INSTITUTO POTOSINO DE INVESTIGACIÓN CIENTÍFICA Y TECNOLÓGICA, A.C.

POSGRADO EN CIENCIAS APLICADAS

Theory of vertically transverse functions and their application to control of critical mechanical systems

Tesis que presenta

José Miguel Sosa Zúñiga

Para obtener el grado de

Doctor en Ciencias Aplicadas

En la opción de

Control y Sistemas Dinámicos

Director de la Tesis:

Dr. David Antonio Lizárraga Navarro

San Luis Potosí, S.L.P. octubre de 2009.

Constancia de aprobación de la tesis

La tesis "***Theory of vertically transverse functions and their application to control of critical mechanical systems***" presentada para obtener el Grado de Doctor en Ciencias Aplicadas en la opción de Control y Sistemas Dinámicos fue elaborada por **José Miguel Sosa Zúñiga** y aprobada el **30 de octubre del 2009** por los suscritos, designados por el Colegio de Profesores de la División de Matemáticas Aplicadas del Instituto Potosino de Investigación Científica y Tecnológica, A.C.

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Esta tesis fue elaborada en la División de Matemáticas Aplicadas del Instituto Potosino de Investigación Científica y Tecnológica, A.C., bajo la dirección del Dr. David Antonio Lizárraga Navarro.

Durante la realización del trabajo el autor recibió una beca académica del Consejo Nacional de Ciencia y Tecnología con número de registro 182435 cubriendo el periodo de agosto de 2005 a julio de 2008.

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DOCTOR EN CIENCIAS APLICADAS
EN LA OPCIÓN DE CONTROL Y SISTEMAS DINÁMICOS

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Resumen

En esta tesis se desarrolla y analiza una metodología teórica para atacar la estabilización práctica de configuraciones para sistemas de segundo orden descritos en grupos de Lie. En particular, esta clase de sistemas surgen de la formulación de Euler-Lagrange que describe la dinámica de sistemas mecánicos. La estabilización de esta clase de sistemas resulta ser no trivial dado que esta clase incluye sistemas mecánicos posiblemente restringidos y subactuados. Además, esta clase de sistemas incluye sistemas que no admiten reducciones cinemáticas, sistemas cuya linealización en puntos de equilibrio es no controlable y sistemas para los que no existe retroalimentación continua en el estado que estabilice por ejemplo, puntos de equilibrio. Ejemplos de esta clase de sistemas son los manipuladores mecánicos subactuados, cuerpos rígidos en el espacio, vehículos a ruedas y vehículos acuáticos subactuados. Es interesante notar que estos sistemas de control son sistemas afines en el control para los cuales el campo vectorial de deriva juega un papel importante para determinar la accesibilidad local del sistema de control.

La metodología analizada en esta tesis, originalmente propuesta en (Sosa [2005]), tiene como objetivo extender la metodología de estabilización propuesta por Morin y Samson [2003] para atacar la estabilización práctica de configuraciones para sistemas de segundo orden. Las contribuciones principales de esta tesis se centran, primero, en analizar la cero-dinámica de lazo cerrado para determinar el comportamiento a largo plazo de las trayectorias del sistema en lazo cerrado y segundo, en modificar el algoritmo de control propuesto con el objetivo de *moldear* las trayectorias de la cero-dinámica para obtener resultados de estabilidad requeridos en aplicaciones prácticas. El desarrollo y el análisis de la metodología teórica presentados en esta tesis sugieren problemas complejos que deben ser resueltos para obtener una metodología unificada, sistemática y general para el control de sistemas mecánicos subactuados via funciones verticalmente transversas.

Abstract

In this dissertation we analyze a theoretical framework to address practical stabilization of fixed configurations for second-order systems on tangent Lie groups based in vertical transversality (initially proposed in Sosa [2005]). In particular we are interested in control systems arising from the Euler-Lagrange formulation for mechanical systems. Stabilization of this class of systems results nontrivial given that this class encompasses, possibly constrained, underactuated mechanical systems. Within this class one may encounter systems that are not kinematic reductions of mechanical systems, systems whose linearization at equilibria is non-controllable, and control systems that cannot be stabilized by means of continuous state feedback. Examples of such systems include underactuated mechanical manipulators, rigid body systems in space, wheeled vehicles and underactuated underwater vehicles. It is interesting to remark that these control systems are affine control systems for which the drift vector field plays a key role in determining important properties such as local accessibility.

The framework analyzed in this thesis, which was initially proposed in (Sosa [2005]), is intended to provide an extension to the stabilization procedure proposed by Morin and Samson [2003] to deal with the practical stabilization of configurations for second-order systems. The main contributions of the thesis center on two important issues. First, in analyzing the closed-loop zero-dynamics to assess the long-term behavior of the trajectories of the closed-loop system and, second, in modifying the proposed control algorithm with the objective of *shaping* the zero-dynamics trajectories to obtain stability results urged by practical applications. However, the analysis done is not conclusive towards the developing of a unified and systematically applicable theoretical framework to address configuration stabilization for general mechanical system via vertically transverse functions.

Contents

Chapter 1

Introduction

This thesis comprises the analysis and characterization of a theoretical framework, which relies on tools from differential geometry, to practically stabilize fixed configurations for underactuated mechanical systems evolving on Lie groups, based on vertical transversality. It also comprises a modification to this control framework with the aim of having suitable stability results required in typical control applications.

The presented control framework is an extension, to the case of second-order systems, of the transverse function approach (TFA) to control proposed by Morin and Samson [2003], which addresses practical point stabilization and trajectory tracking for controllable driftless systems.

The class of systems targeted by this extension are second-order control systems on tangent Lie groups of the form

$$\dot{v} = S_v + \sum_{i=1}^{m} u^i X_{i,v}^{\mathrm{lift}}, \tag{1.1}$$

where S is a second-order vector field on a tangent Lie group TG, $X_1^{\mathrm{lift}}, \dots, X_m^{\mathrm{lift}}$ are vertical lifts of left-invariant vector fields on G satisfying the LARC (Lie Algebra Rank Condition) at some point $g \in G$, and $m \leq n = \dim(G)$. All manifolds, mappings, vector fields and related constructs defined in this thesis are assumed to be smooth unless otherwise stated. (The Reader may refer to the Appendix for notation and basic concepts used in this thesis). In the Euler-Lagrange formulation for mechanical systems, S corresponds to the vector field given by $S = S^{\mathscr{G}} - (dV^\sharp)^{\mathrm{lift}}$, that is, the sum of the geodesic spray associated with a Riemannian metric $\mathscr{G}$ on G, and minus the vertical lift of the vector field corresponding to the gradient of a potential energy function $V : G \longrightarrow \mathbb{R}$. The control vector fields X_i^{lift}, $i = 1, \dots, m$, correspond to vertical lifts of the vector fields $(F^i)^\sharp$ determined by m 1-forms F^i related, in a physical sense, to forces or torques applied to the system.

The class of systems given by (1.1) encompasses possibly constrained, underactuated mechanical systems. In particular, it contains systems that are not kinematic reductions of mechanical systems in the sense of Bullo and Lewis [2005, chap. 4], systems whose linearization at equilibria are non-controllable, and critical control systems.

By a *critical control system* we refer to a system that do not satisfies generalizations of Brockett [1983] necessary condition for the stabilization of equilibrium points by means of continuous pure state feedback. An extension to Brockett's condition states the following. Let $\dot{x} = f(x,u)$ be a control system on a n-dimensional manifold Q, such that $f : Q \times \mathbb{R}^m \longrightarrow TQ$ is continuous and satisfies $f(q,0) = 0$ for some $q \in Q$. Then, a necessary condition for the asymptotic stabilization of q by means of a continuous feedback $\alpha : Q \longrightarrow \mathbb{R}^m$ is that f be open at $(q,0)$. That is, f, viewed as a mapping between topological spaces, maps open neighborhoods of $(q,0) \in Q \times \mathbb{R}^m$ into open neighborhoods of $f((q,0)) \in TQ$. Furthermore, Coron and Rosier [1994] have shown that Brockett's condition still holds even if one allows the use of discontinuous feedback, provided that the solutions are interpreted in the sense of Filippov.

However, it is possible to overcome obstructions to regular feedback stabilization by means of time-varying feedback $(q,t) \mapsto \alpha(q,t)$, as was first shown by Samson [1991] for a particular critical system, the model of a nonholonomic wheeled-cart. Later, Coron [1992] showed that continuous and periodic time-varying feedback can be used to stabilize global accessible driftless systems, but no explicit construction method was provided.

The fact that "underactuated" controllable driftless systems, i.e., those for which the number of inputs is smaller than the dimension of the state space, are critical (a straightforward result that follows from Brockett's condition) motivated further research on time-varying stabilizers and on providing explicit construction methods to design such feedback laws. For instance, Pomet [1992] reported an explicit method to design differentiable time-varying feedback to asymptotically stabilize equilibria for driftless control systems based on the introduction of dissipation.

As an example of this type of feedback, consider the kinematic model of a unicycle-type wheeled mobile robot, schematically depicted in Figure 1.1:

$$\begin{aligned} \dot{q}_1 &= v_1 \cos(q_3) \\ \dot{q}_2 &= v_1 \sin(q_3) \\ \dot{q}_3 &= v_2, \end{aligned} \tag{1.2}$$

where v_1 is the forward velocity of the midpoint between the rear wheels, and v_2 is the velocity of the "steering" angle q_3. This model is obtained under the assumption that the wheels do not slip, that is, the unicycle is allowed to move instantaneously in a direction parallel to the plane of the wheels. By means of the feedback transformation, $x = \phi(q)$, $u = \psi(v,q)$, given by

$$\begin{aligned} x = \phi(q) &:= (q_3, q_1 \cos(q_3) + q_2 \sin(q_3), q_1 \sin(q_3) - q_2 \cos(q_3)), \\ u = \psi(v,q) &:= (v_2, v_1 - (q_1 \sin(q_3) - q_2 \cos(q_3))\, v_2), \end{aligned}$$

System (1.2) can be transformed into the 3-dimensional, 2-input chained form (3-CF),

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= u_1 x_2. \end{aligned} \tag{1.3}$$

The 3-CF is a controllable system which is critical. In fact, the mapping $f : \mathbb{R}^3 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $f : (x,u) \mapsto (u_1, u_2, u_1 x_2)$ is not open at $(0,0) \in \mathbb{R}^3 \times \mathbb{R}^2$ since no point of the form $(0,0,\varepsilon) \in \mathbb{R}^3$ for any $\varepsilon > 0$ belongs to the image of f. Therefore there exists no *pure-state* feedback that renders $0 \in \mathbb{R}^3$ asymptotically stable for (1.3), but a time-varying feedback

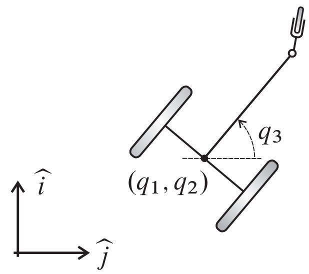


Figure 1.1: Schematic diagram of the unicycle-type wheeled robot of system (1.2).

may stabilize $0 \in \mathbb{R}^3$. Consider for instance the time-varying differentiable feedback reported by M'Closkey and Murray [1997],

$$\begin{aligned} u_1(x,t) &= -x_1 + x_3 \cos(t) \\ u_2(x,t) &= -x_2 + x_3^2 \sin(t), \end{aligned} \tag{1.4}$$

which makes the origin globally asymptotically stable for (1.3). Figure 1.2 (left) shows a numerical simulation of system (1.3) under this feedback law for a given initial condition.

However, it was observed that differentiable time-varying feedback leads to slow convergence rates, as reported by Samson and Ait-Abderrahim [1991]. Furthermore, Murray et al. [1992] showed that no point $q \in Q$ can be exponentially stabilized for controllable driftless systems by means of differentiable time-varying feedback.

Then, Gurvits and Li [1992] showed that no time-varying feedback which is locally Lipschitz with respect to the state can exponentially stabilize equilibria for controllable driftless systems. In order to improve the convergence rate for the trajectories in closed-loop, the time-varying feedback laws must be at most Hölder-continuous at the point to stabilize. A way to achieve faster convergence rates for the closed-loop trajectories is to set the closed-loop system $\dot{q} = F(q,t) := f(q, \alpha(q,t))$ homogeneous of degree zero with respect to some generalized notion of homogeneity, as was shown by M'Closkey and Murray [1993] and M'Closkey and Murray [1997].

Let $\lambda > 0$. Define a *weight vector* to be $r = (r_1, \dots, r_n) \in \mathbb{R}^n$ such that $r_1, \dots, r_n > 0$. A dilation on $\mathbb{R}^n$ with weight r is a map $\Delta^r_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $\Delta^r_\lambda(x) = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n)$. Then, E_r, the Euler vector field corresponding to the dilation Δ^r_λ, is defined in coordinates by $E_r = \sum_{i=1}^n r_i x_i \, \partial/\partial x_i$. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be homogeneous of degree τ (with weight vector r) if $L_{E_r} f = \tau f$ and a vector field $X \in \Gamma(T\mathbb{R}^n)$ is called homogeneous of degree τ (with weight vector r) if $[X, E_r] = \tau X$. In coordinates, f is homogeneous of degree τ iff it satisfies $f(\Delta^r_\lambda(x)) = \lambda^\tau f(x)$, and X is homogeneous of degree τ iff each of its components X_i, viewed as a mapping $\mathbb{R}^n \longrightarrow \mathbb{R}$, is homogeneous of degree $\tau + r_i$.

It is readily verified that the 3-CF system $\dot{x} = X_x$ with $X_x = (u_1, u_2, u_1 x_2)$ is homogeneous of degree 1 with weight vector $r = (1, 1, 2)$. In reference to the 3-CF (1.3), consider the time-varying feedback

$$\begin{aligned} u_1(x,t) &= -x_1 + \operatorname{sign}(x_3)\sqrt{|x_3|}\,\cos(t) \\ u_2(x,t) &= -x_2 + \sqrt{|x_3|}\,\sin(t), \end{aligned} \tag{1.5}$$

which makes the closed-loop system homogeneous of degree zero and makes of the origin globally exponentially stable. Note that the feedback defined in (1.5) is locally Lipschitz, in fact smooth, for every $x \in \mathbb{R}^3$ except at $x = 0$. However, it can be proved, (see M'Closkey and Murray [1997]), that the closed-loop system has unique solution determined by the initial conditions (x_0, t_0). Figure 1.2 (right) shows a numerical simulation of system (1.3) with feedback law (1.5) and same initial condition as in the simulation of Figure 1.2 (left).

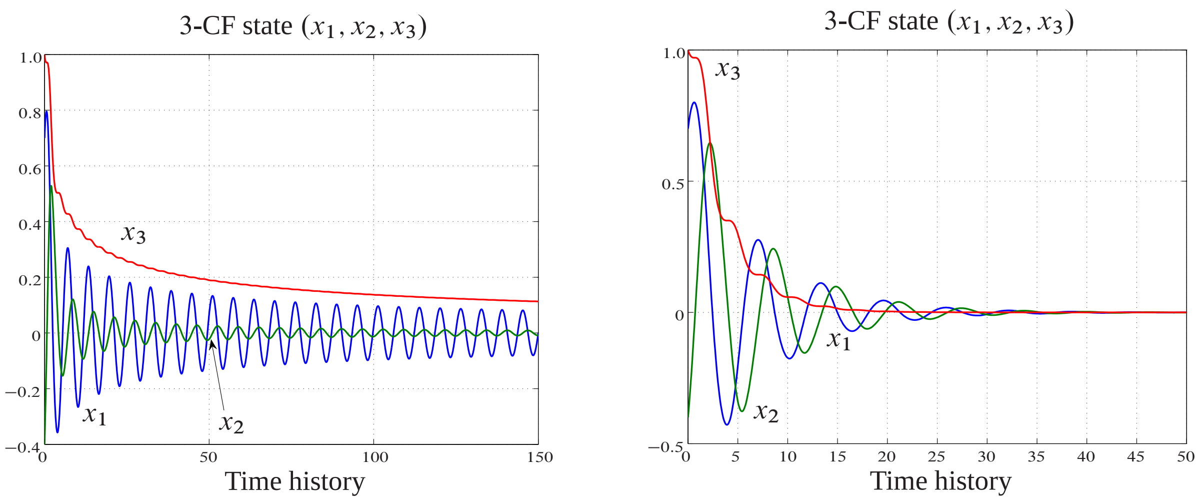


Figure 1.2: Numerical simulation of system (1.3) under feedbacks (1.4) (left) and (1.5) (right), with initial condition $x_0 = (0.7, -0.4, 1.0)$.

Zero-degree homogeneity is a convenient property given that, as was established by Rosier [1992], if the closed-loop system $\dot{q} = F(q,t)$ is homogeneous of degree zero and admits a locally asymptotically stable point $q \in Q$, then q is also globally exponentially stable. Following this line of ideas, Pomet [1992] proposed a methodology to extend differentiable time-varying stabilizers, such as those proposed by M'Closkey and Murray [1997] to obtain homogeneous time-varying stabilizers for homogeneous driftless control systems. In fact, the convergence rate thus obtained is said to be ρ-exponential for some homogeneous norm ρ. That is, if $x(t)$ is the trajectory at time t, of the closed-loop system, with initial condition (t_0, x_0), then it satisfies $\rho(x(t)) \leq \beta\rho(x_0)e^{-\alpha(t-t_0)}$ for some positive reals α, β (cf. M'Closkey and Murray [1997]).

A major result in the construction of time-varying feedback for driftless systems for point-stabilization was proposed by Morin et al. [1999]. The authors, based on the results obtained by Sussmann and Liu [1991] and M'Closkey and Murray [1997], generalized open-loop stabilizers proposed by Sussmann and Liu [1991] and derived a general and constructive method to stabilize equilibria for controllable (possibly not homogeneous) real-analytic driftless systems by means of homogeneous time-varying feedback. The desired equilibrium point turns out to be globally exponentially stable if the closed-loop system is homogeneous, or locally exponentially stable otherwise. In spite of the generality of the method, the construction and structure of such stabilizers can be rather involved even for "simple" control systems, so further research followed to construct stabilizers for canonical systems such as the ones given by Goursat canonical forms or chained systems (cf. Morin and Samson [2000]; Lizárraga et al. [2001]).

Regardless of the improvement of convergence rates, systems controlled by homogeneous, Hölder-continuous feedback exhibit non-robustness against small perturbations of system parameters and model uncertainties, as shown by Lizárraga et al. [1999]. Hence, given that differentiable time-varying feedback yields slow convergence, and homogeneous time-varying feedback presents non-robustness issues, in addition to results enounced by Lizárraga [2003], which points out that constructing "universal" stabilizers to asymptotically stabilize arbitrary system trajectories is a hopeless goal for general driftless systems, one is led to conclude that the asymptotic stabilization of general trajectories for critical systems seems to be too ambitious a control objective.

In this respect, the transverse function approach (TFA), proposed by Morin and Samson [2003], relaxes asymptotic point stabilization to practical point stabilization, a result that seems reasonable when dealing with control of critical systems. Roughly speaking, the term *practical stabilization* refers to the fact that a specified neighborhood of an equilibrium for the closed-loop system is rendered stable and that the trajectories ultimately enter this neighborhood.

The TFA offers some advantages over typical time-varying feedback. For instance, it may achieve faster convergence rates for the closed-loop system trajectories compared with the polynomial convergence rate of differentiable time-varying feedback. Additionally, the feedback laws derived from that approach are smooth, so they do not exhibit some of the non-robustness issues alluded to by Lizárraga et al. [1999]. Moreover, the TFA is able to deal with the stabilization of admissible and non-admissible trajectories of driftless controllable systems in a set-up similar to that for point-stabilization and more recently, it has also been enhanced to deal, under certain conditions, with the asymptotic stabilization of fixed points for driftless controllable systems (see Morin and Samson [2004]).

The TFA, in its original formulation, is applicable to control systems of the form

$$\dot{q} = X_0(t, q) + \sum_{i=1}^{m} u^i X_{i,q}, \tag{1.6}$$

where $X_1, \ldots, X_m \in \Gamma(TQ)$ are vector fields on an n-dimensional manifold Q such that the distribution spanned by $\mathrm{Lie}(\boldsymbol{X}) = \mathrm{Lie}\{X_1, \ldots, X_m\} \subset \Gamma(TQ)$ is completely nonintegrable at some point $q \in Q$. The (possibly null) drift term $X_0(t, \cdot) \in \Gamma(TQ)$ ($t \in \mathbb{R}$) may represent model uncertainties or terms that typically arise in trajectory tracking problems. The non-integrability of $\mathrm{Lie}(\boldsymbol{X})$ implies the local accessibility of (1.6) at q. Morin and Samson showed that the non-integrability of $\mathrm{Lie}(\boldsymbol{X})$ at some $q \in Q$ implies, for every neighborhood $\mathcal{U} \subset Q$ of q, the existence of a mapping $f : \mathbb{T}^\kappa \longrightarrow Q$, $\kappa \geq n - m$, whose image is contained in $\mathcal{U}$ and such that

$$T_{f(\theta)} Q = T_\theta f(T_\theta \mathbb{T}^\kappa) + \mathrm{span}_{\mathbb{R}} \left\{ X_{1,f(\theta)}, \ldots, X_{m,f(\theta)} \right\}, \tag{1.7}$$

for every $\theta \in \mathbb{T}^\kappa$. Any such a mapping f is called a (Morin–Samson) transverse function for $\boldsymbol{X}$ near q (transverse in a sense that differs from the usual notion of transversality in differential topology). Morin and Samson [2003] describe an explicit method to construct transverse functions for driftless systems on Lie groups.

Let us briefly describe the TFA. Suppose that Q is a simply connected n-dimensional Lie group with group composition $(q, p) \mapsto \widehat{\mu}(q, p)$ and that the vector fields in $\boldsymbol{X}$ are left-invariant. Also suppose, by simplicity, that X_0 is zero in (1.6). Also assume that $Q = G$ is a Lie group and that the sum in (1.7) is direct (i.e., $\kappa = n - m$). To design a feedback via the TFA one proceeds by selecting a global frame $\boldsymbol{\Omega} = \{\Omega_1, \dots, \Omega_\kappa\} \subset \Gamma(T\mathbb{T}^\kappa)$ for $T\mathbb{T}^\kappa$, and by defining an auxiliary control system $\dot{\theta} = \sum_{j=1}^{\kappa} w^j \Omega_j(\theta)$, and an error signal $z(t) = \widehat{\mu}(q(t), f(\theta)^{-1})$. Given the fact that f is transverse, for any vector field $D \in \Gamma(TQ)$ admitting $e \in Q$ as an asymptotically stable equilibrium point, there exists a smooth feedback, depending on the state of the composite system $(q(t), \theta(t))$, $\alpha : Q \times \mathbb{T}^\kappa \longrightarrow \mathbb{R}^n$ such that, along the trajectories of the composite system $\dot{q} = \sum_{i=1}^{m} \alpha^i(z, \theta) X_{i,q}$ and $\dot{\theta} = \sum_{j=1}^{\kappa} \alpha^{j+m}(z, \theta)^i \Omega_{j,\theta}$, the error satisfies $z(t) = D_{z(t)}$. Hence, in closed loop, $q(t)$ converges to $f(\theta(t))$ (exponentially if the vector field D is adequately selected) and therefore, there exists $T \in \mathbb{R}_{>0}$ such that $x(t) \in \mathcal{U}$ for every $t \geq T$.

As an example, consider the 3-CF in (1.3). The state manifold $\mathbb{R}^3$ can be endowed with a Lie group composition defined, for $x, y \in \mathbb{R}^3$, by $x \cdot y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_2 y_1)$ with identity element $0 \in \mathbb{R}^3$. The set of control vector fields $\boldsymbol{X} = \{X_1, X_2\} \subset \Gamma(T\mathbb{R}^3)$, defining system (1.3), is left-invariant with respect to this group composition. An example of a transverse function $f : \mathbb{T} \longrightarrow \mathbb{R}^3$ for $\boldsymbol{X}$ near $0 \in \mathbb{R}^3$ is given by $f(\theta) = (\varepsilon \sin(\theta), \varepsilon \cos(\theta), \frac{1}{4}\varepsilon^2 \sin(2\theta))$ with ε strictly positive. Note that for any given neighborhood $\mathcal{U}$ of 0_e, the set $f(\mathbb{T}^\kappa)$ can be made to be contained in $\mathcal{U}$ by appropriately selecting ε. Consider the auxiliary system on $\mathbb{T}$ given by $\dot{\theta} = w$. Then the derivative of the error $z = x \cdot f(\theta)^{-1}$, along the trajectories of the 3-CF and the auxiliary system, can be made to have $0 \in \mathbb{R}^3$ as an exponentially stable equilibrium by means of smooth feedback $(u_1(x, \theta), u_2(x, \theta), w(x, \theta))$. Figure 1.3 shows a numerical simulation for the 3-CF under the feedback obtained with $\varepsilon = 0.05$. However, the TFA is not applicable, in its original formulation, to critical systems for which the drift vector field is necessary to assure accessibility, as is the case for mechanical systems that cannot be kinematically reduced in the sense discussed by Bullo and Lewis [2005, chap. 4]. Instances of this control systems include planar underactuated manipulators, blimp-like systems and underwater vehicles. The extension of the TFA to this class of second-order systems is not immediate nor trivial. Two approaches have been independently developed. The first one, by the original authors of the TFA (cf. Morin and Samson [2005, 2006]). The second approach, initially addressed in the author's M.Sc. thesis (Sosa [2005]), is further described and analyzed in this thesis and allows one to formulate practical point stabilization problems for second-order systems, in particular those defined on (tangent) Lie groups.

Vertically transverse functions for control

The extension of the TFA, the main subject of this thesis, is called the *vertically transverse function approach* (VTFA) and allows one to formulate practical configuration stabilization for second-order systems on tangent Lie groups. This framework relies on the fact that tangent mappings of transverse functions satisfy *vertical transversality*, a property that is similar to Morin and

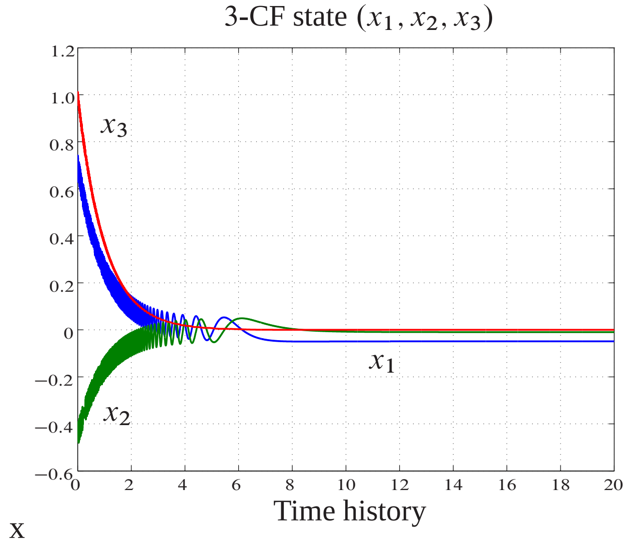


Figure 1.3: Numerical simulation of system (1.2) under the feedback designed via the TFA with $\varepsilon = 0.05$ and initial condition $x_0 = (0.7, -0.4, 1.0)$.

Samson transversality but with relevance to second-order systems. To be precise, the tangent mapping $Tf : T\mathbb{T}^\kappa \longrightarrow TQ$ of a transverse function $f : \mathbb{T}^\kappa \longrightarrow Q$ for $\boldsymbol{X} = \{X_1, \dots, X_m\}$ near $e \in Q$, satisfies, for every $\omega \in T\mathbb{T}^\kappa$,

$$T_{Tf(\omega)}TQ^{\mathrm{vert}} = T_\omega Tf((T\mathbb{T}^\kappa)^{\mathrm{vert}}) + \mathrm{span}_{\mathbb{R}}\left\{X^{\mathrm{lift}}_{1,Tf(\omega)}, \dots, X^{\mathrm{lift}}_{m,Tf(\omega)}\right\}.$$

The VTFA is as follows. Suppose that $Q = G$ is a Lie group ($\dim(G) = n$) and that the vector fields in $\boldsymbol{X}$ are left-invariant. Then consider a second-order system evolving on the tangent Lie group TG, the *target system*, defined by

$$\dot{v} = S_v + \sum_{i=1}^{m} u^i X^{\mathrm{lift}}_{i,v}, \tag{1.8}$$

where $S \in \Gamma(TTG)$ is second-order and $X^{\mathrm{lift}}_1, \dots, X^{\mathrm{lift}}_m \in \Gamma(TTG^{\mathrm{vert}})$ are vertical lifts of the vector fields contained in $\boldsymbol{X}$. Given that the control vector fields of (1.8) are vertical lifts, the drift vector field S determines, to a large extent, the accessibility of (1.8).

Under these conditions, set $\kappa = n - m$ and define a second-order auxiliary control system on $T\mathbb{T}^\kappa$,

$$\dot{\omega} = \Delta_\omega + \sum_{i=1}^{\kappa} w^i \Omega^{\mathrm{lift}}_{i,\omega}, \tag{1.9}$$

where $\Delta \in \Gamma(TT\mathbb{T}^\kappa)$ is second-order and the set $\boldsymbol{\Omega} = \{\Omega_1, \dots, \Omega_m\} \subset \Gamma(TG)$ is a global frame for $T\mathbb{T}^\kappa$. Mimicking the TFA, define an *error* function based on the tangent Lie group operation, $z = v \cdot Tf(\omega)^{-1}$, whose purpose is to quantify the difference between the states of the target and auxiliary systems (1.8)-(1.9). Then it can be shown that the dynamics of this error function, $\dot{z}(t)$, can be assigned arbitrarily by means of smooth feedback in terms of the compound state, i.e., the target system state in addition to the auxiliary system state. If the error is set to have a positive-complete dynamics admitting 0_e, the zero vector in $T_e G$, as an asymptotically stable equilibrium point, the configuration coordinates ultimately enter a predefined neighborhood of the desired configuration provided that the solution of the closed-loop system exists.

Nevertheless, the overall behavior of the closed-loop compound system is not readily assessed. For instance, the long-term behavior of the fiber (or "velocity") coordinates is ultimately characterized by a nontrivial *zero dynamics*. The latter must be analyzed in order to establish stability of the closed-loop system. In general, this closed-loop zero dynamics may not be positive complete and thus may yield undesirable behavior of the system trajectories.

The compound zero-dynamics may be viewed as the result of constraining the trajectories of the target system to be contained in the immersed manifold $Tf(T\mathbb{T}^\kappa)$ by means of the feedback in zero dynamics. That is, if the initial condition of the target system lies in $Tf(T\mathbb{T}^\kappa)$, then it will remain in $Tf(T\mathbb{T}^\kappa)$ for every $t \geq t_0$ whenever the solution of the compound system is defined. However, the zero-error feedback may not satisfy the Lagrange-d'Alembert principle, i.e., it may affect the total energy of the compound zero dynamics. The feedback in zero dynamics has a particular structure in the sense that if the target drift vector field equals the sum of a spray and a vertical vector field (usually equal to the lift of minus the gradient of the potential energy when dealing with mechanical systems), then the auxiliary zero dynamics is itself defined by the sum of a spray and a vertical vector field. In other words, if the target system is an affine-connection control system, then so is the zero-dynamics auxiliary system. In general, the torsionless connection associated to the zero dynamics may not be the Levi-Civita connection of any metric. The problem of determining whether an affine connection admits a (pseudo-) Riemannian metric is, in general, untractable, given the overdetermined nature of the differential equations to solve, as shown by Eisenhart and Veblen [1922]. For the case when the target system is underactuated by one control, we establish a necessary and sufficient condition to determine whether the resulting zero dynamics admits a (pseudo-) Riemannian metric. For the more general cases it remains an open problem.

Assuming that the closed-loop zero dynamics admits a metric, we show that the closed-loop compound system controlled by the VTFA has the set $\{0_e\} \times Z(T\mathbb{T}^\kappa)$ as locally uniformly stable. Roughly speaking, this implies that if the initial value of the error signal is sufficiently close to the target configuration at zero velocity, 0_e, and the auxiliary system's initial velocities are sufficiently small, then the solution of the controlled system is defined for all $t \geq t_0$, the target velocities remain small and the error decays exponentially.

In addition to determining whether the zero dynamics admits a (pseudo-) Riemannian metric, another significant issue is to "introduce dissipation" into the zero dynamics in order to make the fiber coordinates asymptotically vanish, as required in typical applications. This issue is addressed by the potential application of *generalized* vertically transverse functions to modify the closed-loop zero dynamics. Generalized transverse functions (GTF) were introduced by Morin and Samson [2004] to achieve practical and *asymptotic* stabilization of points and general trajectories for driftless control systems. In essence, a GTF for $\boldsymbol{X} = \{X_1, \ldots, X_m\} \subset \Gamma(TG)$ near $e \in G$ is a function $f : \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2} \longrightarrow G$ such that $f(\cdot, \beta) : \alpha \mapsto f(\alpha, \beta) \in G$ is transverse for $\boldsymbol{X}$

near e for every $\beta \in \mathbb{T}^{\kappa_1}$. We present a straightforward generalization of the VTFA in the case *generalized* vertically transverse functions are used. The interest in this class of functions is that its application to control leads to *non-autonomous* zero dynamics with additional control inputs that may be used to influence the behavior of the trajectories in zero dynamics. The central objective is to design these additional control inputs in order to make the zero-section of the zero dynamics asymptotically stable or, at least, locally attractive. In this work, we explore the possibility of designing the additional control inputs using time-varying feedback by means of high-order averaging (Vela [2003]; Sarychev [2001]; Agračhev and Gramkrelidze [1979]) since it has proved to provide useful insights on the construction of time-varying feedback to stabilize driftless systems and second-order systems. High-order averaging (Sarychev [2001]; Vela [2003]) is based on the formalism of chronological calculus developed by Agračhev and Gramkrelidze [1979]. In essence, the latter aims at reducing the qualitative analysis of the flow of a periodic non-autonomous (i.e., time-varying) vector field to the analysis of an autonomous (i.e., time-invariant) vector field by means of asymptotic expansions.

Outline of the thesis

The outline of this thesis is as follows. In Chapter 2 we review the TFA for practical stabilization of equilibrium points for driftless controllable systems exposed in (Morin and Samson [2003]). A section of this chapter encompasses the definition of a (Morin–Samson) transverse function, the description of the constructive method to obtain a transverse function for driftless systems on Lie groups and examples of transverse functions for some given control systems. In this chapter it is also presented the definition of vertically transverse functions and a result which establishes that the tangent mapping of a transverse function is vertically transverse. Finally, it is outlined the proposed set-up to practically stabilize configurations for second-order systems on (tangent) Lie groups by means of vertically transverse functions (VTF).

Chapter 3 includes an analysis for the zero dynamics that results from the application of the VTFA. In particular, it is proven that target and auxiliary zero dynamics are related and that both zero dynamics preserve the structure of the target system. That is, the zero dynamics has the structure of an affine connection control system. Furthermore, a necessary and sufficient condition is given for the existence of a metric for the zero dynamics of systems controlled by the VTFA underactuated by one control. Also stated and proved is an interesting result concerning the stability of systems controlled by the VTFA for which the zero dynamics admits a metric. An example is developed in this chapter in order to clarify these notions and results.

Chapter 4 concerns the use of generalized transverse functions to introduce dissipation into the zero dynamics. A generalization of the VTFA is presented for the case that *generalized* vertically transverse functions are used. We propose an explicit way to construct generalized transverse functions from a given transverse function. In this chapter it is also shown that, whenever the application of the VTFA yields a zero dynamics admitting a metric, there is a GTF such that the resulting non-autonomous zero dynamics also admits a metric. Also reviewed in this chapter is the theory of high-order averaging to render the zero-section locally attractive. An example and a numerical simulation are developed to illustrate the application and performance of generalized vertically transverse functions.

In Chapter 5 we give some concluding remarks concerning the results presented in this thesis and the foreseen scope and significance. We also discuss possible future directions of research.

An Appendix at the end of this document serves to set the notation and to define some basic concepts from differential geometry, the Lagrangian formulation of mechanical systems, and control theory. The Reader unfamiliar with these concepts may also consult the references cited therein. The results gathered in this thesis have been partially reported by Lizárraga and Sosa [2005, 2008], Sosa [2005] and Sosa and Lizárraga [2006, 2008].

Chapter 2

Vertically Transverse Functions and their application to control

In this chapter we outline the proposed framework based on vertically transverse functions, which aims at practically stabilizing configurations for second-order systems on tangent Lie groups. This control approach was initially proposed in the M.Sc. thesis (Sosa [2005]) and is further developed in Chapters 3 and 4 of this thesis. The control systems that this framework addresses are second-order systems evolving on a tangent Lie group TG of the form

$$\dot{x} = S_x + \sum_{i=1}^{m} u^i X_{i,x}^{\text{lift}}, \tag{2.1}$$

where $S \in \Gamma(TTG)$ is second-order, and the set $\boldsymbol{X} = \{X_1, \dots, X_m\} \subset \Gamma(TG)$ contains left-invariant vector fields on G, $m \leq n = \dim(G)$. Recall that every manifold, mapping, vector field or any related construct defined in this thesis are assumed to be smooth unless otherwise stated. Assume that the distribution spanned by $\text{Lie}(\boldsymbol{X})$ is completely nonintegrable at $e \in G$ and, typically, one requires that S be such that the set $\{S, \boldsymbol{X}^{\text{lift}}\} = \{S, X_1^{\text{lift}}, \dots, X_m^{\text{lift}}\}$ is accessible at $0_e \in TG$. System (2.1) is in the sequel referred to as the *target system*.

Section 2.1 of this chapter contains some results concerning the transverse function approach, including the definition and construction of (Morin–Samson) transverse functions for controllable driftless systems and some examples. Section 2.2 contains the definition of vertical transversality and the result that tangent mappings of transverse functions are vertically transverse. Finally, Section 2.3 describes the proposed framework to control second-order systems.

2.1. Morin–Samson transverse functions for driftless systems

2.1.1. Definition and construction of Morin–Samson transverse functions

Consider a set of vector fields $\boldsymbol{X} = \{X_1, \dots, X_m\} \subset \Gamma(TQ)$, $m \leq n = \dim(Q)$ and a point $q \in TQ$. A map $f : \mathbb{T}^\kappa \longrightarrow Q$, with $\kappa \geq n - m$, is said to be ***transverse for*** $\boldsymbol{X}$ ***(near*** q***)*** if there exists a neighborhood $\mathcal{U}$ of q such that $f(\mathbb{T}^\kappa) \subset \mathcal{U}$, and for every $\theta \in \mathbb{T}^\kappa$,

$$T_{f(\theta)}Q = Tf(T_\theta \mathbb{T}^\kappa) + \operatorname{span}_{\mathbb{R}} \left\{X_{1,f(\theta)}, \ldots, X_{m,f(\theta)}\right\}. \tag{2.2}$$

Theorem 2.1 (Morin and Samson [2003])**.** *The driftless system defined by the set of control vector fields $\boldsymbol{X} = \{X_1, \ldots, X_m\} \subset \Gamma(TQ)$ is locally accessible at a point $q \in Q$ iff there exists a transverse function $f : \mathbb{T}^\kappa \longrightarrow Q$ for $\boldsymbol{X}$ near q for some $\kappa \geq n - m$.*

Let $f : \mathbb{T}^\kappa \longrightarrow Q$ be a transverse function for $\boldsymbol{X}$ near $q \in Q$ and let (U, θ) and (V, q) be local coordinates for $\mathbb{T}^\kappa$ and Q such that q and $f(U)$ are contained in V. Then it can be readily shown that (Morin-Samson) transversality (2.2) translates into

$$\mathbb{R}^n = \operatorname{span}_{\mathbb{R}} \left\{X_{1,f(\theta)}, \ldots, X_{m,f(\theta)}, \frac{\partial f}{\partial \theta^1}(\theta), \ldots, \frac{\partial f}{\partial \theta^\kappa}(\theta)\right\}. \tag{2.3}$$

In other words, the matrix having by columns the elements $X_{1,f(\theta)}, \ldots, X_{m,f(\theta)}$, $(\partial f/\partial\theta^1)(\theta), \ldots,$ $(\partial f/\partial\theta^\kappa)(\theta)$ is invertible for every $\theta \in U$. When $Q = G$ is an n-dimensional Lie group, and the elements in $\boldsymbol{X}$ are left-invariant, κ, in the previous theorem, can be chosen equal to $n - m$. Thus, $f : \mathbb{T}^{n-m} \longrightarrow G$ satisfies, for every $\theta \in \mathbb{T}^{n-m}$,

$$T_{f(\theta)}G = Tf(T_\theta \mathbb{T}^{n-m}) \oplus \operatorname{span}_{\mathbb{R}} \left\{X_{1,f(\theta)}, \ldots, X_{m,f(\theta)}\right\}. \tag{2.4}$$

In the latter case, f may be explicitly obtained by the construction method given by Morin and Samson [2003], which is next outlined. Let $\mathfrak{g}$ denote the Lie algebra of G and let $\xi_1, \ldots, \xi_m \in \mathfrak{g}$ to be related to the left-invariant vector fields $X_1, \ldots, X_m$ (i.e., $\xi_i = X_{i,e}$ for $i = 1, \ldots, m$). Define inductively a family $\{G_k\}_{k\in\mathbb{N}}$ of subspaces of $\mathfrak{g}$, by setting $G_0 = \operatorname{span}_{\mathbb{R}}\{\xi_1, \ldots, \xi_m\}$ and $G_k = G_{k-1} + [G_0, G_{k-1}]$ for $k \geq 1$. Then consider mappings $\lambda, \rho : \{m+1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ and an ordered basis $\{\zeta_1, \ldots, \zeta_m\}$ of $\mathfrak{g}$ such that

I. $G_k = \operatorname{span}_{\mathbb{R}}\{\zeta_1, \ldots, \zeta_{\dim(G_k)}\}$ for $k = 1, \ldots, \min\{k : G_k = \mathfrak{g}\}$.

II. Whenever $k \geq 2$ and $\dim(G_{k-1}) \leq i \leq \dim(G_k)$, one has $\zeta_i = \left[\zeta_{\lambda_i}, \zeta_{\rho_i}\right]$, with $\zeta_{\lambda(i)} \in G_a$, $\zeta_{\rho_i} \in G_b$ and $a + b = k$.

The set $\{\zeta_1 \ldots, \zeta_n\}$, together with the mappings λ and ρ, constitute a graded basis for $\mathfrak{g}$. Next, associate with such a basis a weight vector $r = (r_1, \ldots, r_n) \in \mathbb{R}^n$ such that $r_i = k$ iff $\zeta_i \in G_k \backslash G_{k-1}$. Given such a graded basis and weight vector, the construction of a transverse function proceeds by selecting strictly positive reals $\varepsilon_{m+1}, \ldots, \varepsilon_n$ and by defining mappings $f_i : \mathbb{T} \longrightarrow G$ for $i = m + 1, \ldots, n$, as follows

$$f_i(\theta) = \exp\left(\varepsilon_i^{r_{\lambda(i)}} \sin(\theta)\zeta_{\lambda(i)} + \varepsilon_i^{r_{\rho(i)}} \cos(\theta)\zeta_{\rho(i)}\right).$$

Then a transverse function $f : \mathbb{T}^{n-m} \longrightarrow G$ is obtained by setting

$$f(\theta_{m+1}, \ldots, \theta_n) = f_n(\theta_n) f_{n-1}(\theta_{n-1}) \cdots f_{m+1}(\theta_{m+1}). \tag{2.5}$$

2.1.2. Examples of Morin–Samson transverse functions

Transverse function for chained forms

Chained forms (CF) are canonical systems which are well known to describe kinematics models of certain mechanical systems. Examples of such systems are the rolling penny, the kinematic model of a unicycle-type wheeled robot and the kinematic model of a car towing n trailers with "on axle-hitching" (see for instance Murray [1994]). Chained forms are driftless systems for which the Lie algebra of the set of control vector fields is nilpotent and spans a distribution which is completely nonintegrable at every point. The n-dimensional, 2-input chained form system (n-CF) is given by

$$
\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= u_1 x_2 \\
&\vdots \\
\dot{x}_n &= u_1 x_{n-1}.
\end{aligned}
$$

A transverse function for the 3-CF

Consider the 3-CF evolving on $\mathbb{R}^3$, $\dot{x} = u^1 X_{1,x} + u^2 X_{2,x}$, where the control vector fields in $\boldsymbol{X} = \{X_1, X_2\} \subset \Gamma(T\mathbb{R}^3)$ are given by $X_{1,x} = \partial/\partial x^1 + x_2\, \partial/\partial x^2$, and $X_{2,x} = \partial/\partial x^3$. Note that $\mathbb{R}^3$ is a Lie group with group composition defined, for every $x, y \in \mathbb{R}^3$, by

$$\widehat{\mu}(x, y) = x \cdot y = (x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3 + x_2 y_1)\,. \tag{2.6}$$

Following the procedure in the previous subsection we have $G_0 = \mathrm{span}_{\mathbb{R}}\{X_1, X_2\}$ and $G_1 = \mathrm{span}_{\mathbb{R}}\{X_1, X_2, X_3\}$ with $X_3 = [X_1, X_2]$. Hence, a graded basis for the Lie algebra of $\boldsymbol{X}$ is given by the set $\{X_1, X_2, X_3\}$ along with the mappings $\lambda, \rho : \{3\} \longrightarrow \{1, 2, 3\}$ defined by $\lambda(3) = 1$ and $\rho(3) = 2$. The associated weight vector is given by $r = (1, 1, 2)$. Consider a coordinate system (U, θ) on $\mathbb{T} \simeq S^1 \subset \mathbb{R}^2$, for instance $U = S^1 \setminus \{(0, 1)\}$ and $\theta(p) = 2\arctan(\frac{p_1}{1-p_2})$. Then, the construction procedure yields the following transverse function.

$$f(\theta) = \left(\varepsilon \sin(\theta), \varepsilon \cos(\theta), \frac{1}{4}\varepsilon^2 \sin(2\theta)\right), \qquad \varepsilon > 0. \tag{2.7}$$

The latter can be defined for every element of $\mathbb{T}$ by continuity. In coordinates, condition (2.4) reduces to guaranteeing that the determinant of the matrix

$$M := \left[X_{1,f(\theta)}, X_{2,f(\theta)}, \frac{\partial f}{\partial \theta}(\theta)\right] = \begin{pmatrix} 1 & 0 & \varepsilon \cos(\theta) \\ 0 & 1 & -\varepsilon \sin(\theta) \\ \varepsilon \cos(\theta) & 0 & \frac{1}{2}\varepsilon^2 \cos(2\theta) \end{pmatrix}$$

is different from zero. A computation readily shows that $\det(M) = \frac{1}{2}\varepsilon^2$, which is non-zero provided that $\varepsilon > 0$.

A transverse function for the 4-CF

Consider the 4-CF system evolving on $\mathbb{R}^4$,

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= u_1 x_2\\ \dot{x}_4 &= u_1 x_3.\end{aligned}$$

This system can be written as $\dot{x} = \sum_{i=1}^{2} u^i X_{i,x}$, with $X_{1,x} = \partial/\partial x^1 + x_2\,\partial/\partial x^3 + x_3\,\partial/\partial x^4$, and $X_{2,x} = \partial/\partial x^2$. The distribution on $T\mathbb{R}^4$ spanned by $\boldsymbol{X} = \{X_1, X_2\}$ is completely nonintegrable at every $x \in \mathbb{R}^4$, which implies that the 4-ECF is (globally) accessible and hence, for any $x \in \mathbb{R}^4$ and any neighborhood $\mathcal{U}$ of x, there exists a transverse function as defined in (2.2). The set $\{X_1, X_2, X_3, X_4\}$ with $X_3 = [X_1, X_2] = -\partial/\partial x^3$ and $X_4 = [X_1, X_3] = \partial/\partial x^4$, along with the weight vector $r = (1, 1, 2, 3)$, is a graded basis constructed in the sense described above. $\mathbb{R}^4$ can be endowed with a differentiable group composition given, for every $x, y \in \mathbb{R}^4$, by

$$x \cdot y = \left(x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3 + x_2 y_1,\ x_4 + y_4 + x_3 y_1 + \tfrac{1}{2} x_2 {y_1}^2 \right)$$

In what follows of this subsubsection, as we shall often do in the sequel, we use the convention $\sin = \mathrm{s}$ and $\cos = \mathrm{c}$. Let $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_{>0}$. The construction procedure yields the function $f : \mathbb{T}^2 \longrightarrow \mathbb{R}^4$ given by

$$\begin{aligned} f(\theta) \;=\; \Big(& \varepsilon_1 \mathrm{s}(\theta_1) + \varepsilon_2 \mathrm{s}(\theta_2),\ \varepsilon_1 \mathrm{c}(\theta_1),\ \frac{1}{4}{\varepsilon_1}^2 \mathrm{s}(2\,\theta_1) - {\varepsilon_2}^2 \mathrm{c}(\theta_2),\\ & \frac{1}{6}{\varepsilon_1}^3 \mathrm{s}^2(\theta_1)\mathrm{c}(\theta_1) - \frac{1}{4}{\varepsilon_2}^3 \mathrm{s}(2\,\theta_2) - \varepsilon_1 {\varepsilon_2}^2 \mathrm{c}(\theta_2)\mathrm{s}(\theta_1) \Big), \end{aligned}$$

which can be shown to be transverse, for instance, by choosing $\varepsilon_2 = k\,\varepsilon_1$ with $k > 1.5$.

Transverse function for a blimp-like system

Consider a blimp-like system, depicted in Figure 2.1, which may represent, for instance, a hovercraft or a space satellite. This system consists of a planar rigid body moving in $SE(2)$ with a thruster to adjust its pose. The force impelled by this thruster is modeled by an ordered pair (τ_1, τ_2) which actuates at a fixed point assumed to be located along the system body's axis, at a distance l from the center of mass. The configuration of the blimp is determined by $(q_1, q_2, q_3) \in SE(2) \simeq \mathbb{R}^2 \times S^1$, where $(q_1, q_2) \in \mathbb{R}^2$ is the position of the center of mass of the body and $q_3 \in S^1$ is its orientation with respect to a fixed basis. The Euler-Lagrange equations describing the motion of the blimp system yield

$$\begin{aligned} m\ddot{q}_1 - ml\sin(q_3)\,\ddot{q}_3 - ml\cos(q_3)\,\dot{q}_3^2 &= \tau_1\\ m\ddot{q}_2 + ml\cos(q_3)\,\ddot{q}_3 - ml\sin(q_3)\,\dot{q}_3^2 &= \tau_2\\ J\,\ddot{q}_3 - ml\sin(q_3)\,\ddot{q}_1 + ml\cos(q_3)\,\ddot{q}_2 &= 0, \end{aligned} \tag{2.8}$$

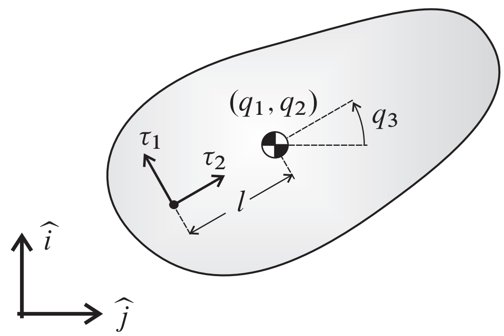


Figure 2.1: Schematic representation of a blimp-like system.

where m is the mass of the rigid body and J is the moment of inertia with respect to the center of mass. Consider the following transformation of the control inputs (which amounts to choosing a different basis of the control distribution):

$$\begin{aligned}\tau_1 &= m\cos(q_3)\,u_1 - \tfrac{m}{J}\left(J - ml^2\right)\,\sin(q_3)\,u_2,\\ \tau_2 &= m\sin(q_3)\,u_1 + \tfrac{m}{J}\left(J - ml^2\right)\,\cos(q_3)\,u_2.\end{aligned}$$

After the transformation and rearrangement of the equations, system (2.8) can be rewritten as

$$\begin{aligned}\ddot{q}_1 &= l\cos(q_3)\dot{q}_3^2 + \cos(q_3)u_1 - \sin(q_3)u_2\\ \ddot{q}_2 &= l\sin(q_3)\dot{q}_3^2 + \sin(q_3)u_1 + \cos(q_3)u_2\\ \ddot{q}_3 &= a\,u_2,\end{aligned} \tag{2.9}$$

where $a = -ml/J$ is a non-zero (negative) constant defined in terms of the system parameters. By relabeling variables, $x_i = q_i$ and $x_{i+3} = \dot{q}_i$ for $i = 1, \ldots, 3$, we may rewrite (2.9) in the form (2.1), that is:

$$\dot{x} = S_x + u^1 X_{1,x}^{\text{lift}} + u^2 X_{2,x}^{\text{lift}},$$

where $S_x = \sum_{i=1}^{3} x_{i+n}\,\partial/\partial x^i + l\,\mathrm{c}(x_3)x_6^2\,\partial/\partial x^4 + l\,\mathrm{s}(x_3)x_6^2\,\partial/\partial x^5$ is the geodesic spray associated with the Riemannian metric given by the inertia tensor of the mechanical system, and the vector fields X_1 and X_2 are given by

$$\begin{aligned}X_{1,x} &= \mathrm{c}(x_3)\frac{\partial}{\partial x^1} + \mathrm{s}(x_3)\frac{\partial}{\partial x^2},\\ X_{2,x} &= -\mathrm{s}(x_3)\frac{\partial}{\partial x^1} + \mathrm{c}(x_3)\frac{\partial}{\partial x^2} + a\frac{\partial}{\partial x^3}.\end{aligned}$$

Given that the configuration space of the system is $\mathbb{R}^2 \times S^1 \simeq SE(2)$, it can be equipped with the Lie group structure of the *special Euclidean group* $SE(2)$. Let $G = SE(2) \simeq \mathbb{R}^2 \times S^1$. The Lie group composition $\hat{\mu} : G \longrightarrow G$ is defined, for every $x, y \in G$, by

$$\hat{\mu}(x, y) = (\mathrm{c}(x_3)y_1 - \mathrm{s}(x_3)y_2 + x_1,\ \mathrm{s}(x_3)y_1 + \mathrm{c}(x_3)y_2 + x_2,\ x_3 + y_3)\,. \tag{2.10}$$

It can be shown that the unlifted control vector fields X_1, X_2 are left-invariant with respect to this group composition and that the distribution spanned by $\mathrm{Lie}\{X_1, X_2\}$ is completely nonintegrable at any point. Let us construct a transverse function $f : \mathbb{T} \longrightarrow G$ for the set of unlifted control vector fields $\{X_1, X_2\}$ by following the procedure described above. A graded basis for the Lie algebra generated by the set $\{X_1, X_2\}$ is the set $\{X_1, X_2, X_3\}$ with $X_3 = [X_1, X_2] = a\mathrm{s}(x_3)\,\partial/\partial x_1 - a\mathrm{c}(x_3)\,\partial/\partial x_2$, along with the weight vector $r = (1, 1, 2)$. The construction of the transverse function yields,

$$f(\theta) = \left(\frac{\mathrm{c}(a\varepsilon\,\mathrm{c}(\theta) - \theta) - \mathrm{c}(\theta)}{a\,\mathrm{c}(\theta)}, \frac{\mathrm{s}(a\varepsilon\,\mathrm{c}(\theta) - \theta) + \mathrm{s}(\theta)}{a\,\mathrm{c}(\theta)}, a\varepsilon\,\mathrm{c}(\theta)\right),$$

which can be defined, by continuity, for every $\theta \in [-\pi, \pi]$. The transversality condition is determined by the invertibility of the matrix $D(\theta) = \left[X_{1,f(\theta)}, X_{2,f(\theta)}, \partial_\theta f(\theta)\right]$. The determinant of the latter may be shown to be

$$\det(D(\theta)) = -2\frac{\mathrm{c}(a\varepsilon\,\mathrm{c}(\theta)) - 1}{\mathrm{c}(2\,\theta) + 1},$$

which can be defined, again by continuity, for $\theta \in [-\pi, \pi]$, and is positive for every $\varepsilon > 0$ and every $\theta \in [-\pi, \pi]$.

2.2. Vertically Transverse Functions

If a mapping $f : \mathbb{T}^\kappa \longrightarrow Q$ is transverse in the sense of Morin and Samson for $\boldsymbol{X}$ near a given $q \in Q$, then $Tf : T\mathbb{T}^\kappa \longrightarrow TQ$ is vertically transverse for $\boldsymbol{X}^{\mathrm{lift}}$ in a sense that is made clear in Definition 2.1 and Theorem 2.2. However, prior to dwelling on this notion, let us state the following lemma concerning tangent mappings, which is useful in the proof of Theorem 2.2.

Lemma 2.1. *Let Q and P be manifolds and $f : Q \longrightarrow P$ a C^2 mapping. Then*

I. *TTf maps vertical vectors to vertical vectors.*

II. *If $v, w \in TQ$ satisfy $\pi_Q(v) = \pi_Q(w)$, then $TTf(\mathrm{lift}(v, w)) = \mathrm{lift}(Tf(v), Tf(w))$.*

III. *The mapping $\mathrm{lift}(v, \cdot) : T_{\pi_Q(v)}Q \longrightarrow T_vTQ^{\mathrm{vert}}$ is a vector space isomorphism for every $v \in TQ$.*

Proof. **I.** Let $\alpha \in TTQ^{\mathrm{vert}}$, so that $T\pi_Q(\alpha) = 0$. Given that the following diagram commutes,

$$\begin{tikzcd} TQ \arrow[r, "Tf"] \arrow[d, "\pi_Q"'] & TP \arrow[d, "\pi_P"] \\ Q \arrow[r, "f"] & P \end{tikzcd}$$

and in view of the chain rule, we have $T\pi_P \circ TTf = Tf \circ T\pi_Q$. Then $T\pi_P \circ TTf(\alpha) = Tf \circ T\pi_Q(\alpha) = 0$, hence $TTf(\alpha) \in TTP^{\text{vert}}$, that is, TTf maps vertical vectors to vertical vectors.

II. Let $v, w \in TQ$ satisfy $\pi_Q(v) = \pi_Q(w)$ and define the curve $\gamma_{v,w}(t) = v + tw$, so $\text{lift}(v, w) = T_0\gamma_{v,w}\left(\frac{\partial}{\partial r}\big|_0\right)$. Then, in view of the linearity of Tf on restriction to fibers, $(Tf \circ \gamma_{v,w})(t) = Tf(v + wt) = Tf(v) + tTf(w) = \gamma_{Tf(v),Tf(w)}$. Therefore

$$\begin{aligned} TTf(\text{lift}(v, w)) &= T_vTf \circ T_0\gamma_{v,w}\left(\tfrac{\partial}{\partial r}\big|_0\right) = T_0(Tf \circ \gamma_{v,w})\left(\tfrac{\partial}{\partial r}\big|_0\right) \\ &= \text{lift}(Tf(v), Tf(w)), \end{aligned}$$

which establishes the claim.

III. Let $v \in T_qQ$ for some $q \in Q$. By (**I.**) in this Lemma, the image of $\text{lift}(v, \cdot)$ is contained in T_vTQ^{vert}. To show that $\text{lift}(v, \cdot)$ is linear and bijective, we use a coordinate chart on Q and the naturally induced coordinates on TQ. If, in such coordinates, $v = (q, \overline{v})$ and $w = (q, \overline{w})$ for any $w \in T_qQ$, then, the curve defined in **II.** of this Lemma is given by $\gamma_{v,w}(t) = (q, \overline{v} + t\overline{w})$. Thus $\text{lift}(v, w) = ((q, \overline{v}), (0, \overline{w}))$, so $\text{lift}(v, \cdot)$ is linear and injective. Also, if $\alpha \in T_vTQ$ has coordinate expression $\alpha = ((q, \overline{v}), (\alpha_L, \alpha_H))$, then $T\pi_Q(\alpha) = (q, \alpha_L)$, hence $u \in \ker(T\pi_Q)$ if, and only if, $\alpha_L = 0$. Thus, $\text{lift}(v, \cdot)$ is also surjective and, consequently, an isomorphism. ■

Definition 2.1 (Vertically Transverse Function)**.** *Let $\boldsymbol{X} = \{X_1, \ldots, X_m\} \subset \Gamma(TQ)$ be a set of vector fields on Q. A bundle mapping $F : T\mathbb{T}^\kappa \longrightarrow TQ$ is said to be vertically transverse for the vertical distribution spanned by $\boldsymbol{X}^{\text{lift}}$ if, for every $\omega \in T\mathbb{T}^\kappa$,*

$$T_{F(\omega)}TQ^{\text{vert}} = TF((T_\omega T\mathbb{T}^\kappa)^{\text{vert}}) + \text{span}_{\mathbb{R}}\left\{X^{\text{lift}}_{1,F(\omega)}, \ldots, X^{\text{lift}}_{m,F(\omega)}\right\}. \tag{2.11}$$

Theorem 2.2. *Let $\boldsymbol{X} = \{X_1, \ldots, X_m\} \subset \Gamma(TQ)$ be a set of vector fields and $f : \mathbb{T}^\kappa \longrightarrow Q$ a transverse function for $\boldsymbol{X}$ near $q \in Q$. Then, for every $\omega \in T\mathbb{T}^\kappa$,*

$$T_{Tf(\omega)}TQ^{\text{vert}} = TTf\left((T_\omega T\mathbb{T}^\kappa)^{\text{vert}}\right) + \text{span}_{\mathbb{R}}\left\{X^{\text{lift}}_{1,Tf(\omega)}, \ldots, X^{\text{lift}}_{m,Tf(\omega)}\right\}, \tag{2.12}$$

that is, Tf is vertically transverse for $\boldsymbol{X}^{\text{lift}}$. Moreover, if $\kappa = n - m$, i.e., if the sum in (2.2) is direct, then so is the sum in (2.12).

Proof. Let $\theta \in \mathbb{T}^\kappa$ and $\omega \in T_\theta\mathbb{T}^\kappa$ and assume that $v \in T_{Tf(\omega)}TQ^{\text{vert}}$. Given that v is vertical, by Lemma 2.1-(**III**), there exists $\widetilde{v} \in T_{f(\theta)}Q$ such that $v = \text{lift}(Tf(\omega), \widetilde{v})$. In addition, from equation (2.2) we deduce the existence of a vector $\widetilde{\omega} \in T_\theta\mathbb{T}^\kappa$ and real numbers $a_1, \ldots, a_m$ such that $\widetilde{v} = Tf(\widetilde{\omega}) + \sum_{i=1}^m a^i X_{i,f(\theta)}$. By applying the linear mapping $\text{lift}(Tf(\omega), \cdot)$ to both members of this equation and using Lemma 2.1-(**II**), we get

$$\begin{aligned} v &= \text{lift}\,(Tf(\omega), Tf(\widetilde{\omega})) + \text{lift}\left(Tf(\omega), \sum_{i=1}^m a^i X_{i,f(\theta)}\right) \\ &= TTf(\text{lift}(\omega, \widetilde{\omega})) + \sum_{i=1}^m a^i X^{\text{lift}}_{i,Tf(\omega)}. \end{aligned}$$

Since $\mathrm{lift}(\omega, \widetilde{\omega}) \in (T_\omega T\mathbb{T}^\kappa)^{\mathrm{vert}}$, this proves that (2.12) holds. Now suppose that $\kappa = n - m$, i.e., equation (2.4) is satisfied, and assume that

$$v \in TTf\left((T_\omega T\mathbb{T}^\kappa)^{\mathrm{vert}}\right) \cap \mathrm{span}_{\mathbb{R}}\left\{X^{\mathrm{lift}}_{1,Tf(\omega)}, \ldots, X^{\mathrm{lift}}_{m,Tf(\omega)}\right\}.$$

We shall prove that $v = 0$. By assumption, there exists $\alpha \in (T_\omega T\mathbb{T}^\kappa)^{\mathrm{vert}}$ and real numbers $a^1, \ldots, a^m$ such that $v = TTf(\alpha) = \sum_{i=1}^m a^i X^{\mathrm{lift}}_{i,Tf(\omega)}$. Since α is a vertical vector, Lemma 2.1-(III) implies that α is given by the vertical lift by ω of a vector $\widetilde{\omega} \in T_\theta\mathbb{T}^\kappa$, namely $\alpha = \mathrm{lift}(\omega, \widetilde{\omega})$. The mapping $\mathrm{lift}(Tf(\omega), \cdot)$ is linear, hence

$$\begin{aligned}
\mathrm{lift}\left(Tf(\omega), \sum_{i=1}^m a^i X_{i,f(\theta)}\right) &= \sum_{i=1}^m a^i \,\mathrm{lift}\left(Tf(\omega), X_{i,f(\theta)}\right) \\
&= \sum_{i=1}^m a^i X^{\mathrm{lift}}_{i,Tf(\omega)} \\
&= \mathrm{lift}(Tf(\omega), Tf(\widetilde{\omega})).
\end{aligned}$$

Since $\mathrm{lift}(Tf(\omega), \cdot)$ is injective by virtue of Lemma 2.1-(III), this implies that $\sum_{i=1}^m a^i X_{i,f(\theta)} = Tf(\widetilde{\omega})$. But the sum in (2.2) is direct, by assumption, hence $\sum_{i=1}^m a^i X_{i,f(\theta)} = Tf(\widetilde{\omega}) = 0$. Using the linearity of $\mathrm{lift}(Tf(\omega), \cdot)$, we conclude that $v = 0$, which completes the proof. ∎

Let (U, θ) and (V, q) be local coordinate charts on $\mathbb{T}^\kappa$ and on G such that $f(U) \in V$. Consider naturally induced coordinate charts for $T\mathbb{T}^\kappa$ and $TT\mathbb{T}^\kappa$ and for TG and TTG, correspondingly. Vertical transversality in (2.12) implies that for any $\alpha \in (T_{Tf(\omega)}TQ)^{\mathrm{vert}}$, there exist $\sigma \in (T_\omega T\mathbb{T}^\kappa)^{\mathrm{vert}}$ and reals $a^1, \ldots, a^m \in \mathbb{R}$ such that $\alpha = TTf(\sigma) + \sum_{i=1}^m a^i X_{i,Tf(\pi_{TQ}(\sigma))}$. Assume that, in coordinates $\sigma = (\theta, \dot{\theta}, 0, \sigma_H)$. Given that in coordinates $TTf(\sigma) = (f(\theta), \dot{\theta}\,(\partial f/\partial\theta)(\theta), 0, \sigma_H\,(\partial f/\partial\theta)(\theta))$, $X_{i,q} = (q, \widehat{X}_i(q))$ for $i = 1, \ldots, m$ and $\alpha = (f(\theta), \dot{\theta}\,(\partial f/\partial\theta)(\theta), 0, \alpha_H)$, one has

$$\alpha_H = \sum_{i=1}^m a^i \widehat{X}_i(x) + \frac{\partial f}{\partial\theta}(\theta)\,\sigma_H.$$

That is, the "vertical transversality" condition (2.12) reduces to

$$\mathbb{R}^n = \mathrm{span}_{\mathbb{R}}\left\{X_{1,f(\theta)}, \ldots, X_{m,f(\theta)}\right\} + \mathrm{span}_{\mathbb{R}}\left\{\frac{\partial f}{\partial\theta^1}(\theta), \ldots, \frac{\partial f}{\partial\theta^\kappa}(\theta)\right\}. \tag{2.13}$$

The coordinate expression of vertical transversality is the same as the expression for (Morin–Samson) transversality (see equation (2.3)). However, vertical transversality, compared to Morin–Samson transversality, is a property that occurs in a high-order tangent level (i.e., in TTQ), therefore, a natural question which arises is whether the transverse function approach can be extended to include second-order systems. The next section contains a possible set-up to practically stabilize configurations for second-order systems based on vertical transversality.

2.3. Framework for practical point-stabilization using vertically transverse functions

Consider the target system in (2.1) and assume that the set $\boldsymbol{X}$ contains left-invariant vector fields on G, an n-dimensional Lie group. Also assume that the distribution spanned by $\boldsymbol{X}$ is completely nonintegrable at some point, say $e \in G$ without loss of generality. Under these conditions, the accessibility of the target system depends, to a large extent, on the drift vector field $S \in \Gamma(TTG)$. When dealing with simple mechanical systems (SMS) (Bullo and Lewis [2005]), the drift vector field S is defined in advance and corresponds to the vector field given by $S = S^g - (dV^\sharp)^{\text{lift}}$, that is, the sum of the geodesic spray associated with a Riemannian metric g, and the vertical lift of vector field corresponding to the gradient of the potential energy function.

Under the assumptions made, by Theorems 2.1 and 2.2, there exists a transverse function $f : \mathbb{T}^{n-m} \longrightarrow G$ near $e \in TG$, and $Tf : T\mathbb{T}^{n-m} \longrightarrow TG$ is vertically transverse for $\boldsymbol{X}^{\text{lift}}$. Thus, for every $\omega \in T\mathbb{T}^{n-m}$, Tf satisfies

$$T_{Tf(\omega)}TG^{\text{vert}} = TTf\left((T_\omega T\mathbb{T}^\kappa)^{\text{vert}}\right) \oplus \operatorname{span}_{\mathbb{R}}\left\{X^{\text{lift}}_{1,Tf(\omega)}, \ldots, X^{\text{lift}}_{m,Tf(\omega)}\right\}. \tag{2.14}$$

The control approach herein presented is much in the spirit of the approach reported by Morin and Samson [2003]. First, one proceeds by setting $\kappa = n - m$ and by selecting a global frame $\boldsymbol{\Omega} = \{\Omega_1, \ldots, \Omega_\kappa\} \subset \Gamma(TT\mathbb{T}^\kappa)$ for $TT\mathbb{T}^\kappa$. The existence of such a global frame is assured since $T\mathbb{T}^\kappa$ is a trivial bundle. If $\{\Lambda_1, \ldots, \Lambda_\kappa\} \subset \Gamma(T\mathbb{T}^\kappa)$ is global frame for $T\mathbb{T}^\kappa$, then $\boldsymbol{\Omega}$ can be set as $\{\Lambda_1^{\text{lift}}, \ldots, \Lambda_\kappa^{\text{lift}}\} \subset \Gamma(TT\mathbb{T}^\kappa)$ in view of the result in Lemma 2.1-(III). Select a second-order vector field $\Delta \in \Gamma(TT\mathbb{T}^\kappa)$, typically the geodesic spray associated with a flat metric on $\mathbb{T}^\kappa$. Then define an auxiliary control system on $T\mathbb{T}^\kappa$ by

$$\dot{\omega} = \Delta_\omega + \sum_{i=1}^{\kappa} w^i \Omega_{i,\omega}, \tag{2.15}$$

where $w^1, \ldots, w^\kappa$ are viewed as control inputs. Next, define an error signal $z(t)$ along the trajectories of the *compound system* $(x(t), \omega(t))$, that is, the trajectories of the target system (2.1) and the auxiliary system (2.15), by using the tangent Lie group composition on TG, $z = \mu(x, Tf(\omega)^{-1})$, which we also write as $z = x \cdot Tf(\omega)^{-1}$. This error signal is used to quantify the difference between the state of the target system and the image by Tf of the state of the auxiliary system. By forcing the *error dynamics* $\dot{z}(t)$ to satisfy a second-order differential equation having 0_e as an asymptotically stable equilibrium by means of smooth feedback, the *error* $z(t)$ approaches zero as t increases. This, in turn, forces the target system trajectory $x(t)$ to approach $Tf(T\mathbb{T}^\kappa)$. Hence, the projection $\pi_G(x(t))$ of the target state to the base manifold (the target system configuration) approaches the set $f(\mathbb{T}^\kappa)$ and, given that the image of f is contained in a prespecified neighborhood $\mathcal{U}$ of the target configuration $e \in G$, the configuration $\pi_G(x(t))$ ultimately enters $\mathcal{U}$, which entails practical point-stabilization.

Prior to obtaining the expression for the error dynamics $\dot{z}(t)$ in terms of the state of the compound system, let us state the following auxiliary proposition, which provides us with an explicit expression for the derivative of the composition $a \cdot b^{-1}$, giving the structure of the error signal.

Proposition 2.1. *Let TG be a tangent Lie group, $A \in \Gamma(TTG)$ be second-order, and B a second-order vector field defined along a given curve $b : (t_0, t_1) \longrightarrow TG$ by $\dot{b}(t) = B_{b(t)}$. Then, if $a : (t_0, t_1) \longrightarrow TG$ is an integral curve of A, the curve $c = \mu(a, b^{-1}) = a \cdot b^{-1}$ satisfies, for $t \in (t_0, t_1)$,*

$$\dot{c}(t) = TR_{b^{-1}(t)} \left(A_{a(t)} - TL_{c(t)} \left(B_{b(t)} \right) \right), \tag{2.16}$$

which defines a second-order differential equation on TG.

Proof. Define $\widetilde{B}$ as the vector field along the curve b^{-1}, that is $\widetilde{B}_{b^{-1}(t)} = \frac{d}{dt} b^{-1}(t)$. By differentiating $e = \mu(b^{-1}(t), b(t))$ one easily concludes that $\widetilde{B}_{b^{-1}(t)} = -TR_{b^{-1}(t)} \circ TL_{b^{-1}(t)}(B_{b(t)})$, and by differentiating $c(t) = \mu(a(t), b^{-1}(t))$ one finds that $\dot{c}(t) = TL_{a(t)}(\widetilde{B}_{b^{-1}(t)}) + TR_{b^{-1}(t)}(A_{a(t)})$. Hence, given the fact that $L_{a(t)} \circ R_{b^{-1}(t)} \circ L_{b^{-1}}(t) = L_{c(t)} \circ R_{b^{-1}(t)} = R_{b^{-1}(t)} \circ L_{c(t)}$, we get

$$\begin{aligned} \dot{c}(t) &= TL_{a(t)} \left(-TR_{b^{-1}(t)} \circ TL_{b^{-1}(t)}(B_{b(t)}) \right) + TR_{b^{-1}(t)}(A_{a(t)}) \\ &= TR_{b^{-1}(t)} \left(A_{a(t)} - TL_{c(t)} \left(B_{b(t)} \right) \right), \end{aligned}$$

which coincides with (2.16). Next we prove that $\dot{c}(t)$ defines a second-order equation. For each β in the image of b, define $C_\beta : \gamma \mapsto TR_{\beta^{-1}} \left(A_{\gamma\beta} - TL_\gamma \left(B_\beta \right) \right)$. It is straightforward to verify that for $t \in \mathbb{R}$ and $\gamma \in TG$ one has $C_{b(t)}(\gamma) \in T_\gamma TG$, that is, $C_{b(t)}$ is a section of TTG and thus a (time-varying) vector field on TG. It remains to show that $T\pi_G \circ C_\beta = \mathrm{id}_{TG}$ for every $\beta \in b((t_0, t_1)) \subset TG$. Choose any such a β and note that $T(\pi_G \circ R_{\beta^{-1}}) = T(\widehat{R}_{\pi_G(\beta^{-1})} \circ \pi_G)$ and $T(\pi_G \circ L_\gamma) = T(\widehat{L}_{\pi_G(\gamma)} \circ \pi_G)$, which, for $\gamma \in TG$, entails that

$$\begin{aligned} (T\pi_G \circ C_\beta)(\gamma) &= T\pi_G \left(TR_{\beta^{-1}} \left(A_{\gamma\beta} - TL_\gamma (B_\beta) \right) \right) \\ &= T\widehat{R}_{\pi_G(\beta^{-1})} \circ T\pi_G \left(A_{\gamma\beta} - TL_\gamma (B_\beta) \right) \\ &= T\widehat{R}_{\pi_G(\beta^{-1})} \Big(T\pi_G(A_{\gamma\beta}) - T\widehat{L}_{\pi_G(\gamma)} \circ T\pi_G(B_\beta) \Big). \end{aligned}$$

However, A and B are second-order , thus $T\pi_G(A_{\gamma\beta}) = \gamma\beta$ and $T\pi_G(B_\beta) = \beta$. Using these equations, along with $\gamma\beta - T\widehat{L}_{\pi_G(\gamma)}(\beta) = T\widehat{R}_{\pi_G(\beta)}(\gamma)$, we obtain $(T\pi_G \circ C_\beta)(\gamma) = T\widehat{R}_{\pi_G(\beta^{-1})}(\gamma\beta - T\widehat{L}_{\pi_G(\gamma)}(\beta)) = \gamma$, as required. ■

In order to apply Proposition 2.1 to obtain the error dynamics, set the curves $a(t)$ and $b(t)$ as the states of the target and auxiliary systems respectively, i.e., $a(t) = x(t)$ and $b(t) = Tf \circ \omega(t)$, so, the vector fields along a and b are, respectively,

$$A_x = S_x + \sum_{i=1}^{m} u^i X_{i,x}^{\mathrm{lift}} \qquad \text{and} \qquad B_{Tf(\omega)} = TTf \left(\Delta_\omega + \sum_{i=1}^{\kappa} w^i \Omega_{i,\omega} \right).$$

Therefore

$$\dot{z} = TR_{Tf(\omega)^{-1}} \left(S_x + \sum_{i=1}^{m} u^i X_{i,x}^{\mathrm{lift}} - TL_z \circ TTf \left(\Delta_\omega + \sum_{i=1}^{\kappa} w^i \Omega_{i,\omega} \right) \right).$$

By grouping the drift and controlled vector fields and using $x = z \cdot Tf(\omega)$ as well as the left-invariance of $X_1^{\text{lift}}, \ldots, X_m^{\text{lift}}$, we obtain the following error dynamics which, by Proposition 2.1, defines a second-order differential equation

$$\begin{aligned}\dot{z} = TR_{Tf(\omega)^{-1}} &\left(S_{z\cdot Tf(\omega)} - TL_z \circ TTf(\Delta_\omega)\right) \\ &+ TR_{Tf(\omega)^{-1}} \circ TL_z \left(\sum_{i=1}^{m} u^i X_{i,Tf(\omega)}^{\text{lift}} - \sum_{i=1}^{\kappa} w^i TTf(\Omega_{i,\omega})\right). \quad (2.17)\end{aligned}$$

We now address how vertical transversality may be used for control purposes. The main idea is that, for second-order systems, the control inputs can only shape the second-order time derivatives of the base trajectories, accelerations when referring to SMS, which amounts to assigning them values in the vertical subbundle. The relevance of requiring Tf to be vertically transverse is that, as stated in equation (2.14), the vertical subbundle is spanned by the control distribution and the image of TTf. This fact provides one with full control over the error system and therefore $\dot{z}(t)$ in (2.17) can be made to satisfy an arbitrarily given smooth second-order dynamics by means of smooth feedback in terms of the compound system trajectory, as stated in the following theorem.

Theorem 2.3 (Existence of a feedback to set the error dynamics)**.** *Given any smooth, second-order vector field $Y \in \Gamma(TTG)$, there exists a smooth feedback law $\alpha = (\alpha^1, \ldots, \alpha^n) : TG \times T\mathbb{T}^\kappa \longrightarrow \mathbb{R}^n$ such that the error $z = x \cdot Tf(\omega)^{-1}$ satisfies $\dot{z} = Y(z)$ along the trajectories of the compound system,*

$$(\dot{x}, \dot{\omega}) = \left(S_x + \sum_{i=1}^{m} \alpha^i \left(x \cdot Tf(\omega)^{-1}, \omega\right) X_{i,x}^{\text{lift}}, \Delta_\omega + \sum_{i=1}^{\kappa} \alpha^{i+m} \left(x \cdot Tf(\omega)^{-1}, \omega\right) \Omega_{i,\omega}\right). \quad (2.18)$$

Proof. Finding the required feedback amounts to setting the right-hand side of (2.17) equal to Y_z, solving the resulting equation for $u^1, \ldots, u^m, w^1, \ldots, w^{n-m}$ in terms of (z, ω), and then checking that the solutions define a smooth mapping $\alpha : TG \times T\mathbb{T}^\kappa \longrightarrow \mathbb{R}^n$. The first step leads to

$$\begin{aligned}\sum_{i=1}^{m} u^i X_{i,Tf(\omega)}^{\text{lift}} - \sum_{i=1}^{\kappa} w^i T_\omega Tf(\Omega_{i,\omega}) &= (TR_{Tf(\omega)^{-1}} \circ TL_z)^{-1}(Y_z - (D_\omega)_z) \\ &= TL_{z^{-1}} \circ TR_{Tf(\omega)}(Y_z - (D_\omega)_z),\end{aligned}$$

where, for each $\omega \in T\mathbb{T}^\kappa$, we have defined the vector field $D_\omega \in \Gamma(TTG)$ by setting

$$D_\omega : z \mapsto TR_{Tf(\omega)^{-1}}(S_{z\cdot Tf(\omega)} - TL_z \circ TTf(\Delta_\omega)).$$

Notice that, since (2.17) is second-order, so is D_ω for every $\omega \in T\mathbb{T}^\kappa$ and, by linearity of $T\pi_G$ on restriction to the fibers, $T\pi_G \circ (Y_z - (D_\omega)_z) = T\pi_G(Y_z) - T\pi_G((D_\omega)_z) = z - z = 0$, which shows that $Y - D_\omega$ is vertical. On the other hand, it is straightforward to check that one has $T\pi_G \circ TL_\xi = T\widehat{L}_{\pi_G(\xi)} \circ T\pi_G$ and $T\pi_G \circ TR_\xi = T\widehat{R}_{\pi_G(\xi)} \circ T\pi_G$ for every $\xi \in TTG$, and both of these equations imply that $TL_{z^{-1}} \circ TR_{Tf(\omega)}(Y_z - (D_\omega)_z)$ is vertical.

Given that Tf is vertically transverse, it satisfies

$$(T_{Tf(\omega)}TG)^{\mathrm{vert}} = \mathrm{span}_{\mathbb{R}}\left\{X^{\mathrm{lift}}_{1,Tf(\omega)},\ldots,X^{\mathrm{lift}}_{m,Tf(\omega)}\right\} \oplus T_\omega Tf((T_\omega T\mathbb{T}^{n-m})^{\mathrm{vert}})$$

for all $\omega \in T\mathbb{T}^\kappa$. Taking into account the assumption that $\{\Omega_1,\ldots,\Omega_m\}$ is a global frame for $(TT\mathbb{T}^\kappa)^{\mathrm{vert}}$, we conclude that there exists a unique mapping $\alpha : TG \times T\mathbb{T}^\kappa \longrightarrow \mathbb{R}^n$ such that, for every $(z,\omega) \in TG \times T\mathbb{T}^\kappa$,

$$\sum_{i=1}^{m} \alpha^i(z,\omega)X^{\mathrm{lift}}_{i,Tf(\omega)} - \sum_{i=1}^{\kappa} \alpha^{i+m}(z,\omega)T_\omega Tf(\Omega_{i,\omega}) = TL_{z^{-1}} \circ TR_{Tf(\omega)}(Y_z - (D_\omega)_z). \quad (2.19)$$

Let (U,z) and (V,θ) be coordinate charts on G and on $\mathbb{T}^\kappa$ respectively, and $(\pi_G^{-1}(U),(z,\dot{z}))$ and $(\pi_{\mathbb{T}^\kappa}^{-1}(V),(\theta,\dot{\theta}))$ the naturally induced coordinate charts on TG and $T\mathbb{T}^\kappa$, respectively. Then, by means of (2.13), the left-hand side of equation (2.19) translates into the matrix product $M(\theta)\,\alpha(\theta,\dot{\theta},z,\dot{z})$, where

$$M(\theta) := \left[X_{1,f(\theta)},\ldots,X_{m,f(\theta)},\frac{\partial f}{\partial\theta^1}(\theta),\ldots,\frac{\partial f}{\partial\theta^\kappa}(\theta)\right],$$

and α is the vector $\alpha = (\alpha^1,\ldots,\alpha^n) \in \mathbb{R}^n$. Given that the composition of smooth functions is smooth, the right-hand side of equation (2.19) is smooth provided that Y and D are smooth vector fields, as previously was assumed. Therefore, the smoothness of α follows from the invertibility of $M(\theta)$ for every $\theta \in V$, a fact that is easily established given that f is transverse. ■

Chapter 3

Analysis of the Zero Dynamics resulting from the VTFA

In this chapter we present analysis results regarding the nature of the closed-loop zero dynamics resulting from the application of the VTF approach proposed in Chapter 2. In particular, in this chapter we show that the zero dynamics is an affine-connection system (in the sense of Bullo and Lewis [2005]). We also present a necessary and sufficient condition to determine whether this affine connection is the Levi-Civita connection of some (pseudo–) Riemannian metric in the case when the target system is underactuated by one control ($\kappa = 1$). Then we establish some results which serve to prove the next important result of this chapter, that is, the stability of the closed-loop system provided that the zero dynamics admits a metric. Finally we present an example of application of the proposed control methodology which aims at illustrating the results in the present and the previous chapter.

3.1. Structure of the zero dynamics

In the previous chapter we have seen how to formulate the point-stabilization problem for second-order systems based on vertically transverse functions. Its application yields an error dynamics which can be arbitrarily set by means of smooth feedback. In particular, we shall assume throughout this chapter that a feedback law $\alpha = (\alpha^1, \ldots, \alpha^n) : TG \times T\mathbb{T}^\kappa \longrightarrow \mathbb{R}^n$ is determined, according to Theorem 2.3, by a given of a vector field $Y \in \Gamma(TTG)$ for which $0_e \in TG$ is an asymptotically stable equilibrium point, so that the error dynamics (2.17):

$$\begin{aligned}\dot{z} = TR_{Tf(\omega)^{-1}} &\left(S_{z \cdot Tf(\omega)} - TL_z \circ TTf(\Delta_\omega)\right) \\ &+ TR_{Tf(\omega)^{-1}} \circ TL_z \left(\sum_{i=1}^{m} u^i X^{\text{lift}}_{i,Tf(\omega)} - \sum_{i=1}^{\kappa} w^i TTf(\Omega_{i,\omega})\right). \quad (3.1)\end{aligned}$$

with feedback $\alpha(z, \omega)$, writes as $\dot{z} = Y_z$. In this case, if the auxiliary state $\omega(t)$ ultimately remains in a compact subset of $T\mathbb{T}^\kappa$, then one may conclude that the target system state $x(t)$ converges to $Tf(\omega(t))$ as $t \to \infty$. This means, in particular, that there exists $T \in \mathbb{R}_{>0}$ such that the base coordinates $\pi_G(x(t)) \in \mathcal{U}$ for $t > T$, where $\mathcal{U}$ is a neighborhood of the desired configuration $e \in G$ determined by the transverse function f. Hence the target system configuration ultimately approaches the desired configuration e to within a prescribed tolerance determined by $\mathcal{U}$. However, further investigation is required to assess the behavior of the compound closed-loop system (2.18):

$$(\dot{x}, \dot{\omega}) = \left(S_x + \sum_{i=1}^{m} \alpha^i \left(x \cdot Tf(\omega)^{-1}, \omega \right) X_{i,x}^{\text{lift}}, \Delta_\omega + \sum_{i=1}^{\kappa} \alpha^{i+m} \left(x \cdot Tf(\omega)^{-1}, \omega \right) \Omega_{i,\omega} \right).$$

The latter may even fail to be positively complete, i.e., the maximum intervals of existence of some of its solutions may be bounded in $\mathbb{R}$. The main concern in this chapter is the study of the zero dynamics, i.e., the restriction of the compound closed-loop dynamics (2.18) to points (x, ω) such that $z = x \cdot Tf(\omega)^{-1} = 0_e$. This enables us to characterize the long-term behavior of the target system.

Under the assumptions made, the closed-loop zero dynamics is obtained by setting $z = 0_e$, $u^i = \alpha^i(0_e, \omega)$ and $w^j = \alpha^{j+m}(0_e, \omega)$, $i = 1, \dots, m,\ j = 1, \dots, n-m$, in (3.1). This yields, for $\omega \in T\mathbb{T}^\kappa$,

$$\begin{aligned} 0_e = {} & TR_{Tf(\omega)^{-1}} \left(S_{Tf(\omega)} - TL_{0_e} \circ TTf(\Delta_\omega) \right) \\ & + TR_{Tf(\omega)^{-1}} \circ TL_{0_e} \left(\sum_{i=1}^{m} \alpha^i \, (0_e, \omega) \, X_{i,Tf(\omega)}^{\text{lift}} - \sum_{i=1}^{\kappa} \alpha^{i+m} (0_e, \omega) \, TTf(\Omega_{i,\omega}) \right), \end{aligned}$$

or, equivalently, the condition

$$S \circ Tf + \sum_{i=1}^{m} \nu^i (X_i^{\text{lift}} \circ Tf) = TTf \left(\Delta + \sum_{j=1}^{\kappa} \nu^{j+m} \Omega_j \right), \tag{3.2}$$

where the mapping $\nu : T\mathbb{T}^\kappa \longrightarrow \mathbb{R}^n$ is defined by $\nu(\omega) = \alpha(0_e, \omega)$. At first sight, equation (3.2) may suggest that the target and auxiliary systems are Tf-related; strictly speaking, however, the target and auxiliary systems are not Tf-related since the mapping $\widehat{Z} := S \circ Tf + \sum_{i=1}^{m} \nu^i X_i^{\text{lift}} \circ Tf$ is not a vector field on TG. The mapping $\widehat{Z}$ can be regarded as a section of the pullback bundle defined by the underlying set $Tf^*(TTG) = \{(\omega, \alpha) \in T\mathbb{T}^\kappa \times TTG : Tf(\omega) = \pi_G(\alpha)\}$ together with the differentiable structure naturally inherited,

$$\begin{array}{ccccc} TT\mathbb{T}^\kappa & \xrightarrow{\mathrm{id}_{T\mathbb{T}^\kappa} \times TTf} & f^*(TTG) & \xrightarrow{F} & TTG \\ \Big\downarrow{\scriptstyle \pi_{T\mathbb{T}^\kappa}} & & \Big\downarrow{\scriptstyle \rho} & & \Big\downarrow{\scriptstyle \pi_G} \\ T\mathbb{T}^\kappa & \xrightarrow{\mathrm{id}_{T\mathbb{T}^\kappa}} & T\mathbb{T}^\kappa & \xrightarrow{Tf} & TG. \end{array}$$

On the other hand, the compound zero dynamics can be viewed as the result of constraining the trajectories of the target system to be contained in the immersed manifold $Tf(T\mathbb{T}^\kappa)$ by means of the zero-error feedback $\nu(\omega)$. The latter adds terms that may be interpreted as forces imposing a holonomic constraint on the target system, namely, if the initial condition of the target system

$x(t_0)$ lies in $Tf(T\mathbb{T}^\kappa)$, then the state $x(t)$ will remain in $Tf(T\mathbb{T}^\kappa)$ for every $t \geq t_0$ for which the solution of the compound system is defined.

The zero-error feedback $\nu(\omega)$ has a particular structure. It is such that it guarantees that if the target drift vector field equals $S = Z + P$ where Z is a spray and P is a vertical vector field (usually equal to the lift of minus the gradient of the potential energy when dealing with SMS), then the auxiliary zero dynamics is itself defined by $\Sigma + \Pi$, where Σ is a spray and Π is a vertical vector field. Namely, if the target system is an affine-connection control system, then so is the zero-dynamics auxiliary system, as stated in Theorem 3.1.

Prior to formally establishing this result, note that given that the sum in (2.14) is exact:

$$T_{Tf(\omega)}TG^{\text{vert}} = TTf\left((T_\omega T\mathbb{T}^\kappa)^{\text{vert}}\right) \oplus \text{span}_{\mathbb{R}}\left\{X^{\text{lift}}_{1,Tf(\omega)}, \ldots, X^{\text{lift}}_{m,Tf(\omega)}\right\},$$

there exists a projector $\mathcal{P}$ which maps vectors in TTG^{vert} lying over points in $N := Tf(T\mathbb{T}^\kappa)$, to vectors in $TTf((TT\mathbb{T}^\kappa)^{\text{vert}})$. Thus, if $w \in TTG^{\text{vert}}|_N$, there exist $\omega \in T\mathbb{T}^\kappa$, $u \in TTf((T_\omega T\mathbb{T}^\kappa)^{\text{vert}})$ and $v \in \text{span}_{\mathbb{R}}\{X^{\text{lift}}_{1,Tf(\omega)}, \ldots, X^{\text{lift}}_{m,Tf(\omega)}\}$, all uniquely determined, such that $w = u + v$. The projector $\mathcal{P} : TTG^{\text{vert}}|_N \longrightarrow TTf((TT\mathbb{T}^\kappa)^{\text{vert}})$ is defined such that $\mathcal{P}(w) = u$. Correspondingly, if $P \in \Gamma(TTG^{\text{vert}})$ is a vertical vector field, then there is a unique vertical vector field $\Pi \in \Gamma((TT\mathbb{T}^\kappa)^{\text{vert}})$ which satisfies $\mathcal{P} \circ P \circ Tf = TTf \circ \Pi$.

Let $\{\Lambda_1, \ldots, \Lambda_\kappa\} \subset \Gamma(T\mathbb{T}^\kappa)$ be a global frame for $T\mathbb{T}^\kappa$. Given that for every manifold Q and any $v \in TQ$ the mapping $\text{lift}(v, \cdot) : T_{\pi_G(v)}G \longrightarrow T_vTG^{\text{vert}}$ is an isomorphism, (by Lemma 2.1 (iii)), then the set $\{\Lambda_1^{\text{lift}}, \ldots, \Lambda_\kappa^{\text{lift}}\} \subset \Gamma(TT\mathbb{T}^\kappa)$ is a global frame for $(TT\mathbb{T}^\kappa)^{\text{vert}}$. As a consequence of vertical transversality there exists a mapping $a = (a^1, \ldots, a^n) : T\mathbb{T}^\kappa \longrightarrow \mathbb{R}^n$ such that

$$P \circ Tf = \sum_{i=1}^{m} a^i \left(X_i^{\text{lift}} \circ Tf\right) + \sum_{j=1}^{\kappa} TTf \circ \left(a^{j+m}\Lambda_j^{\text{lift}}\right).$$

Then $\mathcal{P} \circ P \circ Tf = \sum_{j=1}^{\kappa} TTf \circ \left(a^{j+m}\Lambda_j^{\text{lift}}\right)$.

Theorem 3.1. *Let $\nu : T\mathbb{T}^\kappa \longrightarrow \mathbb{R}^m$ be such that (3.2) holds. Assume that $S = Z + P$, where $Z \in \Gamma(TTG)$ is a spray and $P \in \Gamma(TTG^{\text{vert}})$. Let $\Pi \in \Gamma((TT\mathbb{T}^\kappa)^{\text{vert}})$ be the vector field that satisfies $\mathcal{P} \circ P \circ Tf = TTf \circ \Pi$. Then there exist a spray $\Sigma \in \Gamma(TT\mathbb{T}^\kappa)$ and a vertical vector field $\Pi \in \Gamma((TT\mathbb{T}^\kappa)^{\text{vert}})$ such that*

$$\Delta + \sum_{j=1}^{\kappa} \nu^{j+m}\Omega_j = \Sigma + \Pi.$$

Proof. Let us first recall a standard procedure, in preparation for the sequel of the proof (cf. e.g. [Warner, 1983, Prop. 1.35]), which locally extends mappings defined along immersed manifolds. Consider manifolds L, M, N and mappings $F : L \longrightarrow M$ and $h : L \longrightarrow N$, and assume that T_pF is injective for some $p \in L$. Let d_L and d_M denote the dimensions of L and M, respectively. Then there exists an open neighborhood $U \subset L$ of p such that $F|_U$ is injective, and there exists a cubic-centered coordinate system (V, φ) for M about $F(p)$ for which $F(U)$ is a slice, that is,

(V,φ) satisfies $F(p) \in V$, $\varphi(F(p)) = 0 \in \mathbb{R}^{d_M}$ and $\varphi(F(U)) = (-\varepsilon,\varepsilon)^{d_L} \times \{0\} \subset \mathbb{R}^{d_L} \times \mathbb{R}^{d_M-d_L}$ for some $\varepsilon > 0$. Now, if $\pi : \mathbb{R}^{d_M} \longrightarrow \mathbb{R}^{d_M}$ denotes the projector $\pi((x^1,\ldots,x^{d_M})) = (x^1,\ldots,x^{d_L},0,\ldots,0)$, then the mapping $\widehat{h} = h \circ (F|_U)^{-1} \circ \varphi^{-1} \circ \pi \circ \varphi : V \longrightarrow N$ is smooth and satisfies

$$\begin{aligned}\widehat{h}(F(U)) &= h \circ (F|_U)^{-1} \circ \varphi^{-1} \circ \pi \circ \varphi(F(U)) \\ &= h \circ (F|_U)^{-1} \circ F(U) \\ &= h(U),\end{aligned}$$

since $\varphi^{-1} \circ \pi \circ \varphi(F(U)) = F(U)$. In other words, the mapping $\widehat{h}$ is explicitly constructed to "extend" h so that the following diagram commute

$$\begin{tikzcd} U \subset L \arrow[rr, "F|_U"] \arrow[dr, "h|_U"'] & & V \subset M \arrow[dl, "\widehat{h}"] \\ & N & \end{tikzcd}$$

Given that $f : \mathbb{T}^\kappa \longrightarrow G$ is transverse for $\boldsymbol{X} = \{X_1,\ldots,X_m\}$ near $e \in G$, $Tf : T\mathbb{T}^\kappa \longrightarrow TG$ is vertically transverse to $\{X_1^{\text{lift}},\ldots,X_m^{\text{lift}}\}$ and satisfies (2.14) for every $\omega \in T\mathbb{T}^\kappa$. Let $\{\Lambda_1,\ldots,\Lambda_\kappa\} \subset \Gamma(T\mathbb{T}^\kappa)$ be a global frame for $T\mathbb{T}^\kappa$. In view of Lemma 2.1, the set $\{\Lambda_1^{\text{lift}},\ldots,\Lambda_\kappa^{\text{lift}}\}$ is a global frame for the vertical subbundle $(TT\mathbb{T}^\kappa)^{\text{vert}}$. Since Ω_j is vertical and smooth, there exist smooth functions $\lambda_j^i : T\mathbb{T}^\kappa \longrightarrow \mathbb{R}$, $i,j = 1,\ldots,\kappa$, such that $\Omega_j = \sum_{i=1}^{\kappa} \lambda_j^i \Lambda_i^{\text{lift}}$, for $j = 1,\ldots,\kappa$.

Let $\omega \in T\mathbb{T}^\kappa$ and set $\theta = \pi_{\mathbb{T}^\kappa}(\omega)$. Applying the aforementioned extension procedure, with $F = f$, $h = Tf \circ \Lambda_j$, $L = \mathbb{T}^\kappa$, $M = G$ and $N = TG$, and using the assumption that $T_\theta f$ is injective, one deduces the existence of open sets $U \subset \mathbb{T}^\kappa$ and $V \subset G$, as well as vector fields $\widehat{\Lambda}_j$ defined on V, such that $\theta \in U$ and

$$\widehat{\Lambda}_j \circ f|_U = Tf \circ \Lambda_j|_U, \quad j = 1,\ldots,\kappa. \tag{3.3}$$

That is, the following diagram commutes.

$$\begin{tikzcd} U \subset \mathbb{T}^\kappa \arrow[rr, "f|_U"] \arrow[dr, "Tf\circ\Lambda_j|_U"'] & & V \subset G \arrow[dl, "\widehat{\Lambda}_j"] \\ & TG & \end{tikzcd}$$

In the terminology of [Warner, 1983, Def. 1.51], $\widehat{\Lambda}_j$ is a local C^∞ extension of Λ_j. Moreover, by continuity of f, U can be taken sufficiently small that, by virtue of the transversality property (2.4), $T_qG = \text{span}_{\mathbb{R}}\{X_{1,q},\ldots,X_{m,q},\widehat{\Lambda}_{1,q},\ldots,\widehat{\Lambda}_{\kappa,q}\}$ for every $q \in V$. It follows from Lemma 2.1 (III) that, together with $X_1^{\text{lift}},\ldots,X_m^{\text{lift}}$, the lifted vector fields $\widehat{\Lambda}_1^{\text{lift}},\ldots,\widehat{\Lambda}_\kappa^{\text{lift}}$, defined on $\widehat{W} = \pi_G^{-1}(V) \subset TG$, constitute a frame for the vertical bundle over $\widehat{W}$:

$$T_vTG^{\text{vert}} = \text{span}_{\mathbb{R}}\left\{X_{1,v}^{\text{lift}},\ldots,X_{m,v}^{\text{lift}},\widehat{\Lambda}_{1,v}^{\text{lift}},\ldots,\widehat{\Lambda}_{\kappa,v}^{\text{lift}}\right\}, \quad \forall\, v \in \widehat{W}. \tag{3.4}$$

Let $\sigma : T\mathbb{T}^\kappa \longrightarrow \mathbb{R}^n$ be the mapping with components given by $\sigma^i = \nu^i$ for $i = 1, \ldots, m$ and $\sigma^{j+m} = \sum_{k=1}^{\kappa} \nu^{k+m} \lambda_k^j$, $j = 1, \ldots, \kappa$. In the extension procedure described above we take $h = \sigma$, $L = T\mathbb{T}^\kappa$, $M = TG$ and $N = \mathbb{R}^n$, and replace F by Tf, the tangent mapping of f which is injective as remarked above, to deduce the existence of open neighborhoods $\widehat{U}$ of ω and $\widehat{V}$ of $Tf(\omega)$, as well as a mapping $\widehat{\sigma} : \widehat{V} \longrightarrow \mathbb{R}^n$ such that

$$\widehat{\sigma}^i \circ Tf|_{\widehat{U}} = \sigma^i|_{\widehat{U}}, \quad i = 1, \ldots, n.$$

That is, the following diagram commutes.

$$\begin{tikzcd}
\widetilde{U} \subset T\mathbb{T}^\kappa \arrow[rr, "Tf|_{\widetilde{U}}"] \arrow[dr, "\sigma^i|_U"'] & & \widetilde{V} \subset TG \arrow[dl, "\widehat{\sigma}^i"] \\
& \mathbb{R}^n &
\end{tikzcd}$$

Again, by continuity of Tf, $\widehat{U}$ can be taken so small that $\widehat{V} \subset \widehat{W}$, so that $\widehat{\Lambda}_1^{\mathrm{lift}}, \ldots, \widehat{\Lambda}_\kappa^{\mathrm{lift}}$ and the functions $\widehat{\sigma}^i$ are defined on $\widehat{V} \subset TG$. Using the ingredients above, in particular the verticality and smoothness of P, along with (3.4), we ascertain the existence of a smooth mapping $\widehat{a} = (\widehat{a}^1, \ldots, \widehat{a}^n) : \widehat{V} \longrightarrow \mathbb{R}^n$ such that

$$P = \sum_{i=1}^{m} \widehat{a}^i X_i^{\mathrm{lift}} + \sum_{j=1}^{\kappa} \widehat{a}^{j+m} \widehat{\Lambda}_j^{\mathrm{lift}}. \tag{3.5}$$

Let $C \in \Gamma(TTG^{\mathrm{vert}})$ denote the Liouville vector field associated with G. From the definition above and the fact that $[C, X_i^{\mathrm{lift}}] = -X_i^{\mathrm{lift}}$ for $i = 1, \ldots, m$, it follows that

$$\begin{aligned}
[C, P] &= \sum_{i=1}^{m} (-\widehat{a}^i + C(\widehat{a}^i)) X_i^{\mathrm{lift}} + \sum_{j=1}^{\kappa} (-\widehat{a}^{j+m} + C(\widehat{a}^{j+m})) \widehat{\Lambda}_j^{\mathrm{lift}} \\
&= -P + \sum_{i=1}^{m} C(\widehat{a}^i) X_i^{\mathrm{lift}} + \sum_{j=1}^{\kappa} C(\widehat{a}^{j+m}) \widehat{\Lambda}_j^{\mathrm{lift}}.
\end{aligned} \tag{3.6}$$

Now, we claim that the vector fields

$$\Sigma = \Delta + \sum_{j=1}^{\kappa} (\sigma^{j+m} - a^{j+m}) \Lambda_j^{\mathrm{lift}} \quad \text{and} \quad \Pi = \sum_{j=1}^{\kappa} a^{j+m} \Lambda_j^{\mathrm{lift}}, \tag{3.7}$$

with $a = \widehat{a} \circ Tf|_{\widehat{U}}$, satisfy the properties in the statement. By definition of $\mathcal{P} \circ P \circ Tf = TTf \circ \Pi$, the proof reduces to showing that Σ is a spray. Since Σ is a second-order vector field, as follows immediately from its definition, it suffices to prove that $[\widehat{C}, \Sigma] = \Sigma$, where $\widehat{C} = C^{\mathbb{T}^\kappa}$ denotes the Liouville vector field associated with $\mathbb{T}^\kappa$. Using the definition of Σ and the fact that Δ is a spray we obtain

$$
\begin{aligned}
[\widehat{C}, \Sigma] &= [\widehat{C}, \Delta] + \sum_{j=1}^{\kappa} \left[\widehat{C}, (\sigma^{j+m} - a^{j+m})\Lambda_j^{\text{lift}}\right] \\
&= \Delta + \sum_{j=1}^{\kappa} (-(\sigma^{j+m} - a^{j+m})\Lambda_j^{\text{lift}} + \widehat{C}(\sigma^{j+m} - a^{j+m})\Lambda_j^{\text{lift}}). \qquad (3.8)
\end{aligned}
$$

As suggested by this equation, in order to prove the claim we shall find an expression for $\widehat{C}(\sigma^{j+m} - a^{j+m})$, the Lie derivative of $(\sigma^{j+m} - a^{j+m})$ in the direction of $\widehat{C}$, $j = 1, \ldots, \kappa$. Note that we have

$$
\begin{aligned}
\left[C, S + \textstyle\sum_{i=1}^{m} \widehat{\sigma}^i X_i^{\text{lift}}\right] \circ Tf|_{\widehat{U}} &= [C, Z + P] \circ Tf|_{\widehat{U}} + \sum_{i=1}^{m} \left[C, \widehat{\sigma}^i X_i^{\text{lift}}\right] \circ Tf|_{\widehat{U}} \\
&= Z \circ Tf|_{\widehat{U}} + [C, P] \circ Tf|_{\widehat{U}} \\
&\quad + \sum_{i=1}^{m} ((-\widehat{\sigma}^i + C(\widehat{\sigma}^i)) X_i^{\text{lift}}) \circ Tf|_{\widehat{U}}, \qquad (3.9)
\end{aligned}
$$

and

$$
\begin{aligned}
TTf \circ \left[\widehat{C}, \Delta + \sum_{j=1}^{\kappa} \sigma^{j+m} \Lambda_j^{\text{lift}}\right] &= TTf \circ \left([\widehat{C}, \Delta] + \sum_{j=1}^{\kappa} \left(\sigma^{j+m}[\widehat{C}, \Lambda_j^{\text{lift}}] + \widehat{C}(\sigma^{j+m})\Lambda_j^{\text{lift}}\right)\right) \\
&= TTf \circ \Delta \\
&\quad + \sum_{j=1}^{\kappa} TTf \circ \left((-\sigma^{j+m} + \widehat{C}(\sigma^{j+m}))\Lambda_j^{\text{lift}}\right). \qquad (3.10)
\end{aligned}
$$

Now, from (3.2) and the definition of $\widehat{\sigma}^i$ it follows that $[\widehat{C}, \Delta + \sum_{j=1}^{\kappa} \sigma^{j+m} \Lambda_j^{\text{lift}}]$ and $[C, S + \sum_{i=1}^{m} \widehat{\sigma}^i X_i^{\text{lift}}]$ are Tf-related, hence the respective members of (3.9) and (3.10) are equal. Equating the right-hand sides of (3.9) and (3.10), and then replacing P and $[C, P]$ by their equivalent expressions as given by (3.5) and (3.6), respectively, we obtain

$$
\begin{aligned}
&\left(Z + \sum_{i=1}^{m} (C(\widehat{a}_i + \widehat{\sigma}_i) - (\widehat{a}_i + \widehat{\sigma}_i))\, X_i^{\text{lift}} + \sum_{j=1}^{\kappa} \left(C(\widehat{a}^{j+m}) - \widehat{a}^{j+m}\right) \widehat{\Lambda}_j^{\text{lift}}\right) \circ Tf|_{\widehat{U}} = \\
&\qquad TTf \circ \Delta + \sum_{j=1}^{\kappa} TTf \circ \left((-\sigma^{j+m} + \widehat{C}(\sigma^{j+m}))\Lambda_j^{\text{lift}}\right). \qquad (3.11)
\end{aligned}
$$

Now, by the equality of (3.2), the substraction of $\left(Z \circ Tf + P \circ Tf + \sum_{i=1}^{m} \widetilde{\sigma}^i (X_i^{\text{lift}} \circ Tf)\right)$ to the left-hand member of (3.11) and the substraction of $TTf\left(\Delta + \sum_{j=1}^{\kappa} \sigma^{j+m} \Omega_j\right)$ to its right-hand member, must yield an equivalent equation. After simplifying we obtain

$$\sum_{i=1}^{m}((C(\widehat{a}^i+\widehat{\sigma}^i)-2(\widehat{a}^i+\widehat{\sigma}^i))X_i^{\mathrm{lift}})\circ Tf|_{\widehat{U}} \\ +\sum_{j=1}^{\kappa}((C(\widehat{a}^{j+m})-2\widehat{a}^{j+m})\widehat{\Lambda}_j^{\mathrm{lift}})\circ Tf|_{\widehat{U}} = \\ TTf\circ\left(\sum_{j=1}^{\kappa}(\widehat{C}(\sigma^{j+m})-2\sigma^{j+m})\Lambda_j^{\mathrm{lift}}\right). \quad (3.12)$$

Using the fact that $\widehat{C}$ and C are Tf-related, along with the definitions of tangent mapping and of $a=\widehat{a}\circ Tf|_{\widehat{U}}$, one has for $j=1,\ldots,\kappa$,

$$\begin{aligned} C(\widehat{a}^{j+m})\circ Tf|_{\widehat{U}} &= (TTf\circ\widehat{C})(\widehat{a}^{j+m}) \\ &= \widehat{C}(\widehat{a}^{j+m}\circ Tf|_{\widehat{U}}) \\ &= \widehat{C}(a^{j+m}). \end{aligned}$$

Moreover, by definition of lift of a vector field and using (3.3) and Lemma 2.1 one obtains

$$\begin{aligned} \widehat{\Lambda}_j^{\mathrm{lift}}\circ Tf|_{\widehat{U}}(\omega) &= \mathrm{lift}(Tf(\omega),\widehat{\Lambda}_{j,f(\theta)}) \\ &= \mathrm{lift}(Tf(\omega),Tf(\Lambda_{j,\theta})) \\ &= TTf(\mathrm{lift}(\omega,\Lambda_{j,\theta})) \\ &= TTf(\Lambda_{j,\omega}^{\mathrm{lift}}), \end{aligned}$$

thus $\widehat{\Lambda}_j^{\mathrm{lift}}\circ Tf|_{\widehat{U}}=TTf\circ\Lambda_j^{\mathrm{lift}}$, $j=1,\ldots,\kappa$. These expressions, along with the fiberwise linearity of TTf, enable us to write (3.12) as

$$\sum_{i=1}^{m}\left((C(\widehat{a}^i+\widehat{\sigma}^i)-2(\widehat{a}^i+\widehat{\sigma}^i))X_i^{\mathrm{lift}}\right)\circ Tf|_{\widehat{U}} \\ +\sum_{j=1}^{\kappa}\left(\widehat{C}(a^{j+m}-\sigma^{j+m})-2(a^{j+m}-\sigma^{j+m})\right)\left(TTf\circ\Lambda_j^{\mathrm{lift}}\right)=0.$$

In view of the vertical transversality condition (2.14), the coefficient of $TTf\circ\Lambda_j^{\mathrm{lift}}$ must be zero, which implies that

$$\widehat{C}(a^{j+m}-\sigma^{j+m})=2(a^{j+m}-\sigma^{j+m}),$$

for $j=1,\ldots,\kappa$. Finally, we substitute these equations in (3.8) to conclude that $[\widehat{C},\Sigma]=\Sigma$, as was to be shown. ■

As the previous theorem establishes, the long-term behavior of the compound system can be determined by studying the solutions of the auxiliary zero dynamics given by $\dot{\omega} = \Sigma_\omega + \Pi_\omega$. The latter inherits the structure of the target system in the sense that it is defined by the spray associated to an affine connection plus a vertical vector field whenever the target system has this same structure. An interesting fact is also the effect of the vertical vector field P on the zero dynamics. The definition (3.7) of Σ and Π reveals that *only the projection $\mathcal{P}(P \circ Tf)$ has an effect on the zero dynamics*, while the effects of the complementary component of P are "absorbed" by the control scheme.

3.2. Existence of a metric for the case of mechanical systems underactuated by one control

A natural issue to address next is to determine if the zero dynamics is positive-complete, i.e., whether each of its solutions can be extended to be defined on an interval $[t_0, \infty) \subset \mathbb{R}$, and if so, whether its solutions remain in a compact neighborhood of the zero-section of $T\mathbb{T}^\kappa$. For the general case there are not conclusive results, however, in the case of target systems underactuated by one control we give a sufficient and necessary condition.

Assume, for simplicity, that $S = Z + P$ satisfies the assumptions of Theorem 3.1 and, in addition, $\mathcal{P}(P \circ Tf) = 0$, so that the zero dynamics writes as $\dot{\omega} = \Sigma_\omega$. By virtue of Theorem 3.1, Σ is a spray, so it determines a unique torsionless affine connection ∇^Σ on $T\mathbb{T}^\kappa$. If ∇^Σ is the Levi-Civita connection of some (pseudo–) Riemannian metric g^Σ on $\mathbb{T}^\kappa$, then one can deduce the completeness of the zero dynamics since every compact (pseudo–) Riemannian manifold is geodesically complete (cf. Kobayashi and Nomizu [1996]). Moreover, since in that case the energy would be constant along the solutions, these would evolve in a relatively compact neighborhood of the zero-section of $T\mathbb{T}^\kappa$, that is, the corresponding velocity coordinates would remain bounded. However, determining the existence of a (pseudo-) Riemannian metric for a given torsionless connection is an untractable problem given the overdetermined nature of the Levi-Civita metric differential equations as stated by Eisenhart and Veblen [1922]. The problem has been addressed formerly, for instance by Schmidt [1973], who shows that, although integrability conditions can be drawn from the equations relating the metric with the corresponding Christoffel symbols, geometric conditions can also be stated in terms of the holonomy group of the connection. Thus, for $\theta \in \mathbb{T}^\kappa$, Schmidt considers $\Phi(\theta)$, the *holonomy group with reference point θ*, whose elements are endomorphisms of $T_\theta\mathbb{T}^\kappa$ obtained via parallel transport along all piecewise-smooth loops on the base having θ as endpoints. More precisely, for every loop $\gamma : [0,1] \longrightarrow \mathbb{T}^\kappa$ satisfying $\gamma(0) = \gamma(1) = \theta$, there is a linear mapping $L_\gamma : T_\theta\mathbb{T}^\kappa \longrightarrow T_\theta\mathbb{T}^\kappa$ in $\Phi(\theta)$ which maps $\omega \in T_\theta\mathbb{T}^\kappa$ to its parallel translation along γ. Equivalently, $L_\gamma(\omega) = u(1)$, where $u : [0,1] \longrightarrow T\mathbb{T}^\kappa$ is the unique curve that satisfies $\pi_{\mathbb{T}^\kappa} \circ u = \gamma$, as well as the initial value problem

$$\nabla^\Sigma_{\dot{\gamma}(t)} u(t) = 0, \quad t \in [0,1], \qquad u(0) = \omega.$$

The group structure on $\Phi(\theta)$ is defined by considering parallel translation along concatenated curves on the base, as well as along curves traversed in "reverse." Moreover, since $\mathbb{T}^\kappa$ is connected,

$\Phi(\theta)$ and $\Phi(\theta')$ are isomorphic for all $\theta, \theta' \in \mathbb{T}^\kappa$, a fact that is easily established. Schmidt proves a general version of the following result.

Proposition 3.1 (*Schmidt [1973]*). *A connection ∇^Σ on $\mathbb{T}^\kappa$ is the Levi-Civita connection of a metric on $\mathbb{T}^\kappa$ with signature (p, q) iff there exists a non-degenerate quadratic form g on $T_\theta\mathbb{T}^\kappa$, with signature (p, q), which is invariant under $\Phi(\theta)$.*

Let us remark that, in the statement of the previous result, the invariance condition on g means that, for every $L \in \Phi(\theta)$ and all $\omega, \omega' \in T_\theta\mathbb{T}^\kappa$,

$$g(\omega, \omega') = g(L(\omega), L(\omega')).$$

For practical purposes, determining whether a given connection ∇^Σ satisfies the assumptions of Proposition 3.1 is a rather involved task, especially due to the facts that the connection ∇^Σ needs not be flat and that $\mathbb{T}^\kappa$ is not simply connected. In the particular case of a system underactuated by one control, that is, $\kappa = n - m = 1$, a simple condition can be stated, as we now show. Let $\mathrm{Conn}(T\mathbb{T})$ denote the set of affine connections on the tangent bundle $T\mathbb{T}$ and, for every nowhere vanishing vector field $s \in \Gamma(T\mathbb{T})$, define a mapping $\kappa_s : \mathrm{Conn}(T\mathbb{T}) \longrightarrow \Omega^1(\mathbb{T})$ by setting $\kappa_s(\nabla) = A$, where A is the unique differential one-form determined by $A \otimes s = \nabla s$.

Proposition 3.2. I. *There exists a unique mapping κ of Conn$(T\mathbb{T})$ into $H^1(\mathbb{T})$, the first de Rham cohomology group of $\mathbb{T}$, such that for every global frame $s \in \Gamma(T\mathbb{T})$ the following diagram commutes*

$$\begin{tikzcd} Conn(T\mathbb{T}) \arrow[r, "\kappa_s"] \arrow[dr, "\kappa"'] & \Omega^1(\mathbb{T}) \arrow[d, "\pi"] \\ & H^1(\mathbb{T}) \end{tikzcd}$$

II. *An affine, torsionless connection ∇ on $\mathbb{T}$ is the Levi-Civita connection of a pseudo-Riemannian metric iff $\kappa(\nabla) = 0$.*

Proof. I. Given that $\mathbb{T}$ is one-dimensional, $\Omega^1(\mathbb{T})$ is contained in the kernel of d, so we have only to prove that if $s, s' \in \Gamma(T\mathbb{T})$ are two nowhere vanishing vector fields and $\kappa_s(\nabla) = A$ and $\kappa_{s'}(\nabla) = A'$, then $A - A' = df$ for some $f \in C^\infty(\mathbb{T})$. Since s and s' are nowhere vanishing vector fields, there exists a nowhere vanishing function $h \in C^\infty(\mathbb{T})$ such that $s = hs'$. Then, using the distributivity properties of tensor products of sections as well as the Leibniz property for connections we get

$$\begin{aligned} A \otimes s &= \nabla(hs') \\ &= dh \otimes s' + h\nabla s' \\ &= dh \otimes s' + hA' \otimes s' \\ &= \big(dh/h + A'\big) \otimes s, \end{aligned}$$

whence $A - A' = dh/h$. But h is nowhere null, hence $|h|$ and $\ln \circ |h|$ are smooth and satisfy $dh/h = d(\ln \circ |h|)$, which establishes the claim.

II. Let $\nabla \in \mathrm{Conn}(T\mathbb{T})$ be torsionless, let $s \in \Gamma(T\mathbb{T})$ be a nowhere vanishing vector field, and pick an orientation for $\mathbb{T}$ and a point $\theta \in \mathbb{T}$. Since A is a one-form on an oriented one-dimensional manifold, it makes sense to define the integral $I_A := \int_{\mathbb{T}} A$. Moreover, given a loop $\gamma : [a,b] \longrightarrow \mathbb{T}$ such that $\gamma(a) = \gamma(b)$ and a vector $\omega = ks(\theta)$ in $T_\theta\mathbb{T}$, its parallel transport around γ is given by $L_\gamma(\omega) = \exp(I_A)\, ks(\theta)$. Since $\mathbb{T}$ is one-dimensional, a simple computation shows that the curvature tensor R is zero, so ∇ is a flat connection. Therefore, the holonomy around a loop $\gamma : [a,b] \longrightarrow \mathbb{T}$ depends only on its homotopy class $[\gamma]$ in $\pi(\mathbb{T},\theta)$, the fundamental group of $\mathbb{T}$ based at θ (cf. [Kobayashi and Nomizu, 1996, Chap. II-9]). But $\pi(\mathbb{T},\theta) \simeq \mathbb{Z}$, consequently for every such loop γ there exists $n \in \mathbb{Z}$ such that $L_\gamma(\omega) = \exp(nI_A)\, ks(\theta)$. Given a quadratic non-degenerate form g on $T_\theta\mathbb{T}$, there exists $G \neq 0$ such that $g(s(\theta), s(\theta)) = G$. Thus g is preserved by $\Phi(\theta)$ if and only if, for every $k, k' \in \mathbb{R}$ and every $n \in \mathbb{Z}$, one has

$$\begin{aligned} Gkk' &= g(ks(\theta), k's(\theta)) \\ &= g\left(k \exp(nI_A)\, s(\theta), k' \exp(nI_A)\, s(\theta)\right) \\ &= G\, (\exp(nI_A))^2\, kk', \end{aligned}$$

that is, if and only if $|\exp(nI_A)| = 1$ for all $n \in \mathbb{Z}$. But this is equivalent to $I_A = \int_{\mathbb{T}} A = 0$ and, in turn, since $dA = 0$, to the existence of a function $f \in C^\infty(\mathbb{T})$ such that $A := \kappa_s(\nabla) = df$, that is $\kappa(\nabla) = 0$. On applying Proposition 3.1 one obtains the required result. ∎

3.3. Long-term behavior of the compound system

In this section we study the analysis of the closed-loop system under the assumption that the zero dynamics is determined by a spray that admits a (pseudo–) Riemannian metric. We shall show that if the error dynamics has 0_e as a locally exponentially stable point (in a sense defined below), and the zero dynamics admits a kinetic energy function, then the set $\{0_e\} \times Z(T\mathbb{T}^\kappa)$ is uniformly stable for the compound system, where $Z(T\mathbb{T}^\kappa)$ denotes the zero-section of $T\mathbb{T}^\kappa$. Roughly speaking, this implies that if the initial value of the error signal is sufficiently close to 0_e, and the auxiliary system's initial velocities are sufficiently small, then the solution of the controlled system is defined for all $t \geq t_0$, its velocities remain small and the error decays exponentially.

Before dwelling on this result, precisely stated in Theorem 3.4, we establish some results in preparation for its proof. Theorem 3.3 establishes positive completeness and uniform stability for a certain class of systems $\dot{x} = f(x,t)$, where $f(x,t)$ can be decomposed in two factors, one of which is exponentially decreasing with respect to time. Lemma 3.1 establishes that, given a covering manifold $\widetilde{M}$ for M, $T\widetilde{M}$ is a covering manifold for TM, and finally, Lemma 3.2 establishes that there exists a covering isomorphism $\widetilde{\mathbb{T}^\kappa} \longrightarrow \mathbb{R}^\kappa$ where $\widetilde{\mathbb{T}^\kappa}$ is a covering for $\mathbb{T}^\kappa$ and $\mathbb{R}^\kappa$ is a covering for $\mathbb{R}^\kappa/\mathbb{Z}^\kappa$.

The following is a basic theorem (cf. Coddington and Levison [1984]), concerning the existence of solutions for differential equations defined by continuous (not necessarily Lipschitz) functions, which is to be used in the proof for Theorem 3.3.

Theorem 3.2 (Cauchy-Peano Existence Theorem)**.** *Consider the initial value problem*

$$\dot{x} = f(x,t), \qquad (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}, \tag{3.13}$$

where $f : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$. If f is continuous on $R = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : 0 \le t - t_0 \le a, ||x - x_0|| \le b\}$ for $a, b \in \mathbb{R}_{>0}$, then (3.13) admits a differentiable solution $\phi_{(x_0,t_0)} : t \mapsto \phi_{(x_0,t_0)}(t)$ defined in $[t_0, t_0 + \alpha]$, where $\alpha = \min\left(a, \frac{b}{M}\right)$ and $M = \max\{||f(x,t)|| : (x,t) \in R\}$. Moreover $||\phi_{(x_0,t_0)}(t) - x_0|| \le M(t - t_0)$ for $(x,t) \in R$.

Theorem 3.3. *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous mapping, and let K be strictly positive. Then $(0,0)$ is uniformly stable under the dynamics defined on $\mathbb{R}^{n+1}$ by*

$$\dot{x} = z\,F(x) \tag{3.14}$$
$$\dot{z} = -K\,z \tag{3.15}$$

Proof. It is straightforward to verify that $\{(x_0, 0) \in \mathbb{R}^n \times \mathbb{R}\}$ is a set of equilibria for (3.14)-(3.15) and, in particular, $(0,0) \in \mathbb{R}^n \times \mathbb{R}$ belongs to this set. In order to establish uniform stability of this equilibrium point we shall show that, for any neighborhood $V \subset \mathbb{R}^n \times \mathbb{R}$ of $(0,0)$, there exists a neighborhood $U \subset \mathbb{R}^n \times \mathbb{R}$ of $(0,0)$ such that for any $t_0 \in \mathbb{R}$, if $(x(t_0), z(t_0)) = (x_0, z_0) \in U$ then any solution $\psi_{(x_0,z_0)} : t \mapsto \psi_{(x_0,z_0)}(t)$ of (3.14)–(3.15) is defined and satisfies $\psi_{(x_0,z_0)}(t) \in V$ for every $t \ge t_0$.

In the sequel given $a, b \in \mathbb{R}$ with $a < b$, and an integer $q > 0$, we let $[a,b]^q$ and $(a,b)^q$ denote the q-fold Cartesian products of the interval $[a,b]$ and (a,b) respectively. Let $t_0 \in \mathbb{R}$ be given, and let $V \subset \mathbb{R}^n \times \mathbb{R}$ be a neighborhood of $(0,0)$. By definition of V, there exists $\varepsilon > 0$ such that $[-\varepsilon, \varepsilon]^{(n+1)} \subset V$. It is straightforward to verify that $\mathbb{R}^n \times \{0\}$ is a continuum of equilibria for (3.14)-(3.15), and each of the corresponding constant solutions is uniquely determined by the initial condition. Hence, if $x_0 \in (-\varepsilon, \varepsilon)^n$, the trajectory of (3.14)-(3.15) issued from $(x_0, 0)$ at time t_0 belongs to V for all $t \ge t_0$. Let us now assume that $z_0 \neq 0$, so that the solution of (3.15), $z(t) = z_0 e^{-K(t-t_0)}$, is defined for every $t \ge t_0$. In such a case, (3.14) can be written as

$$\dot{x} = z_0 e^{-K(t-t_0)} F(x). \tag{3.16}$$

Let $F_{\max} = \max\{||F(x)|| : x \in [-\varepsilon, \varepsilon]^n\}$. Clearly, $F_{\max}$ exists since F is continuous and $[-\varepsilon, \varepsilon]^n$ is compact. If $F_{\max} = 0$, then the restriction of F to $[-\varepsilon, \varepsilon]^n$, which determines the derivative of the component $x(t)$ of the solution, is identically zero. In this case, if $x_0 \in (-\varepsilon, \varepsilon)^n$, the unique trajectory issued from (x_0, z_0) writes as $t \mapsto (x_0, z_0 e^{-K(t-t_0)})$, so it suffices to take $z_0 \in (-\varepsilon, \varepsilon)$ to ensure that such solution remains in V for every $t \ge t_0$. On the other hand, assume that $F_{\max} \neq 0$ and set

$$\delta = \frac{(2^K - 1)\,\varepsilon}{2^K} \min\left\{1, \frac{1}{2^K\,F_{\max}}\right\}. \tag{3.17}$$

Clearly, δ is strictly positive since $K > 0$; moreover $\delta < \varepsilon$ for $K > 0$. Choose any $(x_0, z_0) \in (-\delta, \delta)^{(n+1)}$. Then $|z(t)| < \delta e^{-K(t-t_0)} \le \delta$ for every $t \ge t_0$ and hence, $z(t) \in (-\varepsilon, \varepsilon)$ for every $t_0 \in \mathbb{R}$ and every $t \ge t_0$. Consider the sequence $(b_m)_{m=0}^{\infty}$, where

$$b_m = \frac{(2^K - 1)\,\varepsilon}{2^{K(m+1)}},$$

as well as the series $B_m = \sum_{i=0}^{m} b_i$. One readily checks that $\sum_{m=0}^{\infty} b_m = \lim_{m\to\infty} B_m = \varepsilon$. For each $m \geq 1$, define the closed set $R_m = \{(x,t) : x \in [-\varepsilon,\varepsilon]^n, t \geq t_0 + m - 1\}$. Since the function $(x,t) \mapsto z_0 e^{-K(t-t_0)} F(x)$ is continuous on R_m and strictly decreasing with respect to its second argument, $M_m = \max\{||z_0 e^{-K(t-t_0)} F(x)|| : (x,t) \in R_m\}$ exists and satisfies $M_m = |z_0| F_{\max}\, e^{-K(m-1)}$.

Given $p \in [-B_{m-1}, B_{m-1}]^n$, from the Cauchy-Peano existence Theorem 3.2, one concludes that there exists a solution $\psi_{(t_0+m-1,p)} : t \mapsto \psi_{(t_0+m-1,p)}(t)$ for (3.16), defined on the interval $[t_0 + m - 1, t_0 + m]$, provided that $\frac{b_m}{M_m} \geq 1$, that is

$$\frac{b_m}{M_m} = \frac{(2^K - 1)\,\varepsilon}{2^{K(m+1)}\,|z_0| F_{\max}\, e^{-K(m-1)}} \geq 1.$$

But this inequality is satisfied for every $z_0 \in (-\delta,\delta)$ and every $n \geq 1$, for if

$$|z_0| < \delta \leq \frac{(2^K - 1)\varepsilon}{4^K F_{\max}},$$

and $n \geq 1$, then

$$\frac{b_m}{M_m} = \frac{(2^K - 1)\,\varepsilon}{2^{K(m+1)}\,|z_0| F_{\max}\, e^{-K(m-1)}} > \frac{4^K}{2^{K(m+1)} e^{-K(m-1)}} \geq \frac{4^K}{2^{2K}} = 1.$$

Therefore, for every m and every $p \in [B_{m-1}, B_m]^n$, there exists a solution $x_m : t \mapsto \psi_{(t_0+m-1,p)}(t)$ for (3.16) defined in $[t_0 + m - 1, t_0 + m]$. In addition this solution satisfies $||x_m(t) - p|| \leq M_m(t - (t_0 + m - 1))$ for all $t \in [t_0 + m - 1, t_0 + m]$. Hence, taking $t = t_0 + m$ one gets $||x_m(t_0 + m) - p|| \leq M_m$. Given that $b_m \geq M_m$, it follows that $||x_m(t_0 + m) - p|| \leq b_m$. But $||p|| \leq B_{m-1}$, hence $||x_m(t_0 + m) - p|| \leq B_m$. We have thus shown that, for every $m \leq 1$, if $\psi_{(t_0,x)}$ is defined on $[t_0, t_0 + m - 1]$ and satisfies $||\psi_{(t_0,x)}(t_0 + m - 1)|| \leq B_{n-1}$, then $\psi_{(t_0,x)}$ can be extended to be defined on $[t_0, t_0 + m]$ and satisfies $||\psi_{(t_0,x)}(t_0 + m)|| \leq B_n$. Since $||x_0|| < \delta = B_0$, by induction on m we deduce that the solution $\psi_{(x_0,t_0)}$ is defined on $[t_0, \infty)$ and satisfies $||x(t)|| \leq \sup\{B_m : m \geq 1\} = \varepsilon$ for all $t \geq t_0$. Therefore, for every $t_0 \in \mathbb{R}$ and every neighborhood V of $(0,0)$, if (x_0, z_0) is in $(-\delta,\delta)^{(n+1)}$ (with δ given in (3.17)), the trajectories of (3.14)–(3.15) exist and satisfy $(x(t), z(t)) \in V$ for every $t \geq t_0$, as was to be shown. ∎

Theorem 3.3 essentially states that if the initial conditions x_0, z_0 for system (3.14)–(3.15) are sufficiently small, so that $x(t)$ does not grow "too quickly," then the exponentially decaying factor in the derivative of x forces the solution $x(t)$ to remain bounded (and possibly even converge). However, if the initial conditions are not small enough, $x(t)$ may grow *unbounded* despite the exponentially vanishing nature of $|z|$, and even do so *in finite time*. It is worth pointing out that the mapping F is not assumed to be locally Lipschitz–the solution to (3.13) may not be uniquely defined–or to have zero as an asymptotically stable equilibrium. The following corollary is a straightforward result from the previous theorem that is directly applicable in the proof of Theorem 3.4.

Corollary 3.1. *If $F : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and $K > 0$, then $(0,0) \in \mathbb{R}^2$ is uniformly stable under the dynamics defined by $\{\dot{x} = |z| F(x);\ \ \dot{z} = -Kz\}$.*

In order to illustrate the result in the previous corollary, consider, for example, the autonomous system

$$\begin{aligned} \dot{x} &= |z|\, x^2 \\ \dot{z} &= -z. \end{aligned}$$

Indeed, the solution with initial condition (x_0, z_0) at $t = 0$, given by

$$x(t) = \frac{x_0}{1 - (1 - e^{-t})\, x_0 z_0}, \quad z(t) = e^{-t} z_0,$$

is not defined for any t such that the denominator $1 - (1 - e^{-t})x_0 z_0$ vanishes, namely for $t_1 = \ln\left(\frac{x_0 z_0}{x_0 z_0 - 1}\right)$. If $x_0 z_0 < 1$ then either $t_1 < 0$ or $\ln\left(\frac{x_0 z_0}{x_0 z_0 - 1}\right)$ is undefined, thus the solution $x(t)$ exists for $[0, \infty)$ and $\lim_{t\to\infty} x(t) = \frac{x_0}{1 - x_0 z_0}$, so the solution converges and is therefore bounded. If $x_0 z_0 = 1$, then $x(t) = x_0 e^t$, so $\lim_{t\to\infty} |x(t)| \to \infty$, that is, the solution is defined on $[0, \infty)$ but it grows unbounded. If $x_0 z_0 > 1$, then $t_1 > 0$, so the solution is defined on $[0, t_1)$ and leaves any compact interval as $t \to t_1$.

Lemma 3.1. *Let M be a manifold and let $\widetilde{M}$ be a smooth covering space of M with smooth covering map $p : \widetilde{M} \longrightarrow M$. Then $T\widetilde{M}$ is a smooth covering space of TM with smooth covering map $Tp : T\widetilde{M} \longrightarrow TM$.*

Proof. Let $\pi : TM \longrightarrow M$ and $\widetilde{\pi} : T\widetilde{M} \longrightarrow \widetilde{M}$ denote the tangent bundle projections. We shall show that $T\widetilde{M}$ is isomorphic to the pullback covering space $\pi^*(\widetilde{M})$, which is uniquely defined up to isomorphism of covering spaces (cf. Godbillon [1971] or Spanier [1989] for basic definitions and properties of covering spaces). As a set, $\pi^*(\widetilde{M}) = \{(v, q) \in TM \times \widetilde{M} : v \in T_{p(q)}M\}$, and the covering map is $p_1 : (v, q) \mapsto v$. Now, since $T_q p : T_q\widetilde{M} \longrightarrow T_{p(q)}M$ is an isomorphism for every $q \in \widetilde{M}$, one has a map $\Phi : \pi^*(\widetilde{M}) \longrightarrow T\widetilde{M}$ given by $\Phi(v, q) = (T_q p)^{-1}(v)$ which is smooth since p is a local diffeomorphism. In addition, Φ is injective, for if $(T_q p)^{-1}(v) = (T_{q'} p)^{-1}(v')$ for some $(v, q), (v', q') \in \pi^*(\widetilde{M})$, then $q = q'$ since $(T_q p)^{-1}(v)$ belongs to both $T_q\widetilde{M}$ and $T_{q'}\widetilde{M}$. Hence $v = v'$. Also, Φ is surjective, for if $w \in T\widetilde{M}$, then there exists $q \in \widetilde{M}$ such that $w \in T_q\widetilde{M}$, and hence $\Phi(T_q p(w), q) = w$. Thus Φ admits an inverse $\Phi^{-1} : w \mapsto (T_{\widetilde{\pi}(w)} p(w), \widetilde{\pi}(w))$ which is clearly smooth, so Φ is a diffeomorphism. Moreover, $Tp \circ \Phi(v, q) = T_q p((T_q p)^{-1}(v)) = v = \mathrm{id}_{TM} \circ p_1(v, q)$, so the following diagram commutes:

$$\begin{tikzcd} \pi^*(\widetilde{M}) \arrow[r, "\Phi"] \arrow[d, "p_1"'] & T\widetilde{M} \arrow[d, "Tp"] \\ TM \arrow[r, "\mathrm{id}_{TM}"] & TM \end{tikzcd}$$

Let $v \in TM$. By definition of covering space and the fact that $\pi^*(\widetilde{M})$ is one for TM, there exists an open neighborhood $V \subset TM$ of v, a discrete manifold F (i.e., $\dim(F) = 0$) and a diffeomorphism $\Psi : p_1^{-1}(V) \longrightarrow V \times F$. We claim that $\Phi(p_1^{-1}(V)) = (Tp)^{-1}(V)$. Indeed, if

$w \in \Phi(p_1^{-1}(V))$ then, since Φ is a diffeomorphism, there exists $a \in p_1^{-1}(V)$ such that $w = \Phi(a)$, so $Tp(w) = Tp(\Phi(a)) = p_1(a) \in V$, hence $w \in (Tp)^{-1}(V)$. To prove the opposite inclusion, assume that $w \in (Tp)^{-1}(V)$. Then $p_1(\Phi^{-1}(w)) = Tp(w) \in V$, so $w \in \Phi(p_1^{-1}(V))$. Therefore the claim is true. Now, $\Psi \circ \Phi^{-1}((Tp)^{-1}(V)) = \Psi(p_1^{-1}(V)) = V \times F$, which shows that $T\widetilde{M}$ is a covering space for TM. (The fact that $p_1 = Tp \circ \Phi$, with Φ a diffeomorphism, implies that $\Phi : \pi^*(\widetilde{M}) \longrightarrow T\widetilde{M}$ is an *isomorphism of covering spaces* over the identity id_{TM}.) ■

Lemma 3.2. *Consider universal coverings $p : \widetilde{\mathbb{T}}^\kappa \longrightarrow \mathbb{T}^\kappa$ and $\pi : \mathbb{R}^\kappa \longrightarrow \mathbb{R}^\kappa/\mathbb{Z}^\kappa$, and let α be a covering isomorphism over α' as in the commutative diagram*

$$\begin{tikzcd}
\widetilde{\mathbb{T}}^\kappa \arrow[r, "\alpha"] \arrow[d, "p"'] & \mathbb{R}^\kappa \arrow[d, "\pi"] \\
\mathbb{T}^\kappa \arrow[r, "\alpha'"] & \mathbb{R}^\kappa/\mathbb{Z}^\kappa
\end{tikzcd} \tag{3.18}$$

Given the covering $P = Tp : T\widetilde{\mathbb{T}}^\kappa \longrightarrow T\mathbb{T}^\kappa$, endow $T\widetilde{\mathbb{T}}^\kappa$ with the global chart $(T\widetilde{\mathbb{T}}^\kappa, \psi)$ naturally induced by α. Then:

I. *For every neighborhood $U \subset T\mathbb{T}^\kappa$ of the zero-section $Z(T\mathbb{T}^\kappa)$, there exists a compact neighborhood $V \subset \mathbb{R}^\kappa$ of 0 such that $P \circ \psi^{-1}(\mathbb{R}^\kappa \times V)$ is a neighborhood of $Z(T\mathbb{T}^\kappa)$ contained in U.*

II. *The push-forward $P_*\partial/\partial\psi^i$ is well defined for $i = 1, \dots, 2\kappa$.*

III. *Given $f \in C^\infty(T\mathbb{T}^\kappa)$, a compact set $V \subset \mathbb{R}^\kappa$, and an iterated differential operator of the form $D = \partial/\partial\psi^{i_1} \cdots \partial/\partial\psi^{i_k}$, $i_1, \dots, i_k = 1, \dots, 2\kappa$, $k \geq 0$, the function $D(P^* f) \circ \psi^{-1}$ attains its maximum at some point in $\mathbb{R}^\kappa \times V$.*

Proof. Given $k \in \mathbb{Z}^\kappa$, the mappings $\rho_k : x \mapsto x + k$ and $\widetilde{\rho}_k = \alpha^{-1} \circ \rho_k \circ \alpha$ define smooth, right actions of $\mathbb{Z}^\kappa$ on $\mathbb{R}^\kappa$ and on $\widetilde{\mathbb{T}}^\kappa$, respectively. Obviously, $p \circ \widetilde{\rho}_k = p$ and hence $Tp \circ T\widetilde{\rho}_k = Tp$. Note that if $q, \overline{q} \in T\widetilde{\mathbb{T}}^\kappa$ satisfy $Tp(q) = Tp(\overline{q})$, then there exist $s \in \widetilde{\mathbb{T}}^\kappa$ and $k \in \mathbb{Z}^\kappa$ such that $q \in T_s\widetilde{\mathbb{T}}^\kappa$ and $\overline{q} \in T_{\widetilde{\rho}_k(s)}\widetilde{\mathbb{T}}^\kappa$. Hence $Tp(T\widetilde{\rho}_k(q) - \overline{q}) = Tp \circ T\widetilde{\rho}_k(q) - Tp(\overline{q}) = 0$ and, since p is a local diffeomorphism, $\overline{q} = T\widetilde{\rho}_k(q)$.

I. If U is a neighborhood of $Z(T\mathbb{T}^\kappa)$, then $\psi \circ P^{-1}(U) \subset \mathbb{R}^\kappa \times \mathbb{R}^\kappa$ is a neighborhood of $\mathbb{R}^\kappa \times \{0\}$. Thus for every $x \in [0,1]^\kappa \subset \mathbb{R}^\kappa$ there is $\delta_x > 0$ such that $V_x = B_{\delta_x}(x) \times B_{\delta_x}(0) \subset \mathbb{R}^\kappa \times \mathbb{R}^\kappa$ is contained in $\psi \circ P^{-1}(U)$ (where $B_{\delta_x}(y)$ is the open ball of radius δ_x centered on y). The collection $(V_x)_{x \in [0,1]^\kappa}$ is an open cover of the compact set $C = [0,1]^\kappa \times \{0\}$, so there exists a finite set $F \subset [0,1]^\kappa$ such that $(V_x)_{x \in F}$ still covers C. But then, setting $\delta = \min\{\delta_x : x \in F\}$, $\mathbb{R}^\kappa \times B_\delta(0)$ is a neighborhood of C contained in $\psi \circ P^{-1}(U)$. Since $\psi^{-1}([0,1]^\kappa \times \{0\})$ is projected onto $Z(T\mathbb{T}^\kappa)$ by P, the set $V = B_\delta(0)$ satisfies the required condition.

II. If $X \in \Gamma(TT\widetilde{\mathbb{T}}^\kappa)$ is $T\widetilde{\rho}_k$-related to itself, i.e., $TT\widetilde{\rho}_k \circ X = X \circ T\widetilde{\rho}_k$, then the push-forward $P_*X \in \Gamma(TT\mathbb{T}^\kappa)$ is well defined by the relation $(P_*X)_{P(q)} = T_qP(X_q)$. To see this, note that $TP(X_{\overline{q}}) = TTp(X_{T\widetilde{\rho}_k(q)}) = TTp(TT\widetilde{\rho}_k(X_q)) = T(Tp \circ T\widetilde{\rho}_k)(X_q) = TTp(X_q) = TP(X_q)$ which, along with the fact that P is surjective, implies that P_*X is indeed defined. To show that this is also the case for $P_*\partial/\partial\psi^i$, it suffices to show that $\partial/\partial\psi^i$ is $T\widetilde{\rho}_k$-related to itself. To this purpose, let $f \in C^\infty(T\widetilde{\mathbb{T}}^\kappa)$ and define representatives $F = f \circ \psi^{-1}$ and $G = \psi \circ T\widetilde{\rho}_k \circ \psi^{-1}$ of f and $T\widetilde{\rho}_k$, respectively. Then, for $q \in T\widetilde{\mathbb{T}}^\kappa$ and $i = 1,\ldots,2\kappa$, $\partial/\partial\psi^i|_q(f \circ T\widetilde{\rho}_k) = \partial/\partial r^i|_{\psi(q)}(f \circ T\widetilde{\rho}_k \circ \psi^{-1}) = \partial/\partial r^i|_{\psi(q)}(F \circ G) = \sum_j \partial F/\partial r^j(\psi \circ T\widetilde{\rho}_k(q)) \cdot \partial G^j/\partial r^i(\psi(q)) = \partial F/\partial r^i(\psi \circ T\widetilde{\rho}_k(q)) = \partial/\partial\psi^i|_{T\widetilde{\rho}_k(q)}(f)$, that is, $TT\widetilde{\rho}_k \circ \partial/\partial\psi^i = \partial/\partial\psi^i \circ T\widetilde{\rho}_k$. Therefore $P_*\partial/\partial\psi^i$ is defined.

III. To establish this claim it is sufficient to prove the following two facts: **(a)** If $f \in C^\infty(T\mathbb{T}^\kappa)$ and $i = 1,\ldots,2\kappa$, then $\partial/\partial\psi^i(P^*f) = P_*\partial/\partial\psi^i(f) \circ P$; and **(b)** If $V \subset \mathbb{R}^\kappa$ is compact, then $P \circ \psi^{-1}(\mathbb{R}^\kappa \times V)$ is compact in $T\mathbb{T}^\kappa$. Indeed, provided these two facts hold, any smooth function of the form $\partial/\partial\psi^i(P^*f) \circ \psi^{-1}$ factors through a smooth function on $T\mathbb{T}^\kappa$, which attains its maximum on the compact set $P \circ \psi^{-1}(\mathbb{R}^\kappa \times V)$, and an obvious induction argument then completes the proof. **(a)** If $q \in T\widetilde{\mathbb{T}}^\kappa$, then $\partial/\partial\psi^i|_q(P^*f) = \partial/\partial\psi^i|_q(f \circ P) = TP(\partial/\partial\psi^i|_q)(f) = (P_*\partial/\partial\psi^i)_{P(q)}(f)$, that is, $\partial/\partial\psi^i(P^*f) = P_*\partial/\partial\psi^i(f) \circ P$, as claimed. **(b)** Let $(U_\lambda)_{\lambda\in\Lambda}$ be an open cover of $P \circ \psi^{-1}(\mathbb{R}^\kappa \times V)$. Then $(\psi \circ P^{-1}(U_\lambda))_{\lambda\in\Lambda}$ is an open cover of $\mathbb{R}^\kappa \times V$ and, since $[0,1]^\kappa \times V$ is a compact subset of the latter, there is a finite set $J \subset \Lambda$ such that $(\psi \circ P^{-1}(U_j))_{j\in J}$ covers it. But the images by $P \circ \psi^{-1}$ of $[0,1]^\kappa \times V$ and $\mathbb{R}^\kappa \times V$ coincide, hence $(U_j)_{j\in J}$ is a finite subcover of $P \circ \psi^{-1}(\mathbb{R}^\kappa \times V)$, so the latter is compact. ■

Theorem 3.4. *Let $Y \in \Gamma(TTG)$ be a vector field that admits $0_e \in TG$ as a locally exponentially stable equilibrium and assume that a feedback law is applied to the controlled system (2.1) so that the combined error and auxiliary dynamics writes as*

$$(\dot z, \dot\omega) = \left(Y_z,\ \Delta_\omega + \sum_{i=1}^{n-m} \alpha^{i+m}(z,\omega)\,\Omega_{i,\omega}\right). \tag{3.19}$$

Assume, furthermore, that the auxiliary zero dynamics is given by $\dot\omega = \Sigma_\omega = \Delta_\omega + \sum_{i=1}^{n-m}\alpha^{i+m}(0_e,\omega)\Omega_{i,\omega}$, with $\Sigma \in \Gamma(TT\mathbb{T}^{n-m})$ a spray, and that there exists a positive-definite metric tensor $\mathcal{G}$ on $\mathbb{T}^{n-m}$ such that the function $K : TG \times T\mathbb{T}^{n-m} \longrightarrow \mathbb{R}$ defined by $K(z,\omega) = \frac{1}{2}\mathcal{G}(\omega,\omega)$ is constant along the trajectories of $\dot\omega = \Sigma_\omega$. Then the set $\{0_e\} \times Z(T\mathbb{T}^{n-m})$, where $Z(T\mathbb{T}^{n-m})$ denotes the zero-section of $T\mathbb{T}^{n-m}$, is uniformly stable under the dynamics defined by (3.19).

Under the assumptions of Theorem 3.4, if the error decreases "exponentially fast" and the zero dynamics is conservative, thus the conclusion appears to be intuitively clear. However, the proof of the result is slightly involved due to two facts. First, the stability notion involved pertains to a set which cannot be covered by a single coordinate chart, so the analysis is carried out, instead, by "lifting" the system to an appropriate covering manifold. Second, the limiting system that determines the asymptotic properties of the trajectories—the zero dynamics—does not admit an exponentially stable equilibrium, thus ruling out the application of many of the well-known theorems regarding stability in the presence of disturbances. To palliate this difficulty, we reduce the problem to proving stability of a point for a system that satisfies the assumption of Theorem 3.3.

Proof. Let $\kappa = n - m$, $M = TG \times T\mathbb{T}^\kappa$, and define $A \in \Gamma(TM)$ to be the vector field whose value at $(z,\omega) \in M$ is given by the right-hand side of (3.19), with the usual identification $T(TG \times$

$T\mathbb{T}^\kappa) \simeq TTG \times TT\mathbb{T}^\kappa$. (In the sequel of the proof we shall appeal to similar identifications without explicit mention.) The set $\{0_e\} \times Z(T\mathbb{T}^\kappa)$ is invariant under A; indeed, if $\theta \in \mathbb{T}^\kappa$, then any constant function of the form $t \mapsto (0_e, 0_\theta)$ is an equilibrium solution of (3.19), for $Y_{0_e} = 0$, by the assumption that 0_e is an equilibrium of Y, and $\Sigma_{0_\theta} = 0$, since Σ is assumed to be a spray. We shall show that, for every neighborhood V of $\{0_e\} \times Z(T\mathbb{T}^\kappa)$, there exists a neighborhood U of the same set such that (3.19) is U-positive-complete and, for every $x_0 \in U$, one has $\phi(t, 0, x_0) \in V$ for all $t \in [0, +\infty)$. Once that claim is established, the result will follow from the fact that the dynamics (3.19) is autonomous (i.e., it does not depend explicitly on time, so U can be taken to be independent of t_0).

Let $\widehat{Y}, \widehat{\Sigma} \in \Gamma(TM)$ be the vector fields given by

$$\widehat{Y}_{(z,\omega)} = (Y, 0_\omega)\,, \qquad \widehat{\Sigma}_{(z,\omega)} = (0_z, \Sigma_\omega)\,,$$

where $0_z \in T_zTG$ and $0_\omega \in T_\omega T\mathbb{T}^\kappa$ denote the zero vectors in their respective tangent spaces. Similarly, define a vector field $\xi \in \Gamma(TM)$ by setting

$$\xi_{(z,\omega)} = \left(0_z,\ \sum_{i=1}^{\kappa} (\alpha^{i+m}(z,\omega) - \alpha^{i+m}(0_e,\omega))\Omega_{i,\omega} \right).$$

With such objects defined, $A = \widehat{Y} + \widehat{\Sigma} + \xi$, so the closed-loop error dynamics (3.19) writes as

$$(\dot{z}, \dot{\omega}) = A_{(z,\omega)} = \widehat{Y}_{(z,\omega)} + \widehat{\Sigma}_{(z,\omega)} + \xi_{(z,\omega)}. \tag{3.20}$$

The ensuing analysis shall be carried out by pulling the dynamics (3.20) back to a covering manifold of M. For definitions and properties of covering spaces and projections, the Reader may wish to consult e.g. [Godbillon, 1971, Chap. X.3]). Trivially, $\mathrm{id}_{TG} : TG \longrightarrow TG$ is a covering of TG whereas the *universal* covering $\widetilde{\mathbb{T}}^\kappa$ of $\mathbb{T}^\kappa$, endowed with the unique differentiable structure that makes the covering projection $p : \widetilde{\mathbb{T}}^\kappa \longrightarrow \mathbb{T}^\kappa$ differentiable and of maximal rank, is diffeomorphic to $\mathbb{R}^\kappa$. Using Lemma 3.1 we see that $Tp : T\widetilde{\mathbb{T}}^\kappa \longrightarrow T\mathbb{T}^\kappa$ is (diffeomorphic to) the universal covering space of $T\mathbb{T}^\kappa$. Since the product of covering spaces is a covering space in the obvious way, if we set $\widetilde{M} = TG \times T\widetilde{\mathbb{T}}^\kappa$, then $P : \widetilde{M} \longrightarrow M$ is a covering manifold of M, with covering projection $P : (z, q) \mapsto (z, Tp(q))$. By definition, P is a local diffeomorphism, so $T_{(z,q)}P$ is an isomorphism for every $(z, q) \in M$; as a consequence, if $f \in C^\infty(M)$ and $X \in \Gamma(TM)$, then the pullbacks $P^*f \in C^\infty(\widetilde{M})$ and $P^*X \in \Gamma(T\widetilde{M})$ are well defined by setting

$$P^*f = f \circ P \quad \text{and} \quad (P^*X)_{(z,q)} = (T_{(z,q)}P)^{-1}\left(X_{(z,Tp(q))}\right).$$

Moreover, these definitions entail that $P^*X(P^*f) = P^*(X(f))$. System (3.20), together with the projection P, induce a system on $\widetilde{M}$ given by

$$(\dot{z}, \dot{q}) = P^*A_{(z,q)} = P^*\widehat{Y}_{(z,q)} + P^*\widehat{\Sigma}_{(z,q)} + P^*\xi_{(z,q)}, \tag{3.21}$$

and, since A is P-related to P^*A, every trajectory of (3.21) projects by P to a trajectory of (3.20) (one also says that trajectories of (3.21) are *liftings* by P of trajectories of (3.20)). Since K does not depend explicitly on z, so $\widehat{Y}(K) = 0$, and is invariant under Σ, so $\widehat{\Sigma}(K) = 0$, one has

$$P^*A(P^*K) = P^*((\widehat{Y} + \widehat{\Sigma} + \xi)(K)) = P^*(\xi(K)) = P^*\xi(P^*K). \tag{3.22}$$

Let (O, ϕ) be a coordinate chart on TG about 0_e such that $\phi(0_e) = 0$. As is easily checked, the canonical projection $\pi : \mathbb{R}^\kappa \longrightarrow \mathbb{R}^\kappa/\mathbb{Z}^\kappa$ defines a covering space isomorphic to $p : \widetilde{\mathbb{T}}^\kappa \longrightarrow \mathbb{T}^\kappa$, so there exist diffeomorphisms $\alpha : \widetilde{\mathbb{T}}^\kappa \longrightarrow \mathbb{R}^\kappa$ and $\alpha' : \mathbb{T}^\kappa \longrightarrow \mathbb{R}^\kappa/\mathbb{Z}^\kappa$ such that $\pi \circ \alpha = \alpha' \circ p$. Clearly, $(\widetilde{\mathbb{T}}^\kappa, \alpha)$ is a global coordinate chart, which then induces naturally a global chart $(T\widetilde{\mathbb{T}}^\kappa, \widehat{\alpha})$ on $T\widetilde{\mathbb{T}}^\kappa$. The coordinates ϕ and $\widehat{\alpha}$ induce a chart $(O \times T\widetilde{\mathbb{T}}^\kappa, \psi)$ on $\widetilde{M}$, with $\psi : O \times T\widetilde{\mathbb{T}}^\kappa \longrightarrow \mathbb{R}^{2n+2\kappa}$ given by $\psi(z, q) = (\phi(z), \widehat{\alpha}(q))$. We shall label these coordinates as $\widehat{z} = (\widehat{z}^1, \ldots, \widehat{z}^{2n}) = (\psi^1, \ldots, \psi^{2n})$, $\theta = (\theta^1, \ldots, \theta^\kappa) = (\psi^{2n+1}, \ldots, \psi^{2n+\kappa})$ and $\dot{\theta} = (\dot{\theta}^1, \ldots, \dot{\theta}^\kappa) = (\psi^{2n+\kappa+1}, \ldots, \psi^{2n+2\kappa})$. (The choice of these coordinates and labels corresponds, of course, to their intuitive interpretation, the θ^is representing angle-like functions and the $\dot{\theta}^i$s "angular velocities.") Clearly, the points in $\widetilde{M}$ with coordinates $\widehat{z} = 0$ and $\dot{\theta} = 0$ project onto $\{0_e\} \times Z(T\mathbb{T}^\kappa)$, i.e., $P \circ \psi^{-1}(\{0\} \times \mathbb{R}^\kappa \times \{0\}) = \{0_e\} \times Z(T\mathbb{T}^\kappa)$. Given points $(z_0, \omega_0) \in M$ and $(z_0, q_0) \in P^{-1}(\{(z, \omega)\})$, we let $t \mapsto (z(t), \omega(t))$ and $t \mapsto (z(t), q(t))$ denote the solutions of (3.20) and (3.21) initialized, for $t = 0$, at (z_0, ω_0) and (z_0, q_0), respectively. The representative of $t \mapsto (z(t), q(t))$ in the coordinates ψ shall be denoted by $t \mapsto (\widehat{z}(t), \theta(t), \dot{\theta}(t))$.

At this point, the proof reduces to showing that the set $\{0\} \times \mathbb{R}^\kappa \times \{0\}$ is uniformly stable under the representative of system (3.21) in the coordinates ψ, i.e., under the pullback vector field $\widehat{A} := (P \circ \psi^{-1})^* A \in \Gamma(T\mathbb{R}^{2n+2\kappa})$. Indeed, suppose this claim holds, and let $V \subset M$ be a neighborhood of $\{0_e\} \times Z(T\mathbb{T}^\kappa)$. By virtue of Lemma 3.2-(**I**), there exist convex, compact neighborhoods $V_1 \subset \mathbb{R}^{2n}$ and $V_2 \subset \mathbb{R}^\kappa$ of the respective origins, such that $P \circ \psi^{-1}(V_1 \times \mathbb{R}^\kappa \times V_2)$ is a neighborhood of $\{0_e\} \times Z(T\mathbb{T}^\kappa)$ contained in V. Since $\{0\} \times \mathbb{R}^\kappa \times \{0\}$ is assumed to be uniformly stable under $\widehat{A}$, there exists an open neighborhood $U_1 \subset \mathbb{R}^{2n+2\kappa}$ of $\{0\} \times \mathbb{R}^\kappa \times \{0\}$ such that $\widehat{A}$ is U_1-positive-complete and such that, for any initial condition $(z_0, q_0) \in \widetilde{M}$ such that $\psi(z_0, q_0) \in U_1$, the integral curve $\psi(z(t), q(t))$ of $\widehat{A}$, initialized at $\psi(z_0, q_0)$, satisfies $\psi(z(t), q(t)) \in V_1 \times \mathbb{R}^\kappa \times V_2$ for all $t \geq 0$. Then the set $U = P \circ \psi^{-1}(U_1) \subset M$ has the required property that if $(z_0, \omega_0) \in U$ then $(z(t), \omega(t)) \in V$ for all $t \geq 0$, for in that case $(z(t), \omega(t)) = P(z(t), q(t)) \in P \circ \psi^{-1}(V_1 \times \mathbb{R}^\kappa \times V_2) = V$ for all $t \geq 0$.

By smoothness of the right-hand side of (3.21), $(z(t), q(t))$ is defined for t in some neighborhood of 0; for any such t, the derivative of $W : t \mapsto P^*K(z(t), q(t))$ can then be computed as follows

$$\begin{aligned} W'(t) &= P^*A_{(z(t),q(t))}(P^*K) \\ &= P^*\xi_{(z(t),q(t))}(P^*K). \end{aligned}$$

And, since ξ_z is vertical, one has,

$$\begin{aligned} W'(t) &= \left. \sum_{i=1}^{\kappa} \left(\frac{\partial(P^*K)}{\partial \theta^i} \cdot 0 + \frac{\partial(P^*K)}{\partial \dot{\theta}^i} (P^*\xi)^{2n+\kappa+i} \right) \right|_{\psi^{-1}(\widehat{z}(t), \theta(t), \dot{\theta}(t))} \\ &=: \mu \circ \psi^{-1}(\widehat{z}(t), \theta(t), \dot{\theta}(t)), \end{aligned}$$

with $\mu \in C^\infty(\widetilde{M})$ given by $\mu = \sum_{i=1}^{\kappa} \partial(P^*K)/\partial\dot{\theta}^i \cdot (P^*\xi)^{2n+\kappa+i}$. Given the choice of coordinates, there exist functions $g_{i,j} : \mathbb{R}^\kappa \longrightarrow \mathbb{R}$ such that $P^*K \circ \psi^{-1}$ (i.e., the representative of P^*K in the ψ coordinates) writes as

$$P^*K \circ \psi^{-1}(\widehat{z}, \theta, \dot{\theta}) = \frac{1}{2} g_{i,j}(\theta) \dot{\theta}^i \dot{\theta}^j.$$

(Here and in the sequel of the proof the summation convention is in place.) Also, since $\xi_{(0,\omega)} = 0$ for all $\omega \in T\mathbb{T}^\kappa$, then $P^*\xi_{(0,q)} = 0$ for all $q \in T\widetilde{\mathbb{T}}^\kappa$, whence $\mu \circ \psi^{-1}(0, \cdot, \cdot) = 0$. Therefore, using a Taylor expansion with remainder, along with the fact that the representative $P^*K \circ \psi^{-1}$ is quadratic in the components of $\dot\theta$, one obtains, after straightforward computations:

$$\begin{aligned} W'(t) \;=\; \Bigg(\Bigg(&\frac{1}{2}\frac{\partial^2\mu}{\partial\dot\theta^k\partial\widehat{z}^i} \circ \psi^{-1}(0, \theta(t), 0) + \frac{1}{6}\frac{\partial^3\mu}{\partial\dot\theta^k\partial\widehat{z}^j\partial\widehat{z}^i} \circ \psi^{-1}(c_1\widehat{z}(t), \theta(t), 0)\cdot\widehat{z}^j(t) \\ &+ \frac{1}{6}\frac{\partial^3\mu}{\partial\dot\theta^\ell\partial\dot\theta^k\partial\widehat{z}^i} \circ \psi^{-1}(0, \theta(t), c_2\dot\theta(t))\cdot\dot\theta^\ell(t) \\ &+ \frac{1}{24}\frac{\partial^4\mu}{\partial\dot\theta^\ell\partial\dot\theta^k\partial\widehat{z}^j\partial\widehat{z}^i} \circ \psi^{-1}(c_1\widehat{z}(t), \theta(t), c_2\dot\theta(t))\cdot\dot\theta^\ell(t)\cdot\widehat{z}^j(t)\Bigg)\dot\theta^k(t)\Bigg)\widehat{z}^i(t), \end{aligned}$$

for some reals $c_1, c_2 \in (0, 1)$. Without loss of generality, one may assume that the coordinates ϕ were selected in such a way that, in view of the assumption of local exponential stability of 0_e for Y, there exist reals $C_1, C_2 > 0$ such that $|\widehat{z}^i(t)| \le \|\widehat{z}(t)\| \le C_1\|\widehat{z}_0\|e^{-C_2 t}$ for $i = 1, \ldots, n$. Moreover, since $P^*K \circ \psi^{-1}$ is positive-definite, quadratic in the $\dot\theta^i$s, and independent of $\widehat{z}$, there exists $C_3 > 0$ such that $|\dot\theta^k(t)| \le C_3(P^*K\circ\psi^{-1}(\widehat{z}(t), \theta(t), \dot\theta(t)))^{\frac{1}{2}} = C_3 W(t)^{\frac{1}{2}}$ for $k = 1, \ldots, \kappa$ and $t \ge 0$. For $i = 1, \ldots, n$ and $k = 1, \ldots, \kappa$ let

$$\begin{aligned} N_{i,k} \;=\; \max\Bigg\{\Bigg|&\frac{1}{2}\frac{\partial^2\mu}{\partial\dot\theta^k\partial\widehat{z}^i} \circ \psi^{-1}(0, \theta, 0) + \frac{1}{6}\frac{\partial^3\mu}{\partial\dot\theta^k\partial\widehat{z}^j\partial\widehat{z}^i} \circ \psi^{-1}(\widehat{z}, \theta, 0)\cdot\widehat{z}^j \\ &+ \frac{1}{6}\frac{\partial^3\mu}{\partial\dot\theta^\ell\partial\dot\theta^k\partial\widehat{z}^i} \circ \psi^{-1}(0, \theta, \dot\theta)\cdot\dot\theta^\ell \\ &+ \frac{1}{24}\frac{\partial^4\mu}{\partial\dot\theta^\ell\partial\dot\theta^k\partial\widehat{z}^j\partial\widehat{z}^i} \circ \psi^{-1}(\widehat{z}, \theta, \dot\theta)\cdot\dot\theta^\ell\cdot\widehat{z}^j\Bigg| : (\widehat{z}, \theta, \dot\theta) \in V_1 \times \mathbb{R}^\kappa \times V_2\Bigg\}. \end{aligned}$$

That these maxima exist is easily deduced by an obvious extension of Lemma 3.2-(III) and the compactness of V_1 and V_2. Let $C_4 = \max\{N_{i,k} : i = 1, \ldots, n,\ k = 1, \ldots, \kappa\}$. Therefore

$$\begin{aligned} W'(t) &\le C_4|\dot\theta^k(t)||\widehat{z}^i(t)| \\ &\le C_3C_4(W(t))^{\frac{1}{2}}|\widehat{z}^i(t)| \\ &\le C_1C_3C_4(W(t))^{\frac{1}{2}}\|\widehat{z}_0\|e^{-C_2 t}. \end{aligned}$$

Given a neighborhood $V \subset \mathbb{R}^{2n} \times \mathbb{R}^\kappa \times \mathbb{R}^\kappa$ of $\{0\} \times \mathbb{R}^\kappa \times \{0\}$, consider neighborhoods $V_1 \subset \mathbb{R}^{2n}$ and $V_2 \subset \mathbb{R}^\kappa$ of the respective origins. Let $\varepsilon > 0$ be such that $\{\widehat{z} \in \mathbb{R}^{2n} : \|\widehat{z}\| < \varepsilon\} \subset V_1$ and $\{\dot\theta \in \mathbb{R}^\kappa : \|\dot\theta\| < \varepsilon\} \subset V_2$. By assumption, if $\|\widehat{z}_0\| < \delta_1 := \varepsilon/C_1$, then $\widehat{z}(t) < \varepsilon$ for $t \ge 0$. Moreover, by applying Theorem 3.3, along with the *Comparison Lemma* [Khalil, 1996, Lemma 2.5], we deduce that there exists $\delta_2 > 0$ such that, if $W(0) < \delta_2$, then $(W(t))^{\frac{1}{2}} < \varepsilon/(C_3\sqrt{\kappa})$, that is, $\|\dot\theta(t)\| < \varepsilon$ for all $t \ge 0$. It follows that if the initial condition $(z_0, \theta_0, \dot\theta_0)$ is such that $\|\widehat{z}_0\| < \delta_1$ and $\|\dot\theta\| < \delta_2$, then $(\widehat{z}(t), \theta(t), \dot\theta(t)) \in V_1 \times \mathbb{R}^\kappa \times V_2$ for all $t \ge 0$, thus completing the proof. ■

3.4. Example: The 3-ECF

In this section we apply the methodology proposed in Section 2.3 to a particular example of second-order system in order to illustrate some of the concepts and results presented in this chapter. Consider the 3-dimensional Extended Chained Form (ECF), which is a system on $T\mathbb{R}^3 \simeq \mathbb{R}^6$ described by the following equations.

$$\begin{aligned} \ddot{x}_1 &= u_1 \\ \ddot{x}_2 &= u_2 \\ \ddot{x}_3 &= u_1 x_2. \end{aligned}$$

The latter can be viewed as a SMS of the form (2.1),

$$\dot{x} = S_x + u_1 X_{1,x}^{\mathrm{lift}} + u_2 X_{2,x}^{\mathrm{lift}}, \tag{3.23}$$

where $S_x = \sum_{i=1}^{3} x_{i+3}\, \partial/\partial x^i$ is the geodesic spray given by the Euclidean metric on $\mathbb{R}^3$, and $X_{1,x}^{\mathrm{lift}} = \partial/\partial x^4 + x_2\, \partial/\partial x^6$, $X_{2,x}^{\mathrm{lift}} = \partial/\partial x^5$ are vertical vector fields on $T\mathbb{R}^3 \simeq \mathbb{R}^6$.

It is well known (cf. Imura et al. [1996]) that the ECF is static-feedback equivalent to the model of a planar, underactuated, horizontal PPR (Prismatic-Prismatic-Rotational) manipulator, the rotational joint of which is passive. In this model, friction and gravity effects are not considered. The schematics of this mechanical manipulator is depicted in Figure 3.1. The configuration manifold

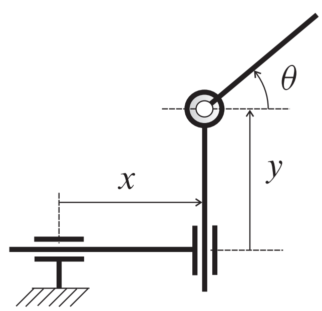


Figure 3.1: Schematic representation of the underactuated PPR manipulator of system (3.24).

for this system is $SE(2) \simeq \mathbb{R}^2 \times S^1$ and, using coordinates $(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})$ for $T(SE(2))$, the Euler-Lagrange equations yield the dynamic model

$$\begin{aligned} M_1\ddot{x} - m_3 l\,\mathrm{s}(\theta)\ddot{\theta} - m_3 l\,\mathrm{c}(\theta)\dot{\theta}^2 &= \tau_1 \\ M_2\ddot{y} + m_3 l\,\mathrm{c}(\theta)\ddot{\theta} - m_3 l\,\mathrm{s}(\theta)\dot{\theta}^2 &= \tau_2 \\ -m_3 l\,\mathrm{s}(\theta)\ddot{x} + m_3 l\,\mathrm{c}(\theta)\ddot{y} + J\ddot{\theta} &= 0, \end{aligned} \tag{3.24}$$

where m_i is the mass of the i-th link, $M_i = \sum_{j=i}^{3} m_j$, and for the third link, J and l are the moment of inertia and the position of the center of mass, both with respect to the joint axis. τ_i is the force applied to the i-th link, $i = 1, 2$. (In the above model, as we shall often do in the sequel, we use the convention sin = s and cos = c.) The VTF approach can be directly applied to system (3.24) given that it can be rewritten of the form (2.1) with left-invariant control vector fields on the tangent Lie group of $SE(2)$ (with group composition (2.10)). However, in order to deal with simpler computations that convey the main ideas more transparently, we apply the VTFA to the ECF (3.23).

Consider the Lie group $\mathbb{R}^3$ with composition law given, for every $x, y \in \mathbb{R}^3$, by

$$\widehat{\mu}(x, y) = (\ x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3 + x_2 y_1\ ).$$

Consider natural (global) coordinates on the tangent Lie group $T\mathbb{R}^3$ tangent group TG. Using these coordinates, the (tangent) Lie group operation on $T\mathbb{R}^3$ is given by

$$\begin{aligned} \mu(x, y) = x \cdot y \ &= \ (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_2 y_1, \\ &\qquad x_4 + y_4, x_5 + y_5, x_6 + y_6 + x_2 y_4 + x_5 y_1) \end{aligned} \tag{3.25}$$

It is straightforward to check that both X_i and X_i^{lift} ($i = 1, 2$) are left-invariant under the respective left translations in G and TG. Also note that $[X_1, X_2] = -\partial/\partial q_3$, so that $\text{Lie}\{X_1, X_2\}(q) = T_q G$ for every $q \in G$. Consequently one can apply the procedure of Morin and Samson [2003], recalled in Section 2.1, to construct a transverse function $f : \mathbb{T} \longrightarrow \mathbb{R}^3$ for the 3-CF system $\dot{x} = u^1 X_{1,x} + u^2 X_{2,x}$ near zero. Take for instance the function in equation (2.7) of Subsection 2.1.2,

$$f(\theta) = \left(\varepsilon\ \sin(\theta), \varepsilon\ \cos(\theta), \frac{1}{4}\varepsilon^2\ \sin(2\theta)\right), \tag{3.26}$$

with $\varepsilon > 0$ arbitrary, and extended to every element in $\mathbb{T}$ by continuity. The transversality condition (2.4) is equivalent to the non-vanishing of the determinant of the matrix with columns $X_{1,f(\theta)}$, $X_{2,f(\theta)}$ and $f'(\theta)$, i.e.,

$$M(\theta) := \left[X_{1,f(\theta)}, X_{2,f(\theta)}, \frac{\partial f}{\partial \theta}(\theta)\right] = \begin{pmatrix} 1 & 0 & \varepsilon\ \cos(\theta) \\ 0 & 1 & -\varepsilon\ \sin(\theta) \\ \varepsilon\ \cos(\theta) & 0 & \frac{1}{2}\varepsilon^2\ \cos(2\theta) \end{pmatrix}.$$

A simple computation shows that $\det(M(\theta)) = -\frac{1}{2}\varepsilon^2$, hence f is transverse.

Using natural coordinates $(\theta, \dot{\theta})$ for $T\mathbb{T}$, the tangent mapping Tf is defined, for every $\omega = (\theta, \dot{\theta}) \in T_\theta\mathbb{T}$, by

$$T_\theta f(\omega) = \big(\varepsilon \text{s}(\theta),\ \varepsilon \text{c}(\theta),\ \tfrac{1}{4}\varepsilon^2 \text{s}(2\theta),\ \varepsilon \text{c}(\theta)\dot{\theta},\ -\varepsilon \text{s}(\theta)\dot{\theta},\ \tfrac{1}{2}\varepsilon^2 \text{c}(2\theta)\dot{\theta}\big).$$

The vertical transversality condition (2.14) is easy to establish. Consider natural coordinates $(\theta, \dot{\theta}, \alpha_L, \alpha_H)$ for $TT\mathbb{T}$. First evaluate *the tangent* of Tf at a vertical vector $\alpha \in (T_\omega T\mathbb{T})^{\text{vert}}$ (i.e., $\alpha \in \ker(T_\omega \pi_{\mathbb{T}})$), which, by the result in Lemma 2.1 yields a vertical vector. Since $T_\omega \pi_{\mathbb{T}}$ maps $(\theta, \dot{\theta}, \alpha_L, \alpha_H)$ to (θ, α_L), α is in the kernel of $T_\omega \pi_{\mathbb{T}}$ if and only if it has the form $\alpha = (\theta, \dot{\theta}, 0, \alpha_H)$, so for simplicity we take $\tilde{\alpha} = (\theta, \dot{\theta}, 0, 1)$. Carrying out the operations one obtains

$$T_\omega T f(\tilde{\alpha}) = \big(0,\ 0,\ 0,\ \varepsilon \text{c}(\theta), -\varepsilon \text{s}(\theta),\ \tfrac{1}{2}\varepsilon^2 \text{c}(2\theta)\big)\,.$$

Now let us check that X_1^{lift}, X_2^{lift} and $T_\omega Tf(\tilde{\alpha})$ span the vertical subspace $(T_{Tf(\omega)}TG)^{\text{vert}}$. In this case, any vector in the latter is of the form $\sum_{i=1}^{3} \alpha_i \, \partial/\partial \dot{q}_i$, that is, its first three components are zero. Hence the verification reduces to computing the determinant of the submatrix consisting of the lower three rows of the matrix with columns $X_{1,Tf(\omega)}^{\text{lift}}$, $X_{2,Tf(\omega)}^{\text{lift}}$ and $T_\omega Tf(\tilde{\alpha})$, but this is exactly the matrix $M(\theta)$ defined above, the determinant of which equals $-\frac{1}{2}\varepsilon^2$, so Tf indeed satisfies (2.14).

Define an auxiliary system evolving on $T\mathbb{T}$ by

$$\ddot{\theta} = w, \tag{3.27}$$

where $w \in \mathbb{R}$ is a control input. This system can be viewed as a (flat) affine-connection control system on $\mathbb{T}$; in fact, the affine connection which defines it is the Levi-Civita connection of a Euclidean on $\mathbb{T}$. Note that (3.27) can be written in the form of (2.15) if we set $\Delta_\omega = \dot{\theta}\, \partial/\partial\theta$, $\kappa = 1$ and $\Omega_{1,\omega} = \partial/\partial\dot{\theta}$. Define also the corresponding error function to be $z = \mu(x, Tf(\omega)^{-1})$,

$$\begin{aligned} z \;\; = \;\; \Big(& x_1 - \varepsilon \mathrm{s}(\theta),\; x_2 - \varepsilon \mathrm{c}(\theta),\; x_3 + \tfrac{1}{4}\varepsilon^2 \mathrm{s}(2\theta) - x_2 \varepsilon \mathrm{s}(\theta), \\ & x_4 - \varepsilon \mathrm{c}(\theta)\dot{\theta},\; x_5 + \varepsilon \mathrm{s}(\theta)\dot{\theta},\; x_3 - x_5 \varepsilon \mathrm{s}(\theta) - x_2 \varepsilon \mathrm{c}(\theta)\dot{\theta} + \tfrac{1}{2}\varepsilon^2 \mathrm{c}(2\theta)\dot{\theta} \Big). \end{aligned}$$

By differentiating this expression we get the error dynamics

$$\dot{z} = F(z, \omega) + \sum_{i=1}^{3} u_i G_i(z, \omega), \tag{3.28}$$

where $u_3 = w$, $G_1(z,\omega) = \partial/\partial x^4 + (z_2 + \varepsilon \mathrm{c}(\theta))\partial/\partial x^6$, $G_2(z,\omega) = \partial/\partial x^5 - \varepsilon \mathrm{s}(\theta)\partial/\partial x^6$, $G_3(z,\omega) = -\varepsilon \mathrm{c}(\theta)\partial/\partial x^4 + \varepsilon \mathrm{s}(\theta)\partial/\partial x^5 + (-\frac{1}{2}\varepsilon^2 - \varepsilon \mathrm{c}(\theta) z_2)\partial/\partial x^6$, and

$$F(z,\omega) = \Big(z_4,\; z_5,\; z_6,\; \varepsilon \mathrm{s}(\theta)\dot{\theta}^2,\; \varepsilon \mathrm{c}(\theta)\dot{\theta}^2,\; \tfrac{1}{2}\varepsilon^2 \mathrm{s}(2\theta)\dot{\theta}^2 + \varepsilon \mathrm{s}(\theta) z_2 \dot{\theta}^2 - 2\varepsilon \mathrm{c}(\theta) z_5 \dot{\theta} \Big).$$

Each of the objects G_i for $i = 1, 2, 3$, as well as F, may be seen as families of vector fields on TG indexed by $\omega = (\theta, \dot{\theta}) \in T_\theta \mathbb{T}$. Moreover, it is clear that $F(\cdot, \omega)$ is second-order whereas G_1, G_2 and G_3 are vertical, thus the error dynamics (3.28) is second-order for every $\omega \in T\mathbb{T}$.

In order to construct a control law as outlined in Theorem 3.4 we select for the desired dynamics a second-order vector field $S \in \Gamma(TTG)$ which has 0 as exponentially stable equilibrium point, for instance

$$S_z = (z_4,\; z_5,\; z_6,\; -k_1 z_1 - k_2 z_4,\; -k_1 z_2 - k_2 z_5,\; -k_1 z_3 - k_2 z_6),$$

where the control gains k_1, k_2 are strictly positive. From this point on, the control design translates into the search for a function $u : TG \times T\mathbb{T} \longrightarrow \mathbb{R}^3$ such that

$$F(z,\omega) + \sum_{i=1}^{3} u_i(z,\omega)\; G_i(z,\omega) = S_z$$

for all $(z, \omega) \in TG \times T\mathbb{T}$. Inspecting the structure of the error dynamics (3.28) one easily deduces that this problem is equivalent to solving (2.19), which in this case boils down to solving for u in the following matrix equation

$$\begin{pmatrix} 1 & 0 & -\varepsilon \mathrm{c}(\theta) \\ 0 & 1 & \varepsilon \mathrm{s}(\theta) \\ z_2 + \varepsilon \mathrm{c}(\theta) & -\varepsilon \mathrm{s}(\theta) & -\frac{1}{2}\varepsilon^2 - \varepsilon \mathrm{c}(\theta) z_2 \end{pmatrix} u = \begin{pmatrix} -\dot{\theta}^2 \varepsilon \mathrm{s}(\theta) - k_1 z_1 - k_2 z_4 \\ -\dot{\theta}^2 \varepsilon \mathrm{c}(\theta) - k_1 z_2 - k_2 z_5 \\ -\frac{1}{2}\varepsilon^2 \mathrm{s}(2\theta)\dot{\theta}^2 - \varepsilon \mathrm{s}(\theta) z_2 \dot{\theta}^2 + 2\varepsilon \mathrm{c}(\theta) z_5 \dot{\theta} - k_1 z_3 - k_2 z_6 \end{pmatrix}.$$

This equation is solvable since the invertibility of the coefficient of u is equivalent to the invertibility of the matrix which ensures the vertical transversality of Tf; its determinant, in particular, is equal to $\frac{1}{2}\varepsilon^2$. In order to illustrate the time evolution for the error and target systems, we include a simple numerical simulation with $x_0 = (1.5, -1.0, 2.2, 0, 0, 0)$, $\omega_0 = (\theta_0, \dot{\theta}_0) = (0, 0)$, $\varepsilon = 0.45$ and control gains $k_1 = 0.08$, $k_2 = 0.4$. Figure 3.2 shows that the error tends to zero whereas the

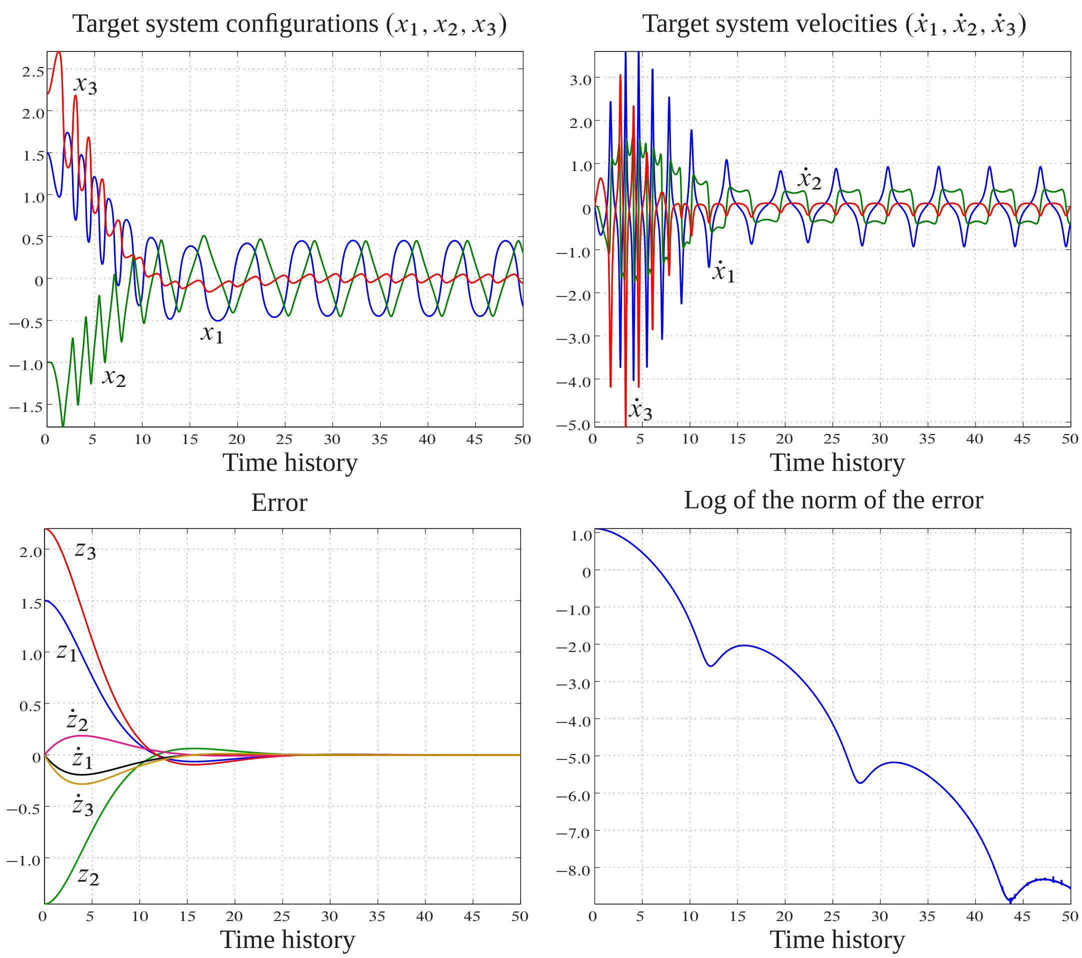


Figure 3.2: Numerical simulation of the ECF under the VTFA with initial conditions $x_0 = (1.5, -1.0, 2.2, 0, 0, 0)$, $\theta_0 = 0$, $\dot{\theta}_0 = 0$, $\varepsilon = 0.45$ and $k_1 = 0.08$, $k_2 = 0.4$.

logarithm of its norm decays sublinearly, so that $z(t) \to 0$ exponentially. On the other hand, the configuration of the ECF (x_1, x_2, x_3) and the velocities (x_4, x_5, x_6) seem to converge to a periodic motion whereas the base curve $q(t) = (\pi_G \circ v)(t)$ ultimately converges to a bounded set, the extent of which can be made arbitrarily small by decreasing ε. Taking smaller values of ε, however, typically leads to increments in the peak excursions of the control signals and in the frequency of the steady-state oscillations. Concerning the velocities $\dot{q}(t)$, they also converge to a bounded set, but the extent of that set depends on the initial conditions (v_0, ω_0), hence one cannot specify it in advance.

Let us now turn to the zero dynamics of the compound system, which, in accordance with Theorem 3.1, is determined by a spray Σ:

$$\dot{\omega} = \Sigma_\omega = \begin{pmatrix} \dot{\theta} \\ -\sin(2\theta)\dot{\theta}^2 \end{pmatrix}. \tag{3.29}$$

This vector field Σ determines a torsionless connection $\nabla : \Gamma(T\mathbb{T}) \longrightarrow \Gamma(T^*\mathbb{T} \otimes T\mathbb{T})$ with Christoffel symbol $\Gamma(\theta) = \sin(2\theta)$, so that $\nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\theta} = \Gamma(\theta)\frac{\partial}{\partial\theta}$. Using the notation of the paragraph preceding Proposition 3.2, if $s = \frac{\partial}{\partial\theta}$ is a local section on the domain of (U, θ), then $\kappa_s(\nabla) = \sin(2\theta)d\theta$, the cohomology class of which is zero. By Proposition 3.2, ∇ is the Levi-Civita connection of a family of metrics on $T\mathbb{T}$, namely

$$g_\theta = Ae^{-\cos(2\theta)} d\theta \otimes d\theta, \quad A > 0.$$

By defining the Lagrangian function $L : T\mathbb{T} \longrightarrow \mathbb{R}$ as

$$L(\omega) = \frac{1}{2} g_\theta(\dot{\theta}, \dot{\theta}) = \frac{1}{2} Ae^{-\cos(2\theta)}\dot{\theta}^2,$$

one readily checks that the associated Euler-Lagrange equation $\nabla_{\dot{\theta}}\dot{\theta} = 0$ precisely coincides with the zero dynamics (3.29), irrespective of the value of A. As pointed out above, the zero dynamics is itself a simple mechanical system, in this case with zero potential. Since in this case $(L \circ \omega)' = 0$, the (kinetic) energy is a conserved quantity and, given that it is bounded with respect to θ and depends quadratically on $\dot{\theta}$, it follows that $\dot{\theta}(t)$ remains bounded for all $t \in [t_0, \infty)$. Consequently, both $Tf(\omega(t))$ and $v(t)$ converge to a bounded neighborhood of the zero-section in TG. In this case it is also clear that such neighborhood depends on the initial conditions. Intuitively, one can think of the error $z(t)$ as converging to 0 exponentially at the expense of a gradual increase in the kinetic energy "stored" in the auxiliary system.

Chapter 4

Generalized Transverse Functions

The application of vertically transverse functions to control, as proposed in Chapter 2, poses two issues that deserve additional analysis. One of these issues consists in finding conditions to determine whether the resulting zero dynamics admits a (pseudo-) Riemannian metric in the general situation, for in such case stability is ensured for the closed-loop system as established in Theorem 3.4. The other issue concerns the introduction of dissipation into the zero dynamics to ensure that the system velocities (i.e., the fiber coordinates) vanish asymptotically, as would be required in typical applications. The main purpose of this chapter is to explore the potential application of generalized transverse functions to “inject” dissipation into the resulting zero dynamics. Generalized transverse functions (GTF) were introduced by Morin and Samson [2004] to achieve practical and *asymptotic* stabilization of points and general trajectories for driftless control systems. Basically, a GTF for a distribution spanned by a set $\boldsymbol{X} = \{X_1, \ldots, X_m\} \subset \Gamma(TG)$ near $e \in G$ is a function $f : \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2} \longrightarrow G$ such that $f(\cdot, \beta) : \alpha \mapsto f(\alpha, \beta) \in G$, is transverse for $\boldsymbol{X}$ near e (in the sense defined by equation (2.2)) for every $\beta \in \mathbb{T}^{\kappa_1}$. In Section 4.1 we make precise this definition and show that the tangent mapping of a GTF is *generalized vertically transverse*. Section 4.2 presents a straightforward generalization of the vertically transverse function approach for the case of vertically transverse functions derived from generalized transverse functions. The interest in this class of functions is that its application to control leads to a *non-autonomous* zero dynamics with additional control inputs which may be used to influence the behavior of the trajectories in zero dynamics. The central objective is to design these additional control inputs in order to make the zero-section of the zero dynamics asymptotically stable or, at least, locally attractive. Two approaches may be followed: the introduction of dissipation *à la* Jurdjevic-Quinn given that the resulting non-controlled zero dynamics admits a metric, and the design of time-varying feedback via high-order averaging (Sarychev [2001]; Agračhev and Gramkrelidze [1979]) as proposed by Vela [2003]. Particulary, the latter is developed for the ECF, for which the numerical simulations presented below show promising results.

4.1. Generalized transverse functions

Generalized transverse functions (GTF) were introduced by Morin and Samson [2004]. Their application to control provides the closed-loop system with extra control inputs that can be used, under appropriate conditions, to address complementary control objectives such as asymptotic rather than practical stabilization of points and trajectories for driftless systems. In this thesis, tangent mappings of generalized transverse functions are used to endow the closed-loop zero dynamics with additional inputs which may be designed, much in the spirit of Morin and Samson [2004], to render the zero-section locally attractive and, therefore, to ensure that the velocities vanish asymptotically. The following definition is a slightly weakened version of the definition introduced by Morin and Samson [2004] which is suitable for its application to second-order systems.

Definition 4.1 (Generalized transverse functions). *Let Q be an n-dimensional manifold. Given a neighborhood $\mathcal{U}$ of $q \in Q$, and a set of vector fields $\{X_1, \ldots, X_m\} \subset \Gamma(TQ)$, a **generalized transverse function (GTF) near** $q \in Q$ is a mapping $f : \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2} \longrightarrow Q$, with $\kappa_1 \geq n-m$ and $\kappa_2 \geq 1$, such that*

a) $f(\mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2}) \subset \mathcal{U}$,
b) $T_{f(\sigma)}Q = \operatorname{span}_{\mathbb{R}} \left\{X_{1,f(\sigma)}, \ldots, X_{m,f(\sigma)}\right\} + T_\alpha f_\beta(T_\alpha \mathbb{T}^{\kappa_1})$, *for every* $\sigma = (\alpha, \beta) \in \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2}$,

where $\left(f_\beta\right)_{\beta \in \mathbb{T}^{\kappa_2}}$ is the family of maps, indexed by β, defined by $f_\beta(\alpha) = f(\alpha, \beta)$.

In particular, in contrast with the original definition, we do not require that $f(0, \beta) = e \in G$ for every $\beta \in \mathbb{T}^{\kappa_2}$ as is required in (Morin and Samson [2004]). When computed in local coordinates, for a given coordinate chart $\sigma = (\alpha, \beta) = (\alpha^i, \beta^j)$ of $\mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2}$, condition *b)* translates into

$$\mathbb{R}^n = \operatorname{span}_{\mathbb{R}} \left\{X_{1,f(\sigma)}, \ldots, X_{m,f(\sigma)}\right\} + \operatorname{span}_{\mathbb{R}} \left\{\frac{\partial f_\beta}{\partial \alpha^1}(\alpha), \ldots, \frac{\partial f_\beta}{\partial \alpha^{\kappa_1}}(\alpha)\right\},$$

namely, f_β is transverse, in the sense of equation (2.2), for every $\beta \in \mathbb{T}^{\kappa_2}$.

A generalized transverse function can be constructed from any given transverse function. For instance, the following proposition provides us with an explicit way to construct a generalized transverse function given a transverse function for a set of left-invariant vector fields on a Lie group.

Proposition 4.1 (Construction of a GTF). *Let G be an n-dimensional Lie group and $g : \mathbb{T}^{\kappa_1} \longrightarrow G$ with $\kappa_1 = n - m$, transverse (near $e \in G$) for a set $\{X_1, \ldots, X_m\}$ of left-invariant vector fields on G. Then, for every $(\alpha, \beta) \in \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2}$, the following choices for $f : \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2} \longrightarrow G$, yield generalized transverse functions.*

a) $f(\alpha, \beta) = g(\alpha + \chi(\beta))$,
b) $f(\alpha, \beta) = g(\chi(\beta))^{-1} \cdot g(\alpha + \chi(\beta))$,

where "$+$" is the Lie group composition in $\mathbb{T}^{\kappa_1}$ and $\chi : \mathbb{T}^{\kappa_2} \longrightarrow \mathbb{T}^{\kappa_1}$ is any mapping.

Proof. Let $\sigma = (\alpha, \beta) \in \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2}$. For any choice, *a)* or *b)*, of $f : \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2} \longrightarrow G$, we must prove that for any $w = T_{f(\sigma)}G$ there exist reals $a^1, \ldots, a^m$ and $\omega \in T_\alpha \mathbb{T}^{\kappa_1}$ such that

$$w = \sum_{i=1}^{m} a^i X_{i,f(\sigma)} + T_\alpha f_\beta(\omega).$$

a) Since $g : \mathbb{T}^{\kappa_1} \longrightarrow G$ is transverse, there exist reals $b^1, \ldots, b^m$ and $\widetilde{\omega} \in T_{\alpha+\chi(\beta)}\mathbb{T}^{\kappa_1}$ such that

$$w = \sum_{i=1}^{m} b^i X_{i,g(\alpha+\chi(\beta))} + T_{\alpha+\chi(\beta)}g(\widetilde{\omega}).$$

Let $\widehat{R}_\theta : \mathbb{T}^{\kappa_1} \longrightarrow \mathbb{T}^{\kappa_1}$ denote the right translation in $\mathbb{T}^{\kappa_1}$ by an element $\theta \in \mathbb{T}^{\kappa_1}$. Since $T\widehat{R}_\theta : T\mathbb{T}^{\kappa_1} \longrightarrow T\mathbb{T}^{\kappa_1}$ is a diffeomorphism for every $\theta \in \mathbb{T}^{\kappa_1}$ there exists $\varpi \in T_\alpha \mathbb{T}^{\kappa_1}$ such that $\widetilde{\omega} = T_\alpha \widehat{R}_{\chi(\beta)}(\varpi)$. Then, by using the chain rule and the fact that $f_\beta(\alpha) = g \circ \widehat{R}_{\chi(\beta)}(\alpha)$ for any $(\alpha, \beta) \in \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2}$, one obtains

$$\begin{aligned} w &= \sum_{i=1}^{m} b^i X_{i,f(\alpha,\beta)} + T_{\alpha+\chi(\beta)}g \circ T_\alpha \widehat{R}_{\chi(\beta)}(\varpi), \\ &= \sum_{i=1}^{m} b^i X_{i,f(\alpha,\beta)} + T_\alpha \left(g \circ \widehat{R}_{\chi(\beta)}\right)(\varpi), \\ &= \sum_{i=1}^{m} b^i X_{i,f(\alpha,\beta)} + T_\alpha f_\beta(\varpi), \end{aligned}$$

as was required.

b) Let $L_x : TG \longrightarrow TG$ be the left translation in the Lie group G by an element $x \in G$. Given that TL_x is a diffeomorphism for every $x \in G$, there exists $v \in T_{g(\alpha+\chi(\beta))}G$ such that $w = T_{g(\alpha+\chi(\beta))}L_{g(\chi(\beta))^{-1}}$. By the transversality of $g : \mathbb{T}^{\kappa_1} \longrightarrow G$, there exist reals $b^1, \ldots, b^m$ and $\widetilde{\omega} \in T_{\alpha+\chi(\beta)}\mathbb{T}^{\kappa_1}$ such that

$$v = \sum_{i=1}^{m} b^i X_{i,g(\alpha+\chi(\beta))} + T_{\alpha+\chi(\beta)}g(\widetilde{\omega}).$$

Then

$$w = T_{g(\alpha+\chi(\beta))}L_{g(\chi(\beta))^{-1}}\left(\sum_{i=1}^{m} b^i X_{i,g(\alpha+\chi(\beta))} + T_{\alpha+\chi(\beta)}g(\widetilde{\omega})\right).$$

By using the chain rule and the left-invariance of X_i, $i = 1, \ldots, m$, one obtains

$$w = \sum_{i=1}^{m} b^i X_{i,f(\alpha,\beta)} + T_{\alpha+\chi(\beta)}\left(L_{g(\chi(\beta))^{-1}} \circ g\right)(\widetilde{\omega}).$$

By using the argument in the proof for *a)* above, there exists $\overline{\omega} \in T_\alpha \mathbb{T}^{\kappa_1}$ such that $\widetilde{\omega} = T_\alpha \widehat{R}_{\chi(\beta)}(\overline{\omega})$, and so, by using the chain rule, one obtains

$$\begin{aligned} w &= \sum_{i=1}^{m} b^i X_{i,f(\alpha,\beta)} + T_{\alpha+\chi(\beta)} \left(L_{g(\chi(\beta))^{-1}} \circ g\right) (T_\alpha \widehat{R}_{\chi(\beta)}(\overline{\omega})), \\ &= \sum_{i=1}^{m} b^i X_{i,f(\alpha,\beta)} + T_\alpha \left(L_{g(\chi(\beta))^{-1}} \circ g \circ \widehat{R}_{\chi(\beta)}\right) (\overline{\omega})), \\ &= \sum_{i=1}^{m} b^i X_{i,f(\alpha,\beta)} + T_\alpha f_\beta(\overline{\omega})), \end{aligned}$$

as was to be shown. ■

If $\kappa_1 = \kappa_2$ then a GTF can be built simply by setting $f(\alpha, \beta) = (g(\beta))^{-1} \cdot g(\alpha + \beta)$. Straightforward examples of GTF, for the case $\kappa_1 = 1$ and $\kappa_2 \geq 1$, are $g(\theta) = g(\theta_1 + \cdots + \theta_{\kappa_2+1})$ and $g(\theta) = g(\theta_2 + \cdots + \theta_{\kappa_2+1})^{-1} \cdot g(\theta_1 + \cdots + \theta_{\kappa_2+1})$ for every $\theta = (\theta_1, \ldots, \theta_{\kappa_2+1}) \in \mathbb{T} \times \mathbb{T}^{\kappa_2}$.

It is clear, as a result from Theorem 2.2, that the tangent mapping of any transverse function f_β for some fixed $\beta \in \mathbb{T}^{\kappa_2}$ is vertically transverse. However we define the tangent map of a generalized transverse function to be generalized vertically transverse in the sense made clear in the following proposition.

Proposition 4.2 (Generalized Vertical Transversality)**.** *Let $f : \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2} \longrightarrow G$ be generalized transverse for the set of vector fields $\{X_1, \ldots, X_m\} \subset \Gamma(TG)$ near $e \in G$. Then $Tf : T\mathbb{T}^{\kappa_1+\kappa_2} \longrightarrow TG$ is* ***generalized vertically transverse*** *for $\left\{X_1^{\text{lift}}, \ldots, X_m^{\text{lift}}\right\}$ in the sense that, for every $\nu \in T\mathbb{T}^{\kappa_1+\kappa_2}$, it satisfies*

$$T_{Tf(\nu)}TG^{\text{vert}} = \text{span}_{\mathbb{R}} \left\{X_{1,Tf(\nu)}^{\text{lift}}, \ldots, X_{m,Tf(\nu)}^{\text{lift}}\right\} + TTf \circ H \left((T_\omega T\mathbb{T}^{\kappa_1})^{\text{vert}} \times \{0\}_{\overline{h}(\nu)}\right), \quad (4.1)$$

where $H : TT\mathbb{T}^{\kappa_1} \times TT\mathbb{T}^{\kappa_2} \longrightarrow TT\mathbb{T}^{\kappa_1+\kappa_2}$ and $\overline{h} : T\mathbb{T}^{\kappa_1+\kappa_2} \longrightarrow T\mathbb{T}^{\kappa_2}$ are natural diffeomorphisms.

Proof. Let $v \in T_{Tf(\nu)}TG^{\text{vert}}$. Since v is vertical, there exists $x \in T_{f(\sigma)}G$ such that $v = \text{lift}\,(Tf(\nu), x)$. Given that f is generalized transverse (cf. Definition 4.1), there exist $a^i \in \mathbb{R}$, for $i = 1, \ldots, m$, and $\widetilde{\omega} \in T_\alpha \mathbb{T}^{\kappa_1}$ such that $x = \sum_{i=1}^m a^i X_{i,f(\sigma)} + T_\alpha f_\beta(\widetilde{\omega})$. Thus,

$$\begin{aligned} v &= \text{lift}\left(Tf(\nu), \sum_{i=1}^{m} a^i X_{i,f(\sigma)} + T_\alpha f_\beta(\widetilde{\omega})\right) \\ &= \text{lift}\left(Tf(\nu), \sum_{i=1}^{m} a^i X_{i,f(\sigma)}\right) + \text{lift}\left(Tf(\nu), T_\alpha f_\beta(\widetilde{\omega})\right) \end{aligned}$$

Note that $T_\alpha f_\beta(\widetilde{\omega}) = Tf \circ h(\widetilde{\omega}, 0_\beta)$ where $h : T\mathbb{T}^{\kappa_1} \times T\mathbb{T}^{\kappa_2} \longrightarrow T\left(\mathbb{T}^{\kappa_1+\kappa_2}\right)$ denotes the natural identification defined, in coordinates, by $h\left((\theta_1, \omega_1), (\theta_2, \omega_2)\right) = (\theta_1, \theta_2, \omega_1, \omega_2)$. Hence

$$\begin{aligned} v &= \sum_{i=1}^{m} a^i X^{\text{lift}}_{i,Tf(v)} + \text{lift}\left(Tf(v), Tf \circ h(\widetilde{\omega}, 0_\beta)\right) \\ &= \sum_{i=1}^{m} a^i X^{\text{lift}}_{i,Tf(v)} + TTf\left(\text{lift}\left(v, h(\widetilde{\omega}, 0_\beta)\right)\right). \end{aligned}$$

Note that $\text{lift}\left(v, h(\widetilde{\omega}, 0_\beta)\right)$ is in $H\left((T_\omega T\mathbb{T}^{\kappa_1})^{\text{vert}} \times \{0\}_{\overline{h}(v)}\right)$, where $\overline{h} : T\mathbb{T}^{\kappa_1+\kappa_2} \longrightarrow T\mathbb{T}^{\kappa_2}$ is defined in coordinates by $\overline{h}(\theta_1, \theta_2, \omega_1, \omega_2) = (\theta_2, \omega_2)$, which completes the proof. ■

4.2. Control framework using generalized VTF

This section formalizes the way generalized vertically transverse functions are used for control purposes. The main objective is to obtain an expression for the closed-loop zero dynamics which, under the proposed methodology, shall be endowed with additional control inputs that may influence the target zero dynamics. Consider the target system (2.1) (rewritten as (4.2) below)

$$\dot{x} = S_x + \sum_{i=1}^{m} u^i X^{\text{lift}}_{i,x}, \tag{4.2}$$

and assume that $\{X_1, \ldots, X_m\} \subset \Gamma(TG)$ are left-invariant vector fields on G, an n-dimensional Lie group, which span a distribution that is completely nonintegrable at some point, say $e \in G$ without loss of generality. Under these conditions there exists a generalized transverse function $f : \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2} \longrightarrow G$ for $\{X_1, \ldots, X_m\}$ near $e \in G$ with $\kappa_1 = n - m$ (in the sequel we set $\kappa_1 = n - m$) and $\kappa_2 \geq 1$. So $Tf : T(\mathbb{T}^{\kappa_1+\kappa_2}) \longrightarrow G$ satisfies (4.1). The procedure herein developed is analogous to the methodology described in Section 2.3 and Chapter 3. Consider an auxiliary control system evolving on $T\mathbb{T}^{\kappa_1} \times T\mathbb{T}^{\kappa_2}$ of the form

$$\dot{\omega} = \Theta_\omega + \sum_{j=1}^{\kappa_1} v^j \Upsilon_{j,\omega}, \tag{4.3}$$

$$\dot{\eta} = \Pi_\eta + \sum_{k=1}^{\kappa_2} w^k \Psi_{k,\eta}, \tag{4.4}$$

where $\Theta \in \Gamma(TT\mathbb{T}^{\kappa_1})$ and $\Pi \in \Gamma(TT\mathbb{T}^{\kappa_2})$ are second-order vector fields, typically sprays that define flat, torsionless, affine connections on $T\mathbb{T}^{\kappa_1}$ and $T\mathbb{T}^{\kappa_2}$ respectively. The sets of vector fields $\{\Upsilon_1, \ldots, \Upsilon_{\kappa_1}\}$ and $\{\Psi_1, \ldots, \Psi_{\kappa_2}\}$ constitute global frames for $(TT\mathbb{T}^{\kappa_1})^{\text{vert}}$ and $(TT\mathbb{T}^{\kappa_2})^{\text{vert}}$, respectively. The auxiliary system (4.3)-(4.4) can be rewritten as a system evolving on $T\mathbb{T}^{\kappa_1+\kappa_2}$

by means of the natural diffeomorphism $h : T\mathbb{T}^{\kappa_1} \times T\mathbb{T}^{\kappa_2} \longrightarrow T\mathbb{T}^{\kappa_1+\kappa_2}$. For instance, consider the following commutative diagram:

$$\begin{array}{ccccc} TT\mathbb{T}^{\kappa_1} \times TT\mathbb{T}^{\kappa_2} & \xrightarrow{\sim} & T\left(\mathbb{T}^{\kappa_1} \times T\mathbb{T}^{\kappa_2}\right) & \xrightarrow{Th} & TT\mathbb{T}^{\kappa_1+\kappa_2} \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\pi_{T\mathbb{T}^{\kappa_1}\times T\mathbb{T}^{\kappa_2}}} & & \downarrow{\scriptstyle\pi_{T\mathbb{T}^{\kappa_1+\kappa_2}}} \\ T\mathbb{T}^{\kappa_1} \times T\mathbb{T}^{\kappa_2} & \xrightarrow{\mathrm{id}_{T\mathbb{T}^{\kappa_1}\times T\mathbb{T}^{\kappa_2}}} & T\mathbb{T}^{\kappa_1} \times T\mathbb{T}^{\kappa_2} & \xrightarrow{h} & T\mathbb{T}^{\kappa_1+\kappa_2} \end{array} \tag{4.5}$$

Let $\overline{\Delta} := (\Theta, \Pi)$ and define $\Delta = h_*\overline{\Delta}$. It is straightforward to show that Δ so defined is a second-order vector field on $T\mathbb{T}^{\kappa_1+\kappa_2}$; in fact it is a spray which defines a flat, torsionless, affine connection whenever Θ and Π are sprays defining flat, torsionless, affine connections. In an analogous way, set $\overline{\Upsilon}_j(\omega, \varpi) = \left(\Upsilon_{j,\omega}, 0_{\varpi}\right)$ for $j = 1, \ldots, \kappa_1$, and $\overline{\Psi}_k(\omega, \varpi) = (0_\omega, \Psi_{k,\varpi})$ for $k = 1, \ldots, \kappa_2$, for every $(\omega, \varpi) \in T\mathbb{T}^{\kappa_1} \times T\mathbb{T}^{\kappa_2}$, and define vertical vectors fields on $T\mathbb{T}^{\kappa_1+\kappa_2}$, $\Omega_j := h_*\overline{\Upsilon}_j$, $\Phi_k := h_*\overline{\Psi}_k$, for $j = 1, \ldots, \kappa_1$ and $k = 1, \ldots, \kappa_2$. Note that the set $\{\Omega_j, \Phi_k, j = 1, \ldots, \kappa_1, k = 1, \ldots, \kappa_2\} \subset \Gamma((TT\mathbb{T}^{\kappa_1+\kappa_2})^{\mathrm{vert}})$ defines a global frame for $\left(TT\mathbb{T}^{\kappa_1+\kappa_2}\right)^{\mathrm{vert}}$. Let $\nu = h(\omega, \eta)$, then system (4.3)-(4.4) can be written as

$$\dot{\nu} = \Delta_\nu + \sum_{i=1}^{\kappa_1} v^i \Omega_{i,\nu} + \sum_{j=1}^{\kappa_2} w^j \Phi_{j,\nu}. \tag{4.6}$$

Define an error signal $z := x \cdot Tf(\nu)^{-1}$ along the trajectories of the auxiliary and target systems (4.2)-(4.6). By differentiating along the trajectories of the compound system (4.2)-(4.6) (by virtue of Proposition 2.1), one checks that

$$\dot{z}(t) = TR_{Tf(\nu(t))^{-1}} \left(\dot{x}(t) - TL_z \circ TTf(\dot{\nu}(t)) \right),$$

for every $t \in \mathbb{R}$ for which the trajectories of the compound system are defined. Hence

$$\dot{z} = TR_{Tf(\nu)^{-1}} \left(\left(S_x + \sum_{i=1}^{m} u^i X_{i,x}^{\mathrm{lift}} \right) - TL_z \circ TTf \left(\Delta_\nu + \sum_{i=1}^{\kappa_1} v^i \Omega_{i,\nu} + \sum_{j=1}^{\kappa_2} w^j \Phi_{j,\nu} \right) \right).$$

Define a non-autonomous second-order vector field $D_{(\nu,w)} \in \Gamma(TTG)$, for each $(\nu, w) \in T\mathbb{T}^{\kappa_1+\kappa_2} \times \mathbb{R}^{\kappa_2}$, by

$$D_{(\nu,w)} : z \mapsto TR_{Tf(\nu)^{-1}} \left(S_{z\cdot Tf(\nu)} - TL_z \circ TTf \left(\Delta_\nu + \sum_{j=1}^{\kappa_2} w^j \Phi_{j,\nu} \right) \right).$$

Then, by making use of the left-invariance of the vector fields $X_i^{\mathrm{lift}},\ i = 1, \ldots, m$, and the fiberwise linearity of the tangent maps we obtain

$$\dot{z} = D_{(\nu,w)}(z) + TR_{Tf(\nu)^{-1}} \circ TL_z \left(\sum_{i=1}^{m} u^i X_{i,Tf(\nu)}^{\mathrm{lift}} - \sum_{i=1}^{\kappa_1} v^i\, TTf\, (\Omega_{i,\nu}) \right). \tag{4.7}$$

Proposition 4.3. *Given any second-order vector field $Y \in \Gamma(TTG)$ and $w = (w^1, \ldots, w^{\kappa_2}) \in \mathbb{R}^{\kappa_2}$, there is a unique smooth feedback law $\alpha = (\alpha^1, \ldots, \alpha^n) : TG \times T\mathbb{T}^{\kappa_1+\kappa_2} \times \mathbb{R}^{\kappa_2} \longrightarrow \mathbb{R}^n$ such that the error dynamics (4.7), satisfies $\dot{z}(t) = Y_{z(t)}$.*

Proof. The proof boils down to setting the right-hand side of (4.7) equal to Y_z and solving for $(u, v) = (u^1, \ldots, u^m, v^1, \ldots v^{\kappa_1})$. This yields

$$D_{(\nu,w)}(z) + TR_{Tf(\nu)^{-1}} \circ TL_z \left(\sum_{i=1}^{m} u^i X^{\text{lift}}_{i,Tf(\nu)} - \sum_{i=1}^{\kappa_1} v^i \, TTf \, (\Omega_{i,\nu}) \right) = Y_z.$$

Rearranging one has

$$TL_{z^{-1}} \circ TR_{Tf(\nu)} \left(Y_z - D_{(\nu,w)}(z) \right) = \left(\sum_{i=1}^{m} u^i X^{\text{lift}}_{i,Tf(\nu)} - TTf \left(\sum_{i=1}^{\kappa_1} v^i \, \Omega_{i,\nu} \right) \right).$$

Note that, in reference to the commutative diagram (4.5), $\Omega_i = h_* \bar{\Upsilon}_j = H(\Upsilon_j, 0_{\varpi})$, for $j = 1, \ldots, \kappa_1$. Hence, given that Tf satisfies (4.1) and $\{\Upsilon_1, \ldots, \Upsilon_{\kappa_1}\}$ is a global frame for $(TT\mathbb{T}^{\kappa_1})^{\text{vert}}$, there exists a mapping $\alpha : TG \times T\mathbb{T}^{\kappa_1+\kappa_2} \times \mathbb{R}^{\kappa_2} \longrightarrow \mathbb{R}^n$ such that, for every $(z, \nu) \in TG \times T\mathbb{T}^{\kappa_1+\kappa_2}$,

$$\left(\sum_{i=1}^{m} \alpha^i(z, \nu, w) X^{\text{lift}}_{i,Tf(\nu)} - \sum_{i=1}^{\kappa_1} \alpha^{i+m}(z, \nu, w) \, TTf \circ \Omega_{i,\nu} \right) = TL_{z^{-1}} \circ TR_{Tf(\nu)} \left(Y_z - D_{(\nu,w)}(z) \right).$$

One can easily show that α so defined is smooth. ∎

In accordance with the previous proposition, by appropriately selecting a feedback $\alpha = (\alpha^1, \ldots, \alpha^n)$ for (4.2)-(4.6) in terms of the error state $z(t)$, the state of the auxiliary system $\nu(t)$ and of the additional control inputs $w = (w^1, \ldots, w^{\kappa_2})$, the error dynamics can be arbitrarily set. In particular, assume, as in Chapter 3, that such a feedback is obtained by requiring that the error dynamics (4.7) satisfy a second-order differential equation defined by $Y \in \Gamma(TTG)$ and that the latter admits $0_e \in TG$ as an asymptotically stable equilibrium point.

In such a case, the behavior of the closed-loop system is qualitatively analogous to the behavior of the closed-loop system under the VTFA, in the sense that the state of the target system (4.2) converges to the image by Tf of the trajectory of the auxiliary system (4.6), provided that the solutions of the closed-loop system are defined. Given that, by definition, $f(\mathbb{T}^{\kappa_1+\kappa_2})$ is contained in a predefined neighborhood $\mathcal{U}$ of e, the configuration coordinates of the target system approach the desired equilibrium configuration, independently of how the extra control inputs $w = (w^1, \ldots, w^{\kappa_2})$ are chosen. The ensuing objective is to design these additional inputs, in terms of the auxiliary system state $\nu \in T\mathbb{T}^{\kappa_1+\kappa_2}$, such that the zero-section of the auxiliary zero-dynamics $Z(T\mathbb{T}^{\kappa_1+\kappa_2})$ is locally attractive for, in that case, the target system velocities vanish asymptotically.

The closed-loop zero dynamics is obtained by setting $z = 0_e$ in (4.7),

$$\left(S_{Tf(\nu)} + \sum_{i=1}^{m} \alpha^i(0_e, \nu, w) X^{\mathrm{lift}}_{i,Tf(\nu)}\right) = TTf\left(\Delta_\nu + \sum_{i=1}^{\kappa_1} \alpha^{i+m}(0_e, \nu, w)\Omega_{i,\nu} + \sum_{j=1}^{\kappa_2} w^j \Phi_{j,\nu}\right). \tag{4.8}$$

Observe that the target zero dynamics is completely determined by the auxiliary zero dynamics. The latter is given by the right-hand member of equation (4.8),

$$\dot{\nu} = \Delta_\nu + \sum_{i=1}^{\kappa_1} \alpha^{i+m}(0_e, \nu, w)\Omega_{i,\nu} + \sum_{j=1}^{\kappa_2} w^j \Phi_{j,\nu}. \tag{4.9}$$

The auxiliary zero dynamics can be realized as a control system on $T\mathbb{T}^{\kappa_1+\kappa_2}$ with control inputs w^j, $j = 1, \ldots, \kappa_2$. Note that these control inputs also determine the zero-error feedback terms α^{i+m}, $i = 1, \ldots, \kappa_1$. If one manages to design $w^1, \ldots, w^{\kappa_2}$ in terms of the auxiliary system state ν such that the zero-section of $T(\mathbb{T}^{\kappa_1+\kappa_2})$ is (locally) attractive, the effect should be that the velocities of the auxiliary and target systems vanish asymptotically. Although stabilizing only a subset, e.g. the zero-section, of the state manifold may seem more relaxed an aim than stabilizing a point, the zero dynamics (4.9) may bear some undesirable properties. For instance, it may be critical and may also fail to be accessible but at generic points. The given structure of the zero dynamics depends, to a large degree, on the structure of the GTF used. Therefore, the goal of having the velocities asymptotically vanish apparently calls for the use of time-varying feedback. In Section 4.3 we explore the application of high-order averaging methods, as exposed in (Vela [2003]; Agračhev and Gramkrelidze [1979]; Sarychev [2001]), with a view toward the design of time-varying feedback to render the zero-section (locally) attractive for the auxiliary zero dynamics. An example is also developed which illustrates, via a numerical simulation, that the asymptotic convergence of the velocities to zero may be achievable by means of time-varying feedback laws.

As an additional approach, one may also try to introduce dissipation into the auxiliary zero dynamics by means of Jurdjevic-Quinn (damping control) method (cf. Bacciotti and Rosier [2001]). The application of this method requires the knowledge of a "weak" Lyapunov function for the uncontrolled system (system (4.9) with $w^1, \ldots, w^{\kappa_2} = 0$). That is, the knowledge of a differentiable, positive-definite function $V : \mathbb{T}^{\kappa_1+\kappa_2} \longrightarrow \mathbb{R}$ such that $V(\nu) = 0$ if $\nu = 0$, and $\mathcal{L}_{\widehat{\Delta}} V(\nu) \leq 0$ for every $\nu \in \mathbb{T}^{\kappa_1+\kappa_2}$, where

$$\widehat{\Delta}_\nu := \Delta_\nu + \sum_{i=1}^{\kappa_1} \alpha^{i+m}(0_e, \nu, 0)\Omega_{i,\nu}. \tag{4.10}$$

In the case that the spray $\widehat{\Delta}$ admits a Riemannian metric $\mathcal{G}$, such a function can be naturally defined by setting $V(\nu) := \frac{1}{2}g(\nu, \nu)$. It is readily verified that V is positive definite and that the derivative of V along $\widehat{\Delta}$ is zero. However, as stated in Chapter 3, determining and finding a (pseudo-) Riemannian metric for a given spray $\widehat{\Delta}$ is, in general, an untractable problem, given the overdetermined nature of the Levi-Civita metric differential equations.

In the case that the VTFA (described in Section 2.3) yields a zero dynamics which admits a (pseudo–) Riemannian metric, there exists a GTF such that the approach described in this section yields a (non-autonomous) zero dynamics whose drift vector field, defined by (4.10), admits a (pseudo–) Riemannian metric. In other words, the resulting non-autonomous zero dynamics is a Levi-Civita connection system for some metric on $\mathbb{T}^{\kappa_1+\kappa_2}$. Let us make this result precise.

Theorem 4.1 (Existence of a metric for a particular case)**.** *Let $g : \mathbb{T}^{\kappa_1} \longrightarrow G$, with $n = \dim(G)$, be transverse for $\{X_1, \ldots, X_m\} \subset \Gamma(TG)$ near $e \in G$ with $\kappa_1 = n - m$. Assume that the application of the VTFA to (4.2), with auxiliary control system (4.3), results in an auxiliary zero dynamics $\dot{\omega} = \Sigma_\omega$ which admits a metric. Then, the use of a GTF $f : \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2} \longrightarrow G$ of the form $f = g \circ \chi$, where $\chi : \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2} \longrightarrow \mathbb{T}^{\kappa_1}$ is smooth, yields a non-autonomous zero dynamics whose drift vector field admits a metric.*

Proof. It is straightforward to show that $f = g \circ \chi : \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_1} \longrightarrow G$ is generalized transverse for every smooth $\chi : \mathbb{T}^{\kappa_1} \times \mathbb{T}^{\kappa_2} \longrightarrow \mathbb{T}^{\kappa_1}$. Let $\sigma = (\sigma^1, \ldots, \sigma^n) : TG \times T\mathbb{T}^{\kappa_1} \longrightarrow \mathbb{R}^n$ be the feedback resulting of applying the VTFA (cf. Theorem 2.3) by requiring the error dynamics satisfy $\dot{\bar{z}} = Y_{\bar{z}}$ with $Y \in \Gamma(TTG)$ a second-order vector field having $0_e \in TG$ as an asymptotically stable point. In other words, $\sigma : TG \times T\mathbb{T}^{\kappa_1} \longrightarrow \mathbb{R}^n$ satisfies

$$\sum_{i=1}^{m} \sigma^i(z, \omega)\, X^{\text{lift}}_{i,Tg(\omega)} - \sum_{i=1}^{\kappa_1} \sigma^{i+m}(z, \omega)\, TTg(\Upsilon_{i,\omega}) = \\ TL_{z^{-1}}\left(TR_{Tg(\omega)}(Y_z) - S_{z\cdot Tg(\omega)}\right) + TTg(\Theta_\omega). \quad (4.11)$$

Define a second-order auxiliary control system on $T\mathbb{T}^{\kappa_1+\kappa_2}$,

$$\dot{\nu} = \Delta_\nu + \sum_{i=1}^{\kappa_1} v^i\, \Omega_{i,\nu} + \sum_{j=1}^{\kappa_2} w^j\, \Phi_{j,\nu}, \quad (4.12)$$

such that Δ is $T\chi$-related to Θ, and Ω_i is $T\chi$-related to Υ_i, for $i = 1, \ldots, \kappa_1$. Also define mappings $\widetilde{\alpha} = (\widetilde{\alpha}^1, \ldots, \widetilde{\alpha}^n) : TG \times T\mathbb{T}^{\kappa_1+\kappa_2} \longrightarrow \mathbb{R}^n$ and $\overline{\alpha} = (\overline{\alpha}^1, \ldots, \overline{\alpha}^n) : T\mathbb{T}^{\kappa_1+\kappa_2} \times \mathbb{R}^{\kappa_2} \longrightarrow \mathbb{R}^n$ such that they satisfy

$$\sum_{i=1}^{m} \widetilde{\alpha}^i(z, \nu) X^{\text{lift}}_{i,Tf(\nu)} - \sum_{i=1}^{\kappa_1} \widetilde{\alpha}^{i+m}(z, \nu)\, TTf\,(\Omega_{i,\nu}) = \\ TL_{z^{-1}}\left(TR_{Tf(\nu)}(Y_z) - S_{z\cdot Tf(\nu)}\right) + TTf(\Delta_\nu), \quad (4.13)$$

and

$$\sum_{i=1}^{m} \overline{\alpha}^i(\nu, w) X^{\text{lift}}_{i,Tf(\nu)} - \sum_{i=1}^{\kappa_1} \overline{\alpha}^{i+m}(\nu, w)\, TTf\,(\Omega_{i,\nu}) = TTf\left(\sum_{j=1}^{\kappa_2} w^j\, \Phi_{j,\nu}\right), \quad (4.14)$$

respectively. It is straightforward to check (cf. Proposition 4.3) that $\alpha = (\widetilde{\alpha} + \overline{\alpha}) : TG \times T\mathbb{T}^{\kappa_1+\kappa_2} \times \mathbb{R}^{\kappa_2} \longrightarrow \mathbb{R}^n$ is such that the error dynamics (4.7) writes as $\dot{z} = Y_z$. In particular, the auxiliary dynamics is given by

$$\dot{\nu} = \Delta_\nu + \sum_{i=1}^{\kappa_1} \widetilde{\alpha}^{i+m}(z,\nu)\Omega_{i,\nu} + \left(\sum_{i=1}^{\kappa_1} \overline{\alpha}^{i+m}(\nu,w)\Omega_{i,\nu} + \sum_{j=1}^{\kappa_2} w^j \Phi_{j,\nu} \right).$$

The long-term behavior is determined by the auxiliary zero dynamics obtained by setting $z = 0$ in the equation above. This yields

$$\dot{\nu} = \widehat{\Delta}_\nu + \sum_{i=1}^{\kappa_1} \overline{\alpha}^{i+m}(\nu,w)\Omega_{i,\nu} + \sum_{j=1}^{\kappa_2} w^j \Phi_{j,\nu}, \tag{4.15}$$

with

$$\widehat{\Delta}_\nu := \Delta_\nu + \sum_{i=1}^{\kappa_1} \widetilde{\alpha}^{i+m}(0_e,\nu)\Omega_{i,\nu}.$$

Note that $\widehat{\Delta} \in \Gamma(TT\mathbb{T}^{\kappa_1+\kappa_2})$, which is second-order, is the drift vector field of the auxiliary zero dynamics since $\overline{\alpha}^{i+m}(\nu,0)$ equals zero by virtue of equation (4.14). Thus we have to prove that $\widehat{\Delta}$ admits a metric. Note that equation (4.13) can be written as

$$\sum_{i=1}^{m} \widetilde{\alpha}^i(z,\nu) X^{\text{lift}}_{i,Tg\circ T\chi(\nu)} - \sum_{i=1}^{\kappa_1} \widetilde{\alpha}^{i+m}(z,\nu)\, TTg \circ TT\chi\,(\Omega_{i,\nu}) =$$
$$TL_{z^{-1}}\left(TR_{Tg\circ T\chi(\nu)}(Y_z) - S_{z\cdot Tg\circ T\chi(\nu)} \right) + TTg \circ TT\chi(\Delta_\nu),$$

By construction of the auxiliary system in (4.12), $TT\chi(\Omega_{i,\nu}) = \Upsilon_{i,T\chi(\nu)}$ and $TT\chi(\Delta_\nu) = \Theta_{T\chi(\nu)}$ so

$$\sum_{i=1}^{m} \widetilde{\alpha}^i(z,\nu) X^{\text{lift}}_{i,Tg\circ T\chi(\nu)} - \sum_{i=1}^{\kappa_1} \widetilde{\alpha}^{i+m}(z,\nu)\, TTg\left(\Upsilon_{i,T\chi(\nu)}\right) =$$
$$TL_{z^{-1}}\left(TR_{Tg\circ T\chi(\nu)}(Y_z) - S_{z\cdot Tg\circ T\chi(\nu)} \right) + TTg\left(\Theta_{i,T\chi(\nu)}\right),$$

Therefore $\widetilde{\alpha}(z, T\chi(\nu))$ coincides with $\sigma(z,\omega)$ in equation (4.11), so the spray Σ and $\widehat{\Delta}$ are $T\chi$-related and, therefore, if Σ admits a metric on $T\mathbb{T}^{\kappa_1}$, $\widehat{\Delta}$ admits a $T\chi$-induced metric on $T\mathbb{T}^{\kappa_1+\kappa_2}$. ■

However it is still an open problem to determine when, for a given generalized transverse function, the resulting non-autonomous zero dynamics satisfies appropriate conditions to enable the introduction of damping so that the velocity coordinates approach zero as time increases (see cf. Bacciotti [1992]).

4.3. Rendering the zero section locally attractive via high-order averaging

In this section we explore the use of high-order averaging to design time-varying feedbacks with the objective of rendering the zero-section locally attractive for the resulting auxiliary zero dynamics (4.9). The choice of a given transverse function plays an important role for it defines the structure of the resulting auxiliary zero dynamics (4.9) and, consequently desirable properties such as local accessibility. In general, the resulting zero dynamics is a second-order control system that may be critical; therefore, dealing with asymptotic stabilization apparently calls for the use of time-varying techniques.

High-order averaging (Sarychev [2001]; Vela [2003]) is based on the formalism of chronological calculus developed by Agračhev and Gramkrelidze [1979]. In essence, the latter aims at reducing the qualitative analysis of the flow of periodic non-autonomous (i.e., time-varying) vector fields to the analysis of autonomous (i.e., time invariant) vector fields by means of asymptotic expansions. High-order averaging has proved to be useful in devising time-varying feedback laws to asymptotically stabilize fixed points for driftless control systems and second-order systems (see Vela [2003]). However, in typical situations, it is difficult to draw conclusions regarding the asymptotic stability of the closed-loop system (see Sarychev [2001]).

In this section we review high-order averaging (Sarychev [2001]; Vela [2003]) and its application to design time-varying feedbacks for stabilization of driftless control systems and second-order control systems as reported by Vela [2003]. We also present an example of its application to design time-varying control laws with the objective of making the zero-section locally attractive for the zero dynamics of the controlled ECF for a given generalized transverse function. We also present a numerical simulation of the closed-loop system which suggests promising results.

4.3.1. High-order averaging theory revisited.

In this subsection we summarize basic concepts of chronological calculus as presented by Agračhev and Gramkrelidze [1979] and high-order averaging theory as presented by Sarychev [2001] and Vela [2003] with a view toward the design of time-varying feedback to render the zero section (locally) attractive for the zero dynamics resulting from the use of GVTF. The Reader may consult (Vela [2003]) for more details on this subject.

Consider a periodic system on $\mathbb{R}^n$ written in the *standard form for averaging*, that is

$$\dot{x} = \lambda X(x,t), \tag{4.16}$$

with X a T-periodic time-varying vector field on $\mathbb{R}^n$, i.e., $X(\cdot\,,t) = X(\cdot\,,t+T)$ for every $t \in \mathbb{R}$, and λ a "small" strictly positive real. The "classical" (*first-order*) average of X (cf. Khalil [1996]) is an autonomous vector field defined, for $x \in \mathbb{R}^n$, by $\overline{X}(x) = \frac{1}{T}\int_{t_0}^{t_0+T} X_\tau(x)\,d\tau$, where $X_\tau := X(\cdot,\tau)$. Under certain conditions $\overline{X}$ may preserve certain properties of X. For instance, assuming that X is continuous and bounded and have continuous and bounded partial derivatives up to second-order with respect to x, if $x^* \in \mathbb{R}^n$ is an equilibrium for (4.16) which is exponentially stable for $\overline{X}$, then there exists $\lambda_0 > 0$ such that for every $\lambda \le \lambda_0$, x^* is locally exponentially stable for (4.16).

High-order averaging (Sarychev [2001]) provides one with more general expressions for averaging computed from truncates of asymptotic series expansions approximating X_τ which may determine stability for cases for which the first-order averaging is not conclusive. These asymptotic expansions are due to Agrač hev and Gramkrelidze [1979] and are based on the Volterra series expansion of the solution of a time-varying differential equation of the form

$$\dot{x}(t) = X_t(x(t)), \tag{4.17}$$

where X_t is a vector field on $\mathbb{R}^n$ for every $t \in \mathbb{R}$ and where it is assumed X to be absolutely continuous on t. (This can be generalized to the case of time-varying vector fields on an n-dimensional smooth manifold M.) Agračhev and Gramkrelidze [1979] address this problem by using a formalism that is "dual" to the standard point of view. *Grosso modo*, the idea is to replace a nonlinear object by a linear, although infinite-dimensional, one. For instance, a point $x \in M$ defines a linear functional $\widehat{x} : C^\infty(M) \longrightarrow \mathbb{R}$ by $\widehat{x}(f) = f(x)$ for $f \in C^\infty(M)$. In turn a vector field $X \in \Gamma^\infty(TM)$ defines a linear functional $\widehat{X} : C^\infty(M) \longrightarrow C^\infty(M)$ such that $\widehat{X}(f)(x) = X_x(f)$ for $x \in M$, and a diffeomorphism $P : M \longrightarrow M$ defines an automorphism $\widehat{P} : C^\infty(M) \longrightarrow C^\infty(M)$ of the algebra $C^\infty(M)$ by $(\widehat{P}(f))(x) = f(P(x))$ for $x \in M$ and $f \in C^\infty(M)$. So, equation (4.17) corresponds to a differential equation given, in the dual formalism, by

$$\dot{\widehat{x}}(t) = \widehat{x}(t) \circ \widehat{X}_t. \tag{4.18}$$

(For further details the Reader may consult the book of Agračhev and Sachkov [2004] and the work of Agračhev and Gramkrelidze [1979]). The solution $\widehat{x}(t)$ to the differential equation above, for a given initial condition $(\widehat{x}_0, t_0)$ at time t, may be obtained by the Volterra series expansion, which in the terminology of Agračhev and Gramkrelidze [1979], is called the *(right) chronological exponential* of the vector field X_t, and is denoted by $\overrightarrow{\exp}\left(\int_{t_0}^t X_\tau \, d\tau\right)$.

Averaging theory concerns with the existence of a (time-parameterized,) autonomous vector field $\overrightarrow{V}_{(t_0,t)}(X_\tau) \in \Gamma(TM)$ such that its flow at unit time, with initial time $t_0' = 0$ coincides with the flow of X in equation (4.17) at time t with initial time t_0, i.e., such that

$$\overrightarrow{\exp}\left(\int_{t_0}^t X_\tau \, d\tau\right) \circ \mathrm{id} = \exp\left(\overrightarrow{V}_{(t_0,t)}(X_\tau)\right).$$

A series expansion exists for this vector field and, assuming convergence, it is called the *logarithm of the chronological exponential*:

$$\overrightarrow{V}_{(t_0,t)}(X_\tau) = \sum_{m=1}^{\infty} \overrightarrow{V}^{(m)}_{(t_0,t)}(X_\tau) = \ln\left(\overrightarrow{\exp}\left(\int_{t_0}^t X_\tau d\tau\right) \circ \mathrm{id}\right). \tag{4.19}$$

The term $\overrightarrow{V}^{(m)}_{(t_0,t)}(X_\tau)$ for $m = 1, \ldots, \infty$, is called the *m-th variation of the identity flow corresponding to the perturbation field X_τ*. The Reader may consult some convergence results given by Agračhev and Gramkrelidze [1979].

In reference to a T-periodic system in the standard form for averaging (4.16), one concerns in obtaining an *averaged system*

$$\dot{z} = Z(z), \tag{4.20}$$

where the autonomous vector field Z is defined by

$$\exp(ZT) = \overrightarrow{\exp}\left(\int_0^T X_\tau \, d\tau\right) \circ \mathrm{id},$$

or equivalently, by

$$Z = \frac{1}{T} \ln \overrightarrow{\exp}\left(\int_0^T X_\tau \, d\tau\right).$$

Z is such that the trajectories of (4.16) coincide with the trajectories of (4.20) up to a *time-periodic* diffeomorphism or flow, i.e., $x(t) = P(t, z(t))$ and $P(t+T, \cdot) = P(t, \cdot)$, where P is called the Floquet mapping, (cf. Vela [2003]). Typically, however, Z is very difficult to compute explicitly, and the way to circumvent this difficulty is by using an infinite series expansion of the form $Z \cong \frac{1}{T}\sum_{i=1}^{\infty} \lambda^i \Lambda^{(i)}(X_\tau)$, where $\Lambda^{(i)}(X_\tau) = \overrightarrow{V}^{(m)}_{(t_0,t)}(X_\tau)$. By analyzing the mth-partial sum (or "truncate") $\mathrm{Trunc}_m(Z) = \frac{1}{T}\sum_{i=1}^{m} \lambda^i \Lambda^{(i)}(X_\tau)$, one may infer some stability properties of (4.16) for sufficiently small values of λ.

The first terms $\Lambda^{(1)}, \ldots, \Lambda^{(3)}$ may be expressed in terms of the Lie bracket (cf. Sarychev [2001] and Vela [2003]).

$$\begin{aligned}
\Lambda^{(1)} &= \overline{X} \\
\Lambda^{(2)} &= \overline{\frac{1}{2}\left[\int_0^t X_\tau \, d\tau, X_t\right]} \\
\Lambda^{(3)} &= \frac{1}{2}T\left[\Lambda^{(1)}, \Lambda^{(2)}\right] + \overline{\frac{1}{3}\left[\int_0^\tau X_{\tau_1}\, d\tau_1, \left[\int_0^\tau X_{\tau_1}\, d\tau_1, X_\tau\right]\right]} \\
\Lambda^{(4)} &= -\frac{1}{12}\overline{\int_0^\tau \left[\int_0^{\tau_1}\left[\int_0^{\tau_2} X_{\tau_3}\, d\tau_3, X_{\tau_2}\right] d\tau_2, [X_{\tau_1}, X_\tau]\right] d\tau_1} \\
&\quad -\frac{1}{12}\overline{\left[\int_0^\tau \left[\int_0^{\tau_1}\left[\int_0^{\tau_2} X_{\tau_3}\, d\tau_3, X_{\tau_2}\right] d\tau_2, X_{\tau_1}\right] d\tau_1, X_\tau\right]} \\
&\quad -\frac{1}{12}\overline{\int_0^\tau \left[\int_0^{\tau_1} X_{\tau_2}\, d\tau_2, \left[\left[\int_0^{\tau_1} X_{\tau_2}\, d\tau_2, X_{\tau_1}\right], X_\tau\right]\right] d\tau_1}
\end{aligned}$$

High-order averaging of driftless control systems

In this subsubsection we give the explicit forms for the first order truncates of averaging, in terms of Lie brackets and integrals of the control inputs, of the series approximating driftless control systems. Consider a driftless control system on $\mathbb{R}^n$ of the form

$$\dot{q} = \sum_{i=1}^{m} u^i X_{i,q}.$$

Assume that $n > m$ and that the system is controllable. Consider time-periodic feedback of the form $u^i(q,t) = f^i(q) + v^i(t/\lambda)$ for $i = 1, \ldots m$, where the terms f^i are intended to stabilize the directly controlled variables of the state. Substitution yields

$$\dot{q} = X_{S,q} + \sum_{i=1}^{m} v^i(t/\lambda) X_{i,q},$$

where $X_S = \sum_{i=1}^{m} f^i(q) X_i(q)$. By making a time transformation $t \mapsto \lambda\tau$ we obtain a system in the standard form for averaging:

$$\frac{dq}{dt} = \lambda\left(X_{S,q} + \sum_{i=1}^{m} v^i(\tau) X_{i,q}\right).$$

Define the following integral terms of the time-varying part of the input proposed.

$$V_{(n)}^{(i)}(t) = \int_{t_0}^{t}\int_{t_0}^{s_{n-1}} \cdots \int_{t_0}^{s_2} v^i(s_1)\, ds_1 \ldots ds_{n-1}.$$

Also define $V_{(n_1,\ldots,n_k)}^{(i_1,\ldots,i_k)}(t) = V_{(n_1)}^{(i_1)}(t) \cdots V_{(n_k)}^{(i_k)}(t)$ for $k \geq 1$, $i, i_k = 1, \ldots, m$ and, in a similar manner, define integrals and averages of these terms. For instance $V_{(\hat{n})}^{(\hat{i})}(t) = \int_{t_0}^{t} V_{(n)}^{(i)}(\tau)\, d\tau$ and $\overline{V_{(n)}^{(i)}(t)} = \frac{1}{T}\int_{t_0}^{t} V_{(n)}^{(i)}(\tau)\, d\tau$ and the difference between the integral and averaged terms by $\widetilde{V}_{(n)}^{(i)} = V_{(n)}^{(i)} - \overline{V_{(n)}^{(i)}}(\tau)$. Then, the averaged system of order $m = 1, 2, 3$ in the time variable t is given by

$$\dot{z} = \frac{1}{\lambda}\mathrm{Trunc}_m(Z),$$

where,

$$\mathrm{Trunc}_1(Z)_z = X_S(z) + \overline{V_{(0)}^{(i)}(t)} Y_i(z),$$

and assuming that the first order time averages of the inputs v^i vanish, i.e., $\overline{V_{(0)}^{(i)}(t)} = 0$, the second and third order averaged truncates are given by

$$\begin{aligned} \mathrm{Trunc}_2(Z)_z &= \mathrm{Trunc}_1(Z)_z + \lambda\, \overline{V_{(1)}^{(i)}(t)}\, [Y_i, X_S]_z + \frac{1}{2}\lambda\, \overline{V_{(1,0)}^{(i,j)}(t)} \left[Y_i, Y_j\right]_z , \end{aligned} \tag{4.21}$$

$$\begin{aligned} \mathrm{Trunc}_3(Z)_z &= \mathrm{Trunc}_2(Z)_z + \lambda^2 \left(\overline{V_{(2)}^{(i)}(t)} - \frac{1}{2} T\, \overline{V_{(1)}^{(i)}(t)} \right) [X_S, [X_S, Y_i]]_z \\ &\quad - \frac{1}{3}\lambda^2 \left(\overline{V_{\widehat{(1,0)}}^{\widehat{(i,j)}}(t)} - \frac{1}{2} T\, \overline{V_{(1,0)}^{(i,j)}(t)} \right) \left[X_S, \left[Y_i, Y_j\right]\right]_z \\ &\quad + \frac{1}{3}\lambda^2 \left(\overline{V_{(1,1)}^{(i,j)}(t)} + \overline{V_{\widehat{(1,0)}}^{\widehat{(i,j)}}(t)} - T\, \overline{V_{(1,0)}^{(i,j)}(t)} \right) \left[Y_i, \left[Y_j, X_S\right]\right]_z \\ &\quad + \frac{1}{3}\lambda^2 \overline{V_{(1,1,0)}^{(i,j,k)}(t)} \left[Y_i, \left[Y_j, Y_k\right]\right]_z . \end{aligned} \tag{4.22}$$

As an example of the application of high-order averaging to intuitively design time-varying feedback, consider the stabilization of zero ($0 \in \mathbb{R}^3$) for the chained form (3-CF).

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= u_1 x_2 . \end{aligned} \tag{4.23}$$

As above, consider a feedback law $u : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}^3$ of the form $u(x,t) = f(x) + v(t/\lambda)$, where f involves stabilizing terms linear in the directly controlled components of the state. For instance choose u to be defined by

$$\begin{aligned} u_1(x,t) &= -k_1 x_1 + \alpha \sin(t/\lambda) \\ u_2(x,t) &= -k_2 x_2 + \beta \cos(t/\lambda), \end{aligned} \tag{4.24}$$

where α and β are to be defined. Given that sin and cos are zero-average, periodic functions, first-order averaging applied to (4.23)-(4.24) yields:

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1 \\ \dot{x}_2 &= -k_2 x_2 \\ \dot{x}_3 &= -k_1 x_1 x_2 . \end{aligned} \tag{4.25}$$

The first-order averaging is an autonomous system for which we cannot conclude stability of zero. However, consider the second-order averaging expression:

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1 \\ \dot{x}_2 &= -k_2 x_2 \\ \dot{x}_3 &= -k_1 x_1 x_2 + \tfrac{1}{2}\lambda\alpha\beta . \end{aligned} \tag{4.26}$$

Note that if we select $\alpha = -2\,\mathrm{sign}(k_3 x_3)\sqrt{|k_3 x_3|}$ and $\beta = \sqrt{|k_3 x_3|}$ with $k_3 > 0$, then zero is exponentially stable for (4.26). In fact, it can be shown, by using homogeneity arguments, that the feedback (4.24) with the chosen α, β, i.e.,

$$\begin{aligned} u_1(x,t) &= -k_1 x_1 - 2\,\mathrm{sign}(k_3 x_3)\sqrt{|k_3 x_3|}\, \sin(t/\lambda) \\ u_2(x,t) &= -k_2 x_2 + \sqrt{|k_3 x_3|}\, \cos(t/\lambda), \end{aligned} \tag{4.27}$$

applied to the 3-CF yields the closed-loop system having zero as asymptotically stable equilibrium point. Figure 4.1 shows a numerical simulation of the averaged system (4.26) with the chosen α and β, and of the closed-loop system (4.23)–(4.27) for an arbitrarily selected initial condition $x_0 = (0.4, -1.2, 1.8)$ and $\lambda = 0.5$.

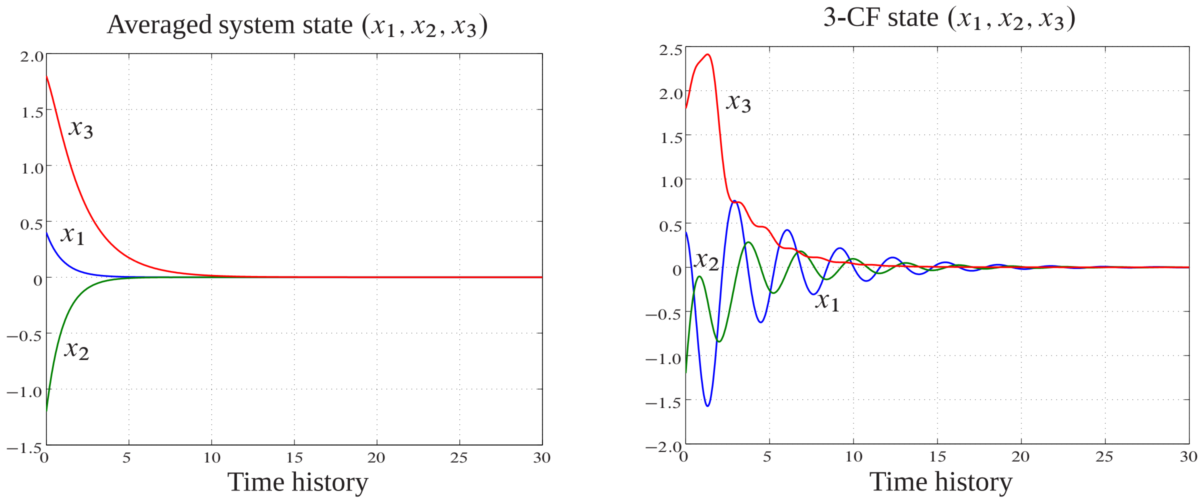


Figure 4.1: Numerical simulation for the averaged system (4.26) (left) and for the closed-loop system (4.23)–(4.27) (right).

High-order averaging for second-order control systems

These ideas can also be used to design time-varying feedback for second-order systems, as proposed by Vela [2003]. Consider the class of systems on $T\mathbb{R}^n$ of the form

$$\dot{x} = X_x + \sum_{i=1}^{m} u^i Y_{i,x}^{\mathrm{lift}}, \tag{4.28}$$

where X is a spray and Y_i, $i = 1, \ldots, m$, a vector field on $\mathbb{R}^n$. The continuous mapping $u^i : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ is assumed to have the form $u^i(x,t) = f^i(x) + \frac{1}{\lambda} v^i(t/\lambda)$ for $i = 1, \ldots, m$, with $\lambda > 0$. Following the procedure in (Vela [2003]), we rescale the time variable and consider a truncate of the "nonlinear variation of constants" for (4.28) which yields a system in the standard form for averaging (4.16):

$$\dot{x} \;=\; \lambda\left(X_{S,x} + V_{(1)}^{(i)}(t) \left[Y_i^{\mathrm{lift}}, X_S\right]_x - \frac{1}{2} V_{(1,1)}^{(i,j)}(t) \left\langle Y_i : Y_j \right\rangle_x^{\mathrm{lift}} \right). \tag{4.29}$$

Here, $X_S = X + \sum_{i=1}^{m} f^i Y_i^{\mathrm{lift}}$ and the operator $\langle \cdot : \cdot \rangle$, a generalization of the notion of symmetric product (cf. Bullo and Lewis [2005]), is given by $\langle X : Y \rangle^{\mathrm{lift}} = \left[X^{\mathrm{lift}}, \left[X_S, Y^{\mathrm{lift}}\right]\right]$.

Assuming that the first order time averages of the inputs v^i vanish, i.e., $\overline{V^{(i)}_{(0)}(t)} = \overline{v^i(t)} = 0$, the first and second-order averaged truncates in terms of Lie brackets and symmetric products of the drift and control vector fields in (4.29) are

$$\mathrm{Trunc}_1(Z)_z \;=\; X_{S,z} + \overline{V^{(i)}_{(1)}(t)}\,\left[Y_i^{\mathrm{lift}}, X_S\right]_z - \frac{1}{2}\overline{V^{(i,j)}_{(1,1)}(t)}\,\left\langle Y_i : Y_j\right\rangle_z^{\mathrm{lift}}, \tag{4.30}$$

$$\begin{aligned}\mathrm{Trunc}_2(Z)_z \;=\; & \mathrm{Trunc}_1(Z)_z + \lambda\left(\overline{V^{(i)}_{(2)}(t)} - \frac{1}{2}T\,\overline{V^{(i)}_{(1)}(t)}\right)\left[\left[Y_i^{\mathrm{lift}}, X_S\right], X_S\right]_z \\ & -\frac{1}{2}\lambda\left(\overline{V^{\widehat{(i,j)}}_{\widehat{(1,1)}}(t)} - \frac{1}{2}T\,\overline{V^{(i,j)}_{(1,1)}(t)}\right)\left[\left\langle Y_i : Y_j\right\rangle^{\mathrm{lift}}, X_S\right]_z \\ & +\frac{1}{2}\lambda\overline{V^{(i,j)}_{(2,1)}(t)}\left[\left[Y_i^{\mathrm{lift}}, X_S\right], \left[Y_j^{\mathrm{lift}}, X_S\right]\right]_z \\ & +\frac{1}{2}\lambda\left(\overline{V^{(i,j,k)}_{(2,1,1)}(t)} - \frac{1}{2}T\,\overline{V^{(i)}_{(1)}(t)V^{(j,k)}_{(1,1)}(t)}\right)\left\langle Y_i : \left\langle Y_j : Y_k\right\rangle\right\rangle_z^{\mathrm{lift}}. \end{aligned} \tag{4.31}$$

Suppose that the objective is to render the zero-section locally attractive (as the final objective for the non-autonomous zero-dynamics (4.9)). The rationale to design $u^i(x,t)$ for system (4.28) is to set $f^i(x)$ such that it stabilizes the state components that are "directly controlled." The inputs v^i are chosen as zero-average functions of t, e.g. $a\sin(\omega t) + b\cos(2\omega t)$, where a and b are to be designed in terms of z in order to render the zero-section locally attractive. This idea is exemplified next with the aim of rendering the zero-section locally attractive for the 3-ECF (3.4). Assume that the time-varying feedback considered is of the form

$$\begin{aligned} u_1(x,t) &= -k_{1,2}x_4 + \frac{1}{\lambda}\alpha_{1,1}\,\sin(t/\lambda) \\ u_2(x,t) &= -k_{2,1}x_2 - k_{2,2}x_5 + \frac{1}{\lambda}\alpha_{2,1}\,\sin(t/\lambda). \end{aligned} \tag{4.32}$$

The first-order averaged system is given by

$$\begin{aligned} \dot{x}_1 &= x_4 \\ \dot{x}_2 &= x_5 \\ \dot{x}_3 &= x_6 \\ \dot{x}_4 &= -k_{12}x_4 \\ \dot{x}_5 &= -k_{21}x_2 - k_{22}x_5 \\ \dot{x}_6 &= -k_{12}x_2x_4 - \tfrac{1}{2}\alpha_{1,1}\alpha_{2,1}. \end{aligned} \tag{4.33}$$

It is thus clear that, in order to have the velocity variables x_4, x_5 and x_6 asymptotically approaching zero for the averaged system, a possible choice for $\alpha_{1,1}$, $\alpha_{2,1}$ is

$$\begin{aligned} \alpha_{1,1} &= k_{32}\mathrm{sign}(x_6)\sqrt{|x_6|} \\ \alpha_{2,1} &= \sqrt{|x_6|}. \end{aligned} \tag{4.34}$$

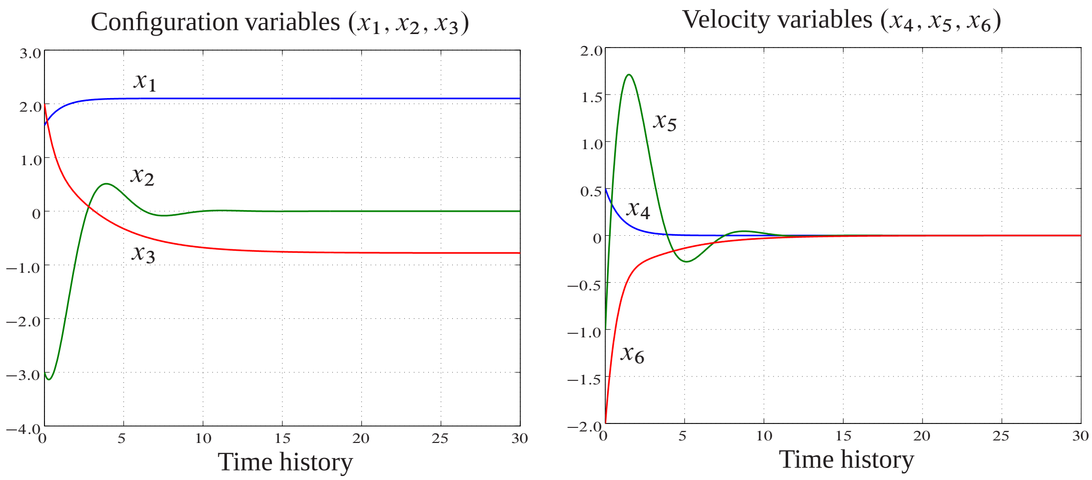


Figure 4.2: Numerical simulation for the 3-ECF under the feedback (4.32)-(4.34).

The Figure 4.2 shows a numerical simulation for the averaged system (4.33) with $\alpha_{1,1}$ and $\alpha_{1,2}$ given by (4.34) and $k_{12} = 1.0$, $k_{21} = 1.0$, $k_{22} = 1.0$, $k_{32} = 0.6$. The initial condition is $x_0 = (1.6, -3, 2, 0.5, -1, -2)$. The Figure 4.3 shows numerical simulations for the 3-ECF (3.4) under the time-varying feedback (4.32), with $\alpha_{1,1}$ and $\alpha_{2,1}$ given by (4.34), for two different values of λ. The values of the parameters k_{12}, k_{21}, k_{22} and k_{32} and the initial condition are the same than in the simulation of Figure 4.2. In the upper two plots of Figure 4.3 $\lambda = 1.0$ and in the lower two plots $\lambda = 0.1$. One may observe that the zero-section is attractive. It is important to remark that, although high-order averaging is a technique that can be used to heuristically design time-varying feedback as shown in the examples above, it is not easy to draw conclusions concerning the stability properties of the closed-loop system.

4.3.2. Application of high-order averaging to the ECF

In this section we illustrate the application of GVTF, along with averaging techniques, to examine the possibility of rendering the zero-section of $T\mathbb{T}^{\kappa_1} \times T\mathbb{T}^{\kappa_2}$ locally attractive under the zero dynamics. Although stabilizing only a subset, e.g. the zero-section, of the state manifold may seem more relaxed an aim than stabilizing a point, it should be kept in mind that the zero dynamics in the form (4.10) may be critical in the sense that it may not satisfy analogous conditions to Brockett's condition for the stabilization of a set via pure-state static feedback.

Consider the 3-dimensional Extended Chained Form (ECF), a SMS with state $(x, \dot{x})$ evolving on $T\mathbb{R}^3 \simeq \mathbb{R}^6$ (also analyzed under the VTFA in Section 3.4) given by:

$$\dot{x} = S_x + u_1 X^{\text{lift}}_{1,x} + u_2 X^{\text{lift}}_{2,x}, \tag{4.35}$$

where $S_x = x_4\, \partial/\partial x^1 + x_5\, \partial/\partial x^2 + x_6\, \partial/\partial x^3$ is the geodesic spray given by the Euclidean metric on $\mathbb{R}^3$ and $X^{\text{lift}}_{1,x} = \partial/\partial x^4 + x_2\, \partial/\partial x^6$, $X^{\text{lift}}_{2,x} = \partial/\partial x^5$ are the control vector fields on $T\mathbb{R}^3 \simeq \mathbb{R}^6$. Since the set $\{X_1, X_2\}$ satisfies the LARC at $e = 0 \in \mathbb{R}^3$, there exists a transverse function near e,

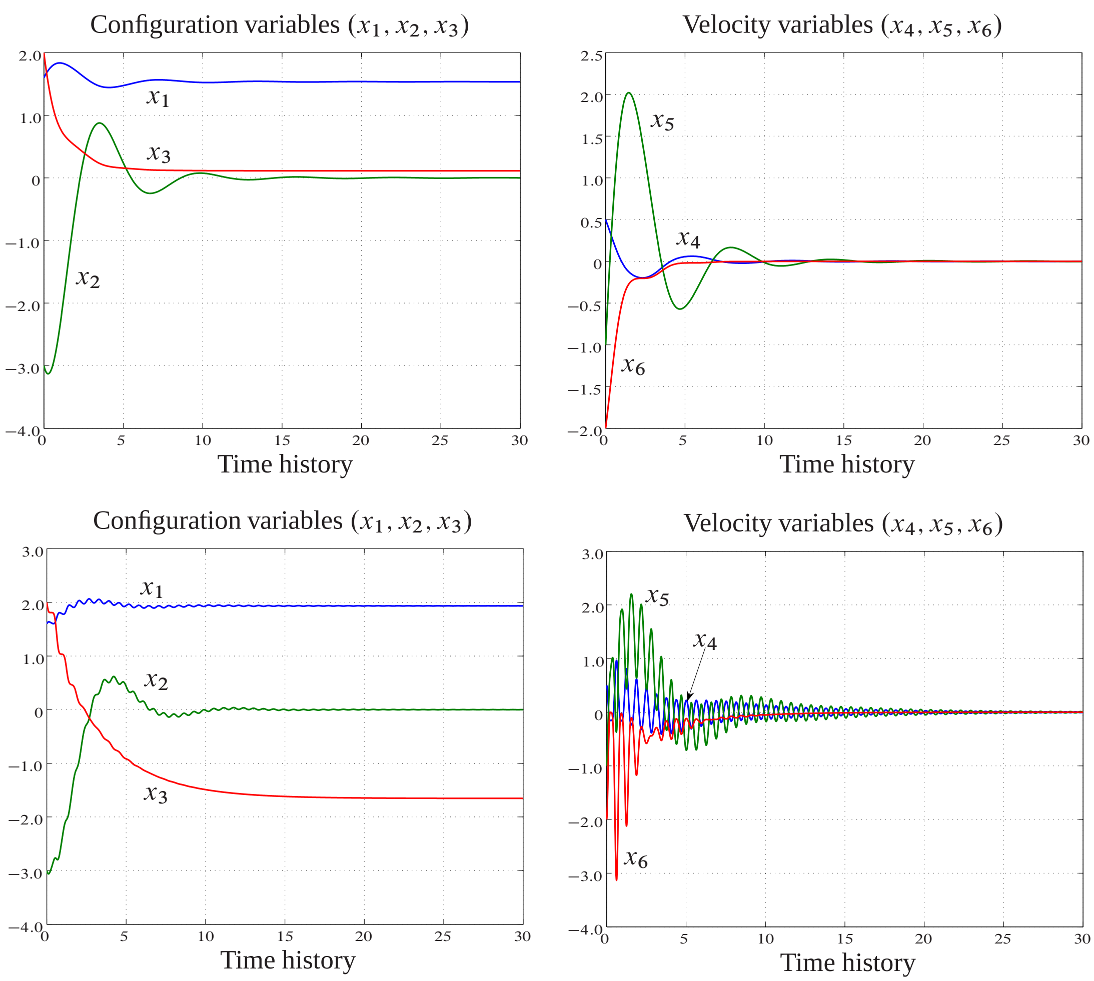


Figure 4.3: Numerical simulation for the 3-ECF under the feedback (4.32)-(4.34).

an instance of which is the map $f : \mathbb{T} \longrightarrow \mathbb{R}^3$ given by

$$f(\theta) = \left(\varepsilon \sin(\theta), \varepsilon \cos(\theta), \frac{1}{4}\varepsilon^2 \sin(2\theta)\right), \qquad \varepsilon > 0. \tag{4.36}$$

Note that, by taking ε sufficiently small, $f(\mathbb{T})$ may be made to lie in an arbitrarily predefined neighborhood $\mathcal{U}$ of e. A GTF $g_1 : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{R}$ can be constructed by defining $g_1(\theta) = f(\theta_2) \cdot f(\theta_1 + \theta_2)^{-1}$ for $\theta = (\theta_1, \theta_2) \in \mathbb{T} \times \mathbb{T}$. Explicitly one has

$$\begin{aligned} g_1(\theta_1, \theta_2) = \Big(& \varepsilon(\mathrm{s}(\theta_1 + \theta_2) - \mathrm{s}(\theta_2)),\ \varepsilon(\mathrm{c}(\theta_1 + \theta_2) - \mathrm{c}(\theta_2)), \\ & \frac{\varepsilon^2}{4}(\mathrm{s}(2\theta_2) + \mathrm{s}(2\theta_1 + 2\theta_2)) - \frac{\varepsilon^2}{2}(\mathrm{s}(2\theta_2 + \theta_1) + \mathrm{s}(\theta_1)) \Big). \end{aligned}$$

It is readily verified that g_1 satisfies the conditions given in Definition 4.1, hence, it is a GTF for (4.35). Consider the auxiliary control system $\ddot{\theta}_1 = u_3$, $\ddot{\theta}_2 = u_4$ on $T\mathbb{T} \times T\mathbb{T}$. Define an error signal $z = x \cdot Tg_1(\theta, \omega)^{-1}$, where $\omega = \dot{\theta}$. Given a second-order vector field, say $S_d(z) = (z_4, z_5, z_6, -z_1 - z_4, -z_2 - z_5, -z_3 - z_6)$, by Proposition 4.3 there exists a unique feedback law $(u_1(x, \theta, \omega, u_4), u_2(x, \theta, \omega, u_4), u_3(x, \theta, \omega, u_4))$ which sets the error dynamics equal to $\dot{z} = S_d(z)$. Thus, in closed-loop, the trajectory $z(t)$ approaches zero exponentially, which in turn forces the state $x(t)$ to approaches $Tg_1(\theta, \omega)$, forcing, in turn, the configuration trajectories to ultimately enter a neighborhood of e. The zero dynamics, however, must be analyzed to determine the evolution of the fiber-coordinates. Moreover, the trajectories should be made to converge to the zero-section. The resulting *controlled* zero dynamics is given by

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \Gamma_1(\theta, \omega) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -2 + 2\,\mathrm{c}(\theta_1) \\ 1 \end{pmatrix} u_4, \tag{4.37}$$

where

$$\begin{aligned} \Gamma_1(\theta, \dot{\theta}) \;=\; & -\mathrm{s}(2\theta_1 + 2\theta_2)\, \dot{\theta}_1^2 - 2\,\mathrm{s}(2\theta_1 + 2\theta_2)\dot{\theta}_1\dot{\theta}_2 - 2\,\mathrm{s}(\theta_1)\dot{\theta}_1\dot{\theta}_2 + 2\,\mathrm{s}(\theta_1 + 2\theta_2)\dot{\theta}_1\dot{\theta}_2 \\ & -\mathrm{s}(2\,\theta_2)\dot{\theta}_2^2 + 2\,\mathrm{s}(\theta_1 + 2\theta_2)\dot{\theta}_2^2 - 2\,\mathrm{s}(2\theta_1 + 2\theta_2)\dot{\theta}_2^2. \end{aligned}$$

Consider an input of the form

$$u_4(\omega, t) = -k_2\omega_2 + \frac{1}{\tau}\left(\alpha_{1,1}\mathrm{s}(t/\tau) + \alpha_{1,2}\mathrm{c}(t/\tau) + \cdots + \alpha_{n,1}\mathrm{s}(n\,t/\tau) + \alpha_{n,2}\mathrm{c}(n\,t/\tau)\right),$$

with $k_2 > 0$. Computing the truncated expressions for averaging, at least for $n \leq 4$, the linearization of the truncated first order averaged system is of the form

$$\begin{aligned} \dot{z}_1 &= z_3 \\ \dot{z}_2 &= z_4 \\ \dot{z}_3 &= a_{1,1}(z)\,{\alpha_{1,1}}^2 + a_{1,2}(z)\,{\alpha_{1,2}}^2 + \cdots + a_{n,1}(z)\,{\alpha_{n,1}}^2 + a_{n,2}(z)\,{\alpha_{n,2}}^2 \\ \dot{z}_4 &= -k_2 z_4, \end{aligned}$$

where the terms $a_{i,j}$ are sums of sines and cosines of functions of z_1 and z_2. Note, however, that it is impossible to solve for $\alpha_{1,1}$ and $\alpha_{1,2}$ in terms of z if one wishes to obtain, say, $\dot{z}_3 = -k_1 z_3$.

By examining next the truncated expression for second order averaging, setting u_4 as in (4.3.2), with $n = 1$, we end up with a system that is no longer second-order. To preserve the second-order nature, one may choose $u_4 = -k_2 z_4 + \alpha_{1,2}\mathrm{c}(t/\tau)$. However, the application of such input yields $\dot{z}_3 = a_{1,2}(z)\alpha_{1,2}^2$ leading, in turn, to the impossibility of designing $\alpha_{1,2}$ to make the zero-section locally attractive.

The choice of a GTF may be essential to determine properties of the resulting zero dynamics, such as accessibility. Therefore, alternative GTFs may yield a controlled zero dynamics which allows one to achieve the required goals. Consider, for instance, the GTF defined on $\mathbb{T}^3$ given by

$$g(\theta) = \left(\varepsilon\mathrm{s}(\theta_1/2) + \varepsilon\mathrm{s}(\theta_3), \varepsilon\mathrm{c}(\theta_1/2), \frac{1}{4}\varepsilon^2\mathrm{s}(\theta_1) + \varepsilon^2\mathrm{s}(\theta_2)\right). \tag{4.38}$$

The resulting controlled zero dynamics is

$$\begin{pmatrix} \dot{\theta} \\ \dot{\omega} \end{pmatrix} = S_{(\theta,\omega)} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4\mathrm{c}(\theta_2) \\ 1 \\ 0 \end{pmatrix} u_4 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -4\mathrm{c}(\theta_1/2)\mathrm{c}(\theta_3) \\ 0 \\ 1 \end{pmatrix} u_5, \tag{4.39}$$

where $S_{(\theta,\omega)} = (\omega_1, \omega_2, \omega_3, \Gamma(\theta,\omega), 0, 0)$ and $\Gamma(\theta,\omega) = -1/2\,\mathrm{s}(\theta_1)\omega_1^2 - 4\,\mathrm{s}(\theta_2)\omega_2^2 + 4\,\mathrm{c}(\theta_1/2)\mathrm{s}(\theta_3)\omega_3^2$. The *unlifted* control vector fields of (4.39), $X_{1,\theta} = 4\,\mathrm{c}(\theta_2)\,\partial/\partial\theta^1 + \partial/\partial\theta^2$ and $X_{2,\theta} = -4\,\mathrm{c}(\theta_1/2)\mathrm{c}(\theta_3)\,\partial/\partial\theta^1 + \partial/\partial\theta^3$ satisfy the LARC at every point in $\mathbb{T}^3$ except, possibly, at points in a subset of measure zero. Hence, depending on the drift vector field S, we may expect (4.39) to be accessible at generic points (θ,ω) in $T\mathbb{T}^3$.

$$\begin{aligned} u_4 &= -k_2 z_2 - k_5 z_5 + 1/\lambda\,\alpha\,\mathrm{s}(t/\lambda) \\ u_5 &= -k_3 z_3 - k_6 z_6 + 1/\lambda\,\beta\,\mathrm{s}(t/\lambda), \end{aligned} \tag{4.40}$$

applied to system (4.39), is given by

$$\begin{aligned} \dot{z}_1 &= z_4 \\ \dot{z}_2 &= z_5 \\ \dot{z}_3 &= z_6 \\ \dot{z}_4 &= a(z,\alpha,\beta) \\ \dot{z}_5 &= -k_2 z_2 - k_5 z_5 \\ \dot{z}_6 &= -k_3 z_3 - k_6 z_6, \end{aligned}$$

where the term $a(z, \alpha, \beta)$ is given by

$$\begin{aligned} a(z,\alpha,\beta) &= -\tfrac{1}{2}\mathrm{s}(z_1){z_4}^2 - 4\,\mathrm{s}(z_2){z_5}^2 + 4\,\mathrm{c}(z_1/2)\mathrm{s}(z_3){z_6}^2 - 4\,k_2 z_2 \mathrm{c}(z_2) - 4\,\mathrm{c}(z_2)k_5 z_5 \\ &\quad +4\,k_3 z_3 \mathrm{c}(z_1/2)\mathrm{c}(z_3) + 4\,\mathrm{c}(z_1/2)\mathrm{c}(z_3)k_6 z_6 - 4\,\alpha^2 \mathrm{s}(z_1)\mathrm{c}^2(z_2) \\ &\quad -4\,\alpha\,\beta\,\mathrm{c}(z_2)\mathrm{s}(z_1/2)\mathrm{c}(z_3) + 8\,\alpha\,\beta\,\mathrm{c}(z_2)\mathrm{s}(z_1)\mathrm{c}(z_1/2)\mathrm{c}(z_3) \\ &\quad +4\,\beta^2 \mathrm{s}(z_1/2)\mathrm{c}^2(z_3)\mathrm{c}(z_1/2) - 4\,\beta^2\mathrm{c}^2(z_1/2)\mathrm{c}^2(z_3)\mathrm{s}(z_1). \end{aligned}$$

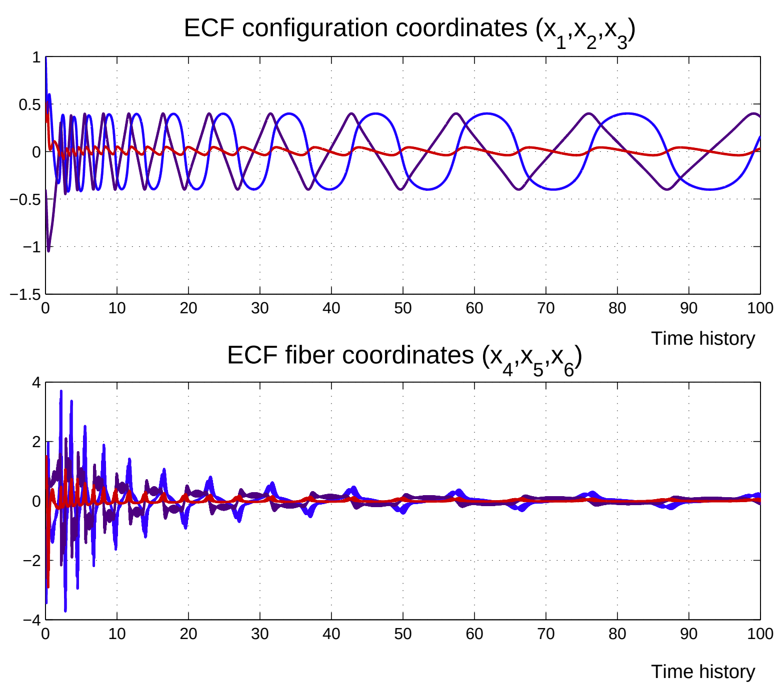
Observe that if k_2, k_5, k_3 and k_6 are strictly positive, then the components of the state z_2, z_5, z_3 and z_6 converge to zero exponentially. Hence we may focus on components z_1 and z_4 of the first order average truncate, under the assumption that the remaining components equal zero, namely

$$\begin{aligned} \dot{z}_1 &= z_4 \\ \dot{z}_4 &= -\tfrac{1}{2}\mathrm{s}(z_1)z_4^2 - 4\alpha^2\,\mathrm{s}(z_1) + 4\alpha\beta\,\mathrm{s}(3z_1/2) - \beta^2\,\mathrm{s}(2z_1). \end{aligned} \tag{4.41}$$

It is not intuitively evident how to design α and β as functions of z such that z_4 converges to zero. However, we may set them such that $\dot{z}_4$ is quadratic in z_4, and such that the term that involves the product $\alpha\beta$ in (4.41) introduces "dissipation." For example, consider

$$\begin{aligned} \alpha &= -k_1\,\mathrm{s}(3z_1/2)\mathrm{sign}(z_4)z_4, \\ \beta &= k_4 z_4, \end{aligned} \tag{4.42}$$

with k_1, k_4 strictly positive. A numerical simulation with appropriately fixed parameters λ, $k_1, \ldots, k_6$, suggests that the zero-section is locally attractive. However, at present, we have no proof of the attractiveness of the zero-section.



Furthermore, numerical simulations for the *compound* system with control feedback $(u_1, u_2, u_3)(z, \theta, \omega, u_4, u_5)$ designed using the generalized vertically transverse function associated with (4.38), and $(u_4, u_5)(z, t)$ designed by means of high-order averaging in equations (4.40) and (4.42), suggest that the zero-section is locally attractive. Indeed, the simulation shown in the figure below is representative of the qualitative behavior exhibited by the compound system with different initial conditions. As depicted, the configuration components seem to enter a prescribed bounded neighborhood of e whereas the velocities seem to vanish asymptotically.

Chapter 5

Concluding Remarks and possible Future Research

In this dissertation we develop and analyze a theoretical framework to address stabilization of configurations for second-order systems on tangent Lie groups. In particular, we are interested in control systems arising from the Euler-Lagrange formulation for mechanical systems. Stabilization of this class of systems happens to be nontrivial, given that this class encompasses possibly constrained, underactuated mechanical systems. Within this class one may encounter systems that are not kinematic reductions of mechanical systems, systems whose linearization at equilibria are non-controllable, and control systems that cannot be stabilized by means of continuous state feedback. Examples of such systems include underactuated mechanical manipulators, rigid body systems in space, wheeled vehicles and underactuated underwater vehicles. It is interesting to remark that these control systems are control-affine systems for which the drift vector field plays a key role in determining important properties, such as local accessibility.

The framework analyzed in this thesis (initially proposed by Sosa [2005] and described in Chapter 2 of this thesis) is intended to provide an extension of the stabilization procedure proposed by Morin and Samson [2003] to deal with the practical stabilization of configurations for second-order systems. The main contributions of the thesis (Chapters 3–4) center on two important issues. First, in analyzing the closed-loop zero dynamics to assess the long-term behavior of the trajectories of the closed-loop system and, second, in modifying the proposed control algorithm with the objective of *shaping* the zero-dynamics trajectories to obtain stability results urged by practical applications. However, the analysis done is not conclusive towards the developing of a unified and systematically applicable theoretical framework to address configuration stabilization for general mechanical system via vertically transverse function, given that the remaining problems may result involved and possibly untractable.

5.1. Conclusions

The existence of a (Morin–Samson) transverse function $f : \mathbb{T}^\kappa \longrightarrow G$ for a set of vector fields $\boldsymbol{X} = \{X_1, \dots, X_m\} \subset \Gamma(TG)$ near a point $q \in G$ is equivalent to the local accessibility of $\boldsymbol{X}$ at that point, as shown by Morin and Samson [2003].

Vertical transversality is a direct consequence of Morin–Samson transversality in the sense that the tangent mapping of every Morin–Samson transverse function $Tf : T\mathbb{T}^\kappa \longrightarrow TG$ is vertically transverse. Based on this property we develop a control methodology intended to practically stabilize configurations for second-order systems evolving on tangent Lie groups.

The procedure of application is as follows. First, we attach an auxiliary system to the *target system*. This auxiliary system is itself a "completely actuated" second-order control system evolving on the tangent bundle of the κ-dimensional torus $T\mathbb{T}^\kappa$. Then, based on the tangent Lie group composition, we define an error function whose derivative along the trajectories of the target and auxiliary system satisfies a second-order differential equation and can be assigned arbitrarily by means of smooth feedback depending on the state of the composite system (target-auxiliary system).

In particular, we make the error dynamics to admit zero as an asymptotically stable equilibrium point. If the initial conditions of the target system do not belong to the image of the vertically transverse function, (i.e., if the initial error is different from zero), the feedback effect is to force the target trajectories to asymptotically approach the image of the vertically transverse function. This entails that the bundle projection of the target trajectories $\pi_G(x(t))$, i.e., the configuration variables, which in a mechanical system represent positions and orientations, ultimately enter into a prespecified neighborhood of the desired configuration.

On the other hand, if the target system has initial conditions in the image of the vertically transverse function, i.e., if the initial error is zero, then the effect of the feedback on the target system may be interpreted, as "forces" that holonomically constrain the target system. These forces may not satisfy d'Alembert principle and so they may introduce non-zero energy into the zero dynamics.

In general, it is difficult to determine whether the closed-loop composite system is positive complete i.e., whether its solutions are defined for every instant time after the initial time. In fact, the velocity variables may grow unbounded or even escape in finite time. In order to single out conditions to ensure stability of the closed-loop system and, therefore, boundedness of the trajectories, one is compelled to study the zero dynamics, that is, the dynamics obtained by restricting the error to be identically zero.

We prove that the resulting control inputs in zero dynamics, the zero-error feedback, has a particular structure. First, the target and auxiliary zero dynamics are related, so it suffices to study the auxiliary dynamics to draw conclusions on the target zero dynamics. Moreover, the auxiliary zero dynamics is an affine connection system that preserves the structure of the target system, that is, the zero dynamics is defined by the sum of a spray $\Sigma \in \Gamma(TT\mathbb{T}^\kappa)$ and a vertical vector field $\Pi \in \Gamma((TT\mathbb{T}^\kappa)^{\mathrm{vert}})$. In particular, this structure may be regarded as that of a mechanical system with the sum of a spray—having terms quadratic in the velocities—plus a vertical vector field that may arise from the gradient of a potential energy function.

The spray Σ defines a torsionless connection. In the general case, the latter is not the Levi-Civita connection of any metric, for most torsionless connections are not Levi-Civita given the overdetermined nature of the necessary conditions (cf. Eisenhart and Veblen [1922]). However, the situation when the zero dynamics admits a Riemannian metric is of interest for, in that case, one may establish positive completeness of the zero dynamics; indeed, every compact (pseudo–) Riemannian manifold, e.g. $\mathbb{T}^\kappa$, is geodesically complete (Kobayashi and Nomizu [1996]).

Although at the present there is not an easily computable characterization of when a given torsionless connection admits a metric, in the case when the target system is underactuated by one control, we establish necessary and sufficient conditions for the existence of a (pseudo–) Riemannian metric.

Considering that the target system is underactuated by one or more controls, and assuming that the zero dynamics admits a (pseudo–) Riemannian metric, we prove stability of the closed-loop system controlled by feedback laws based on vertically transverse functions. This is done by defining a positive-definite function $K : TG \times T\mathbb{T}^\kappa \longrightarrow \mathbb{R}$ which is invariant along the trajectories of the zero dynamics and proving that its trajectories remain bounded. This is rather involved given the topology of $\mathbb{T}^\kappa$, which is not simply connected, and therefore more than one chart would be necessary to cover it and carry out computations in coordinates. An approach based in coordinates, however, would rise some issues at the moment of gluing the local results to give the global one. An alternative method is presented in this thesis which extends the dynamics to a covering manifold and then establishes global uniform stability of the zero-section.

We also propose the use of tangent mappings of generalized transverse functions in order to provide the zero dynamics with additional *control inputs* that may be designed in order to make the velocities converge to zero. Roughly speaking, a generalized transverse function first defined by Morin and Samson [2004], is a transverse function which depends on additional parameters. We present a straightforward generalization of the VTFA for the case of vertically transverse functions derived from generalized transverse functions. The interest in this class of functions is that its application to control leads to *non-autonomous* zero dynamics with additional control inputs which may be used to influence the behavior of the trajectories in zero dynamics. The central objective is to design these additional control inputs in order to make the zero-section of the zero dynamics asymptotically stable or, at least, locally attractive. We prove that if the VTFA yields a zero dynamics admitting a metric, then there is a generalized transverse function that results in a non autonomous zero dynamics that also admits a metric. We explore the use of high-order averaging as presented by Vela [2003] in order to design time-varying feedback to force the velocities to asymptotically approach zero. Particulary, we develop an example for the 3-ECF for which the numerical simulations presented show promising results.

5.2. Future Research

The main problem of practical point stabilization with asymptotically vanishing velocities remains to be solved in the general case under the control methodology proposed. Additional open problems include that of finding an associated metric for the zero dynamics, and constructing generalized transverse functions that lead to non-autonomous, accessible zero dynamics with sprays that admit metrics. When addressing the study of these problems, however, a number of interesting subproblems arise in a natural way. The following are possible directions for future work directly related with the proposed control approach.

- Given that most torsionless affine connections are not Levi-Civita, a possible line of research is to study conditions on transverse functions which yield zero dynamics that admit metrics.
- It seems that specific choice of a generalized transverse function is very important in defining the structure of the resulting non-autonomous zero dynamics. Therefore, it is of interest to study the relation between a given generalized transverse function and the accessibility of the resulting zero dynamics.
- The adaptation of the proposed methodology to tackle practical trajectory stabilization for mechanical systems. However this cannot be immediately done given that there are hard issues to be answered for its application to point-stabilization.

Chapter 6

Appendix

This Appendix is intended to fix the notation and to introduce basic notions from differential geometry, the Lagrangian formulation of mechanical systems, and control theory. The Reader may consult the following references for further details on these topics.

- **Differential Geometry**:

  Warner [1983], Kobayashi and Nomizu [1996], Boothby [2003], Grifone [1972], Godbillon [1971], Hatcher [2001], Hatcher [2003].

- **Mechanical Systems**:

  Bullo and Lewis [2005], Abraham and Marsden [1985].

- **Control Theory**:

  Nijmeijer and van der Schaft [1991], Isidori [1995].

Differential-geometric notions

We shall refer to objects of class C^∞ as *smooth* or differentiable. All manifolds, mappings, vector fields and related constructs are assumed to be smooth unless otherwise stated. We shall sometimes use Einstein summation convention to shorten notation, that is, repeated, doubled indices in quantities multiplied together are implicitly summed. For example $a_i b^i$ for $i = 1, \ldots, n$ denotes the sum $\sum_{i=1}^n a_i b^i$.

Let Q be a Hausdorff, paracompact n-dimensional manifold. For each point q in Q we denote by $T_q Q$ the ***tangent space of*** Q ***at*** q. The first and second tangent bundles of Q are denoted by TQ and TTQ respectively, and their projections onto the base spaces by $\pi_Q : TQ \longrightarrow Q$ and $\pi_{TQ} : TTQ \longrightarrow TQ$. Given two manifolds Q and P, and a map $f : Q \longrightarrow P$, we denote the ***tangent map of*** f ***at*** $q \in Q$ by $T_q f : T_q Q \longrightarrow T_{f(q)} P$ and by $Tf : TQ \longrightarrow TP$ the bundle map covering f. A mapping $f : Q \longrightarrow P$ defines a unique, up to isomorphism vector bundle over Q, $f^*(TP)$ called the ***pullback bundle by*** f defined by $f^*(TP) = \{(q, v) \in Q \times TP : f(q) = \pi_P(v)\}$ with bundle projection $\rho : (q, v) \mapsto q$ along with the differentiable structure naturally inherited.

A map $f : Q \longrightarrow P$ between topological spaces is said to be ***open*** at $q \in Q$ if for any neighborhood $\mathcal{U}$ of q, $f(\mathcal{U})$ is a neighborhood of $f(q)$. The set of vector fields on Q (respectively TQ) is denoted by $\Gamma(TQ)$ (respectively $\Gamma(TTQ)$). Given a vector field X, we write either X_q or $X(q)$ to denote its value at a point q. Let $\boldsymbol{X} = \{X_1, \ldots, X_m\} \subset \Gamma(TQ)$. Lie$(\boldsymbol{X})$ denotes the Lie algebra generated by the set $\boldsymbol{X}$. Let $f : Q \longrightarrow P$ be a map between manifolds and X, Y vector fields on Q and P respectively. X is said to be ***f-related to*** Y if $Y \circ f = Tf \circ X$. A ***coordinate chart*** on Q is denoted by (U, q) where U is an open set in Q and $q = (q^1, \ldots, q^n) : U \longrightarrow \mathbb{R}^n$ is a homeomorphism. Given any such chart there are naturally defined coordinate charts on TQ and on TTQ (see Warner [1983]), defined on $\pi_Q^{-1}(U)$ and on $\pi_{TQ}^{-1} \circ \pi_Q^{-1}(U)$, respectively. These coordinate charts are denoted by $(q, \dot{q}) = (q^i, \dot{q}^i)$ and by $(q, \dot{q}, \alpha_L, \alpha_H) = (q^i, \dot{q}^i, \alpha_L^i, \alpha_H^i)$ and are referred to as ***natural coordinates induced by*** (U, q) on TQ and TTQ respectively. We let $r = (r^1, \ldots, r^n)$ denote the canonical coordinates on $\mathbb{R}^n$. A ***parallelizable n-dimensional manifold*** is an n-dimensional manifold that admits a global frame, i.e., one for which there exists a set of n, linearly independent vector fields $\boldsymbol{X} = \{X_1, \ldots, X_n\} \subset \Gamma(TQ)$ such that $\boldsymbol{X}_q = T_q Q$ at any $q \in Q$. It is straightforward to see that if Q is parallelizable then TQ is parallelizable.

The ***zero-section*** of a tangent bundle TQ is the subbundle, denoted by $Z(TQ)$, of zero-vectors in TQ; as an embedded submanifold, it is diffeomorphic to Q. $\mathbb{T}^\kappa$ denotes the κ-dimension torus. As an instance of an "angular" coordinate system (U, θ) on $\mathbb{T} \simeq S^1 \subset \mathbb{R}^2$ take for example $U = S^1 \setminus \{(0, 1)\}$ and $\theta(p) = 2\arctan(\frac{p_1}{1-p_2})$.

A vector field $X \in \Gamma(TTQ)$ is said to be ***second-order*** if $T\pi_Q \circ X = \mathrm{id}_{TQ}$. In natural coordinates, $X \in \Gamma(TTQ)$ is second-order iff $X(q, \dot{q}) = (q, \dot{q}, \dot{q}, X_H(q, \dot{q}))$. This notion may be extended to vector fields along a curve in TQ as follows. Let X be defined on the image of a curve $\gamma : (t_0, t_1) \longrightarrow TQ$, then X is second-order along γ if for every $t \in (t_0, t_1)$, $T\pi_Q\left(X_{\gamma_t}\right) = \gamma_t$.

Given $v \in TQ$, the ***vertical space over*** v is the subset of $T_v TQ$ given by $T_v TQ^{\mathrm{vert}} = \{\alpha \in T_v TQ : T_v \pi(\alpha) = 0\}$. The disjoint union of the spaces $T_v TQ^{\mathrm{vert}}$, $v \in TQ$, with the differentiable structure naturally induced by TTQ, is called the ***vertical subbundle*** of TTQ and is denoted by TTQ^{vert}. A ***vertical vector*** is an element $\alpha \in TTQ^{\mathrm{vert}}$. In natural coordinates a vertical vector, α, is written as $\alpha = (q, \dot{q}, 0, \alpha_H)$. A section $X \in \Gamma\left(TTQ^{\mathrm{vert}}\right)$ of the vertical subbundle is said to be a ***vertical-valued*** (or simply *vertical*) vector field on TQ. By definition, $X \in \Gamma(TTQ)$ is vertical iff $T\pi_Q \circ X = 0$.

Given $v, w \in TQ$ with $\pi_Q(v) = \pi_Q(w)$ the vector in $T_v TQ$ defined by lift$(v, w) = \frac{d}{dt}\big|_{t=0}(v + tw)$ is called the ***vertical lift of*** w ***by*** v. In natural coordinates, if $v = (x, \dot{x})$ and $v = (x, \dot{y})$ then lift$(v, w) = (x, \dot{x}, 0, \dot{y})$. The ***vertical lift of a vector field*** X on Q is a vector field on TQ given by $X^{\mathrm{lift}}(v) = \mathrm{lift}(v, X_{\pi_Q(v)})$. Let X be expressed in natural coordinates by $X(q) = (q, \widehat{X}(q)$ then $X^{\mathrm{lift}}(q, \dot{q}) = (q, \dot{q}, 0, \widehat{X}(q))$.

The ***Liouville vector field*** C on Q is defined by $C(v) = \mathrm{lift}(v, v)$. The ***canonical almost tangent structure*** J on Q is defined by $J(X)_v = \mathrm{lift}(v, T\pi_Q \circ X_v)$. So defined, J satisfies $[C, J] = -J$. A ***spray*** X is a second-order vector field satisfying $[C, X] = X$. In natural coordinates, $X \in \Gamma(TTQ)$ is a spray iff $X(q, \dot{q}) = (q, \dot{q}, \dot{q}, X_H(q, \dot{q}))$ and the components X_H^i are quadratic in the coordinates $\dot{q}$, i.e. $X_H^i(q, \dot{q}) = \rho^i_{j,k}(q)\dot{q}^j\dot{q}^k$ for some functions $\rho^i_{j,k} \in C^\infty(Q)$, $i, j, k = 1, \ldots, n$. Alternatively, let $X \in \Gamma(TTQ)$. It is then readily verified that X is second-order iff $J(X) = C$ and X is vertical iff $J(X) = 0$. If $Y \in \Gamma(TQ)$, then $\left[C, Y^{\mathrm{lift}}\right] = -Y^{\mathrm{lift}}$.

An (r,s)-***tensor (field)*** t over Q is an r-times contravariant, s-times covariant tensor field on Q, i.e., t is a section of the tensor bundle $T_s^r(TQ) := \bigotimes_{i=1}^r TQ \otimes \bigotimes_{i=1}^s T^*Q$. Let $f, g \in C^\infty(Q)$ and $X, X', Y, Y', Z \subset \Gamma(TQ)$. An ***affine connection*** ∇ on a manifold Q is a mapping $\Gamma(TQ) \times \Gamma(TQ) \longrightarrow \Gamma(TQ)$ denoted by $\nabla : (X,Y) \mapsto \nabla_X Y$ which satisfies ***a)*** $\nabla_{fX+gX'} Y = f(\nabla_X Y) + g(\nabla_{X'} Y)$ and ***b)*** $\nabla_X (fY + gY') = f(\nabla_X Y) + g(\nabla_X Y') + (Xf)Y + (Xg)Y'$. Given $X \in \Gamma(TQ)$, the object defined by $\nabla_X : Y \mapsto \nabla_X Y$ is known as the ***covariant derivative of*** Y ***along*** X.

Let (∂_{q^i}) denote the natural coordinates for the tangent space induced by (q^i) and let X, Y be vector fields on Q with coordinate representation $X(q) = (q^i, \widehat{X}^i(q)\partial_{q^i})$ and $Y(q) = (q^i, \widehat{Y}^i(q)\partial_{q^i})$ respectively. Then, in coordinates, $\nabla_X Y(q) = (q^i, \partial\widehat{Y}^i/\partial q^j\ \widehat{X}^j + \Gamma^i_{j,k}(q)\widehat{X}^j\widehat{Y}^k)$, where $\Gamma^i_{j,k}$ are called the ***Christoffel symbols*** associated with ∇ and are defined by $\nabla_{\partial_{q^j}} \partial_{q^k} = \Gamma^i_{j,k}\partial_{q^i}$. By following related conditions to ***a)*** and ***b)*** given above, it can be defined the covariant derivative of arbitrary-type tensors over Q.

Let X, Y, Z be vector fields on Q. The ***torsion tensor*** T is a $(1,2)$-tensor defined by $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$. The ***curvature tensor*** is a $(1,3)$-tensor given by $R(X,Y)Z = \nabla_X Z - \nabla_Y Z - \nabla_{[X,Y]} Z$. In natural coordinates the components $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ and $R^i_{ijk} = \partial\Gamma^i_{lj}/\partial x^k - \partial\Gamma^i_{kj}/\partial x^l + \sum_m (\Gamma^m_{lj}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm})$ are defined by $T(\partial_{x^j}, \partial_{x^k}) = T^i_{jk}\partial_{x_i}$ and $R(\partial_{x^k}, \partial_{x^l})\partial_{x^j} = R^i_{jkl}\partial_{x^j}$.

A ***pseudo-Riemannian metric*** g on Q is a symmetric, non-degenerate $(0,2)$-tensor over Q. In terms of a local coordinate chart, a $(0,2)$-tensor given by $g = g_{ij}dx^i dx^j$ is symmetric iff $g_{ij} = g_{ji}$, and is non-degenerated iff the matrix g_{ij} is invertible. A ***Riemannian metric*** is a pseudo-Riemannian metric which also satisfies positive-definiteness, that is $g(v,v) > 0$ for every nonzero $v \in TQ$. A manifold together with a given Riemannian (respectively pseudo-Riemannian) metric (Q,g) is said to be a Riemannian (respectively pseudo-Riemannian) manifold. For every (pseudo-) Riemannian manifold (Q,g) there exist canonical mappings $\flat : TQ \longrightarrow T^*Q$ and $\sharp : T^*Q \longrightarrow TQ$, called ***flat*** and ***sharp*** respectively, which are defined by $v^\flat = g(v,\cdot)$ for $v \in TQ$, and by $\sharp = \flat^{-1}$.

Let Q and P be Riemannian manifolds with Riemannian metrics g and h and $f : Q \longrightarrow P$ an immersion. f is said to be ***isometric*** (or alternatively an isometry) if $g(X,Y) = h(f_*X, f_*Y)$ for all $X, Y \in \Gamma(TQ)$.

If $g \in T_2^0(TQ)$ is a (pseudo–) Riemannian metric, there exists a unique affine connection on Q such that the torsion tensor equals zero and that parallel translation with respect to this connection is an isometry. This metric is called the ***Levi-Civita connection*** (or the *metric connection*). In local coordinates, the Christoffel symbols Γ^i_{jk} of a Levi-Civita connection are defined by

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{jl}}{\partial q^k} + \frac{\partial g_{kl}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l} \right).$$

A curve $\gamma : [t_0, t_1] \longrightarrow Q$ is said to be a geodesic if $\nabla_{\gamma'(t)}\gamma'(t) = 0$. The geodesic spray of a metric connection ∇ is a second-order vector field $S \in \Gamma(TTQ)$ such that the solutions to the differential equation $\dot{x} = S_x$ are geodesics. In local coordinates $(q^i, \dot{q}_i)$ for TQ, the geodesic spray is defined by

$$S(q,\dot{q}) = (q^i, \dot{q}^i, \dot{q}^i, -\Gamma^i_{j,k}(q)\dot{q}^j\dot{q}^k).$$

Tangent Lie groups

Assume that G is a Lie group with group composition denoted by $\widehat{\mu}$. We write $\widehat{L}_a, \widehat{R}_a : G \longrightarrow G$ to denote the left and right translations by a on G, respectively. The identity element for $\widehat{\mu}$ is denoted by $e \in G$. A vector field X on G is said to be left-invariant if $X_{gh} = T\widehat{L}_g(X_h)$ for all $g, h \in G$. The set of left-invariant vector fields on G together with the Lie bracket form a Lie algebra $\mathfrak{g}$, which is isomorphic to T_eG. The tangent bundle TG, with composition given by $\mu(x,y) = T\widehat{L}_{\pi_G(x)}(y) + T\widehat{R}_{\pi_G(y)}(x)$, is also a Lie group, usually referred to as the ***tangent Lie group*** of G. The identity element, under this composition, is $0_e \in T_eG$ (the zero vector in T_eG). The inverse element $v^{-1} = -T\widehat{L}_{\pi_G(v)^{-1}} \circ T\widehat{R}_{\pi_G(v)^{-1}}(v) \in T_{\pi_G(v)^{-1}}G$. We let $L_v, R_v : TG \longrightarrow TG$ denote the left and right translations by $v \in TG$, respectively. Sometimes we use $x \cdot y$ or xy in place of $\mu(x,y)$.

Control systems

Let X be a vector field on Q and $t \mapsto \psi_{(x_0,t_0)}(t)$ the solution of the differential equation $\dot{x} = X_x$ with initial condition $x_0 \in Q$, $t_0 \in \mathbb{R}$. The vector field X is said to be ***complete*** if for every $t \in \mathbb{R}$ and for every $x_0 \in Q$, $t_0 \in \mathbb{R}$, $\psi_{(x_0,t_0)}(t)$ is defined. X is said to be ***positive-complete*** if $\psi_{(x_0,t_0)}$ are defined for every $t \in \mathbb{R}_{>0}$. Given a subset U of Q, the vector field X is said to be U***-positive complete*** if for every $t_0 \in \mathbb{R}$, $x_0 \in U$ the trajectories of the system $\dot{x} = X_x$, $t \mapsto \phi(t, t_0, x_0)$ are defined for every t in $[t_0, \infty)$.

Let $\boldsymbol{X} = \{X_1, \ldots, X_m\} \subset \Gamma(TQ)$ and $q \in Q$. $\boldsymbol{X}$ is said to satisfy the LARC (*Lie Algebra Rank Condition*) at q if $T_qQ = \mathrm{span}_{\mathbb{R}}(\mathrm{Lie}(\boldsymbol{X})_q)$, namely, if $\boldsymbol{X}$ is completely nonintegrable. The driftless system $\dot{x} = \sum_{i=1}^m u^i X_{i,x}$ is said to satisfy the LARC at q if $\boldsymbol{X}$ satisfies the LARC at q. An affine-control system $\dot{x} = D_x + \sum_{i=1}^m u^i X_{i,x}$ is said to satisfy the LARC at a point q if the distribution spanned by $\{D, X_1, \ldots, X_m\}$ is completely nonintegrable at q. A ***second-order (control-affine) system*** on TQ is a system $\dot{v} = S_v + \sum_{i=1}^m u^i Y_{i,v}$ where $S \in \Gamma(TTQ)$ is second-order and $\{Y_1, \ldots, Y_m\} \subset \Gamma(TTQ^{\mathrm{vert}})$. A second-order system is said to be an ***affine-connection control system*** if S is the geodesic spray of some affine connection ∇. For such a control system, if ∇ is the Levi-Civita connection for a given (pseudo-) Riemannian metric the system is said to admit a (pseudo-) Riemannian metric.

A ***simple mechanical system (SMS)***, (Bullo and Lewis [2005]), is a system on TQ defined by a Riemannian metric g on Q, a real-valued function $V : Q \longrightarrow \mathbb{R}$ called potential function and a set of $m \leq n$, 1-forms, $F = \{F_1, \ldots, F_m\}$, whose elements represent forces or torques exerted on the system. In natural coordinates $(q^i, \dot{q}^i)$ for TQ, the equations of motion are given by

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i}(q, \dot{q}) - \frac{\partial \mathcal{L}}{\partial q^i}(q, \dot{q}) = \sum_{j=1}^{m} u^j F^i_j(q), \qquad i = 1, \ldots, n;$$

where $\mathcal{L} : TQ \longrightarrow \mathbb{R}$, which is called the ***Lagrangian function*** of the system, is defined by $\mathcal{L}(v) = \frac{1}{2} g_{\pi_Q(v)}(v, v) - V(\pi_Q(v))$. The coordinate-free expression is given by

$$\dot{v} = S_v + \sum_{i=1}^{m} u^i Y^{\text{lift}}_{i,v}$$

where the drift vector field $S = S^g - (dV^\sharp)^{\text{lift}}$ is the sum of the geodesic spray associated with the Riemannian metric g, and minus the vertical lift of vector field corresponding to the potential function. The control vector fields Y_i^{lift} are vertical lifts of the vector fields given by $(F^i)^\sharp$. A particular case of SMS is when the Lagrangian is defined only by the kinetic energy, i.e., $\mathcal{L}(v) = g_{\pi_Q(v)}(v, v)$, then the drift vector field coincides with the geodesic spray associated to the Riemannian metric g. In such a case the system is said to be a ***SMS with zero potential energy***.

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