the following:

Definition 1 *A wavefront is a line or surface in a two- or three-dimensional medium through which waves are passing, being the locus of all adjacent points at which the disturbances are in phase.*

In the two dimensional case, the wavefronts are curves that travel at the group velocity. The concept of wavefront is useful to explain the important Huygens-Fresnel principle, which will be presented in detail later in this chapter.

As said before, the expression (2.4) describes the undulatory motion of an elastic string, for which the oscillation is transversal to the propagation axis. In the case of sound waves, on the other hand, the oscillation is longitudinal. This means that the displacement of the wave front occurs in the same direction of the wave propagation.

The generalization of the argument to two and three dimensional wave phenomena is trivial, although the solution to the equation is much more difficult for several reasons. The local geometrical neighborhood gets more complicated and the Laplacian operator must be written in an appropriate coordinate system. In these higher dimensional cases, there are interesting features to be observed like the nodal curves and the oscillation modes of the vibrating medium.

2.2 Longitudinal Oscillations: The Case of Sound

In the case of sound, the approach to get the wave equation is much similar, though the geometrical considerations refer to a cylinder rather than to a string. The external perturbation is the pressure exerted on a plane normal to the axis of the cylinder. Notice that any bulk material can be perturbed in several ways, either longitudinally or transversally, the combination of these directions corresponds to the general three-dimensional case. The quantity $u(x,t)$, in the case of the cylinder, represents the longitudinal displacement of the differential volume element, $(x+\Delta x)A$. The pressure, defined as $P = \Delta F/A$, can be rewritten as follows:

$$P = \frac{\Delta m\, a}{A}. \tag{2.6}$$

Remembering that the volumetric density is defined as $\rho(x) = m/\Delta x$ and the acceleration a is the second time derivative of the displacement $u(x)$, the previous expression is rewritten as

$$P = \rho(x)\Delta x \frac{\partial^2 u}{\partial t^2}. \tag{2.7}$$

If the cylinder is considered as a continuous spring, then Hooke's law states that $\Delta F = k\partial_x u$. The combination of the previous equations yield the following expression:

$$\frac{k}{A}\left(\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\right) = \rho(x)\Delta x \frac{\partial^2 u}{\partial t^2}, \tag{2.8}$$

where the left hand side of the equation represents the total force exerted onto a volume element and $\partial_x u(x)$ represents the deviation of the slice with respect to its equilibrium position.

Following similar arguments as in the case of the string the wave equation is obtained:

$$\frac{\rho(x)}{k/A}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}. \tag{2.9}$$

The wave velocity is now given by

$$v^2 = \frac{k/A}{\rho(x)},$$

which is usually expressed by

$$v = \sqrt{B/\rho}. \tag{2.10}$$

B is the ratio of effort to unit deformation:

$$B = -\frac{\Delta p}{\Delta V/V}. \tag{2.11}$$

This quantity is the inverse of the compressibility and it is known as the bulk modulus. Worth to mention is the fact that wave propagation is considered an adiabatic process from the thermodynamic point of view.

2.3 Optics and Electrodynamics

As described in the previous section, the first scientific incursions in the field of optics were done very early in human history. Among the most important contributions are those of Newton and Huygens. As a believer of the particle nature of light, Newton developed the basics of geometric optics, whereas his opposer, Huygens, had little opportunity to develop his ideas about the aether as light's vibrational material. Both interpretations yield *correct* and useful results, although the fundamental ideas are not settled even nowadays. As mentioned before, Pierre de Fermat stated the principle of least optical path. This principle is related to the particle nature of light while the contribution of Huygens, later completed by Fresnel and Kirchhoff, accounts for the wave properties of light. The understanding of a wave representation, as mathematically related to a particle picture as possible, is one of the main aims of this work.

Physical optics is a branch of optics that can be considered an application of electrodynamics because the object of study are optical fields in the form of electromagnetic wave fields. Optical waves obey a wave equation of the same type as that for elastic oscillations:

$$\square^2_{\mathrm{elm}} U(\vec{x},t) = 0, \tag{2.12}$$

where $\square^2_{\mathrm{elm}} = \nabla^2 - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}$, and c is the speed of light.

2.3.1 Electromagnetism

Experimentally it can be observed that light is a wave since it undergoes the basic wave phenomena: diffraction and interference. Mathematically, it is easily shown [4, 9, 10] that light is actually a vector wave that has electric and magnetic components. This is briefly presented in the following.

Usually, one starts with Maxwell's equations in electro-magnetism in mks system:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.13}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{2.14}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{2.15}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.16}$$

together with the constitutive relations

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P},$$
$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}.$$

If the medium has no free charges, then the electric charge distribution ρ and the current density $\mathbf{J}$ are equal to zero. If additionally $\mathbf{M} \simeq \mathbf{0}$, as in the case of optical fibers, one can proceed taking the curl of equation (2.13):

$$\nabla \times \nabla \times \mathbf{E} = -\nabla \times \frac{\partial \mathbf{B}}{\partial \mathbf{t}} = -\frac{\mathbf{1}}{\mathbf{c^2}}\frac{\partial^2 \mathbf{E}}{\partial \mathbf{t}^2} - \mu_0 \frac{\partial^2 \mathbf{P}}{\partial \mathbf{t}^2}. \tag{2.17}$$

Applying vector calculus identities, the following electromagnetic wave equation is obtained:

$$\nabla^2 \mathbf{E} - \frac{\mathbf{1}}{\mathbf{c^2}}\frac{\partial^2 \mathbf{E}}{\partial \mathbf{t}^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial \mathbf{t}^2}. \tag{2.18}$$

In vacuum, the polarization term vanishes, and the resulting equation is the simple form of the wave equation.

2.4 Huygens - Fresnel Principle

Although the wave nature of light was first suspected by Huygens, his ideas did not have the deserved attention because of the influence of the famous book *Optics*, written by Newton. Nevertheless, Huygens principle was correct, and was successfully mathematically explained by Fresnel and Kirchhoff. Huygens principle is based on the wave nature of light, it states that each point of a wavefront is a point source itself. A plane wave will emit spherical waves from each and every point of a wavefront. This principle holds for any type of wave phenomenon and analogous results are expected in each one of them.

There are two different approaches to the diffraction problem. On the one hand, the differential description through the d'Alembert equation. The solution of this equation can be considered to be monochromatic, which yields the Helmholtz equation. In turn, Helmholtz equation is solved and suitable approximations are applied to its solution to obtain a description of paraxial propagation. On the other hand, a description in terms of optical paths can be used. The Fermat least optical path principle can be suitably written to obtain, through standard approximations, a description involving Fresnel integrals. These two descriptions of paraxial propagation were shown to be reciprocal [11], more precisely, at the level of the phases of the wavefields.

2.5 Helmholtz Equation

The Helmholtz differential operator is obtained from a separation of variables in the solution of the d'Alembert equation (2.4). The physical meaning of this separation is that the monochromatic set of solutions, i.e. a single propagation frequency, is selected:

$$u(r,t) = \hat{u}(r)e^{i\omega t}. \tag{2.19}$$

If one substitutes (2.19) in equation (2.4) and takes into account the fact that the wave velocity is the central frequency divided by the wave number i.e. $v = \omega_0/k$, the Helmholtz operator is readily obtained. Since the Laplacian operator will only affect the time independent factor, the form of the Helmholtz equation is obtained:

$$\nabla^2 \hat{u}_r(r) + k^2 \hat{u}_r(r) = 0, \qquad k^2 = 4(\pi/\lambda)^2. \tag{2.20}$$

2.6 Quantum Mechanics

In order to obtain the Schrödinger equation it is necessary to review some key concepts. Since this approach deals with the wave nature of matter, an introduction must be done. To begin with, the photoelectric effect will be mentioned, next, a few words will be said about de Broglie's hypothesis, and finally, unavoidable heuristic considerations leading to the Schrödinger equation will be presented.

2.6.1 The Photoelectric Effect

The discovery of the photoelectric effect was made by Hertz in 1887. The effect consists in the observation that metallic surfaces are capable of emitting electrons if light with short enough wavelengths insides on it [12]. In 1905, Einstein used Planck's idea of light quanta to explain the photoelectric effect.This phenomenon was the first evidence of light behaving as a particle, since it was proved that light has a minimum unit called photon.

2.6.2 De Broglie Waves

Opposed to the case of photoelectric effect, the wave nature of light was theoretically proposed before the corresponding experimental results were obtained. Einstein's theoretical explanation was a response to unexplained experimental data. On the other hand, de Broglie's theory was only experimentally confirmed by Davisson and Germer in 1927, four years after the theory was developed. De Broglie suspected that, just as wave phenomena can have particle behavior, particles could have a wave nature. The wave and corpuscular nature of a microscopic particle is through its momentum p.

2.6.3 Schrödinger Equation

If the previous concepts are considered, the wave function of a *particle* can be regarded as a solution to a certain *wave equation* in the most general form:

$$\Psi(\vec{r},t) = A(\vec{r},t)e^{i(\omega t-\vec{k}\cdot\vec{r})}$$

where $k = 2\pi/\lambda$ is the wave number or vector in the two or three-dimensional cases, and ω is the frequency. Since the de Broglie hypothesis is true, these identities follow:

$$k = \frac{p}{\hbar}\,, \qquad\qquad \omega = \frac{E}{\hbar}\,,$$

where p and E are the momentum and energy of the particle respectively, and $\hbar = h/2\pi$ is Planck's constant. The first partial derivative of the one dimensional wave function with respect to position is the following:

$$\frac{\partial \Psi}{\partial x} = \frac{i}{\hbar}\hat{p}\Psi$$

from where one could conclude that the momentum could be regarded as a differential operator

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}\ . \tag{2.21}$$

Notice that the square of the momentum operator is

$$\hat{p}^2 = -\hbar^2\frac{\partial^2}{\partial x^2}.$$

In the same way, if the time partial differential is taken, the energy should be considered as an eigenvalue of the Hamiltonian operator

$$\hat{H} = i\hbar\frac{\partial}{\partial t}. \tag{2.22}$$

This consideration allows to put the energy, by means of the Hamiltonian, at the operatorial level as the momentum.

Since the Hamiltonian of a system is the sum of the kinetic energy operator and a non-operatorial term which describes the potential energy:

$$\hat{H} = \hat{T} + V = \frac{p^2}{2m} + V(x,t)\ ,$$

a simple substitution yields

$$i\hbar\frac{\partial}{\partial t}\Psi = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\Psi. \tag{2.23}$$

The functions Ψ belong to a special type of vector space known as Hilbert space, indicating that every operation is performed in such space. The equation (2.23) is the famous

Schrödinger equation. This equation is, at the microscopic scale, as important as Newton's second law in ordinary mechanics. By choosing the potential V that models different physical systems, many details and properties of the dynamical systems can be inferred. Amongst the simplest quantum mechanical models, the square well was useful to explain the tunnel effect which ultimately led to technological breakthroughs such as the atomic microscope.

2.6.4 The Moshinsky Shutter

The experiment of Davisson and Germer can be regarded as a diffraction experiment, not for photons but for an electron beam. Another famous quantum diffraction investigation has been put forth by Moshinsky in 1952 [13]. He stated the problem as follows: "*A monochromatic beam of noninteracting particles of mass m and energy* $\hbar^2 k^2/2m$ *moves parallel to the x axis from the left to the right. At* $x = 0$ *the beam is stopped by a shutter perpendicular to the beam. If at* $t = 0$ *the shutter is opened, what will be the transient particle current observed at a distance x from the shutter?*"

For simplicity we assume that the shutter acts as a perfect absorber. Then, the wavefunction that initially represents a particle of the beam is given by

$$\psi(x, t = 0) = \Theta(-x)e^{ikx} , \tag{2.24}$$

with $\Theta(y) = 0$ for $y < 0$ and $\Theta(y) = 1$ for $y > 0$. The time evolution of ψ is obtained from the equations

$$\psi(x,t) = \int_{-\infty}^{\infty} dx' K_0(x,t|x',t_0)\psi(x',t_0) , \tag{2.25}$$

where K_0 is the retarded free-particle propagator

$$K_0(x,t|x',t') = \left(\frac{m}{2\pi\hbar(t-t')}\right)^{1/2} \exp\left(\frac{im(x-x')^2}{2\hbar(t-t')}\right) . \tag{2.26}$$

According to the calculations of Moshinsky, the result is:

$$\psi(x, t \geq 0) = M(x; k; (\hbar/m)t) . \tag{2.27}$$

The Moshinsky function can, therefore, be interpreted as the wavefunction of a monochromatic particle, which at $t = 0$ is confined to the left half-space $x \leq 0$. The close relationship of M with the theory of diffraction follows from evaluating $|M|^2$,

$$|M(x;k;(\hbar/m)t)|^2 = \frac{1}{2}\left[\left(\frac{1}{2}+C(u)\right)^2 + \left(\frac{1}{2}+S(u)\right)^2\right] \tag{2.28}$$

which has the familiar form of Fresnel scattering of light by a semiplane.The variable u in the Fresnel integrals is

$$u = \left(\frac{\hbar}{m\pi t}\right)^{1/2}(kt-x)\ . \tag{2.29}$$

Moshinsky function is defined in terms of the complementary error function as follows:

$$M(x;k;t) = \frac{1}{2}e^{i(kx-k^2t/2)}\mathrm{erfc}\left[(x-kt)/\sqrt{2it}\right]\ , \tag{2.30}$$

with

$$\sqrt{i} = \exp(i\pi/4)\ , \quad 1/\sqrt{i} = \exp(-i\pi/4)\ .$$

2.6.5 Quantum Dispersion

Since dispersion is one of the main topics of the following chapter, a brief comment is provided about its mathematical treatment in quantum mechanics. Quantum dispersion and/or quantum scattering is most generally approached by means of the S-matrix. The S-matrix is the unitary matrix $|S|^2 = 1$ which connects asymptotic particle states in the Hilbert space of physical states. This matrix is related to the transition probability amplitude in quantum mechanics and to the cross sections of various interactions. It is worth mentioning that the scattering problem in quantum mechanics is analogue to the study of far field or Fraunhoffer diffraction through Fourier methods.

Chapter 3

Self-Image Effects

In optics, the production of images is usually accomplished with the aid of optical elements (i.e. mirrors, lenses, beam splitters, and others). For instance, in Fraunhofer or far field diffraction, lenses are necessary to focus light and produce sharp images. In the case of near field diffraction, periodic objects have the remarkable property that when a plane monochromatic wave incides on them, exact images are produced without intermediate optical elements. This phenomenon was discovered one hundred and seventy years ago by William Henry Fox Talbot. His discovery was purely experimental and at the time he did not have a mathematical description of the phenomenon.

The next important moment in near field diffraction imaging was Lord Rayleigh's calculation of the reconstruction distance. For a wavelength λ and a periodic object of period a the Talbot distance is $z_{\mathrm{T}} = a^2\lambda^{-1}$ [14]. It was only in 1989, more than one century after Rayleigh's contribution, that the first and only one review on the Talbot effect had been written by Patorski [15]. Michael Berry and Susan Klein discovered in 1996 that the diffraction profile of a Ronchi grating is intimately related to arithmetic structures such as Gauss Sums and other fundamental relationships in number theory [16]. Their investigation was carried out in the mathematical framework based on Helmholtz equation.

In this chapter, after briefly reviewing the results of Berry and Klein, it is shown that analogous results can be obtained for the case of dispersion in linear optical fibers. This is one of the main results in this thesis.

3.1 Paraxial Approximation

In order to describe the wave field due to a Ronchi grating, the paraxial approximation is useful. The approximation can be applied to the general solution of the Helmholtz equation and it is related to the following factorization of the solutions (2.19):

$$\hat{u}_r(r) = u_p(r)e^{ik_z z}.$$

The Laplacian of this function turns out to be

$$\nabla^2 \hat{u}_r(r) = \left(\nabla_\perp^2 u_p(r) + \partial_z^2 u_p(r) + ik_z \partial_z u_p(r) - k_z^2 u_p(r)\right) e^{ik_z z}. \tag{3.1}$$

The paraxial approximation considers two facts

1. The second order partial z-derivative of $u_p(r)$ is considerably smaller than the first order one, i.e. $\partial_z^2 u_p(r) \ll \partial_z u_p(r)$, so that it can be neglected.

2. Small angle propagation: $k_z \approx k$.

Therefore the new equation is

$$\nabla_\perp^2 u_p(r) + i2k\partial_z u_p(r) = 0 \tag{3.2}$$

and the corresponding operator is the paraxial operator

$$\hat{O}_{parax} = \nabla_\perp^2 + i2k\partial_z. \tag{3.3}$$

d'Alembert equation + Monochromatic Sol. : $U(r,t) = U(r)e^{i\omega t}$	$\rightarrow$	Helmholtz equation
Helmholtz equation + Paraxiality : $k \sim k_x$, $2k\partial_z U \gg \partial_{zz} U$	$\rightarrow$	Paraxial equation (Schrödinger-like)

Table 3.1: The procedure to obtain the paraxial equation from the d'Alembert equation is schematically shown.

3.2 Talbot Effect for Helmholtz Scalar Fields

A Ronchi grating is a transmitance periodic grating which consists of rows transmitting or blocking light propagation. A different way of thinking about this type of gratings is to say that they are a periodic array of slits. This section contains a brief review of the work done by Berry and Klein about the diffraction on this type of gratings. Passing to dimensionless transverse and paraxial variables $\xi = x/a$ and $\zeta = z/z_{Tal}$, respectively, the scalar wave solution $u(\xi,\zeta)$ of the Helmholtz equation can be expressed as a convolution in ξ of the Ronchi unit cell square function and the Dirac comb transmittance.

The unit cell of a Ronchi grating is usually expressed through a descriptive function:

$$g(\xi) = \begin{cases} 1 & \xi \in [-\frac{1}{4}, \frac{1}{4}] \,, \\ 0 & \xi \notin [-\frac{1}{4}, \frac{1}{4}] \,. \end{cases} \tag{3.4}$$

The wavefield just behind the grating is thus

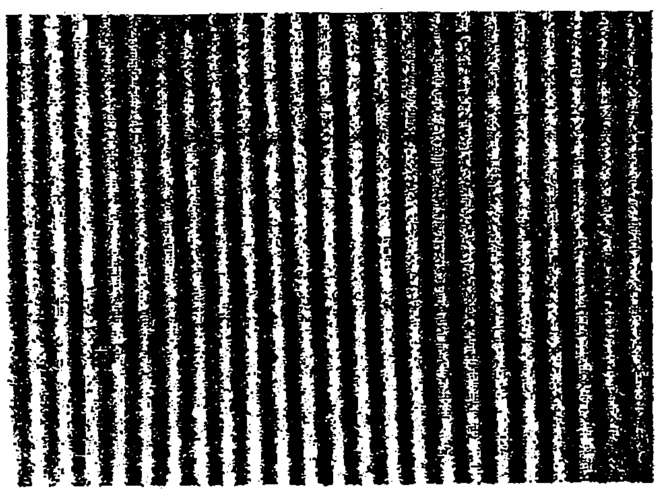

Figure 3.1: Talbot image of a Ronchi grating.

$$u(\xi, \zeta = 0^{+}) = \int_{-1/2}^{+1/2} g(\xi') \left(\sum_{n=-\infty}^{\infty} \exp[i2\pi n(\xi - \xi')] \exp[i\Theta_n(\zeta)] \right) d\xi' \,. \tag{3.5}$$

In the previous expressions, the unit cell is the single spatial period of the grating, which is considered to be centered at the origin and of length equal to unity. The quantity $\Theta_n(\zeta) = 2\pi\zeta\frac{a^2}{\lambda^2}\sqrt{1-\left(\frac{n\lambda}{a}\right)^2}$ is a phase produced by the diffraction of the Dirac comb 'diagonal' rays.The Fresnel approximation for this phase comes from a Taylor expansion up to the second order for the square root. This approximation leads to $\Theta_n(\zeta) \approx -\pi n^2\zeta$. It can be easily shown now that in the Fresnel approximation, equation (3.5) can be written as an infinite sum of phase exponentials in both variables ξ and ζ

$$u_p(\xi,\zeta) = \sum_{n=-\infty}^{\infty} g_n \exp[i2\pi n\xi - i\pi n^2\zeta] \ , \tag{3.6}$$

where the amplitudes g_n are the Fourier modes of the transmittance function of the Ronchi grating

$$g_n = \int_{-1/4}^{+1/4} d\xi' \exp[-i2\pi n\xi'] \ , \tag{3.7}$$

and ψ_p is the paraxial propagator, related to the propagator in quantum mechanics.

The solution $u_p(\xi,\zeta)$ can be readily obtained through the Fourier method applied to the paraxial diffraction equation. The integer Talbot effect occurs when the exponential phase factor in ζ equals one, while the fractional Talbot effect takes place at rational distances of the integer Talbot distance, i.e.

$$z = \frac{p}{q} z_T \, .$$

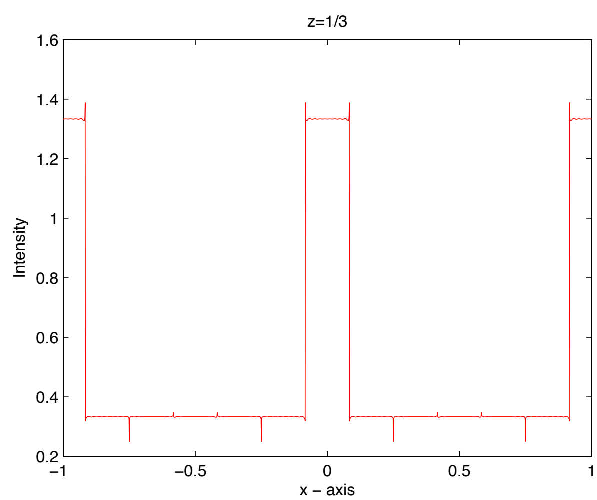

Figure 3.2: $z = 1/3$

The figures show the graph of the Fourier series of the Talbot images at different distances. The interpretation of these images in terms of the numbers p and q as well as the corresponding phases will be discussed in the next chapter.

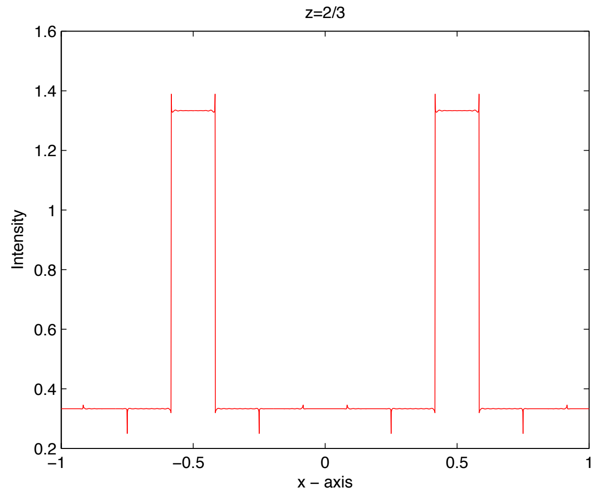

Figure 3.3: $z = 2/3$

3.3 Fiber Dispersion

As it has been previously discussed, the equations that govern light propagation in optical fibers have a close relationship with those that describe diffraction. This idea will be developed in detail emphasizing some of the most important features.

Maxwell's equations applied to optical fibers yield a special type of wave equation In terms of the d'Alambertian operator, this equation is the following:

$$\square^2 \mathbf{E} = \mu_0 \partial_{tt} \mathbf{P}. \tag{3.8}$$

In the optical fiber, the polarization has a causal response to the electric field. This means that the two vector quantities are related through

$$\mathbf{P}(r,t) = \varepsilon_0 \int_{-\infty}^{\infty} \chi(t-\tau)\mathbf{E}(r,\tau)d\tau \tag{3.9}$$

where $\chi(t)$ is zero for negative values of time and represents the electric susceptibility of the material.

If the positive convention of the Fourier transform is employed, i.e.,

$$\mathcal{F}(g(t)) = g_F(\omega) = \int_{-\infty}^{\infty} g(\tau)e^{i\omega\tau}d\tau, \tag{3.10}$$

and the equations (3.8) and (3.9) are combined, one obtains an operator that acts on the electric field in the Fourier domain:

$$\hat{O}_{disp}E_F(r,\omega) \equiv \left[\nabla^2 + \beta^2(\omega)\right] E_F(r,\omega) = 0. \tag{3.11}$$

This operator corresponds to the Helmholtz operator, it is in the Fourier domain and indicates a non-monochromatic propagation given by $\beta(\omega)$. This quantity is called the mode propagation constant and can be expressed in terms of the refractive index as:

$$\beta(\omega) = n(\omega)\frac{\omega}{c}.$$

The propagation constant in the fiber corresponds to the wave number in vacuum propagation and it is useful to describe the frequency dependence of the refraction index through it. It is usually expanded in power series to get the required approximation. The equivalent to the paraxial approximation is, in this case, the slowly varying envelope approximation (SVEA). It refers to the following two considerations

1. The field propagates near a central frequency and therefore a central wavenumber k_0.

2. The field propagates only in the direction of the longitudinal axis z of the optical fiber, i.e. $E_F(r,\omega) = A_F(z,\omega)e^{-ik_0 z}$.

This allows to drop the second order z-derivative of A_F, which is typical from paraxial approximation.

Fiber dispersion is characterized by Rayleigh scattering, which is the determining factor of the fiber loss spectrum in an optical fiber [17]. The minimum of the attenuation curve is around $\lambda_0 = 1.55\mu m$ (figure 3.4). For this reason, it is usual to operate fibers at frequencies near this value. The function $\beta(\omega)$ is Taylor expanded to second order around the central frequency:

$$\beta(\tilde{\omega}) \simeq \beta_0 + \beta_1\tilde{\omega} + \beta_2\tilde{\omega}^2$$

where

$$\tilde{\omega} = \omega - \omega_0 \ , \qquad \beta_n = \frac{d\beta}{d\omega}\Big|_{\omega_0} .$$

The meaning of the expansion coefficients is the following: the zero order coefficient $\beta_0 = k_0$ is the wave number at the central frequency, while the first order one β_1 is the inverse of the group velocity.

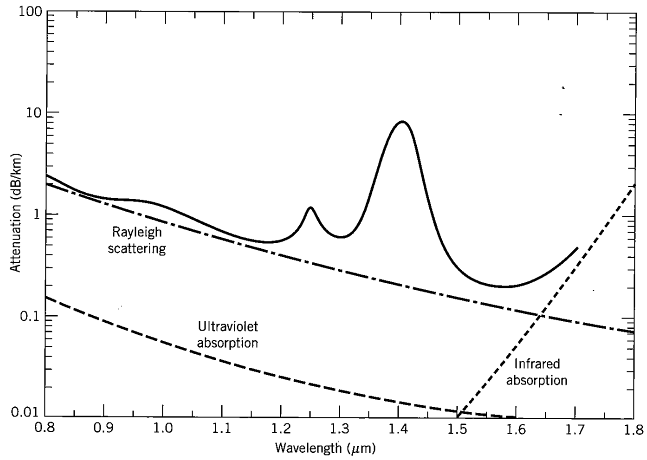

Figure 3.4: Fiber loss spectrum.

Bringing equation (3.11) back to the time domain, the term containing $\beta_1^2 = 1/c^2 \ll 1$ is dropped. After a suitable coordinate transformation, the exact form of the paraxial equation is obtained for a linear optical fiber.

SVEA means decomposing the electromagnetic fields in two factors: a rapidly varying phase component $\exp(-ik_0 z)$ and a slowly varying amplitude field $A(z,\tilde{t})$ enveloping the rapid oscillatory fields. The time $\tilde{t}$ is a time measured in a reference frame that travels at the group velocity. The following Schrödinger-like dispersion equation can be obtained for the slow field in the *SVEA* approximation

$$2i\frac{\partial A}{\partial z} = -\text{sign}(\beta_2)\frac{\partial^2 A}{\partial \tilde{t}^2} \ . \tag{3.12}$$

The coefficient β_2 is expected to be zero where β_1 (or the group reffractive index n_g) reaches its minimum. Because of technological requirements aforementioned, β_2 is usually a negative parameter, $sign(\beta_2) = -1$, which makes the sign in the right hand side of the dispersion equation (3.12) to be positive. The usual wavelength employed in current technologies is beyond the minimum of attenuation only slightly affecting the group velocity. This region (figure 3.5) is known as the anomalous dispersion region.

This is the simplest form of the dispersion equation that one can envision in which actually only a small amount of material dispersion shows up. It can be fulfilled in the practical situation when the dielectric medium has sharp resonances ($\delta\omega_r \ll \omega_r$).

As it can be seen, the SVEA equation has exactly the same mathematical form as the

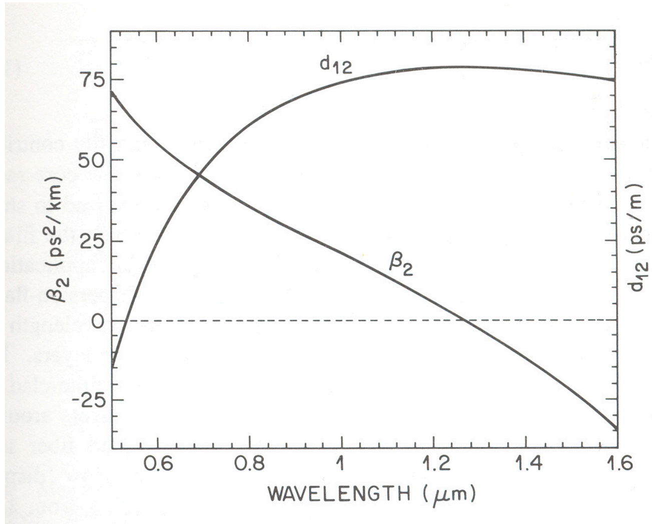

Figure 3.5: Coefficient β_2 is negative near the $1.55\mu m$ region.

diffraction equation in the paraxial approximation, written again here for comparison:

$$2i\frac{\partial u_p}{\partial z} = \frac{\partial^2 u_p}{\partial x^2}\ .$$

Maxwell equations + Physical properties of O.F.	$\rightarrow$	d'Alembert equation with P
d'Alembert equation with P + SVEA: $\beta_0\partial_z A \gg \partial_{zz}A, \quad \beta(\omega) \sim O(\tilde{\omega}^2)$	$\rightarrow$	Dispersion (Schrödinger-like)

Table 3.2: The procedure to obtain the SVEA equation from Maxwell's equations is schematically shown.

3.3.1 Non - Linear Fibers

The case of non-linear optical fibers is thoroughly discussed by Argawal in 1995 [18], thus, the comment here will just refer to the properties of this kind of fibers. The susceptibility is the quantity that carries the non-linearity, it occurs as a tensor instead of a scalar. The Schrödinger-like equation appears with a non-linear term. The solutions to the resulting equations present different self-action properties suitable for technological applications.

3.4 From Diffraction to Fiber Dispersion

Many results in diffraction can be translated to the case of dispersion in fibers by using the following mappings:

$$\text{Diffraction}\left\{\begin{array}{ccc} x & \rightarrow & \tilde{t} \\ y & \rightarrow & \mathbf{r} \\ z & \rightarrow & z \end{array}\right\}\text{Dispersion.}$$

- In the first row, one passes from the grating axis to the time axis of a frame traveling at the group velocity of a pulse.
- In the second row, one passes from the second grating axis that here we consider constant to the transverse section of the optical fiber.
- Finally, the propagation axis remains the same for the two settings.

This *change of variables* will be used here to compare the results obtained in the two frameworks.

The general solution of the *SVEA* dispersion equation (3.12) for the amplitude $A(z,\tilde{t})$ depends on the initial conditions. Since the structures being dealt with are periodic ones, it is natural to write the input signal of an optical fiber as a Fourier series,

$$A(0,\tilde{t}) = \sum_{n=-\infty}^{\infty} C_n^0 e^{-i\omega_n \tilde{t}},$$

where C_n^0 are the Fourier coefficients of the initial pulse.

A single pulse corresponds to the unit cell and therefore the coefficients obtained in both treatments are in correspondence

$$\begin{array}{rcl} g(\xi) & \leftrightarrow & C(\tau) \\ g_n & \leftrightarrow & C_n^0. \end{array}$$

Using the Fourier method to solve the propagation equation (3.12), the form of a dispersed pulse at an arbitrary distance can be written z as follows:

$$A(z,\tilde{t}) = \sum C_n^0 \exp\left[i\frac{\omega_n^2 z}{2} - i\omega_n \tilde{t}\right], \qquad \text{where } \omega_n = 2\pi n/T. \tag{3.13}$$

If the scaled variables $\tau = \tilde{t}/T$ and $\zeta = 2z/z_{\mathrm{T}}$ are employed, $A(z,\tilde{t})$ can be rewritten as

$$A(\zeta,\tau) = \sum C_n^0 \exp\left[i\pi n^2 \zeta - i2\pi n \tau\right]. \tag{3.14}$$

This is possible because the Talbot distance corresponding to this case is $z_{\mathrm{T}} = T^2/\pi$. Just as in the context of diffraction, the convolution of a single pulse with the paraxial dispersion propagator can be equally done before or after the SVEA, equivalent to paraxial approximation in diffraction, is performed. It can be noticed that equation (3.14) can be also written as

$$A(\zeta,\tau) = \int_{-T/2}^{T/2} A(0,\tau')\alpha(\zeta,\tau'-\tau)d\tau' \tag{3.15}$$

since C_n^0, as mentioned before, are the Fourier coefficients of the input signal. In addition, the integral kernel α given by

$$\alpha(\zeta,\tau) = \sum_{n=-\infty}^{\infty} \exp\left[i\pi n^2\zeta - i2\pi n\tau\right] \tag{3.16}$$

can be thought of as the analog of the paraxial diffraction propagator previously denoted as ψ_p used by Berry and Klein [16], and by our group [19].

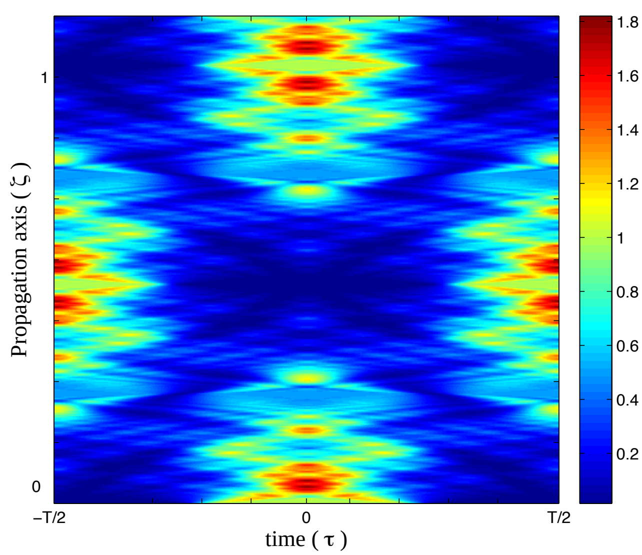

Figure 3.6: Dispersion of a supergaussian pulse $A(0,\tau) = \exp(-\tau^{12})$ through a linear optical fiber.

3.5 Diffraction Through Fresnel Integrals

A different approach to diffraction is by means of the Huygens-Fresnel principle. This theoretical approach explains some effects imposible to describe if it is assumed that light rays travel in straight lines. Though this principle is usually introduced regarding light as a wave, it describes in fact particle aspects. This means that through the Huygens-Fresnel principle one is able to describe classical paths of light.

The Huygens-Fresnel principle states that each point of a wavefront is itself a point source [20]. The simple case of a single slit is usually treated as follows [20]. The area of the slit S is divided into differential parts dS. Each differential part of the surface radiates as a spherical source and contributes to the field in a given point p and the source here is the contribution from every dS in the surface. The field in p is proportional to the field at the source and proportional to the projection of dS on the direction perpendicular to the ray coming from dS to p. The phase of the field is changed in kR and its intensity reduced by a factor $1/R$. Thus the expression reads

$$du(p) \propto u(x,0)\frac{e^{ikR}}{R}\cos\theta dx. \tag{3.17}$$

If a periodic structure is used instead of a single slit, its mathematical expression should be added. As before, such information is introduced as follows:

$$u(x,0) = \sum_{m=-\infty}^{\infty} \exp(i2\pi mx).$$

Recalling that $R = \sqrt{x^2+z^2}$, the complete expression for diffraction reads

$$u(x,y) \propto \int u(x,0)\frac{\exp\left\{ik(x^2+z^2)^{1/2}\right\}}{(x^2+z^2)^{1/2}}\cos\theta dx. \tag{3.18}$$

From this expression, the change to Talbot coordinates, $\xi = x/a$ and $\zeta = z/z_{\mathrm{T}}$, should be done. The resulting integral is known as Fresnel integral and it can be graphically represented through the Cornu spiral. The general form of the Fresnel integrals is the following:

$$F(\xi) = \int_0^{\xi} \exp\left\{i\frac{\pi x^2}{2}\right\} dx$$

or in complex number representation:

$$S(\xi) + iC(\xi) = \int_0^{\xi} \sin\left(\frac{\pi x^2}{2}\right) dx + i\int_0^{\xi} \cos\left(\frac{\pi x^2}{2}\right) dx$$

If the real and imaginary parts are plotted in the corresponding axes with the variable ξ as parameter, the graph of the Cornu spiral is obtained. This spiral is used to evaluate the

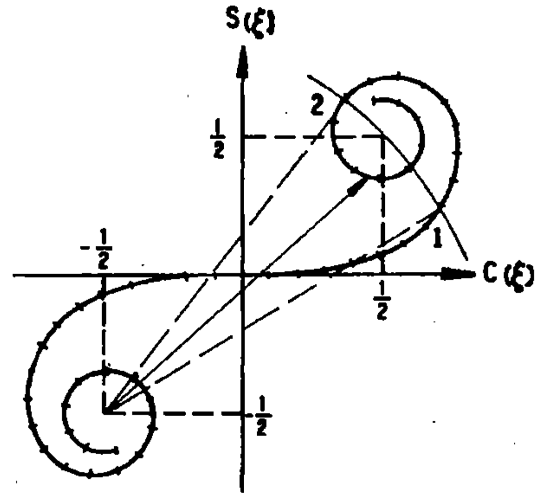

Figure 3.7: This is a plot of the Cornu spiral, useful to calculate the Fresnel integrals.

real and imaginary parts of the above integral. For more detailed information, the reader is referred to a famous paper of Moshinsky [21].

Regarding the dispersion effects in optical fibers, the statement of the Huygens-Fresnel principle can be re-interpreted and an integral description of dispersion obtained. The discussion of such interpretation will be carried out in the next chapter.

3.6 Quantum Revivals

Usually, the description of quantum phenomena is done through the solution of the Schrödinger equation for a particular potential. The solution of this equation is obtained by assuming the posibility of separation of variables which leads to the time independent Schrödinger equation.

The relationship of diffraction and dispersion phenomena with Quantum Mechanics can be obtained relating the propagation variable z of the formers to time in the latter. These two quantities are the evolution variables in each context. It can also be seen that the propagation modes ω_n could be related to the energy levels of a certain quantum system. The transversal variable in diffraction, which corresponds to time in dispersion, can be related to a spatial variable in quantum revivals because in these variables the second order derivative appears.

Chapter 4

Number Theory

As discussed in the literature [11, 16, 19], the phases of the Talbot images are naturally expressed through *quadratic* Gauss sums. This chapter is dedicated to a short introduction to number theory along with the basics of Gauss sums and their properties. Finally, the analytical procedure that leads from Talbot paraxial phases to Gauss sums is developed.

4.1 Basic Number Theory

First, it is appropriate to start with a few definitions and theorems. This section follows the famous text on number theory by Hardy and Wright [22].

Theorem 2 *Every positive integer, except* 1*, is a product of primes.*

Proof: Given a positive integer n, either it is prime or has divisors between 1 and n. The least divisor of n is prime, otherwise it would not be the least one. Then

$$n = p_1 n_1, \quad p_1 \text{ prime.}$$

Now, n_1 is either prime or has divisors between 1 and n_1. Again, the least of them is prime. Continuing recursively picking the least divisor we will find that n_k will be a prime ($n_{k-1} = p_k$), so that the sequence is ended:

$$n = p_1 p_2 \ldots p_k \ .$$

In this sequence of primes, they are not necessarily all different, so that the prime expansion can be properly expressed as

$$n = p_1^{a_1} p_2^{a_2} \dots p_\kappa^{a_\kappa}$$

where the sum of the exponents a_κ is the total number of factors k.

Definition 3 *An integer is said to be expressed in standard form if its prime expansion is written as above.*

In the following, the fundamental theorem of arithmetics will be stated. Nevertheless, the proof will not be given in detail because of its complexity. Instead, the reader is referred to any text on number theory, for instance [22], where the theorem is explained to be a corolary of a simpler theorem.

The notation $a|b$ means that a divides b, this notation will be useful in the following.

Theorem 4 (Euclid's First Theorem-EFT) *If p is prime and $p|ab$, then $p|a$ or $p|b$.*

Theorem 5 (Corolary of EFT) *The prime expansion of a number n is unique.*

Definition 6 *The highest common divisor d of a set of positive integers $a, b, \dots, k$, is the largest positive integer which divides all of them at the same time:*

$$d = (a, b, \dots, k).$$

Theorem 7 *If*

$$a = \prod_p p^\alpha, \quad \textit{and} \quad b = \prod_p p^\beta,$$

then

$$(a, b) = \prod_p p^{min(\alpha, \beta)}.$$

The proof follows from the fundamental theorem of arithmetics and the definition of the maximum common divisor.

Definition 8 (Coprimes) *If $(a, b) = 1$, a and b are said to be coprimes. A set of integers are coprimes if every pair of the set is coprime.*

Definition 9 *If $m|(x-a)$, x is said to be congruent to a to modulus m, this is written as*

$$x \equiv a (\bmod m), \quad or \quad x \equiv_m a.$$

Definition 10 *If $x \equiv_m a$, and $0 < a < m-1$, then a is the least residue of x to modulus m. Thus, two numbers a and b congruent $(\bmod m)$ if they have the same residues $(\bmod m)$.*

Definition 11 *A class of residues $(\bmod m)$ is the class of all the numbers congruent to a given residue $(\bmod m)$, and every member of the class is called a representative of the class.*

Congruences have three basic properties,

1. Symmetry
$$a \equiv b \Longrightarrow b \equiv a,$$
2. Transitivity
$$a \equiv b \quad \text{and} \quad b \equiv c \Longrightarrow a \equiv c$$
3. Additivity:
$$a \equiv b \quad \text{and} \quad a' \equiv b' \Longrightarrow a + b \equiv a' + b'.$$

4.1.1 Gauss sums and their properties

It is trivial to show that
$$\sum_{n=0}^{b-1} e^{i2\pi\frac{n}{b}} = 0,$$
for any integer $b > 1$. However, similar sums of nonlinear powers in n are usually nonzero:
$$\sum_{n=0}^{b-1} e^{i2\pi\frac{n^k}{b}} \neq 0,$$
where $k \geq 2$, b is a prime with $b \equiv 1 \, (\bmod k)$ and the sum is over an arbitrary complete system of residues $(\bmod b)$. Such sums are really difficult to calculate. For $k = 2$, they are known as (quadratic) Gauss sums because two centuries ago Gauss first gave explicit results for them. A paper by Berndt and Evans [23] is an excellent survey of the history of Gauss sums, also the appendix of the work of Merkel *et al* [24] is recomended. Gauss sums having linear phases added to the quadratic ones are of special interest in physical applications such as those discussed in this document. Emphasis is given here on this type of Gauss sums following the papers of Hannay and Berry[25] and also of Matsutani and Ônishi.[11]

Definition 12 *A finite quadratic sum of the form*

$$S(m,n,c) = \frac{1}{n}\sum_{s=0}^{n-1} \exp\left[i\pi\left(\frac{m}{n}s^2 + \frac{c}{n}s\right)\right]$$

is called a Gauss sum.

In the basic definition of the Gauss sum, the summation index runs through the simplest complete set of residues modulo n, however, the index can run through any given set of residues modulo n. The following theorem refers to a multiplicative property of Gauss sums which is possible to relate to Talbot phases.

Theorem 13 *If* $(n,n') = 1$*, then*

$$S(m,nn',c) = S(mn',n,c)S(mn,n',c).$$

Proof: The proof is followed from Hardy and Wright, starting from the right hand side of the expression.

$$\begin{aligned} S(mn',n,c)S(mn,n',c) &= \frac{1}{n}\sum_{s=0}^{n-1} \exp\left[i\pi\left(\frac{mn'}{n}s^2 + \frac{c}{n}s\right)\right] \frac{1}{n'}\sum_{s'=0}^{n'-1} \exp\left[i\pi\left(\frac{mn}{n'}s'^2 + \frac{c}{n'}s'\right)\right] \\ &= \frac{1}{nn'}\sum_{s=0}^{n-1}\sum_{s'=0}^{n'-1} \exp\left[i\pi\left(\frac{mn'^2s^2 + mn^2s'^2 + cn's + cns'}{nn'}\right)\right]. \end{aligned}$$

The double sum is substituted by a single one with index $\sigma = n's + ns'$ which runs through a complete set of residues modulo nn', because of the coprimality of n and n'. Therefore

$$\begin{aligned} S(mn',n,c)S(mn,n',c) &= \frac{1}{nn'}\sum_{\sigma=0}^{nn'-1} \exp\left[i\pi\left(\frac{m}{nn'}\sigma^2 + \frac{c}{nn'}\sigma\right)\right] \\ &= S(m,nn',c). \end{aligned}$$

This theorem can be useful to decompose the Talbot phases into prime factors and in this way, from simple graphs, be able to build any Talbot image. Nevertheless it is not clear how to perform this decomposition.

At rational values of the propagation coordinate the diffraction and dispersion phases can be expressed through the infinite Gaussian sum

$$G_\infty(a,b,c) = \lim_{N\to\infty} \frac{1}{2abN} \sum_{m=-Nab}^{Nab} \exp\left[i\frac{\pi}{b}\left(am^2+cm\right)\right],$$

which is an average over a period $2abN$. If a and b are co-prime numbers, the sum is zero unless c is an integer. Let $m = bn+s$, then the sum is divided into two sums

$$G_\infty(a,b,c) = \lim_{N\to\infty} \frac{1}{2abN} \sum_{n=-Na}^{Na-1} \sum_{s=0}^{b-1} \exp\left[i\frac{\pi}{b}\left(a(bn+s)^2+c(bn+s)\right)\right],$$

The argument of the exponential in the sum becomes

$$i\frac{\pi}{b}\left(a(bn+s)^2+c(bn+s)\right) = i2\pi ans + i\pi n(abn+c) + i\frac{\pi}{b}\left(as^2+bs\right)$$

and if ab and c are both odd or even at the same time, only terms in s survive: if n is even, the term is even and if n is odd, i.e., $n = 2k_1+1$, and also $ab = 2k_2+1$ and $c = 2k_3+1$, the term is

$$\begin{aligned} i\pi n(abn+c) &= i\pi(2k_1+1)[(2k_2+1)(2k_1+1)+(2k_3+1)] \\ &= i\pi(2k_1+1)(4k_1k_2+2k_2+2k_1+2k_3+2) \\ &= i2\pi(2k_1+1)(2k_1k_2+k_2+k_1+k_3+1) \end{aligned}$$

Thus G_∞ is rewriten as

$$G_\infty(a,b,c) = \lim_{N\to\infty} \frac{1}{2abN} \sum_{n=-Na}^{Na-1} \sum_{s=0}^{b-1} \exp\left[i\frac{\pi}{b}\left(as^2+cs\right)\right],$$

and taking the limit, G_∞ coincides with the quadratic Gauss sum

$$G_\infty(a,b,c) = S(a,b,c) = \frac{1}{a} \sum_{m=0}^{a-1} \exp\left[i\frac{\pi}{b}\left(am^2+cm\right)\right].$$

Notice the following property: in the normalization fraction and the upper limit of the sum, a can be interchanged for b simultaneously. We will use this property because it is appropriate from the standpoint of the physical application to run the sum over the denominator of the phase, i.e., over b.

We consider now the various cases according to the odd even character of the parameters a, b and c. Only the first case will be treated in detail.

<u>Case a and c even, b odd (because of coprimality to a)</u>

A factor $\exp(i\pi cmk)$ can be added since c is even. Then

$$S(a,b,c) = \frac{1}{b}\sum_{m=0}^{b-1} e^{i\frac{\pi}{b}am^2} e^{i\frac{\pi}{b}cm(kb+1)}$$

Noticing that $kb+1 \equiv 1(\mathrm{mod}\, b)$ and defining $a\bar{a}_b \equiv 1(\mathrm{mod}\, b)$ we can write

$$S(a,b,c) = \frac{1}{b}\sum_{m=0}^{b-1} \exp\left(\left[i\pi\frac{a}{b}\left(m^2 + cm\bar{a}_b\right)\right]\right)$$

and completing the square in the argument of the exponential:

$$S(a,b,c) \quad = \quad \frac{1}{b}\sum_{m=0}^{b-1} \exp\left(i\pi\frac{a}{b}\left[\left(m + \frac{c\bar{a}_b}{2}\right)^2 - \left(\frac{c\bar{a}_b}{2}\right)^2\right]\right)$$

Next, one should notice the following simpler form:

$$\sum_{m=0}^{b-1} \exp\left[i\pi\frac{a}{b}\left(m + \frac{c\bar{a}_b}{2}\right)^2\right] = \sum_{n=0}^{b-1} \exp\left(i\pi\frac{a}{b}n^2\right)$$

since the linear term plus the independent one in the left hand side exponential lead to a different order of the terms with respect to the right hand side of the equation. Furthermore we note that

$$\sum_{n=0}^{b-1} \exp\left(i\pi\frac{a}{b}n^2\right) = \sum_{n=0}^{b-1}\left[1 + \left(\frac{an/2}{b}\right)_L\right] \exp\left(i\pi\frac{a}{b}n\right)$$

where $\left(\frac{l_1}{l_2}\right)_L$ denotes the Legendre symbol that for l_2 an odd prime number is defined as zero if l_1 is divided by l_2, 1 if l_1 is a quadratic residue mod l_2 (there exists an integer k such that $k^2 = l_1$ (mod l_2)), and -1 if l_1 is not a quadratic residue.

Therefore the Gauss sum is expressed as follows:

$$S(a,b,c) \quad = \quad \frac{1}{b}\exp\left(-i\pi\frac{a}{b}(c\bar{a}_b/2)^2\right)\sum_{n=0}^{b-1}\left[1 + \left(\frac{an/2}{b}\right)_L\right] \exp\left(i\pi\frac{a}{b}n\right)$$

and the latter sum under splitting into two sums yields zero in the first one. Making use of the Legendre symbol multiplicative property in the top argument we get:

$$S(a,b,c)=\frac{1}{b}\left(\frac{a/2}{b}\right)_L \exp\left(-i\pi\frac{a}{b}(c\bar{a}_b/2)^2\right)\sum_{n=0}^{b-1}\left(\frac{n}{b}\right)_L \exp\left(i2\pi\frac{n}{b}\right).$$

The following result from number theory [26] should be used in order to get the most compact result for $S(a,b,c)$:

$$\left(\frac{1/2}{b}\right)_L=\left(\frac{2}{b}\right)_L=(-1)^{(b^2-1)/8}=\exp\left(\pm i\pi\frac{b^2-1}{8}\right)$$

and then

$$\sum_{n=0}^{b-1}\left(\frac{n}{b}\right)_L \exp\left(i2\pi\frac{n}{b}\right)=\sqrt{b}\exp\left(i\pi\frac{(b-1)^2}{8}\right)$$

for $b\equiv 1\,(\bmod 4)$ or $b\equiv 3\,(\bmod 4)$. Then:

$$\begin{aligned} S(a,b,c) &= \frac{1}{b}\left(\frac{a}{b}\right)_L\left(\frac{1/2}{b}\right)_L \exp\left(-i\pi\frac{a}{b}(c\bar{a}_b/2)^2\right)\sum_{n=0}^{b-1}\left(\frac{n}{b}\right)_L \exp\left(i2\pi\frac{n}{b}\right) \\ &= \frac{1}{\sqrt{b}}\left(\frac{a}{b}\right)_L \exp\left(\pm i\pi\frac{(b^2-1)}{8}-i\pi\frac{a}{b}(c\bar{a}_b/2)^2+i\pi\frac{(b-1)^2}{8}\right). \end{aligned}$$

The terms in the exponential with the common factor $i\pi/8$ are rewritten

$$\frac{i\pi}{8}\left(\pm b^2\mp 1+b^2-2b+1\right)=-\frac{i2\pi}{8}(b-1)$$

having chosen the lower sign (there is no final change if the $+$ sign is picked up).

Recalling now that a and c are even we get

$$S(a,b,c)=\frac{1}{\sqrt{b}}\left(\frac{a}{b}\right)_L \exp\left(-i\pi\left[\frac{1}{4}(b-1)+\frac{a}{b}(c\bar{a}_b/2)^2\right]\right).$$

Case a, b, and c odd

Exactly the same expression can be worked out with the only minor modification that a is replaced by $2\bar{2}_b a$, because it is congruent $\bmod\, b$.

Case a odd, b even, c odd

Similar arguments lead to a slightly different result.

The three final results are as follows:

$$S(a,b,c)=\frac{1}{\sqrt{b}}\left(\frac{a}{b}\right)_L \exp\left(-i\frac{\pi}{4}\left[(b-1)+\frac{a}{b}(c\bar{a}_b)^2\right]\right) \qquad a \text{ even, } b \text{ odd, } c \text{ even,}$$

$$S(a,b,c)=\frac{1}{\sqrt{b}}\left(\frac{a}{b}\right)_L \exp\left(-i\frac{\pi}{4}\left[(b-1)+\frac{2a}{b}\bar{2}_b^3(c\bar{a}_b)^2\right]\right) \qquad a \text{ odd, } b \text{ odd, } c \text{ odd,}$$

$$S(a,b,c)=\frac{1}{\sqrt{b}}\left(\frac{b}{a}\right)_L \exp\left(-i\frac{\pi}{4}\left[-a+\frac{a}{b}(c\bar{a}_b)^2\right]\right) \qquad a \text{ odd, } b \text{ even, } c \text{ odd.}$$

These results were obtained under the assumption that b is prime. For the general case where b is not a prime the Legendre symbol should be substituted by the Jacobi symbol, which is a product of Legendre symbols defined in the following way. Let the prime decomposition of b be $b=\prod_{i=1}^{i=n} p_i^{r_i}$. Then, by definition, the Jacobi symbol is

$$\left(\frac{a}{b}\right)_J=\prod_{i=1}^{i=n}\left(\frac{a}{p_i}\right)_L^{r_i}.$$

4.2 From Talbot Effect to Gauss Sums

As discussed before, there are two ways to describe diffraction effects: on the one hand by the paraxial propagator, and on the other hand, by Fresnel integrals. In this section both approaches will be developed and ultimately compared to point out the relationship of them through the reciprocity of Gauss sums.

4.2.1 Wave Representation of Talbot Images

From the results obtained in section 3.5, the relation between the Talbot phases and Gauss sums will be developed in the following. By discretizing the optical propagation axis ζ by means of rational numbers, one can write the rational paraxial field as a shifted delta comb affected by phase factors, which is the main result of Berry and Klein:

$$\psi_{\mathrm{p}}\left(\xi,\frac{p}{q}\right)=\frac{1}{q^{1/2}}\sum_{n=-\infty}^{\infty}\delta\left(\xi_p-\frac{n}{q}\right)\exp[i\Phi_{\mathrm{diffr}}(n;q,p)]\;, \tag{4.1}$$

where $\xi_p = \xi - e_p/2$ and $e_p = 0(1)$ if p is even (odd). The paraxial phases $\exp[i\Phi_{\rm diffr}(n;q,p)]$ are specified in Sec. 4 and appear to be the physical quantities directly connected to number theory. The expression

$$\sum_{n=-\infty}^{\infty} \delta(\xi - n) = \sum_{m=-\infty}^{\infty} \exp(i2\pi m)$$

represents an infinite set of Dirac's Deltas equally spaced. This expression is discussed in Appendix . It is widely used in optics and it is known as the Poisson formula for the Dirac Comb.

The rational approximation (4.2) allows for the following physical interpretation of the paraxial self-imaging process: *in each unit cell of the plane p/q, q images of the grating slits are reproduced with spacing a/q and intensity reduced by* $1/q$. This interpretation has been given by Berry and also by Schleich in different works.

In this expression, the trick is to turn the continuous propagation axis into the rational number axis and also to perform the integer modulo division of n with respect to the rational denominator of the propagation axis, i.e.,

$$\zeta = \frac{p}{q}, \qquad n = lq + s. \tag{4.2}$$

Through this approximation, the sum over n is divided into two sums: one over negative and positive integers l, and the other one over $s \equiv n (\bmod q)$

$$\alpha(\zeta,\tau) = \sum_{l=-\infty}^{\infty} \sum_{s=0}^{q-1} \exp\left[i\pi(lq+s)^2 \frac{p}{q} - i2\pi(lq+s)\tau\right]. \tag{4.3}$$

This form of $\alpha(\zeta,\tau)$ is almost exactly the same as given by Berry & Klein [16] and by Matsutani and Ônishi [11]. The difference is that the sign of the exponent is opposite. Following these authors one can express α in terms of the Poisson formula leading to

$$\alpha(p/q,\tau) = \frac{1}{\sqrt{q}} \sum_{n=-\infty}^{\infty} \left[\frac{1}{\sqrt{q}} \sum_{s=0}^{q-1} \exp\left[i\pi\left(\frac{p}{q}s^2 - 2s\tau\right)\right]\right] \delta(\tau_p + \frac{n}{q}) \ , \tag{4.4}$$

where τ_p is a notation similar to ξ_p. The rest of the calculations are straightforwardly performed though they are lengthy. By algebraic manipulations the phase factor can be easily obtained and we reproduce below the two expressions for direct comparison

$$\Phi_{\rm disp}(n;q,p) = \frac{1}{\sqrt{q}} \sum_{s=0}^{q-1} \exp\left\{i\frac{\pi}{q}\left[ps^2 - 2s(n - \frac{qe_p}{2})\right]\right\} \tag{4.5}$$

$$\Phi_{\rm diffr}(n;q,p)=\frac{1}{\sqrt{q}}\sum_{s=0}^{q-1}\exp\left\{i\frac{\pi}{q}\left[2s(n+\frac{qe_p}{2})-ps^2\right]\right\}. \tag{4.6}$$

Both phases are special types of Gauss sums from the mathematical standpoint. The difference of signs here appears because of the sign convention chosen for the Fourier transform. Not surprisingly, the changes in the mathematical formulation are minimal although the experimental setup is quite different.

If one tries to make computer simulations using the Fourier transform method, the Gibbs phenomenon is unavoidable for discontinuous transmittance functions. However, in the case of fiber dispersion, one class of continuous pulses one could work with are the supergaussian ones, i.e., functions of the following form

$$A(\zeta=0,\tau)=A_0\exp\left[\frac{-\tau^N}{\sigma_0}\right], \tag{4.7}$$

where N is any even number bigger than two. As N increases, the supergaussian pulse resembles more and more a square pulse. Computer simulations of the evolution of supergaussian pulse trains are given in [4].

In this way, the evaluation of the Talbot phases are performed by identification of the parameters: $a=\pm p$, $b=q$, $c=\mp 2(n\mp qe_p/2)$, respectively, where $e_p=0$ (1) if p even (odd). The upper sign is for dispersion and the lower for diffraction. Thus, we get:

p even, q odd:

$$\begin{aligned}\Phi_{\rm disp}(n;p,q) &= \left(\frac{p}{q}\right)_J\exp\left(-i\frac{\pi}{4}\left[(q-1)+\frac{p}{q}(2n\bar{p}_q)^2\right]\right),\\ \Phi_{\rm diffr}(n;p,q) &= \left(\frac{p}{q}\right)_J\exp\left(+i\frac{\pi}{4}\left[(q-1)+\frac{p}{q}(2n\bar{p}_q)^2\right]\right).\end{aligned}$$

p odd, q odd:

$$\begin{aligned}\Phi_{\rm disp}(n;p,q) &= \left(\frac{p}{q}\right)_J\exp\left(-i\frac{\pi}{4}\left[(q-1)+2\bar{2}_q^3\frac{p}{q}((2n-q)\bar{p}_q)^2\right]\right),\\ \Phi_{\rm diffr}(n;p,q) &= \left(\frac{p}{q}\right)_J\exp\left(+i\frac{\pi}{4}\left[(q-1)+2\bar{2}_q^3\frac{p}{q}((2n+q)\bar{p}_q)^2\right]\right).\end{aligned}$$

p odd, q even:

$$\begin{aligned}\Phi_{\rm disp}(n;p,q) &= \left(\frac{q}{p}\right)_J\exp\left(-i\frac{\pi}{4}\left[-p+\frac{p}{q}((2n-q)\bar{p}_q)^2\right]\right),\\ \Phi_{\rm diffr}(n;p,q) &= \left(\frac{q}{p}\right)_J\exp\left(+i\frac{\pi}{4}\left[-p+\frac{p}{q}((2n+q)\bar{p}_q)^2\right]\right).\end{aligned}$$

4.2.2 Path Representation of Talbot Images

In the previous chapter, the Fresnel integral was obtained as a consequence of basic physical principles. In diffraction, the wavefield due to a periodic structure is obtained by this means and the mathematical expression is as follows:

$$u(\xi,z) \propto \int u(\xi',0)\cos(\theta)\frac{\exp\left[ik\sqrt{a^2(\xi-\xi')^2+z^2}\right]}{\sqrt{a^2(\xi-\xi')^2+z^2}}ad\xi' \tag{4.8}$$

Exactly the same expression is allowed to be used in dispersion if the proper variables are employed. This means that, theoretically, the propagation coordinate has a transversal spatial coordinate in which, in this case, the structure will be periodic. As explained in section 3.4, in fiber optics, the usual substitution is $\tilde{t} = t + z/v_g$ where v_g is the speed of light in the optical fiber. The required substitution is $\xi \to \tau$. This easy substitution is permitted because the variable τ is already non-dimensional and the multiplying factor v_g is included through the change of variables. In this way the Huygens-Fresnel principle can be interpreted to give the following integral:

$$A(z,\tau) \propto \int A(\tau',0)\cos(\theta_{disp})\frac{\exp\left[i\kappa\sqrt{a^2(\tau-\tau')^2+z^2}\right]}{\sqrt{a^2(\tau-\tau')^2+z^2}}ad\tau', \tag{4.9}$$

where κ is a proportionality factor and the angle θ_{disp} characterizes the delay in time of the corresponding component. The paraxial approximation in this context means small angle propagation. The corresponding approximation for dispersion means that the pulse components have a small delay compared to the propagation distance, this means that similar considerations in both, dispersion and diffraction can be done. Explicitly, $(\xi-\xi')$ and $(\tau-\tau')$ are small quantities, and the phase can be Taylor expanded to the second order:

$$f(\hat{\eta}) = \sqrt{a^2\hat{\eta}^2+z^2} \approx z + \frac{a^2}{z}\frac{\hat{\eta}^2}{2}.$$

Making use of the paraxial approximation in diffraction, the paraxial field obtained is

$$\hat{u}(\xi,\zeta) \propto \sum_{n=-\infty}^{\infty}\exp\left(i\pi\frac{(\xi+n)^2}{\zeta}\right).$$

The SVEA field is also obtained considering the scaling of the propagation variable,

$$\hat{A}(\zeta,\tau) \propto \sum_{n=-\infty}^{\infty} \exp\left(i\pi\frac{(\tau+n)^2}{\zeta}\right)$$

where $\hat{\eta}$ represents the transversal distance between the nth slit or pulse and the observation point. From the expressions above it is possible to develop the correspondence of wave and particle representations through the reciprocity of Gauss sums. Since both expressions are the same, both propagators will be developed as $\hat{\psi}(\eta,\zeta)$, where η would be rather distance (diffraction) or time (dispersion). The propagation axis is again expressed as p/q, but the summation index n is divided this time into $n = lp+s$ for reciprocity purposes. The infinite sum is thus separated into:

$$\hat{\psi}(\eta, p/q) = \sum_{n=-\infty}^{\infty} \sum_{s=0}^{p-1} \exp\left[i\pi(\eta+s)^2 \frac{q}{p}\right] \delta(\eta_p - \frac{n}{q}). \tag{4.10}$$

The term corresponding to $\Phi(n;q.p)$ yields

$$\hat{\Phi}(n;q,p) = \frac{1}{\sqrt{q}} \sum_{s=0}^{p-1} \exp\left\{i\frac{\pi}{p}\left[qs^s + 2s\left(n \pm \frac{qe_p}{2}\right)\right]\right\} \exp\left\{i\pi\frac{(n \pm qe_p/2)^2}{pq}\right\} \tag{4.11}$$

The form of the Gauss sums here obtained can be shown to be reciprocal. For the case of diffraction, it was obtained by Matsutani and Ônishi, while the case of dispersion is easily obtained if the proper variables are chosen as shown in this work and in the paper by our group [19].

It is worth to point out that in atom interferometry, Dubetsky and Berman [27] used a method introduced by Winthrop and Worthington [28] to provide another type of simplification of Gauss sums.

Chapter 5

Irrational Talbot distances

5.1 Fractal Approach

In the Talbot terminology the self-reconstructed images in the planes $z = (p/q)z_{\mathrm{T}}$ consist of q superposed copies of the grating as already mentioned, completed with discontinuities. Although there is a finite number of images at fractional distances, they still represent an infinitesimal subset of all possible images that occur at the irrational distances.

In the planes located at irrational fractions of the Talbot distance the light intensity is a *fractal function* of the transverse variable. This means that the field intensity has a definite value at every point, but its derivative has no definite value. Such fractal functions are described by a fractal dimension, D, between one (corresponding to a smooth curve) and two (corresponding to a curve so irregular that it occupies a finite area). In the case of Ronchi gratings, for example, the fractal dimension of the diffracted intensity in the irrational transverse planes is $3/2$ [16].

Moreover, the so-called *carpets* were defined by Berry and Klein, such structures are wave intensity rugged patterns on the (ξ,ζ) plane, i.e., the plane of the transverse coordinate and the propagation coordinate. Since they are surfaces, their fractal dimension takes values between two and three. According to Berry and Klein [16], in general for a surface where all directions are equivalent, the fractal dimension of the surface is one unit greater than the dimension of an inclined curve that cuts through it. Taking into account this argument, the carpet's dimension is expected to be $5/2$. However, Talbot landscapes were found not to be isotropic. For fixed ξ, as the intensity varies as a function of distance ζ from the grating, the fractal dimension is found to be $7/4$, one quarter more than in the transverse case.

Therefore the longitudinal fractals are slightly more irregular than the transverse ones. In addition, the intensity is more regular along the bisectrix channel because of the cancelation of large Fourier components that has fractal dimension of $5/4$. The landscape is dominated by the largest of these fractal dimensions (the longitudinal one), and so is a surface of fractal dimension $1+7/4 = 11/4$. In conclusion, the fractal dimension of a non-isotropic intensity landscape is given by its largest fractal dimension.

5.2 Wavelet approach

Wavelet transforms (WT) are known to have various advantages over the Fourier transform and in particular it has been demonstrated they can add up supplementary information on the fractal features [29, 30, 31]. A simple definition of the one-dimensional continuous wavelet transform is

$$f_W(a,b) = \int_{-\infty}^{\infty} f(\mathrm{s}) W_{a,b}^{*}(\mathrm{s}) d\mathrm{s} \tag{5.1}$$

where the asterix denotes complex conjugation and $W_{a,b}$ is a so-called daughter wavelet which is derived from the mother wavelet $W(\mathrm{s})$ by dilation and shift operations:

$$W_{a,b}(\mathrm{s}) = \frac{1}{\sqrt{a}} W\left(\frac{\mathrm{s}-b}{a}\right) \tag{5.2}$$

In the following, the Morlet wavelets are used. These type of wavelets are derived from the typical Gaussian-enveloped mother wavelet which is itself a windowed Fourier transform

$$W(\mathrm{s}) = \exp[-(\mathrm{s}/\mathrm{s}_0)^2] \exp(i2\pi k\mathrm{s}). \tag{5.3}$$

The point is that if the mother wavelet contains a harmonic structure, e.g., in the Morlet case the phase $\exp(i2\pi k\mathrm{s})$), the WT provides both frequency and spatial information of the signal.

In the wavelet framework the expansion of an arbitrary signal $f(t)$ can be written in an orthonormal wavelet basis in the form

$$f(t) = \sum_{\mathrm{m}} \sum_{\mathrm{n}} f_{\mathrm{n}}^{\mathrm{m}} W_{\mathrm{m,n}}(t) \ , \tag{5.4}$$

where the coeficients f_n^m can be calculated as follows.

$$f_{\mathrm{n}}^{\mathrm{m}} = \int_{-\infty}^{\infty} f(t) W_{\mathrm{m,n}}(t) dt \ . \tag{5.5}$$

The orthonormal wavelet basis functions $W_{\mathrm{m,n}}(t)$ fulfill the following dilation-translation property

$$W_{\mathrm{m,n}}(t) = 2^{\mathrm{m}/2} W(2^{\mathrm{m}} t - \mathrm{n}) \ . \tag{5.6}$$

In the wavelet approach the fractal character of a certain signal can be inferred from the behavior of its power spectrum $P(f)$, which is the Fourier transform of the autocovariance (also termed autocorrelation) function and in differential form $P(f)df$ represents the contribution to the variance of a signal from frequencies between f and $f + df$. Indeed, it is known that for self-similar random processes the spectral behavior of the power spectrum is given by [32, 33]

$$P_f(\omega) \sim |\omega|^{-\beta_f} \ , \tag{5.7}$$

where β_f is the spectral parameter of the wave signal and it is not to be confused with the propagation constant. In addition, the variance of the wavelet coefficients possesses the following behavior [33]

$$\mathrm{var}\, f_{\mathrm{n}}^{\mathrm{m}} \approx (2^{\mathrm{m}})^{-\beta_f} \ . \tag{5.8}$$

These formulas are certainly suitable for the Talbot transverse fractals because of the interpretation in terms of the regular superposition of identical and equally spaced grating images. These wavelet formulas have been used within this work for calculations related to the same rational paraxiallity for the two cases of transverse diffraction fields (figure 5.1) and the fiber-dispersed optical fields (figure 5.2), respectively. The inset b) of both figures show a graph that displays the coefficient m in the horizontal axis and the $\log|varx_m^n|$ in the vertical axis, being β_f the slope of the graph. The letter x in the graphs stands for the transversal variable of the optical field. The basic idea is that the above-mentioned formulas can be employed as a checking test of the self-similarity structure of the optical fields. The requirement is to have a constant spectral parameter β_f over many scales. In the case of supergaussian pulses, their dispersed fields turned out not to have the self-similarity property as can be seen by examining figure (5.2) where one can see that the constant slope is not maintained over all scales. In both figures, Morlet wavelets were employed. A great deal of details can be seen in all basic quantities of the diffracted field, namely in the intensity, modulus, and phase. On the other hand, the same wavelet transform applied to the N=12 supergaussian dispersed pulse, although showing a certain similarity to the previous discontinuous case, contains less structure and thus looks more regular. This points to the fact that if in diffraction experiments one uses continuous transmittance gratings the fractal behavior would turn milder.

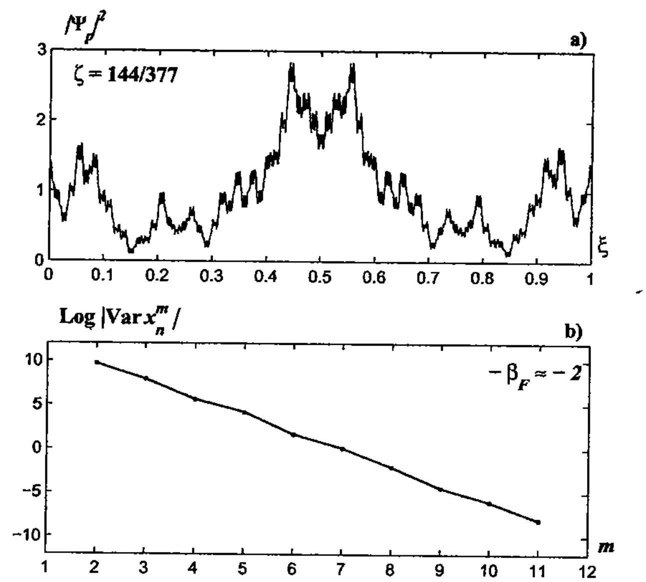

Figure 5.1: Fractal Talbot for Diffraction

More realistically, paraxial waves display electric and magnetic polarization singularities [34]. If the paraxial wavefield is treated as a signal then it is worth pointing out here that detection of signal singularities has been studied in quite detail by the experts in wavelet processing [35]. We plan to study this aspect in future research.

The fractal aspects of the paraxial wavefield have been probed here by means of the wavelet transform for the cases of diffraction and fiber dispersion. In the case of diffraction, the previous results of Berry and Klein are confirmed showing that the wavelet approach can be an equivalent and more informative tool. The same procedure applied to the case of fiber dispersion affecting the paraxial evolution of supergaussian pulses indicates that the self-similar fractal character does not show up in the latter type of axial propagation. This is a consequence of the continuous transmittance function of the supergaussian pulses as opposed to the singular one in the case of Ronchi gratings.

Finally, as a promising perspective, we would like to suggest the following experiment by which irrational distances can be determined. The idea is that the spectral index of the

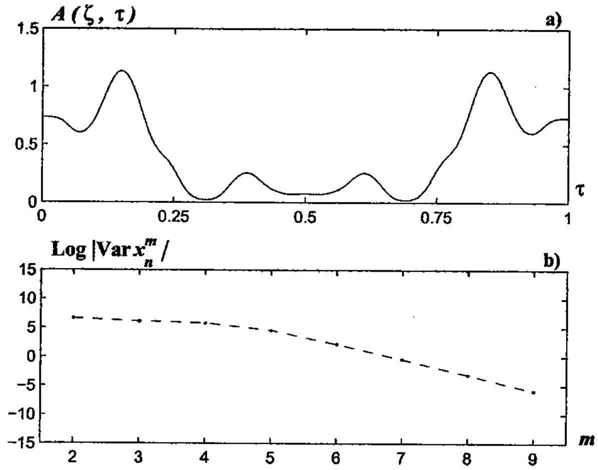

Figure 5.2: Dispersion Talbot (not fractal).

Talbot fractal images can be used as a very precise pointer of rational and irrational distances with respect to the Talbot one. Suppose that behind a Ronchi grating under plane wave illumination a CCD camera is mounted axially by means of a precision screw. The Talbot image at z_{T} can be focused experimentally and can be used to calibrate the whole system. An implemented real time wavelet computer software can perform a rapid determination of the fractal index β_f, which in turn allows the detection of changes of the distance in order to determine if the CCD camera is at rational or irrational multiples of the Talbot distance. To the best of our knowledge, we are not aware of any experimental setup in which irrational distances can be determined in such an accurate way. This also points to high-precision applications in metrology.

Chapter 6

Conclusion

This thesis work includes a historical view diffraction and fiber optics. As mentioned in the main matter of the document, this view is important to understand the development of the theory that explains a given phenomenon. The mathematical theory underlying the optical Talbot Effect applies to every phenomenon with the same description, this means that every physical phenomenon governed by the d'Alembert equation in the paraxial approximation shows this self-reconstruction effect. Two descriptions of this type phenomena are available in this document, and the correct interpretation of them is indeed a matter of basic interest for science.

Reciprocity of Gauss sums has been shown to properly account for the relationship between wavy and particle representations of two phenomena, namely dispersion and diffraction. The step to show the same property in the framework of Quantum Revivals should not be difficult since in this work, the mathematical relation between the three effects is fairly explained. In the particular case of the Talbot Effect, it could be said that the mathematical theory underlying such a remarkable effect has deep implications in the study the complementarity principle, since the reciprocity could be interpreted as a consequence of duality.

Regarding the wavelet transform, the utility of this tool has not been developed in its full capacity. As a possibility for the future it is suggested that the wavelet transform is employed to analyze phase singularities in order to study the behavior of the phase as a function of the denominator q of the rational Talbot distances.

Finally, I would like to point out a possible applications of the optical Talbot Effect. A carpet of light could be useful in quantum computation if the Talbot reconstructions are considered as three dimensional optical traps. If a continuous grating is employed, there are certain *spots* of maximums of intensity surrounding the integer Talbot distance, this spots could possibly be optical traps. This traps, if possible, could be useful to create three dimensional arrays of atoms. The second application is in a completely different field, which is cryptography. The obvious application is to factorize numbers.

The importance of the self-reconstruction effects is not yet envisioned, but for sure in the next few years a lot of new applications will be apparent in a variety of fields of physics and technology.

Appendix A

Biographies

In this appendix, some biographies of relevant people are added. All the information was extracted from the website

`www.howstuffworks.com`

This site is used as a tool for educators and teachers. The information is reliable and was carefully checked.

Alhazen (965-1040) - Born in Iraq as Abu Ali Hasan Ibn al-Haitham, the great Arab physicist is more often known by the Latinized version of his first name, Alhazen. The efforts of Alhazen resulted in over one hundred works, the most famous of which was Kitab-al-Manadhirn, rendered into Latin in the Middle Ages. The translation of the book on optics exerted a great influence upon the science of the western world, most notably on the work of Roger Bacon and Johannes Kepler. A significant observation in the work contradicted the beliefs of many great scientists, such as Ptolemy and Euclid. Alhazen correctly proposed that the eyes passively receive light reflected from objects, rather than emanating light rays themselves.

Jacques Babinet (1794-1872) - Jacques Babinet was a French physicist, mathematician, and astronomer born in Lusignan, who is most famous for his contributions to optics. Among Babinet's accomplishments are the 1827 standardization of the ngstrm unit for measuring light using the red cadmium line's wavelength, and the principle (bearing his name) that similar diffraction patterns are produced by two complementary screens.

Roger Bacon (1214-1294) - Roger Bacon was an English scholastic philosopher who was also considered a scientist because he insisted on observing things for himself instead of depending on what other people had written. Bacon's writings included treatises on optics (then called perspective), mathematics, chemistry, arithmetic, astronomy, the tides, and the reformation of the calendar. His skill in the use of optical and mechanical instruments caused him to be regarded by many as a sorcerer. Bacon was acquainted with the properties of mirrors, knew the powers of steam and gunpowder, had a working knowledge in microscopy, and possessed an instrument very much like a modern telescope.

Niels Bohr (1885-1962) - Building on Ernest Rutherford's work on the nucleus, Bohr developed a new theory of the atom, which he completed in 1913. The work proposed that electrons travel only in certain orbits and that any atom could exist only in a discrete set of stable states. Bohr further held that the outer orbits, which could hold more electrons than the inner ones, determine the atom's chemical properties and conjectured that atoms emit light radiation when an electron jumps from an outer orbit to an inner one. Although Bohr's theory was initially viewed with skepticism, it earned him the Nobel Prize in physics in 1922 and was eventually expanded by other physicists into quantum mechanics.

William Henry Bragg (1862-1942) - Sir William Henry Bragg was a noted British physicist and President of the Royal Society who had numerous research interests, but the work that earned him a rank as one the great leaders in science was his historic advancements in X-ray crystallography. Working with his son William Lawrence Bragg, he developed a method of bombarding single crystals with high-energy X-rays emitted by specially constructed vacuum tubes. By examining the pattern of X-rays diffracted by various crystals, Bragg and his son were able to establish some fundamental mathematical relationships between an atomic crystal structure and its diffraction pattern. For this achievement, William Henry Bragg and William Lawrence Bragg were awarded the Nobel Prize in Physics in 1915.

Tycho Brahe (1546-1601) - Tycho Brahe was a Danish astronomer who made the most accurate observations possible without the aid of a telescope. On November 11, 1572 he observed what seemed to be a bright new star near Cassiopeia and studied it for the next 18 months. Brahe was surprised to find that the star seemed to be further away than the moon and that it intensified in brightness before eventually slowly fading out of view. The event was extremely significant because it would not have been possible if the Aristotelian conception of a harmonious and unchanging universe were correct. Brahe attempted to modify the Ptolemaic theory to coincide with his observations, and proposed the Tychonic system, in which the Earth remained immobile, but the sun served as a secondary center. Although it was an interesting attempt at a compromise between two completely different viewpoints, the Tychonic system never garnered much support.

Louis de Broglie (1892-1987) - During his long and illustrious career, de Broglie worked on various aspects of wave mechanics and published a large number of scientific treatises. He also taught theoretical physics at the Sorbonne in Paris and composed several books exploring the relationship between physics and philosophy. In addition to the Nobel Prize, de Broglie received a large number of other honors, including a number of honorary doctorate degrees, an appointment as an adviser to the French Atomic Energy Commissariat, and election into the French Academy of Sciences and the British Royal Society.

Claude Chappe (1763-1805) - Claude Chappe was an engineer and cleric who invented a device known as the semaphore visual telegraph, an optical signaling system especially important during the French Revolution. In August of 1794, Chappe's semaphore visual telegraph conveyed in less than an hour the news that the Republican army had recaptured Cond-sur-l'Escaut from the Austrians, a feat that would have taken approximately twenty-four hours if transported by courier on horseback. The system was considered a success and another line was soon installed between Paris and Landau, others following in later years.

Marie Alfred Cornu (1841-1902)- Cornu made a wide variety of contributions to the fields of optics and spectroscopy, but is most noted for significantly increasing the accuracy of contemporary calculations of the speed of light. In 1878, Cornu made adjustments to an earlier method of measuring the velocity of light developed by Armand Fizeau in the 1840s. The changes and improved equipment resulted in the most accurate measurement taken up to that time, 299, 990 km per second. Other significant accomplishments of Cornu include a photographic study of ultraviolet radiation and the establishment of a graphical approach, known as the Cornu spiral, for calculating light intensities in Fresnel diffraction.

Louis-Jacques-Mand Daguerre (1787-1851) - Born near Paris, France on November 18, 1787, Louis-Jacques-Mand Daguerre was to become both a painter and the inventor of the first successful form of photography. As an artist, Daguerre was interested in creating realistic renderings and utilized a camera obscura to aid his efforts. In hopes of simplifying the process, he became intrigued with the idea of permanently fixing an image chemically, as were many others during the period. Working with Joseph-Nicephore Niepce, Daguerre developed a photographic process termed the daguerreotype, which enjoyed widespread use in Europe for a limited time during the middle 1800s.

Leonardo da Vinci (1452-1519) - Leonardo da Vinci was a painter, sculptor, architect, engineer, scientist and genius who best represents the ideals of the Renaissance period. Da Vinci was a great engineer and inventor who designed buildings, bridges, canals, forts and war machines. He was also fascinated by birds and flying and drew designs of fantastic flying machines. Da Vinci was also intrigued with the study of optics and conducted extensive investigations and made drawings about the nature of light, reflections, and shadows. Even though it was not until over 100 years later that the first telescope was invented by Hans Lippershey, da Vinci realized the possibility of using lenses and mirrors to view heavenly bodies. Da Vinci was one of the greatest painters of all times. The Last Supper and the Mona Lisa are two of his best-known paintings.

René Descartes (1596-1650) - René Descartes is often referred to as the father of modern philosophy for his revolutionary breach from Aristotelian thought. In its place he attempted to establish a dualistic system that rested on a clear distinction between the mind, the origin of thought, and matter. He is, perhaps, most commonly remembered for his philosophical declaration, ”Cogito, ergo sum” (I think, therefore I am). However, in addition to his many philosophical reflections, Descartes made significant contributions to mathematics and the sciences, including optics.

Thomas Alva Edison (1847-1931) - Thomas Edison was an American inventor who achieved his greatest successes in his Menlo Park laboratory and was called the ”Wizard of Menlo Park.” This research and development laboratory was the first of its kind anywhere; it became a model for later, modern research and development facilities such as Bell Laboratories. It was during this period of his life that Edison and his staff were responsible for many inventions and innovations. More patents were issued to Edison than have been issued to any other single person in United States history, a total of 1,093. Edison is perhaps best known for his invention of the incandescent light bulb.

Albert Einstein (1879-1955) - Albert Einstein was one of the greatest and most famous scientific minds of the 20th century. The eminent physicist is best remembered for his theories of relativity, as well as his revolutionary notion concerning the nature of light. However, his innovative ideas were often misunderstood and he was frequently ridiculed for his vocal involvement in politics and social issues. The birth of the Manhattan Project yielded an inexorable connection between Einstein's name and the atomic age. However, Einstein did not take part in any of the atomic research, instead preferring to concentrate on ways that the use of bombs might be avoided in the future, such as the formation of a world government.

Euclid (325-265 BC) - Though often overshadowed by his mathematical reputation, Euclid is a central figure in the history of optics. He wrote an in-depth study of the phenomenon of visible light in Optica, the earliest surviving treatise concerning optics and light in the western world. Within the work, Euclid maintains the Platonic tradition that vision is caused by rays that emanate from the eye, but also offers an analysis of the eye's perception of distant objects and defines the laws of reflection of light from smooth surfaces. Optica was considered to be of particular importance to astronomy and was often included as part of a compendium of early Greek works in the field. Translated into Latin by a number of writers during the medieval period, the work gained renewed relevance in the fifteenth century when it underpinned the principles of linear perspective.

Pierre de Fermat (1601-1665) - Pierre de Fermat was a lawyer by occupation, but possessed one of the greatest mathematical minds of the seventeenth century. He made major contributions to geometric optics, modern number theory, probability theory, analytic geometry, and is generally considered the father of differential calculus. Through the use of his method for determining minima and maxima, Fermat established what is usually described as the principle of least time in 1658. According to the tenet, a beam of light traveling between two points will follow the path that takes the shortest amount of time to complete. From the principle of least time, the law of refraction and the law of reflection can be deduced. Future scientists, however, demonstrated that Fermat's principle was incomplete or only partially true.

Joseph von Fraunhofer (1787-1826) - In 1813, von Fraunhofer accomplished what is often considered his greatest achievement. He independently rediscovered William Hyde Wollaston's dark lines in the solar spectrum, which are now known as Fraunhofer lines. He described a great number of the 500 or so lines he could see using self-designed instruments, labeling those most prominent with letters, a form of nomenclature that is still in favor. Fraunhofer lines would eventually be used to reveal the chemical composition of the sun's atmosphere.

Augustin-Jean Fresnel (1788-1827) - Augustin-Jean Fresnel, was a nineteenth century French physicist, who is best known for the invention of unique compound lenses designed to produce parallel beams of light, which are still used widely in lighthouses. In the field of optics, Fresnel derived formulas to explain reflection, diffraction, interference, refraction, double refraction, and the polarization of light reflected from a transparent substance.

Dennis Gabor (1900-1979) - In the late 1940s, Dennis Gabor attempted to improve the resolution of the electron microscope using a procedure that he called wavefront reconstruction, but which is now known as holography. Though he was unable to realize his goal at the time, his work was to find much more prolific use years later, after the development of the laser in 1960. Gabor received the Nobel Prize in Physics in 1971 for his foundational holographic research and experimentation.

Galileo Galilei (1564-1642) - Galileo's many and varied accomplishments span the scientific disciplines of astronomy, physics, and optics. He was also an inventor, mathematician, and author who is widely known for his famous experiment dropping different size balls from the Leaning Tower of Pisa that resulted in new ideas about physics and the idea that "laws" of science could, and should, be questioned.

Francesco M Grimaldi, S.J. (1613 to 1663) was born and died in Bologna and was professor of mathematics and physics at the Jesuit college in Bologna for many years. He was one of the great geometer-physicists of his time and was an exact and skilled observer, especially in the field of optics. He discovered the diffraction of light and gave it the name diffraction, which means "breaking up." He laid the groundwork for the later invention of the diffraction grating. He was one of the earliest physicists to suggest that light was wavelike in nature. He formulated a geometrical basis for a wave theory of light in his Physico-mathesis de lumine (1666). It was this treatise which attracted Isaac Newton to the study of optics. Newton deals with the diffraction problems of Grimaldi in Part III of his Opticks (1704), after having first learned of Grimaldi's diffraction from the writings of another Jesuit geometer, Honor Fabri.

Robert Grosseteste (1175-1253) - Grosseteste was particularly interested in astronomy and mathematics, and he asserted that the latter was essential to investigations of natural phenomena. Consequently, his study of light often took a mathematical turn, resulting in a refinement of optical science. In his investigations of rainbows, comets, and other optical phenomena, he notably made use of both observational data and mathematical formulations. Moreover, Grosseteste was an early proponent of the need for experimental support of scientific theories and carried out numerous experiments with mirrors and lenses.

William Rowan Hamilton (1805-1865) - Largely due to his important treatise on systems of rays, William Rowan Hamilton won the position of Royal Astronomer of Ireland while still an undergraduate at Trinity College, but it would be for his prediction of conical refraction that he would achieve even wider recognition in scientific circles. Hamilton later concentrated his efforts on the study of dynamics and produced several important papers in the field. Hamiltonian mechanics became appreciated as the discipline of quantum mechanics began to take shape in the twentieth century.

Robert Hooke (1635-1703) - Robert Hooke was an experimental scientist who lived in seventeenth century England where he made major contributions to the emerging discipline of optical microscopy. Hooke's interest in microscopy and astronomy is exemplified by the treatise Micrographia, his best known work on optical microscopy, and a volume on comets, Cometa detailing his close observation of the comets occurring in

1664 and 1665. Hooke observed a wide diversity of organisms in the microscope, including insects, sponges, bryozoans, diatoms, and bird feathers. Perhaps less well known, Robert Hooke coined the term ”cell”, in a biological context, as he described the microscopic structure of cork like a tiny, bare room or monk’s cell in his landmark discovery of plant cells with cell walls.

Christiaan Huygens (1629-1695) - Christiaan Huygens was a brilliant Dutch mathematician, physicist, and astronomer who lived during the seventeenth century, a period sometimes referred to as the Scientific Revolution. Huygens, a particularly gifted scientist, is best known for his work on the theories of centrifugal force, the wave theory of light, and the pendulum clock. His theories neatly explained the laws of refraction, diffraction, interference, and reflection, and Huygens went on to make major advances in the theories concerning the phenomena of double refraction (birefringence) and polarization of light.

John Kerr (1824-1907) - John Kerr was a Scottish physicist who discovered the electro-optic effect that bears his name and invented the Kerr cell. Pulses of light can be controlled so quickly with a modern Kerr cell that the devices are often used as high-speed shutter systems for photography and are sometimes alternately known as Kerr electro-optical shutters. In addition, Kerr cells have been used to measure the speed of light, are incorporated in some lasers, and are becoming increasingly common in telecommunications devices.

Johannes Kepler (1571-1630) - Johannes Kepler was a sixteenth century German astronomer and student of optics who first delineated many theories of modern optics. In 1609, he published Astronomia Nova delineating his discoveries, which are now called Kepler’s first two laws of planetary motion. This work established Kepler as the ”father of modern science”, documenting how, for the first time, a scientist dealt with a multitude of imperfect data to arrive at a fundamental law of nature.

Gustav Robert Kirchhoff (1824-1887) - Gustav Kirchhoff was a nineteenth century physicist who is well known for his contributions to circuit theory and the understanding of thermal emission, but who also made significant discoveries in optics. His work in the area spectroscopy, much of which was carried out in conjunction with chemist Robert Bunsen, was foundational to the field, as was his study of black body radiation. Kirchhoff’s findings are commonly considered to have been instrumental to Max Planck’s quantum theory of electromagnetic radiation formulated at the beginning of the twentieth century.

Étienne-Louis Malus (1775-1812) - In 1807, Malus commenced experiments on double refraction, the phenomenon that causes a light beam to divide into two orthogonal rays on passing through certain materials, such as Iceland spar. Malus’s findings supported those obtained earlier by Dutch scientist Christiaan Huygens, whose description of double refraction was founded upon the then controversial idea that light is characteristically wavelike. In 1808, Malus discovered that light could be polarized (a term coined by Malus) by reflection as he observed sunlight reflected from the windows of the Luxemburg Palace in Paris through an Iceland spar crystal that he rotated.

James Clerk Maxwell (1831-1879) - James Clerk Maxwell was one of the greatest scientists of the nineteenth century. He is best known for the formulation of the theory of electromagnetism and in making the connection between light and electromagnetic waves. He also made significant contributions in the areas of physics, mathematics, astronomy and engineering. He considered by many as the father of modern physics.

Claudius Ptolemy (Approximately 87-150) - Claudius Ptolemy was one of the most influential Greek astronomers and geographers of his time. Ptolemy propounded the geocentric theory in a form that prevailed for 1400 years. According to historians, Ptolemy was a mathematician of the very highest rank, however others believed that he committed a crime against his fellow scientists by betraying the ethics and integrity of his profession.

Lord Rayleigh (John William Strutt) (1842-1919) - Lord Rayleigh was a British physicist and mathematician who worked in many disciplines including electromagnetics, physical optics, and sound wave theory. The criteria he defined still act as the limits of resolution of a diffraction-limited optical instrument. Rayleigh wrote over 446 scientific papers, but is perhaps best known for his discovery of the inert gas argon, which earned him a Nobel Prize.

Vasco Ronchi (1897-1988) was born on 19th December 1897 in Florence, Italy. He studied from 1915-1919 at the Faculty of Physics of the University of Pisa and was a pupil of L. Puccianti. During that time he served in the 1st World War and was awarded a medal. His doctoral thesis was published in 1919. As Assistant Researcher of Prof. Antonio Garbasso he worked at the Institute of Physics of the University of Florence from 1920. He developed an interest in Optics and in 1922 published his method of testing even large optics with simple equipment. This test is known as the 'Ronchi Test'. This test involves a small diffraction grating with only 20 to 30 lines per mm. Some scientists refer to such a grating as a 'Ronchi'. He made the development during one of his first tasks at the Institute of Physics. He had to test the then largest lens of the Astronomical Observatory of Arcetri, a 28 cm (11 inch) achromatic doublet with 533 cm focal length made by G. B. Amici in 1839. The test applied was the Hartmann-Test which required eight months to be completed. With Ronchi's development the test of large optics could then be made in very short time. The year 1945, soon after the war, Ronchi founded the 'Fondazione Giorgio Ronchi', named for his son who had died in that war in 1944. This organization supported the publication of a new scientific Journal titled 'Atti della Fondazione Giorgio Ronchi'. This was because the 'Bulletin of the Italian Optical Association' and the publication 'Ottica' could not be printed during the war years due to lack of paper. In 1953 Ronchi was elected President of the 'Union Internationale d'Histoire des Sciences' within the UNESCO and was reelected on four subsequent 3-year terms. He resigned from the Directorship of the National Institute of Optics in 1975, aged 78. He had led the Institute for 58 years and was nominated 'Cavaliere di Gran Croce al Merito della Repubblica Italiana'. His about 900 papers and 30 books were published in italian Italian as well as in foreign languages. He was called to be member of the Optical Society of America (1935?) and later nominated 'Fellow' and 'Emeritus Fellow' of this organization. He died on the 31. October 1988.

Erwin Schrdinger (1887-1961) - The Austrian physicist Erwin Schrdinger made fundamental advances in establishing the groundwork of the wave mechanics approach to quantum theory. Influenced by de Broglie's work, which had gained additional weight due to the support of Albert Einstein, Schrdinger attributed the quantum energies of the electron orbits in the atom thought to exist to the vibration frequencies of electron matter waves, now known as de Broglie waves, around the nucleus of the atom. For his significant contributions to science, Schrdinger was bestowed with many honors, including the Nobel Prize for Physics, which he shared with Paul Dirac in 1933.

Willebrord Snell (1580-1626) - Willebrord Snell was an early seventeenth century Dutch mathematician who is best known for determining that transparent materials have different indices of refraction depending upon the composition. Snell discovered that a beam of light would bend as it enters a block of glass, and that the angle of bending was dependent upon the incident angle of the light beam. Light traveling in a straight line into the glass will not bend but, at an angle, the light is bent to a degree proportional to the angle of inclination. In 1621, Snell found a characteristic ratio between the angle of incidence and the angle of refraction. Snell's law demonstrates that every substance has a specific bending ratiothe "refractive index. The greater the angle of refraction, the higher the refractive index for a substance.

Sir William Henry Fox Talbot (1800-1877) - William Fox Talbot, an English chemist, philosopher, mathematician, linguist, and Egyptologist, is best known for the innovative photographic techniques he developed. His work in the mid-1800s is the foundation upon which modern photography is based. Bad timing, however, has left Talbot a footnote to Louis-Jacques-Mand Daguerre who is more popularly known as the founder of modern photography. Daguerre publicly announced his method for creating a plate from which a single photographic print could be made. Only a few weeks after this announcement, Talbot revealed his innovation, the Calotype.

John Tyndall (1820-1893) - From a humble background, John Tyndall rose to great heights, becoming one of the most eminent men of science during his period. The self-made man was a powerful lecturer and an influential writer who published on topics ranging from molecular physics and magnetism to mountaineering, literature, religion, and the motion of glaciers. In optics, he is most famous for his discovery of the phenomenon that came to be known as the Tyndall effect.

Thomas Young (1773-1829) - Thomas Young was an English physician and a physicist who was responsible for many important theories and discoveries in optics and in human anatomy. His best known work is the wave theory of interference. Young was also responsible for postulating how the receptors in the eye perceive colors. He is credited, along with Hermann Ludwig Ferdinand von Helmholtz, for developing the Young-Helmholtz trichromatic theory.

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Appendix B

Dirac Comb

This appendix is dedicated to give a short introduction about the Dirac's Delta Comb and its properties.

B.1 Dirac's Delta

Rigorous treatment of the Dirac delta requires measure theory or the theory of distributions. This mathematical object is actually a functional which takes functions and returns elements from a field. Nevertheless, this mathematical object is usually thought of as a function with a couple of special properties:

$$\delta(t-a) = 0, \text{ for } \quad x \neq 0 \tag{B.1}$$

and

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a). \tag{B.2}$$

The Delta function is usually defined in terms of a sequence of functions. It appears in physics to model a localized source of field. It is useful as an approximation for a tall narrow spike function (an impulse). It is the same type of abstraction as a point charge, point mass or electron point. As distributions, the Heaviside step function is an antiderivative of the Dirac delta distribution.

As a distribution, the Dirac's Delta has compact support, being $\{0\}$ such support. Because of this definition, and the absence of a true function with the properties (B.1) and (B.2), it is important to realize that these properties are just notational convenience. The function here being treated may be interpreted as a probability density function. Its characteristic function is then the unity, as is the moment generating function, so that all moments are zero.

B.2 Delta Comb

Definition 14 *A Delta Comb is an infinite set or array of Dirac's Deltas equally spaced with a period a. It is denoted*

$$\Delta_a(t) = \sum_{n=-\infty}^{\infty} \delta(t - an).$$

It is also recognized as the Sha function for its resemblance to the cyrillic letter sha.

As a periodic function it is natural to express it as a Fourier series:

$$\Delta_a(t) = \sum_{n=-\infty}^{\infty} c_n \exp\{i\omega_n t\} \tag{B.3}$$

where $\omega_n = 2\pi n/a$ are the Fourier modes.

Since $\Delta_a(t) = \Delta_a(t+T)$, the coefficients are calculated as

$$\begin{aligned} c_n &= \frac{1}{T}\int_{-T/2}^{T/2} \Delta_a(t)\exp\{-i2\pi nt/a\}\,dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\exp\{-i2\pi nt/a\}\,dt \\ &= \frac{1}{T}\exp\{-i2\pi n0/a\}\,dt \\ &= \frac{1}{T} \end{aligned}$$

So, every coefficient equals $1/T$, and thus the Fourier series for the Dirac Comb is given by

$$\Delta_a(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \exp\{i\omega_n t\}. \tag{B.4}$$

Such expression is recognized as the Poisson Formula and is widely used in optics as an aid to represent periodic functions such as gratings.

Appendix C

Talbot Images

Graphs

A set of useful graphs of Talbot Images is included. The distances are taken to be rational numbers of the form $z = p/q$ and $q = 1, 2, \ldots, 7$.

$q = 2$

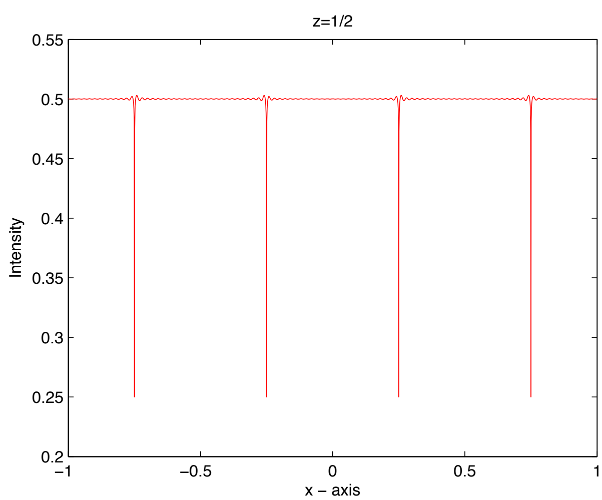

Figure C.1: $z = 1/2$

$q = 3$

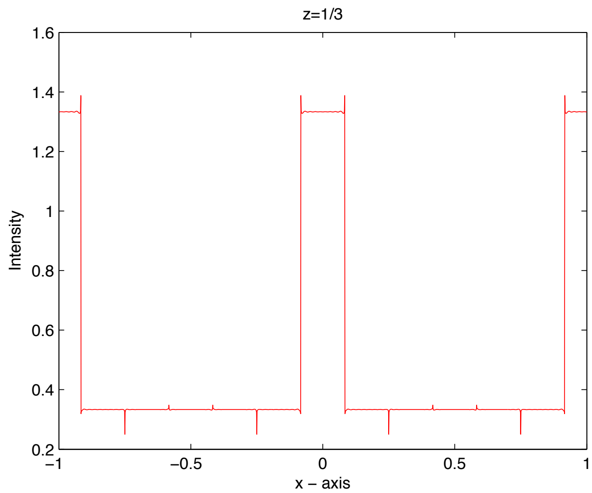

Figure C.2: $z = 1/3$

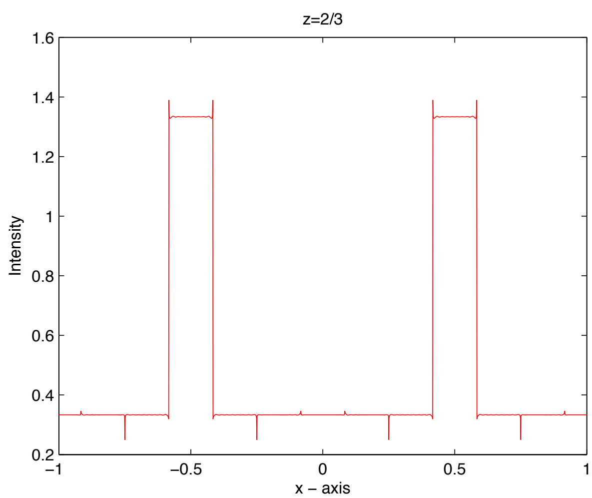

Figure C.3: $z = 2/3$

$q = 4$

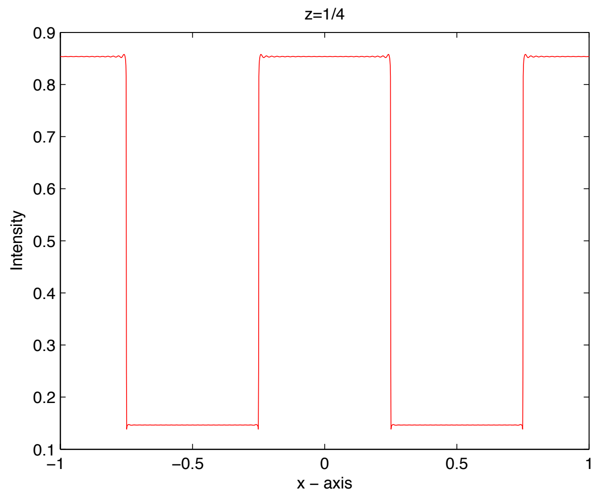

Figure C.4: $z = 1/4$

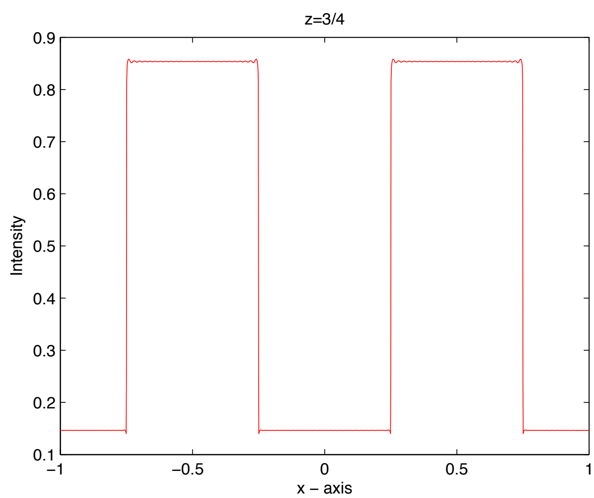

Figure C.5: $z = 3/4$

$q = 5$

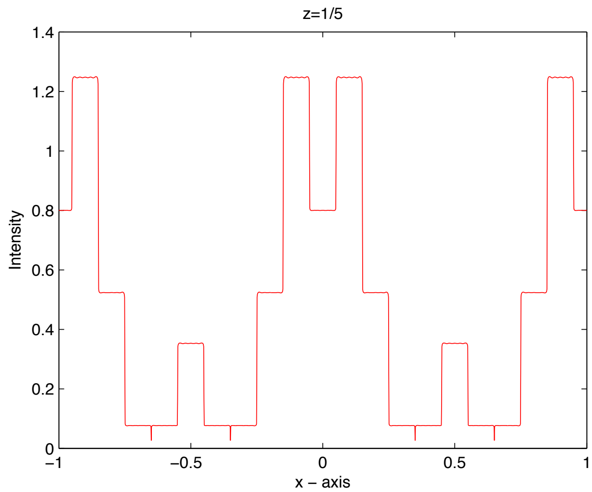

Figure C.6: $z = 1/5$

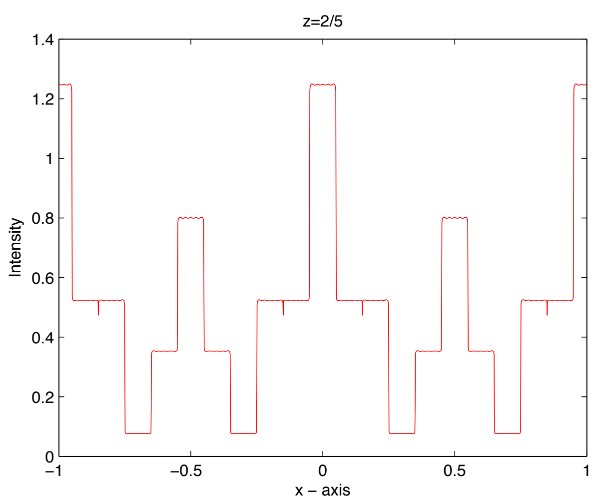

Figure C.7: $z = 2/5$

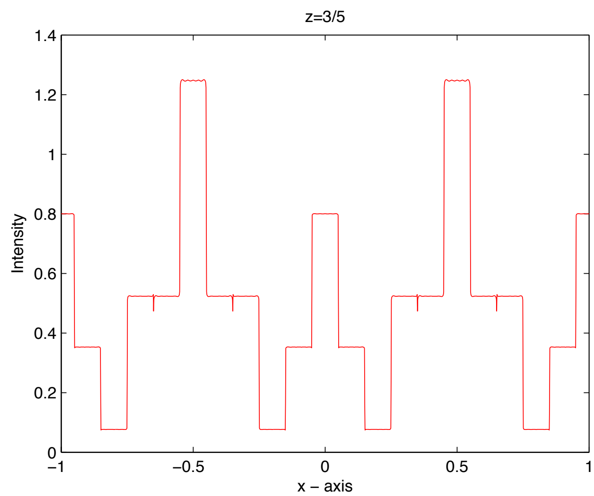

Figure C.8: $z = 3/5$

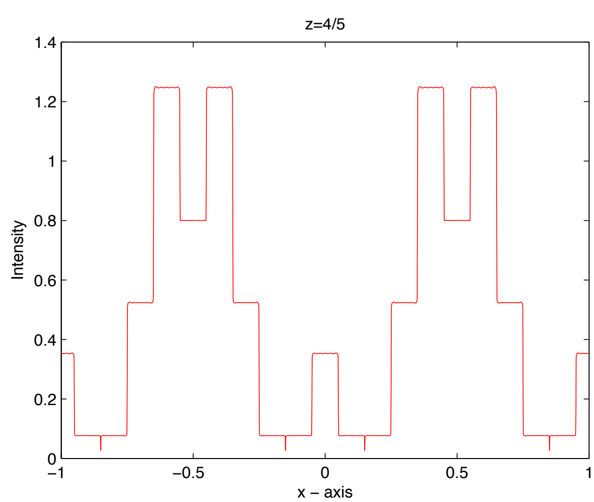

Figure C.9: $z = 4/5$

$q = 6$

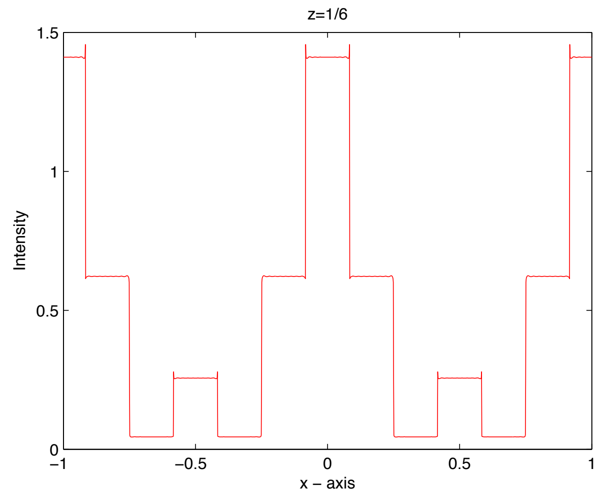

Figure C.10: $z = 1/6$

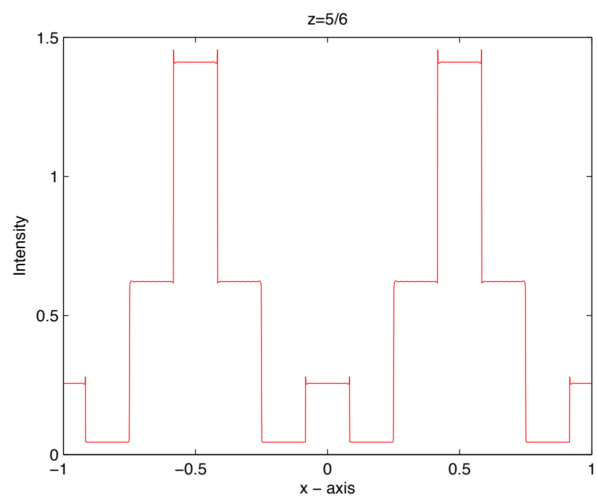

Figure C.11: $z = 5/6$

$q = 7$

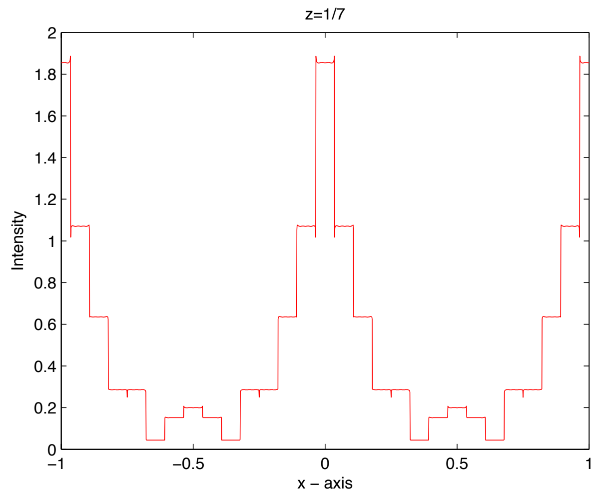

Figure C.12: $z = 1/7$

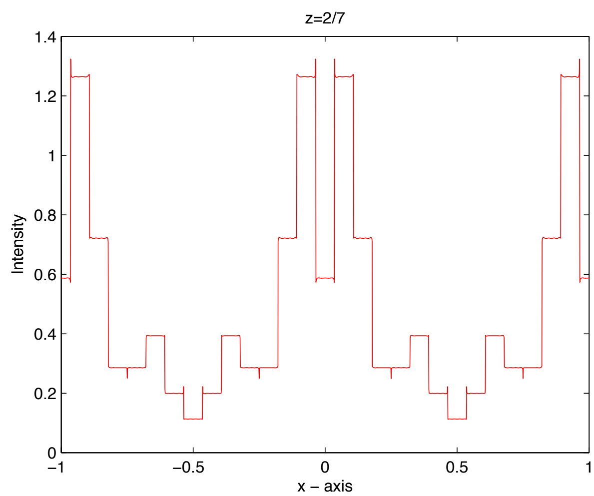

Figure C.13: $z = 2/7$

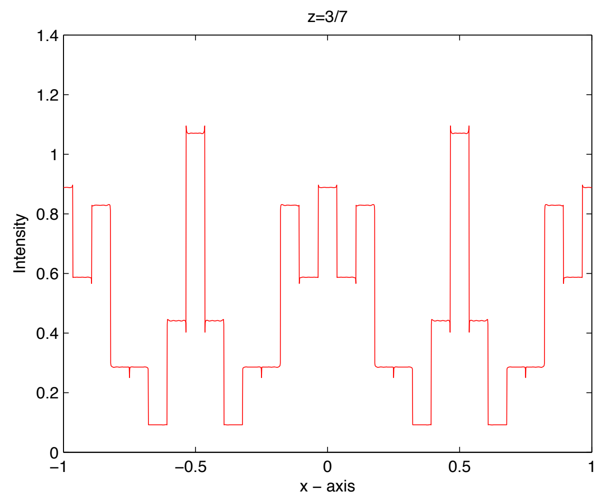

Figure C.14: $z = 3/7$

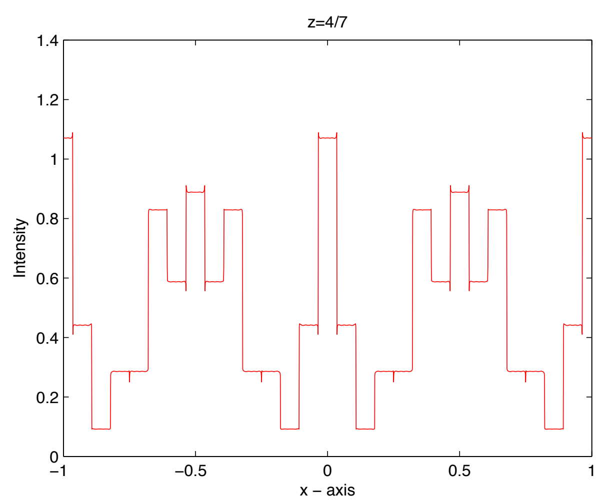

Figure C.15: $z = 4/7$

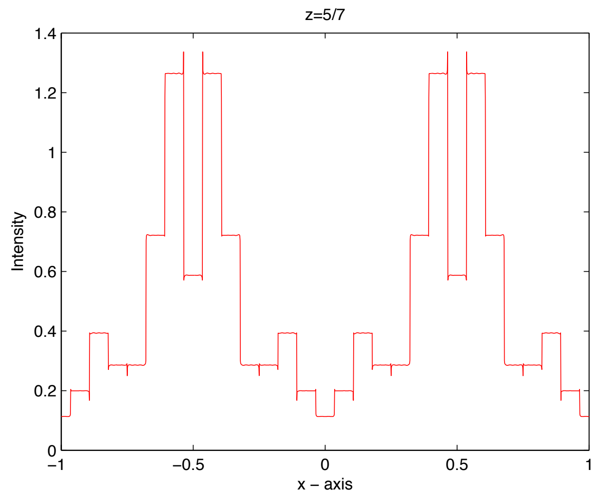

Figure C.16: $z = 5/7$

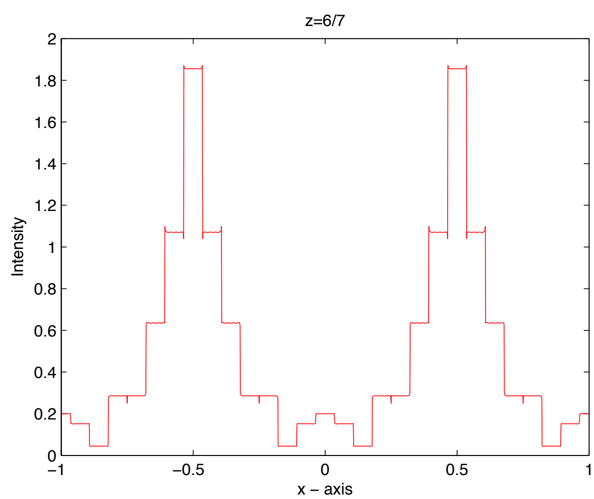

Figure C.17: $z = 6/7$

Code

The graphs were obtained in Matlab with the aid of a program in FORTRAN 90. The code is included:

```
program sumas
implicit none

integer::k,n,arriba,abajo,numid
real::x,AK,pi,z,Scos,Ssin,phik,Psisq
character(len=8)::id
character(16)::archout

pi=3.14159265

! write(6,*) "uno"

do abajo=1,1
do arriba=1,2

! write(6,*)"dos"

numid=arriba*10+abajo
! write(6,*) numid
write(id,101)numid
101 format(1I2.2)
! write(6,*) "tres"

archout='suma'//trim(id)
open(11,file=trim(archout),status='replace')
```

```
z=real(arriba)/real(abajo)


do n=0,2000

x=(real(n)/real(1000))-1
Scos=0.0
Ssin=0.0

do k=0,1024
AK=(-1.0)**real(k)/(2.0*real(k)+1.0)
AK=AK*cos(2.0*pi*x*(2.0*real(k)+1.0))

Scos=Scos+AK*cos(pi*z*((2.0*real(k)+1.0)**2.0))
Ssin=Ssin+AK*sin(pi*z*((2.0*real(k)+1.0)**2.0))
end do

Psisq=(1.0/2.0+(2.0/pi)*Scos)**2.0
Psisq=Psisq+((2.0/pi)*Ssin)**2.0

write(11,*) Psisq

end do

close(11)

end do !numerador
end do !denominador
```

```
end program
```

Appendix D

Gaussian Wave Front

D.1 Wavefronts and rays

A wavefront is defined as the locus (a line or surface in an electromagnetic wave) of points with the same phase. If a monochromatic plane wave is considered, the wavefronts will be infinite planes perpendicular to the propagation direction. If one restricts to two dimensions only, the wavefronts will be thus straight lines with the same orthogonality property. Given a point light source, the propagation direction will be radial and in consequence the wavefronts will be concentric spheres.

The problem of determining the shape of a light source remains an interesting field. An easy way to get the shape of a light source is to find a set orthogonal to the graph of the wavefront. The simplest example is the circular wavefront in two dimensions. Circumferences centered at the origin will be the wavefronts. If half of the wavefront is considered we can readily say that the wavefront is the set $(x, f(x))$ where

$$f(x) = \sqrt{1-x^2}. \tag{D.1}$$

The orthogonal set would be a family of straight lines that satisfy the equation

$$y - f(x_1) = \frac{(x-x_1)}{f'(x_1)}, \tag{D.2}$$

which intersect at

$$y = \frac{-x_1}{f'(x_1)} + f(x_1).$$

By suitable substituting, the intersection is found to be at the origin. The family of straight lines are the rays themselves which come out from a point source.

D.2 Gaussian wavefront

Consider the graph of a function $(x, f(x))$. The orthogonal family of straight lines is obtained by equation (D.2). If $f(x) = e^{-x^2}$ the gaussian function, the family of orthogonal straight lines is

$$y = \frac{x}{2x_1 e^{-x^2}} - \frac{1}{2e^{-x_1^2}} + e^{-x_1^2}. \tag{D.3}$$

with slope $1/2x_1 e^{-x^2}$ and intersection with the vertical axis in $y = -1/2e^{-x_1^2} + e^{-x_1^2}$. It is obvious that the orthogonal line at $(0,1)$ is the y axis itself, and the expression for the slope is undefined. However if one takes the limit of the y intersection (i.e. $b(x_1)$), it exists:

$$\begin{aligned}
\lim_{x_1 \to 0} b(x_1) &= \lim_{x_1 \to 0} -\frac{1}{2e^{-x_1^2}} + e^{-x_1^2} \\
&= -\frac{1}{2} + 1 \\
&= \frac{1}{2}.
\end{aligned}$$

This is illustrated in the plot of Fig. [**?**]. Its interpretation would be that the source is an extended one with a limit point at $y = 1/2$.

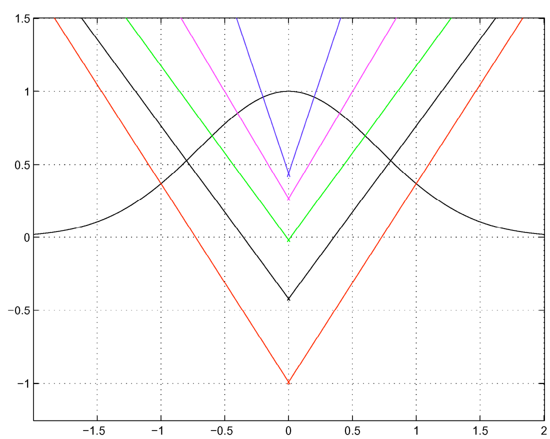

Figure D.1: The lines denote the rays coming out from the *source*

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