Instituto Potosino de Investigación Científica y Tecnológica

STUDY AND APPLICATION OF A DESINGULARIZATION ALGORITHM FOR DRIFTLESS CONTROL SYSTEMS

Tesis que presenta

Ana Cristina Silva Loredo

Para obtener el grado de

Maestra en Control y Sistemas Dinámicos

Director de la tesis

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To the memory of my father Armando: the love that you gave me and the memories you left me will always be my strength and inspiration.

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Resumen

Palabras clave: Sistema regular, restricción no holonómica, punto singular, álgebra de Lie, álgebra de Lie libre.

En este trabajo se aborda el problema de planeación de movimiento (MPP por sus siglas en inglés) en modelos cinemáticos de robots móviles tipo carro con restricciones no holonómicas, también conocido como *state steering*. En particular, se toma en consideración la posible existencia de puntos singulares en dichos modelos y se estudia un algoritmo de *desingularización* propuesto en [Chitour et al., 2013], el cual grantiza que las señales de control que resuelven el MPP en el sistema desingularizado también lo resuelven en el sistema singular y son presumiblemente menos complejas que las que se obtendrían resolviendo el mismo problema para el sistema singular. Además se presentan aplicaciones a sistemas particulares tanto del algoritmo de desingularización como de una metodología de control propuesta en [Chitour et al., 2013] para sistemas sin deriva regulares y nilpotentes.

Abstract

Key words: Singular system, nonholonomic constraint, singular point, free Lie algebra, Lie algebra.

This work addresses the motion planning problem (MPP) for kinematic models of car-like mobile robots with nonholonomic constraints, also known as the *state steering* problem. In particular, the possible existence of singular points for these models is considered and a *desingularization* algorithm, proposed in [Chitour et al., 2013], is studied. This algorithm ensures that the control signals that solve the MPP for the "desingularized" system also solve it for the singular system and are presumably less complex than those that would be obtained by solving the same problem for the singular system. In addition, we present applications of both the desingularization algorithm and a control methodology proposed in [Chitour et al., 2013] for particular examples of regular, nilpotent, driftless systems.

Notations and conventions

$\mathbb{R}$	Set of real numbers
$\mathbb{N}$	Set of natural numbers (without incuding $\{0\}$)
$\mathbb{Z}$	Set of integer numbers
$\mathbb{R}_{>0}$	Set of strictly positive real numbers
$L_{\mathcal{X}}$	Lie algebra generated by $\mathcal{X}$
$L_X Y$	Lie derivative of Y in the direction of X
$\mathcal{L}_B$	Free Lie algebra benerated by B
$\Gamma(B)$	Set of smooth sections of a bundle B
X_p	Vector field X evaluated at a point p
Δ	Smooth distribution
Υ	Smooth co-distribution
T_pM	Tangent space to a manifold M at $p \in M$
T_p^*M	Co-tangent space to a manifold M at $p \in M$
TM	Tangent bundle to a manifold M
T^*M	Co-tangent bundle to a manifold M
Q	Configuration manifold
MPP	Motion planing problem
$LARC(x)$	Lie algebra rank condition at x

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Chapter 1

Introduction

One of the problems frequently studied in control is the motion planning problem (MPP), whose solution consists in obtaining admissible control inputs for a system such that these signals bring the system from an initial state x_0 to a desired final state x_f, generally in a finite time T.

Commonly, when designing control laws for a dynamical system, one works with a model or "mathematical description"of the evolution over time of the system, sometimes simplified by various assumptions. In the case of mechanical systems, this model usually represents the dynamics of the system and is obtained by using Newton's laws of motion; for this reason the inputs for this kind of systems are usually given in the form of "generalized forces or torques", which act instantaneously upon the accelerations.

The dynamic model of a mechanical system is in general a second order system; however, in many cases a mechanical system can be described in some sense by a first-order driftless control-affine system, called a "kinematic reduction", with velocities as inputs. This "reduction" is a mathematical representation of the kinematics of the system, such that every controlled trajectory for the kinematic model can be implemented as a trajectory of the full second-order system under some appropriate control input.

The kinematic reduction is sometimes a justifiable step that makes certain control task, especially motion planning, considerably simpler. However, it is natural to ask when can be a mechanical system kinematically reduced? References like [Bullo and Lewis, 2005], [Lewis, 1999], and [Choset et al., 2005] establish necessary and sufficient conditions that mechanical systems must satisfy in order to be kinematically reducible, among which, if a system is a kinematic reduction of a mechanical system, then all feasible trajectories for the kinematic system are also feasible for the second-order system.

Due to the properties exhibited by this reductions, for mechanical systems that are kinematically reduced one can model their kinematics and solve the MPP in this model (with velocities as inputs), to consequently obtain, using control techniques such as backstepping, control acceleration inputs that solve the MPP in the dynamical model of the system.

An interesting case of mechanical systems that can be kinematically reduced and described by driftless control-affine systems are car-like mobile robots. There exist many structural configurations of these systems, two of which are the car with N trailers and the cart with N trailers; their kinematic modeling and some structural properties such as controlabillity and stabilizability have been studied in some references, e.g., [Jean, 1996].

Regarding the MPP for the kinematic models of the car an the cart with N trailers, and for driftless control-affine systems in general, several algorithms to calculate control laws that solve it have been developed over the years; some of them are focused to solve the MPP for a specific type of driftless system, which makes them rather restrictive. There exist, for example, methods for nilpotentizable systems ([Lafferriere and Sussmann, 1991]), sinusoidal controls for chained form systems ([Murray and Sastry, 1993]), techniques for left invariant systems defined on Lie Groups ([Bullo et al., 2000]), etc.

Other steering techniques have been developed in order to solve the MPP in general driftless control-affine systems. Nevertheless, these and the techniques mentioned in the previous paragraph usually side step a drawback that some systems may have: the existence of singular points. The term singular point will be defined below in terms of differentiable manifolds, Lie algebras and free Lie algebras; however in the following paragraph an intuitive interpretation of the meaning of this concept is presented.

A driftless control-affine system is usually given in the form

$$\dot{x} = \sum_{i=1}^{m} X_i(x)u_i, \tag{1.1}$$

where X_i, for $i = 1, \dots, m$, are vector fields defined on a differentiable manifold Q, and u_i, for $i = 1 \dots, m$, are control inputs. A singular point of (1.1) is a point $q \in Q$ such that the growth vector of $\{X_1, \dots, X_m\}$ is not constant at any neighborhood of q. The concept of growth vector of $\{X_1, \dots, X_m\}$ may be thought of as a measure of the number of dimensions that the vector fields of $\{X_1, \dots, X_m\}$ and their Lie brackets of certain "length" can span when evaluated at a point belonging to Q.

Usually, the length of Lie brackets necessary to span the tangent space to Q at a singular point is larger than those necessary to span it at a point that is not singular (called a *regular point*). This, from the control viewpoint, has the disadvantage that the control laws necessary to steer the system from or to a singular point are somewhat more "involved" or "complex" than those control laws necessary to solve the same problem for regular points.

Taking into account the possible existence of singular points in driftless control-affine systems, the authors of [Chitour et al., 2013] propose control algorithms to solve the MPP in regular chained form and nilpotent systems such as (1.1), and a control algorithm for driftless control-affine systems in general. For these algorithms it is assumed that the system to be controlled is regular, thus the authors also propose a "desingularization algorithm" that allows one to obtain a "lifting" of systems with singular points (singular systems), such that the system obtained does not have any singular point (is a regular system) and the control laws that solve the MMP for the lifted system also solve the MPP for the original singular system as well.

As a useful feature in the control algorithm, some steps of the desingularization algorithm ensure that the system obtained is in "privileged coordinates", which represents an advantage in the design of control laws for the system. However, the authors mention that is not necessary to obtain the lifting in these coordinates, i.e., if these steps are omitted, the system obtained is still a regular system in some other coordinates.

The work presented in this thesis has the following main goals:

1. The study of the desingularization algorithm proposed in [Chitour et al., 2013]. Since privileged coordinates is a topic widely studied in many references, and there exist several methodologies to obtain them, this work does not take into account the steps that ensure the system obtained is in privileged coordinates.

2. The application of the desingularization algorithm to a particular kinematic model of a car-like mobile robot. The search of a singular system in which apply the desingularization algorithm was focused in the kinematic models of the car and the cart with N trailers.

3. The application of the control algorithm for nilpotent regular systems, proposed in [Chitour et al., 2013], to a nilpotent system.

The content of this thesis is organized as follows: Chapter 1 gives a brief introduction to the subject matter along with some motivation for this work. In Chapter 2 we present some preliminary aspects about differentiable manifolds and constrained systems that will be used in the ensuing developments. Chapter 3 is an explanation of the type of systems that were modeled and to which the desingularization algorithm is applied along with the modeling methodology used. Chapter 4 is a brief introduction to Lie algebra and free Lie algebra theory; here we explain some concepts defined for the algorithm, such as the growth vector and the multimonomial P associated with a growth vector. The desingularization algorithm of [Chitour et al., 2013] is presented and explained in Chapter 5; in Chapter 6 the desingularization of a car-like mobile system is developed. In chapter 7 the aplication of the control methodology proposed in [Chitour et al., 2013] for nilpotent systems is presented. Finally, chapter 8 contains our conclusions and future work.

Chapter 2

Preliminaries

This chapter is a brief introduction to some concepts in differentiable manifolds that will be used throughout this document; the reader may wish to consult [Warner, 1983] for detailed definitions of such concepts. In the second part of the chapter, holonomic and non holonomic constraints will be defined, and the use of the term "holonomy" in different contexts will be discussed.

2.1 Differentiable Manifolds. Definitions and conventions.

As defined, e.g. in [Warner, 1983], a d-dimensional ***differentiable manifold*** of class C^k is an ordered pair $(M, \mathcal{F})$, where M is a locally Euclidean space of dimension d, and $\mathcal{F}$ is a differentiable structure of class C^k on M. Elements in $\mathcal{F}$ are ***coordinate systems*** (or "charts") (U, φ), where U is a connected open set and $\varphi : U \longrightarrow \mathbb{R}^d$ is a ***coordinate map***. For each $i = 1, \ldots, d$, the functions $x_i = r_i \circ \varphi$ are called ***coordinate functions***, where, for $a \in \mathbb{R}^d$, $r_i(a) = a_i$. One shall use p to refer to a point in M and x to refer to a point $p \in M$ expressed in coordinates φ, i.e., $x = \varphi(p)$, and hence x can be seen as the "representative" of p in coordinates φ.

Hereafter, T_pM and T_p^*M will denote respectively the ***tangent space*** to M at $p \in M$ and the ***cotangent space*** of M at $p \in M$. Let (U, φ) be a coordinate chart of M, with $\varphi = (x_1, \ldots, x_n)$; it is well known (e.g. [Warner, 1983]) that for every $p \in U$, a natural basis for T_pM is $\left\{ \frac{\partial}{\partial x_1}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p \right\}$ and a natural basis for $T_p{}^*M$ is $\{dx_1\big|_p, \ldots, dx_n\big|_p\}$, where for every $f \in C^\infty(U, \mathbb{R})$, the mappings $\frac{\partial}{\partial x_i}\Big|_p$ and $dx_i\big|_p$ are given by $\frac{\partial}{\partial x_i}\Big|_p(f) = \frac{\partial}{\partial r_i}\Big|_\varphi(p)(f \circ \varphi^{-1})$ and $dx_1\big|_p(\frac{\partial}{\partial x_i}\Big|_p) = \delta^i_j$, respectively.

Let M be a topological space. A ***real vector bundle*** of rank k over M is a triple (M, E, π), with E a topological space and $\pi : E \longrightarrow M$ a surjective continuous map, satisfying the following conditions:

1. For each $p \in M$, the ***fiber*** $E_p = \pi^{-1}(\{p\})$ over p is endowed with the structure of a k-dimensional real vector space.

2. For each $p \in M$, there exist a neighborhood U of p and a homeomorphism $\phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$, called a ***local trivialization*** of E over U, satisfying the following conditions:

   - Let π_U be the natural projection of $U \times \mathbb{R}^k$ in U, i.e. $\pi_U(U \times \mathbb{R}^k) = U$. Then $\pi_U \circ \phi = \pi$.

   - For each $q \in U$, the restriction of ϕ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

By an abuse of notation, a vector bundle (E, M, π) is usually denoted by E alone.

If M and E are smooth manifolds, π is a smooth map, and the local trivializations can be chosen to be diffeomorphisms, then E is called a ***smooth vector bundle***. In this case, the local trivializations that are diffeomorphisms onto their images are called ***smooth local trivializations***. The space E is called the ***total space*** of the bundle, M is called the ***base***, and π is called the ***projection*** (ref. [Lee, 2003]).

Let M be a differential manifold and let us define $TM := \bigsqcup_{p \in M} T_pM$ and $T^*M := \bigsqcup_{p \in M} T_p{}^*M$, with $\bigsqcup$ the disjoint union. Let $\pi_1 : TM \longrightarrow M$ and $\pi_2 : T^*M \longrightarrow M$ be respectively the canonical projection of TM on M which assigns to an element in T_pM the point p for every $p \in M$, and the canonical projection of T^*M on M which assigns to an element in $T_p{}^*M$ the point p for every $p \in M$. TM and T^*M are differential manifolds since both can be equipped with a differential structure inherited by the differential structure of M (ref. [Warner, 1983]). The triple (M, TM, π_1) is a smooth vector bundle of rank $2 \dim M$ over M with fiber T_pM, called the ***tangent bundle*** and denoted by TM. The triple (M, T^*M, π_2) is a smooth vector bundle of rank $2 \dim M$ over M with fiber $T_p{}^*M$, called the ***cotangent bundle***.

A ***vector field*** X on M is a section of TM, that is, a mapping $X : M \longrightarrow TM$ such that $\pi_1 \circ X$ is the identity map on M. One shall use $\Gamma(B)$ to denote the set of smooth sections of a bundle B. The evaluation $X(p)$ of a vector field X at a point $p \in M$ will often be denoted by X_p; X_p is a tangent vector in T_pM. Let X and Y be smooth vector fields, let $f \in C^\infty(M)$ and $p \in M$. One writes $X(f)$ to denote the function whose value at p is $X_p(f)$. A vector field $[X, Y]$, called the ***Lie bracket*** of X and Y, is defined by setting $[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$, for every $f \in C^\infty(M)$.

A smooth curve $x : [a, b] \to M$, with $a, b \in \mathbb{R}$, is an ***integral curve*** of the vector field X if $\dot{x}(t) = X(x(t))$ for each $t \in [a, b]$. A ***distribution*** $\Delta : M \to TM$ of rank c

on a d-dimensional manifold M is the specification of a c-dimensional subspace $\Delta(p)$ of T_pM for each p in M. A vector field X on M is said to lie in the distribution Δ if $X_p \in \Delta(p)$ for each $p \in M$. A distribution Δ is said to be smooth if, for each $p \in M$, there exists a neighborhood U of p and vector fields $X_1, \ldots, X_c$ of class C^∞ on U which span Δ at every point of U. A smooth distribution Δ is called ***involutive*** if $\Gamma(\Delta)$ is closed under the Lie bracket operation, i.e., if $[X, Y]$ takes values in Δ whenever X and Y are smooth vector fields lying in Δ. A c-dimensional ***codistribution*** $\Upsilon : M \to T^*M$ on a d-dimensional manifold M is the specification of a c-dimensional subspace $\Upsilon(p)$ of T_p^*M for each p in M. The annihilator of a codistribution Υ, denoted $\Upsilon^\perp$ is the distribution defined as $\Upsilon^\perp(x) = \{v \in T_xM : (\forall \omega \in \Upsilon(x))(\omega(v) = 0)\}$.

Let $\psi : N \longrightarrow M$ be of class C^∞; the mapping ψ is said to be an immersion if the tangent mapping $T_p\psi$ (which is linear by definition) is injective for each $p \in M$. The pair (N, ψ) is said to be a ***submanifold*** of M if ψ is an injective immersion. A submanifold (N, ψ) of M is an ***integral manifold*** of a distribution $\mathcal{D}$ on M if $T_p\psi(T_nN) = \mathcal{D}(\psi(n))$ for each $n \in N$ (cf. [Warner, 1983]).

Let $\Lambda_k{}^*M = \bigcup \Lambda_k(T_p{}^*M)$ be the ***exterior k-bundle*** over the differentiable manifold M, whose construction is detailed in [Warner, 1983]. A ***differential k-form*** on a manifold M is a C^∞ mapping $\alpha : M \longrightarrow \Lambda_k{}^*M$ whose composition with the canonical projection is the identity map, i.e., α is a section of $\Lambda_k{}^*M$. In that sense, a 0-form is just a smooth real-valued function, and a 1-form is a ***covector field***, i.e., a section $\beta : M \longrightarrow T^*M$ of T^*M.

2.1.1 "Local representatives"

In many cases it is useful to use "local representatives" to express some of the concepts defined in the preceding paragraphs. For example, one usually refers to a vector field $X : M \longrightarrow TM$ through its local "representative" in certain coordinates. Let (U, φ) be a coordinate system and let $\Omega = \varphi(U)$. The local representative of X in coordinates φ is defined as $\hat{X} = T\varphi \circ X \circ \varphi^{-1}$. Note that φ^{-1} exists since φ is an homeomorphism. In this case, $\hat{X}$ is said to be the *push forward* of X by φ, denoted by $\varphi_* X = \hat{X}$, and the following diagram commutes:

$$\begin{array}{ccc} TM & \xrightarrow{T\varphi} & T\Omega \\ {\scriptstyle X}\uparrow & & \uparrow{\scriptstyle \hat{X}} \\ N & \xrightarrow{\varphi} & \Omega \end{array}$$

Let (U, φ), with $\varphi = (x_1, \ldots, x_n)$ be a coordinate chart for M and let $p \in U$. Let X be a vector field on M and α be a differential 1-form on M. Since $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ and $\{dx_1, \ldots, dx_n\}$ yield bases for T_pM and T_p^*M, respectively, there exist $f_1, \ldots, f_n$, $g_1, \ldots,$

$g_n \in C^\infty(M)$ such that $X(p) = f_1(p)\frac{\partial}{\partial x_1}\big|_p + \cdots + f_1(p)\frac{\partial}{\partial x_n}\big|_p$ and $\alpha(p) = g_1(p)dx_1|_p + \cdots + g_n(p)dx_n|_p$. As a consequence, distributions and co-distributions may also be described through local representatives. In a similar sense, the mathematical representations of control systems used in this work are local representatives of the system in certain coordinates.

2.2 Constrained mechanical systems

When the dynamics or the kinematics of a mechanical system is mathematically represented, a ***configuration manifold*** or ***space of configurations*** is usually defined. This space of configurations is a differential manifold Q, where each point in Q represents a specific position of the system, i.e., the points in Q are in a one-to-one correspondence with the set of positions and orientations that the system may attain.

In many interesting cases, the motion of the system is "limited" or "restricted" in some way, i.e., the system is subject to constraints that may arise from the system's structure itself, or from the way in which it is actuated upon.

Some constraints restrict the system's motion in the sense that they reduce the number of degrees of freedom of the system, i.e., they restrict the set of possible configurations of the system. This type of constraints are called ***holonomic constraints*** and, as is explained in the following, they also restrict the speed of the system indirectly. An example of holonomic constraint is present in any rigid body, which is a system of "material" points whose positions are constrained so that the distance between any two points remains constant along the system's motion. Some other constraints restrict the possible values of the velocities of the parts of the system without restricting positions, they are called ***nonholonomic constraints***. An example of this kind of constraint is the "no-slip"condition in the motion of a rolling body, which requires that the relative velocity of the point of contact of the rolling body with the contact surface vanishes identically, i.e., at all times during the motion. Although not explicitly considered in this work, there exist other classification schemes for constraints. For instance, according to [Jarzbowska and Pietrak, 2014], in control there are usually four constraint sources: kinematic constraints, conservation laws, design and control constraints (which often come from "under actuation"), and task-based constraints (such as a trajectory to follow). For the development of this thesis only holonomic and nonholonomic constraints are considered.

2.2.1 Holonomic constraints

The mathematical representation of a holonomic constraint on a system is equivalent to specifying one or more functions $f_i : Q \longrightarrow \mathbb{R}$, for $i = 1, \dots, s$, and defining the intersection of the sets $\{x \in Q : f_i(x) = 0\}$ as the set of allowed configurations of the system. Let $c : \mathbb{R} \longrightarrow Q$ be an admissible trajectory of the system, i.e., for every $t \in \mathbb{R}$ one has $f_i(c(t)) = 0$. Let g be a C^∞ function on a neighborhood of $f(c(t))$ and let $v \in T_{c(t)}Q$; the differential $df_i\big|_{c(t)} : T_{c(t)}Q \longrightarrow \mathbb{R}$, is given by $df_i\big|_{c(t)}(v)(g) = v(g \circ f_i)$, thus $\dot{c}(t)$ must lie in the kernel of the differential $df_i\big|_{c(t)}$, i.e., $df_i\big|_{c(t)}(\dot{c}(t)) = 0$. It follows that the intersection of all the sets $\{v \in TQ : df_i(v) = 0\}$, for $i = 1, \dots, s$, is the set of allowed velocities of the system, furthermore this intersection is a subset of the annihilator of the co-distribution Υ generated by all the 1-forms df_i.

Let Δ be the distribution generated by $\{v \in TQ : df_i(v) = 0,\ i = 1, \dots, s\}$. From the point of view of the theory of differentiable manifolds, a constraint is said to be holonomic if the distribution Δ is involutive, which implies, by the Frobenius theorem (ref.[Warner, 1983]), that Δ is integrable, i.e., each $q \in Q$ is contained in an integral manifold N of Δ, with N a submanifold of Q such that $T_nN = \Delta(n)$, for every $n \in N$.

Example 1. *(Holonomic constraint).* Let us consider the rigid pendulum shown in Figure 2.1. Suppose that one regards $\mathbb{R}^2$ as the configuration manifold and that Σ is a reference frame of some coordinate system in $\mathbb{R}^2$. The position of a point (x, y) is limited by the lenght L since, at any time, (x, y) must be such that $x^2 + y^2 = L^2$. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R} : (x, y) \longmapsto x^2 + y^2 - L^2$. The set $\{x \in \mathbb{R}^2 : f(x) = 0\}$ is the set of allowed positions. Therefore, the simple pendulum may be viewed as a circle holonomically constrained by a rigid rod.

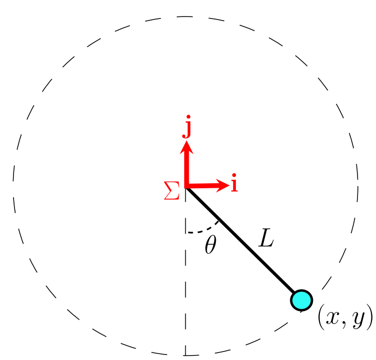

Figure 2.1: Simple pendulum

One way to deal with holonomic constraints is by choosing a suitable configuration manifold Q, whose points represent the set of allowable configurations after the effects of the constraints have been considered. Thus the allowable configurations are implicitly restricted by the nature of Q. For example, suppose that, for the pendulum mentioned in Example 1, one chooses $Q = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = L^2\}$ as the configuration space. Since Q represents a circle and is a differentiable manifold, every possible position of the system belongs to a circle of radius L, hence one need not constrain the configuration space $\mathbb{R}^2$ via the zero set of the function f.

2.2.2 Nonholonomic constraints

The mathematical representation of a nonholonomic constraint for a system is equivalent to specifying one or more differential forms $\alpha_i : TQ \longrightarrow \mathbb{R}$, for $i = 1, \dots, s$, and declaring that the set $\{v \in TQ : v \in \operatorname{ann}(\Upsilon)\}$, with Υ the codistribution generated by the functions α_i, is the set of allowed velocities.

In the theory of differentiable manifolds, a constraint is said to be nonholonomic if the distribution Δ, generated by the set $\{v \in TQ : v \in \operatorname{ann}(\Upsilon)\}$ is noninvolutive, which implies that not every point of Q is contained in an integral manifold of Δ.

Example 2. *(Nonholonomic constraint).* Consider the rolling disk of radius R shown in Figure 2.2, where P is an arbitrary point of the periphery of the disk, P_0 is the contact point of the wheel with the xy–plane, and (x,y) are the coordinates of the point in the plane where the disk touches the plane. Assume that the disk is allowed to roll on the plane without slipping. Let $Q = \mathbb{R}^2 \times (S^1)^2$ be the space of configurations selected for the rolling disk, then a point $q \in Q$ is a 4-tuple (x, y, θ, φ). To satisfy the no slipping condition, the velocity of the point P_0 on the direction of j_1 must be equal to zero, with P_0 and j_1 expressed with respect to the coordinate frame $\Sigma_0 = (i_0, j_0)$. Thus the no slipping condition may be modeled by defining the differential form $\alpha : TQ \to \mathbb{R} : (\dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta}) \mapsto dy - R\sin(\varphi)d\theta$, where α represents the velocity of P_0 in the direction of j_1. Furthermore, every velocity that this system can attain belongs to the aninhilator of the co-distribution $\Upsilon = \operatorname{span}\{\alpha\}$.

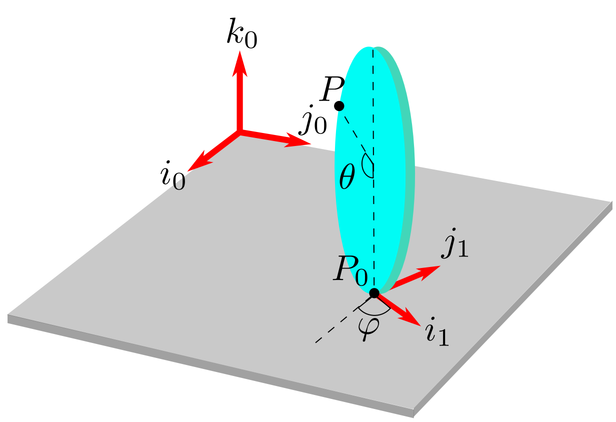

Figure 2.2: Rolling disk

In the sequel, a ***constraint distribution*** will be understood as the distribution Δ generated by the subset of TQ in wich Υ vanishes.

2.3 Holonomy in a more general context

Let us consider the mathematical representation of the kinematics of a mechanical system

$$\dot{x}(t) = \sum_{i=1}^{m} X_i(x) u_i, \tag{2.1}$$

defined on Ω, with Ω a non empty open subset of $\mathbb{R}^n$, where m and n are integers, $u = (u_1, \dots, u_m)$ are control inputs that take values in $\mathbb{R}$, and X_i is a vector field on Ω.

Let Δ be the distribution spanned by vector fields $X_1, \dots, X_m$. The distribution Δ is said to be an ***integrable or holonomic distribution*** if Δ is involutive. As explained previously in this chapter, the involutivity of a distribution is a necessary and sufficient condition for the existence of integral manifolds of Δ through each $x \in \Omega$. From the point of view of control theory, the involutivity of Δ implies that, for system (2.1), all trajectories that start in a point belonging to an integral manifold of Δ cannot leave it, i.e., there are directions in which the system's state cannot be steered regardless of the control input. The distribution Δ is said to be ***nonintegrable or nonholonomic*** if Δ is not involutive. Nonholonomic distributions are especially interesting in control theory since the noninvolutivity of a distribution associated to a system implies that this system can be steered indirectly in some directions, i.e., one may control some state varibles indirectly.

Example 3. *(Nonholonomic Distribution)* Let us consider the following two-input system on $\mathbb{R}^3$:

$$\dot{x} = \begin{pmatrix} 1 \\ 0 \\ x_2 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_2 = X_1(x) u_1 + X_2(x) u_2 \tag{2.2}$$

Let $\Delta_x = \mathrm{span}\{X_1(x), X_2(x)\}$, by simple computation one has:

$$[X_1, X_2](x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.3}$$

Since for each $x \in \mathbb{R}^3$, the tangent vector $[X_1, X_2](x)$ is linearly independent from $X_1(x)$ and $X_2(x)$, i.e., $[X_1, X_2](x)$ does not lie in the distribution Δ, it follows that Δ is not involutive.

In mechanics there also exists the concept of nonholonomy but seen from a different approach: the ***nonholonomic mechanical systems***. In general, what makes a mechanical system nonholonomic is the presence of nonholonomic constraints. Nonholonomic systems are present in a great variety of environments; ranging from Engineering to Robotics, wheeled vehicle and satellite dynamics, manipulation devices and locomotion systems.

Altough there is an extensive theory on the definition and study of mechanical systems, particularly nonholonomic systems, this work only addresses issues related to the kinematics of a particular class of nonholonomic systems; namely car-like wheeled mobile robots. The reader may refer to, e.g., [Bloch, 2003], [Jean, 2014] or [Monforte, 2004] for more comprehensive studies of nonholonomic systems.

Remark 1. It is important to remark that there exists the concept of the nonholonomy of a system for general systems that are not necessarily mechanical. For example the authors of [Chitour et al., 2013] define a nonholonomic system as a driftless control-affine system in the sense that it is usually the case that one has more state variables that control inputs; therefore some variables are controlled indirectly. Since the present work is focused on the study and implementation of a desingularization algorithm for car-like wheeled mobile robots, hereafter the term nonholonomic system will be reserved only for mechanical systems with nonholonomic constraints, and the nonholonomic systems in the sense of [Chitour et al., 2013] will be called simply driftless control-affine systems.

Chapter 3

Car-like wheeled mobile robots and their kinematic modeling

One of the objectives of this chapter is to recall an established methodology for the kinematic modeling of nonholonomic car-like wheeled robots, in which the kinematic model is derived from the nonholonomic constraints of the system. The other objective is to present three models obtained through this methodology, for which the desingularization procedure may be applied.

3.1 Car-like mobile robots with nonholonomic constraints

The mobile robots modeled in this work are systems capable of locomotion on a surface solely through the actuation of wheels mounted on the system that are in contact with a surface. There exist several types of car-like robot structures; for this work we consider only two of these structures: the cart with N trailers and the tricycle with N trailers.

The cart pulling N trailers is shown in Figure 3.1. Two wheels of the car $N+1$ are fixed-direction wheels, i.e., the point where the shaft connects with these wheels is the center of the wheel and the orientation of the wheel plane with respect to the shaft is constant; the third is a "caster wheel" and its function is to provide support. The wheels of the N trailers are fixed-direction wheels. For modeling purposes, in this work it will be assumed that the cart with N trailers moves on a surface $\mathbb{R}^2$ and that each wheel touches the surface only at a point. It is also assumed that the fixed wheels rotate without slipping; as explained below, nonholonomic constraints of this system arise precisely by virtue of this last assumption. Finally, it is assumed that lengths C_i and L_j, for $i = 2, \ldots, N+1$ and $j = 1, \ldots, N+1, \ldots$ are constant.

The car pulling N trailers is shown in Figure 3.2. Two wheels of the tricycle $N+1$

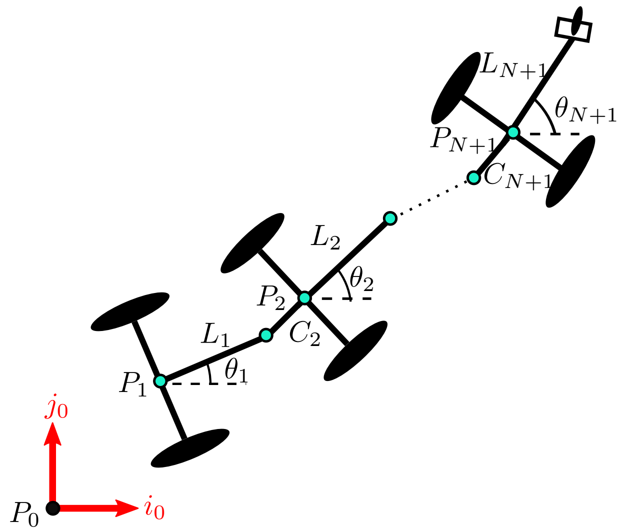

Figure 3.1: Graphic representation of a cart with N trailers.

are fixed wheels; the third wheel is an orientable wheel. Assumptions made for the fixed direction wheels of the cart with N trailers are also made for the two fixed-direction wheels of the tricycle and for the wheels of the N trailers. It is also assumed that the orientable wheel rotates without slipping. As for the cart with N trailers, it is supposed that the tricycle with N trailers moves on a surface $\mathbb{R}^2$ and that lengths C_i and L_j, for $i = 2, \ldots, N+1$ and $j = 1, \ldots, N+1$ are constant.

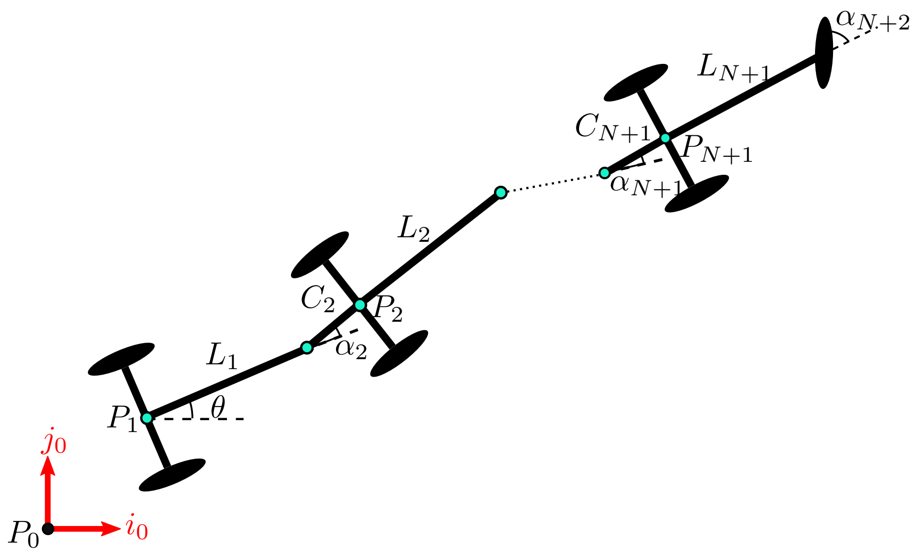

Figure 3.2: Graphic representation of a car with N trailers.

3.2 A methodology for modeling the kinematics of car-like mobile robots

Mathematically, modeling the kinematics of a mechanical system consists in defining the space of configurations Q for the system and defining a distribution Δ on Q, which represents the allowable values that the instantaneous speeds of the system may take at each point of Q. The purpose of this section is to outline an established methodology which allows one to obtain mathematical approximations of the kinematics of car-like robots by considering their nonholonomic constraints. The reader may wish to refer to a robotics reference, for example [Spong et al., 2005], for an extended explanation of this and other modeling procedures.

Roughly speaking, the kinematic modeling of car-like robots presented here involves the definition of the nonholonomic constraints considered for the concerned system; it is from these constraints that one can construct an approximate model of the system kinematics.

3.2.1 Affine spaces and moving frames of reference

As the reader may see in, e.g., [Siciliano et al., 2008], when modeling the kinematics of a rigid body, its position on the plane is expressed in terms of the position of a suitable point P on the body with respect to a fixed reference frame $\Sigma_0 = (i_0, j_0)$, while its orientation is expressed in terms of the components of the unit vectors of a mobile frame whose origin is p. When the rigid body has links, it is necessary to know the relative angles formed between these links too; for example, to determine the exact position in the space of the cart with N trailers shown in figure 3.1, it is sufficient to know the position (x, y) with respect to a fixed reference frame $\Sigma_0 = (i_0, j_0)$ of a selected point p on the cart, and the angles $\theta_1, \alpha_2, \ldots, \alpha_{N+2}$, where $(x, y, \theta_1, \alpha_2, \ldots, \alpha_{N+2})$ are coordinates of the configuration manifold Q.

Altough for simplicity, when modeling the kinematics of a system, fixed and mobile reference frames are frequently studied from the physics point of view, they have an interesting mathematical formulation that will be addressed in this section.

In general, the fixed frame Σ_0 is chosen such that $\{i_0, j_0\}$ is the canonical basis for $\mathbb{R}^2$. Let P be an arbitrary point of a rigid body. For modeling terms, it is frequently supposed that P is moving on $\mathbb{R}^2$, thus the position of P on the (i_0, j_0)-plane is variant over the time. Let $\mathcal{P} : \mathbb{R} \longrightarrow \mathbb{R}^2 : t \longmapsto \mathcal{P}(t)$, where $\mathcal{P}(t)$ is the position of a point P at time t. The following definition proves useful to define a moving frame.

In [Gallier, 2012], an ***affine space*** over a field K is defined as a triple $(V, E, \bar{+})$, where E is a non-empty set, V is a vector space over K, and $\bar{+}$ is a mapping $\bar{+} : E \times V \longrightarrow E$ satisfying, for every $a, b \in E$ and every $u, v \in V$, the following:

1. $a \bar{+} 0 = a$.
2. $(a \bar{+} u) \bar{+} v = a \bar{+} (u \bar{+} v)$.
3. There exists $w \in V$ such that, for every $\bar{w} \in V$, $a \bar{+} w = b$ and $a \bar{+} \bar{w} = b$ if and only if $w = \bar{w}$, i.e., w is unique.

An affine space may be seen as a vector space "without its origin", i.e., without additive identity. One "forgets" about the origin by adding translations to a class of maps defined on the affine space. Nevertheless, there is a simple way to set an origin for E that endows it with the structure of a K-vector space: Let $e \in E$, and define $\varphi_e : V \longrightarrow E$ as the map given, for every $v \in V$, by $\varphi_e(v) = e \bar{+} v$. Let $u, v \in V$ and let us suppose that $\varphi_e(u) = \varphi_e(v)$; it follows that $e \bar{+} u = e \bar{+} v$ and, by definition of $\bar{+}$, one has $u = v$, therefore φ_e is injective. Let $\bar{e} \in E$. By definition of $\bar{+}$, there exists $w \in V$ such that $e \bar{+} w = \bar{e}$, therefore φ_e is surjective. Define $\tilde{+} : E \times E \longrightarrow E$, for every $e_1, e_2 \in E$, by setting $e_1 \tilde{+} e_2 = \varphi_e(\varphi_e^{-1}(e_1) + \varphi_e^{-1}(e_2))$, where $+$ is the sum defined on V. If one defines the multiplication by scalars $\tilde{*}$ on E in a similar way, it is easy to prove that $(E, \tilde{+}, \tilde{a})$ is a vector space; moreover e is the additive identity of $\tilde{+}$. Thus φ_e is a vector space homomorphism, i.e., a linear map.

Let V a real vector space, and let $\mathfrak{o} \in V$. It follows from the previous definition that the triple $(V, V, +)$, with $+$ the sum on V, is an affine space over $\mathbb{R}$. Moreover, $\varphi_{\mathfrak{o}}$ is an homomorphism between $(V, +, *)$ and $(V, \tilde{+}, \tilde{*})$, with $*$ the scalar multiplication defined on V. In that sense, $\Sigma_k = (i_k, j_k)$, is said to be a ***moving frame*** in $\mathbb{R}^2$, if and only if i_k and j_k are the image by $\varphi_{\mathfrak{o}}$ of a basis on $\mathbb{R}^2$, i.e., if there exists $v_1, v_2 \in \mathbb{R}^2$ such that $\{v_1, v_2\}$ is a basis of $\mathbb{R}^2$ and $(i_1, j_1) = (\varphi_{\mathfrak{o}}(v_1), \varphi_{\mathfrak{o}}(v_2))$. It is easy to prove that $\{i_1, j_1\}$ is a basis for $(V, \tilde{+}, \tilde{*})$.

Since $\mathcal{P}(t) \in \mathbb{R}^2$, for every $t \in \mathbb{R}$, $\mathcal{P}(t)$ is a linear combination $\mathcal{P}(t) = p^1(t) i_0 + p^2(t) j_0$, that may be represented by the vector $\begin{pmatrix} p^1(t) \\ p^2(t) \end{pmatrix}$. Since $\mathcal{P} : \mathbb{R} \longrightarrow \mathbb{R}^2$, one has $\dot{\mathcal{P}}(t) \in T_{\mathcal{P}(t)}\mathbb{R}^2$ for every $t \in \mathbb{R}$. To obtain the kinematic model of a car-like robot one may consider the position of the robot at an arbitrary instant of time t; for that reason from here P will be used to denote $\mathcal{P}(t)$ and $\dot{P}$ will be used to denote $\dot{\mathcal{P}}(t)$ for $t \in \mathbb{R}$. In that sense a point P will ve represented by a vector $\begin{pmatrix} p^1 \\ p^2 \end{pmatrix}$

3.2.2 Homogeneous transformation

The topology of $\mathbb{R}^2$, and the differential structure on $\mathbb{R}^2$ and $T\mathbb{R}^2$, allow one to define an homomorphism betwen $\mathbb{R}^2$ and $T\mathbb{R}^2$. Due to the existence of this homomorphism, in Robotics, P and $\dot{P}$ are frequently represented in the canonical basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ with the following notation:

- A position vector $P = (p^1, p^2) \in \mathbb{R}^2$ will be represented by:

$$P = \begin{pmatrix} p^1 \\ p^2 \\ 1 \end{pmatrix}$$

- A velocity vector $\dot{P} = (\dot{p}^1, \dot{p}^2) \in \mathbb{R}^2$ will be represented by:

$$\dot{P} = \begin{pmatrix} \dot{p}^1 \\ \dot{p}^2 \\ 0 \end{pmatrix}$$

As the reader may read in [Spong et al., 2005], homogeneous transformations are affine functions that may represent the change of coordinates between two reference frames; in other words, a homogeneous transofrmation is the matrix representation of the mapping $\varphi_{\mathbf{o}}$ defined previously in order to set an origin on an affine space. The homogeneous transformation that represents the change of coordinates from Σ_1 to Σ_0, which sets P_0 as the origin of the space generated by i_0 and j_0, is given by the matrix

$$^0M_1 = \begin{pmatrix} R(\theta) & d \\ 0 & 1 \end{pmatrix} \tag{3.1}$$

where $R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ is a 2×2 rotation matrix that represents the rotation of Σ_1 with respect to Σ_0 and $d = \begin{pmatrix} p^1 \\ p^2 \end{pmatrix}$ is the vector that represents the position of P with respect to Σ_0.

By convention, a point $P \in \mathbb{R}^2$ expressed with respect to a frame of coordinates Σ_i will be refered to as "P in coordinates i", denoted by ^iP. The same notation will be used for $\dot{P}$.

Example 4. Let us consider Figure 3.3 and let $Q \in \mathbb{R}^2$ and suppose that $^1Q = (q^1, q^2)$. One changes from ^1Q to ^0Q by

$$^0Q = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & p_1 \\ \sin(\theta) & \cos(\theta) & p_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q^1 \\ q^2 \\ 1 \end{pmatrix} = \begin{pmatrix} q^1\cos(\theta) - q^2\sin(\theta) + p^1 \\ q^1\sin(\theta) + q^2\cos(\theta) + p^2 \\ 1 \end{pmatrix}.$$

The velocity ${}^0\dot{Q}$ of ${}^0Q$ is given by

$$
{}^0\dot{Q} = \begin{pmatrix} -q^1\sin(\theta)\dot{\theta} - q^2\cos(\theta)\dot{\theta} + \dot{p}^1 \\ q^1\cos(\theta)\dot{\theta} - q^2\sin(\theta)\dot{\theta} + \dot{p}^2 \\ 0 \end{pmatrix}.
$$

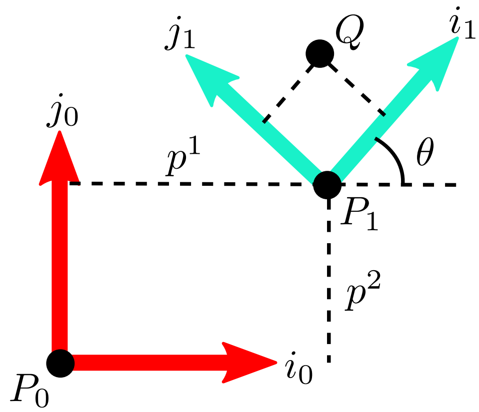

Figure 3.3: Graphic representation of P, Σ_0 and Σ.

3.2.3 Steps of the modeling methodology

- Establish a reference frame fixed on the floor, and "moving frames" fixed to each of the articulated bodies. Compute the corresponding coordinate change matrix between each pair of contiguous frames.

- Define all the differential forms that represent nonholonomic constraints for the system. For the systems modeled in this thesis, the nonholonomic constraints derive from the "no slipping" assumption as follows: since the wheel is rolling on the floor without slipping, the velocity of the contact point of the wheel with the floor is equal to zero, which implies that the components of this velocity parallel and orthogonal to the wheel are equal to zero as well.

- Let Υ be the co-distribution spanned by the set $\{\alpha_1, \ldots, \alpha_s\}$ of all differential forms that represent a nonholonomic constraint. Let $\{v_1, \ldots, v_2\}$ be a basis for the constraint distribution $\Delta = \Upsilon^\perp$ of Υ. A mathematical representation of the kinematics of the system in coordinates x will then be given by $\dot{x} = v_1u_1 + \cdots + v_su_s$, where each u_i is a control input.

To illustrate this methodology, these steps will be detailed through an example in the following section.

3.3 Example: Modeling of the kinematics of the cart with 2 Trailers

Let us consider the cart with 2 trailers, whose graphic representation is shown in figure 3.4. Consider for this system the same assumptions made for the cart with N trailers in Section 3.1. Note that in order to determine the exact position of this system on the plane, it is enough to know x, y, θ_1, θ_2 and θ_3; therefore $Q = \mathbb{R}^2 \times (S^1)^3$ is an admissible configuration space for this system and $(x, y, \theta_1, \theta_2, \theta_3)$ are coordinates on Q.

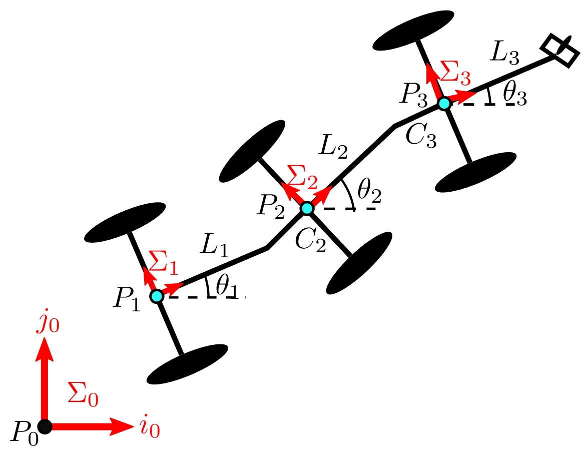

Figure 3.4: Cart with 2 trailers.

Let us suppose that $\{i_0, j_0\}$ is the canonical basis of $\mathbb{R}^2$. Let ${}^kT_l$ be the homogeneous transformation matrix that represents the change of coordinates from frame k to frame l, for $k \in \{0, \ldots, 3\}$ and $l \in \{1, \ldots, 3\}$. Therefore one has the following homogeneous transformation matrices:

$$
{}^0T_1 = \left(\begin{array}{c|c} R(\theta_1) & \begin{array}{c} x \\ y \end{array} \\ \hline 0 & 1 \end{array}\right) \qquad {}^1T_2 = \left(\begin{array}{c|c} R(\theta_2 - \theta_1) & \begin{array}{c} C + L \\ 0 \end{array} \\ \hline 0 & 1 \end{array}\right) \tag{3.2}
$$

$$
{}^2T_3 = \left(\begin{array}{c|c} R(\theta_3 - \theta_2) & \begin{array}{c} C + L \\ 0 \end{array} \\ \hline 0 & 1 \end{array}\right)
$$

Let, for $s = 1, \ldots, 3$, ${}^0i_s := e_1^s$, ${}^0j_s := e_2^s$ and $e_3 := e_3^s$. Let $E_s = \text{span}\{e_1^s, e_2^s, e_3^s\}$. From subsection 3.2.1 it is easy to see that $E_s = (\mathbb{R}^2, \tilde{+}, \tilde{\ast})$, where P_1, P_2 and P_3 are respectively the origin of E_1, E_3 and E_5. Since E_s is a vector space, it is a basic fact of linear algebra (e.g. [Strang, 1988]) that there exists the dual space E_s^* of E_s, spanned by the dual basis $\{\gamma_s^1, \gamma_s^2, \gamma_s^3\}$ of $\{e_1^s, e_2^s, e_3^s\}$, where γ_s^1 γ_s^2 and γ_s^3 are linear maps from E_s onto $\mathbb{R}$, such that $\gamma_s^i(e_j^s) = \delta_j^i$, with $i, j \in \{1, 2, 3\}$. In other words γ_s^r is the function that extracts the e_r^s-th component of a vector $v \in E_s$, i.e., "γ_s^r projects v onto e_r^s".

To satisfy the no slipping conditions on the wheels, it is sufficient to establish that ${}^0\dot{P}_1$ satisfies the following:

- ${}^0\dot{P}_1$ projected in the direction of ${}^0j_1$ vanishes.
- ${}^0\dot{P}_2$ projected in the direction of ${}^0j_3$ vanishes.
- ${}^0\dot{P}_3$ projected in the direction of ${}^0j_5$ vanishes.

In other words, $v = \begin{pmatrix} v^1 \\ v^2 \\ 0 \end{pmatrix}$ is an admissible velocity for this system, only if v satisfies $\gamma_1^2(v) = 0$, $\gamma_3^2(v) = 0$ and $\gamma_5^2(v) = 0$, with γ_r^2, for $r \in \{1, 3, 5\}$, expressed in terms of the dual basis of $\{i_0, j_0, e^3\}$, i.e., in coordinates 0. Let $\{e_1, \ldots, e_3\}$ be the canonical basis of $\mathbb{R}^3$ and $\{\gamma^1, \ldots, \gamma^3\}$ its dual basis.

Example 5. By definition of ${}^0T_1$, one has

$$
{}^0P_1 = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}, \quad {}^0\dot{P}_1 = \begin{pmatrix} \dot{x} \\ \dot{y} \\ 0 \end{pmatrix}, \quad {}^0i_1 = \begin{pmatrix} \cos(\theta_1) \\ \sin(\theta_1) \\ 0 \end{pmatrix} \text{ and } {}^0j_1 = \begin{pmatrix} -\sin(\theta_1) \\ \cos(\theta_1) \\ 0 \end{pmatrix}.
$$

According to the previous notation, $e_1^1 = \cos(\theta_1)e_1 + \sin(\theta_1)e_2$, $e_2^1 = -\sin(\theta_1)e_1 + \cos(\theta_1)e_2$ and $e_3^1 = e_3$. Since $\{e_1^1, e_2^1, e_3^1\}$ is a basis for $\mathbb{R}^3$, and $\{\gamma_1^1, \gamma_1^2, \gamma_1^3\}$ is its dual basis, it is well known from linear algebra that

$$
e_j^1 = ({}^0{T_1}^{-1})_j^i e_i \quad \text{and} \quad \gamma_1^j = ({}^0T_1)_i^j \gamma^i,
$$

where $({}^0{T_1}^{-1})_j^i$ is the (i, j)-th component of ${}^0{T_1}^{-1}$, the inverse matrix of ${}^0T_1$. Therefore

$$
\begin{aligned}
\gamma_1^1 &= \cos(\theta_1)\gamma^1 + \sin(\theta_1)\gamma^2 \\
\gamma_1^2 &= -\sin(\theta_1)\gamma^1 + \cos(\theta_1)\gamma^2 \\
\gamma_1^3 &= \gamma^3
\end{aligned}
$$

By simple computations one has that every admissible velocity $v \in \mathbb{R}^3$ satisfies $\gamma_j^2(v) = 0$, for $j = 1, 2, 3$, which translates into the following conditions:

$$
\begin{aligned}
\gamma_1^2(v) &= -\sin(\theta_1)v^1 + \cos(\theta_1)v^2 = 0\\
\gamma_2^2(v) &= -\sin(\theta_2)v^1 + \cos(\theta_2)v^2 + (C+L)\cos(\theta_2-\theta_1)v^3 = 0\\
\gamma_3^2(v) &= -\sin(\theta_3)v^1 + \cos(\theta_3)v^2 + (L+C)\cos(\theta_3-\theta_1)v^3 + (L+C)\cos(\theta_3-\theta_2)v^4 = 0
\end{aligned}
$$

Let $(x, y, \theta_1, \theta_2, \theta_3) = (x_1, \ldots, x_5)$, therefore $x = (x_1, x_2, x_3, x_4, x_5)$ are coordinates for Q. In order to define the nonholonomic constraints of the system, let us define, from γ_r^s, with $r = 1, 2$ and $s = 1, 2, 3$, the six following 1-forms on Q by:

$$
\begin{aligned}
\alpha_1(x) &= \sin(x_3)dx_1 + \cos(x_3)dx_2\\
\alpha_2(x) &= -\sin(x_4)dx_1 + \cos(x_4)dx_2 + (C+L)\cos(x_4-x_3)dx_3\\
\alpha_3(x) &= -\sin(x_5)dx_1 + \cos(x_5)dx_2 + (L+C)\cos(x_5-x_3)dx_3 + (L+C)\cos(x_5-x_4)dx_4
\end{aligned}
$$

Let Υ be the co-distribution spanned by $\alpha_1, \ldots, \alpha_3$. Therefore $w \in TQ$ is an admissible speed for the system only if w belongs to the constraint distribution $\Delta = \{v \in TQ : v \in \operatorname{ann}(\Upsilon)\}$, i.e., if w simultaneously satisfies $\alpha_s(w) = 0$, for $s = 1, \ldots, 3$.

To obtain a mathematical representation of the system one starts by finding a basis for the distribution Δ. Let

$$
V_1(x) = \begin{pmatrix} \cos(x_3)\cos(x_4-x_3)\cos(x_5-x_4)\\ \sin(x_3)\cos(x_4-x_3)\cos(x_5-x_4)\\ \frac{1}{C+L}\sin(x_4-x_3)\cos(x_5-x_4)\\ \frac{1}{C+L}\sin(x_5-x_4)\\ 0 \end{pmatrix}, \qquad V_2(x) = \begin{pmatrix} 0\\0\\0\\0\\1 \end{pmatrix}. \tag{3.3}
$$

Since $\{V_1, V_2\}$ is free and spans Δ, $\{V_1, V_2\}$ is a basis of Δ, i.e., $\Delta = \operatorname{span}\{V_1, V_2\}$. A mathematical representation of the kinematics of the system shown in Figure 3.4, is given by:

$$
\dot{x} = V_1(x)u_1 + V_2(x)u_2, \tag{3.4}
$$

with u_1 and u_2 considered as arbitrary control inputs.

3.4 Kinematic models for the tricycle and the tricycle with 1 trailer

The tricycle (car without trailers) and the tricycle with one trailer (car with one trailer) are systems of interest for the development of this work and the desingularization algorithm will be applied to one of their models. In this section we recall the mathematical representation of the kinematics of both systems, described in [Lizárraga et al., 2001], and obtained by the modeling methodology previously explained.

Figure 3.5 shows a kinematic representation of the tricycle. Note that an appropriate space of configurations for this system is $Q_1 = \mathbb{R}^2 \times (S^1)^2$. Let $x = (x_1, x_2, x_3, x_4)$ be coordinates for Q_1, with x_1 and x_2 representing the orthogonal projection of point P_1 on the floor, x_3 representing the angle θ and x_4 representing the angle α.

If one assume that the wheels roll on the floor without slipping, the nonholonomic constraints for this system are imposed via the following conditions:

- ${}^0\dot{P}_1$ projected in the direction of j_1 vanishes.
- ${}^0\dot{Q}$ projected in the direction of j_2 vanishes.

Following the methodology of the previous section, one obtains the kinematic model for the tricycle:

$$\dot{x} = \begin{pmatrix} \cos(x_3)\cos(x_4) \\ \sin(x_3)\cos(x_4) \\ \frac{1}{L_1}\sin(x_4) \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_2, \tag{3.5}$$

where u_1 represents the velocity of point P_1 and u_2 represents the rotation speed $\dot{\alpha}$.

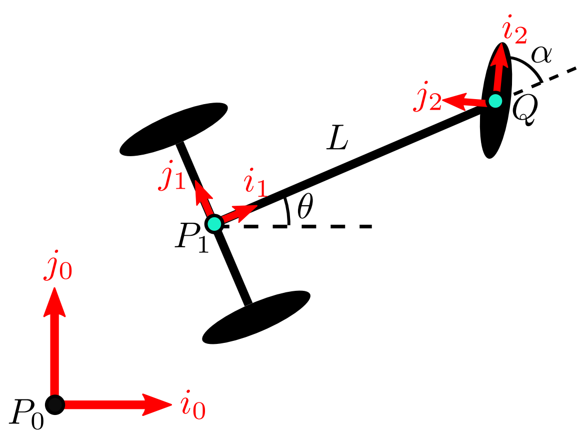

Figure 3.5: Graphic representation of a tricycle.

The kinematic representation of the tricycle with 1 trailer is shown in Figure 3.6. It is clear that $Q_2 = \mathbb{R}^2 \times (S^1)^3$ is a configuration manifold for this system. Let $x = (x_1, x_2, x_3, x_4, x_5)$ be coordinates for Q_2; assume that x_1 and x_2 represent the position of point P_1 on the floor, and that x_3, x_4 and x_5 represent respectively the angles θ_1, α_2 and α_3.

The nonholonomic constraints for this system arise from the following assumptions:

- ${}^0\dot{P}_1$ projected in the direction of j_1 vanishes.
- ${}^0\dot{P}_2$ projected in the direction of j_2 vanishes.
- ${}^0\dot{Q}$ projected in the direction of j_3 vanishes.

By applying the modeling methodology described in this chapter one obtains the following kinematic model for the tricycle with 1 trailer:

$$\dot{x} = \begin{pmatrix} \frac{1}{L_2}\cos(x_3)\left(L_2\cos(x_4)\cos(x_5) + C_2\sin(x_4)\sin(x_5)\right) \\ \frac{1}{L_2}\sin(x_3)\left(L_2\cos(x_4)\cos(x_5) + C_2\sin(x_4)\sin(x_5)\right) \\ \frac{1}{L_1 L_2}\left(L_2\sin(x_4)\cos(x_5) - C_2\cos(x_4)\sin(x_5)\right) \\ \frac{1}{L_1 L_2}\left(L_1\sin(x_5) - L_2\sin(x_4)\cos(x_5) + C_2\cos(x_4)\sin(x_5)\right) \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_2, \qquad (3.6)$$

where u_1 represents the magnitude of the velocity of point P_1, and u_2 represents the speed of rotation $\dot{\alpha}_3$ of the steering wheel.

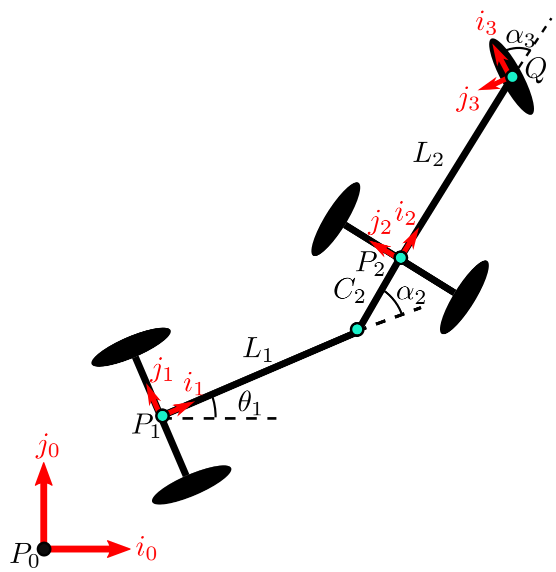

Figure 3.6: Graphic representation of a tricycle with 1 trailer.

Chapter 4

Lie algebras and free Lie algebras

This chapter gives a brief description of two mathematical structures that play an important role in the ensuing development of this work: Free Lie algebras generated by finite sets, and Lie algebras generated by a set of m vector fields. These related concepts are key elements in the desingularization algorithm explained in Chapter 4. The information presented in this section is mainly taken from references [Serre, 1992a] and [Warner, 1983].

4.1 Lie Algebras

A ***Lie algebra*** $\mathfrak{g}$ over $\mathbb{R}$ is a real vector space $\mathfrak{g}$ together with a $\mathbb{R}$-bilinear operator $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called the Lie bracket) such that, for all $x, y, z \in \mathfrak{g}$:

(a) $[x, y] = -[y, x]$ (skew symmetry or anti-commutativity).

(b) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ (Jacobi Identity).

Let M be a differential manifold of dimension n and let $X_1, \ldots, X_m \in \Gamma(TM)$. As can be seen for example in [Warner, 1983], if $(U, \varphi = (x_1, \ldots, x_n))$ is a coordinate system on an open subset $U \subseteq M$, then each vector field X_i can be expressed in coordinates φ as $X_i|U = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}$, with $a_j \in C^\infty(U)$ for each $j = 1, \ldots, n$, i.e., a_j is a C^∞ function on U.

Let $\mathcal{X} = \{X_1, \ldots, X_m\}$, $\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\{\mathcal{X}\} \subseteq \Gamma(TM)$, and define operators $+$ and $\star$ as:

$$\begin{aligned}
+ : \Gamma(TM) \times \Gamma(TM) &\longrightarrow \Gamma(TM) : (X, Y) \longmapsto \sum_{j=1}^{n} (a_j + b_j) \frac{\partial}{\partial x_j}, \\
\star : C^\infty(M) \times \Gamma(TM) &\longrightarrow \Gamma(TM) : (f, X) \longmapsto \sum_{j=1}^{n} (f a_j) \frac{\partial}{\partial x_j}
\end{aligned}$$

with $X = \sum_{j=1}^n a_j \frac{\partial}{\partial x_j}$ and $Y = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j}$. The definitions of $+$ and $\star$ imply that for all $p \in M$, $X, Y \in \Gamma(TM)$, and $f \in C^\infty(M)$, one has $(X+Y)_p = X_p + Y_p$ and $(f \star X)_p = f(p)X_p$. It is easy to prove that the triple $(\mathfrak{g}, +, \star)$, with the domain of $\star$ restricted to the set of constant functions, is a vector space over $\mathbb{R}$.

Consider $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} : (X,Y) \longmapsto [X,Y]$, with $[X,Y]$ denoting the Lie Bracket of vector fields X and Y defined in Section 2.1. It is easy to see that Lie bracket defined in that way satisfies the Jacobi identity and is skew-symmetric. Let $X, Y, Z \in \mathfrak{g}$, $a \in \mathbb{R}$, $p \in M$ and $f \in C^\infty(M)$. By definition:

$$\begin{aligned}
[a \star X, Y]_p(f) &= (a \star X)_p(Yf) - Y_p((a \star X)f) \\
&= a(p)X_p(Yf) - (Y_p(a)X_p(f) + a(p)Y_p(Xf)) \\
&= a(p)(X_p(Yf) - Y_p(Xf)) - Y_p(a)X_p(f) \\
&= a \star [X,Y]_p(f),
\end{aligned}$$

$$\begin{aligned}
[X+Y, Z]_p(f) &= (X+Y)_p(Zf) - Z_p((X+Y)f) \\
&= X_p(Zf) + Y_p(Zf) - Z_p(Xf + Yf) \\
&= X_p(Zf) + Y_p(Zf) - Z_p(Xf) - Z_p(Yf) \\
&= [X,Z]_p(f) + [Y,Z]_p(f),
\end{aligned}$$

hence $[\cdot,\cdot]$ is linear with respect to its first argument. Similarly $[X, a \star Y]_p(f) = a \star [X,Y]_p(f)$ and $[X, Y+Z]_p(f) = [X,Y]_p(f) + [X,Z]_p(f)$, therefore $[\cdot,\cdot]$ is linear with respect to its second argument too. Thus, $[\cdot,\cdot]$ is bilinear and the real vector space $(\mathfrak{g}, +, \star)$ together with $[\cdot,\cdot]$ is a Lie Algebra over $\mathbb{R}$. The Lie Algebra generated by $\mathcal{X}$ will be denoted by $L_{\mathcal{X}}$.

A function $D : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a derivation if D is $\mathbb{R}$-linear and has the property that, for every $X, Y \in \mathfrak{g}$, $D([X,Y]) = [D(X), Y] + [X, D(Y)]$. Let $X \in \mathfrak{g}$ and let us define a map $\mathfrak{L}_X : \mathfrak{g} \longrightarrow \mathfrak{g}$ by $\mathfrak{L}_X(Y) = [X,Y]$, for every $Y \in \mathfrak{g}$. Let $Z \in \mathfrak{g}$, then:

$$\begin{aligned}
\mathfrak{L}_X([Y,Z]) &= [X,[Y,Z]] \\
&= -[Y,[Z,X]] - [Z,[X,Y]] \text{ (by Jacobi identity of } [\ \ ,\ \ ]) \\
&= [[X,Y],Z] + [Y,[X,Z]] \\
&= [\mathfrak{L}_X(Y), Z] + [Y, \mathfrak{L}_X(Z)];
\end{aligned}$$

it follows that $\mathfrak{L}_X$ is a derivation and $\mathfrak{L}_X(Y)$ is called the ***Lie derivative*** of Y in the direction of X.

4.1.1 Further examples of Lie Algebras

The following is a list of other examples of Lie algebras:

- Any vector space V over a field K trivially becomes a Lie Algebra over K if one defines the Lie bracket, for all $v_1, v_2 \in V$, as $[v_1, v_2] = 0$. It is clear that the Lie bracket defined in this way is bilinear, skew-symmetric and satisfies the Jacobi identity. A Lie algebra with an identically vanishing Lie Bracket is said to be commutative or Abelian.

- The vector space $\mathfrak{g}(n, \mathbb{R})$ of all $n \times n$ real matrices forms a Lie algebra over $\mathbb{R}$ with the Lie bracket defined by $[A, B] = AB - BA$. Let $A, B, C \in \mathfrak{g}(n, \mathbb{R})$ and $k \in K$. By definition of the sum and multiplication of matrices one has:

  a) $[A, B] = AB - BA = -(BA - AB) = -[B, A]$,
  b) $[[A, B], C] + [[B, C], A] + [[C, A], B] = [AB - BA, C] + [BC - CB, A] = 0$,
  c) $[kA + B, C] = kAC + BC - kCA - CB = k[A, C] + [B, C]$,
  d) $[A, kB + C] = k(AB - BA) + AC - CA = k[A, B] + [A, C]$,

  therefore the Lie bracket is skew symmetric, satisfies the Jacobi identity and is bilinear.

4.2 Free Lie algebras

This section seeks to explain the construction of the free Lie algebra generated by a set with m elements. Hereafter, K will denote a commutative and associative ring with a unit. Modules and algebras mentioned here are taken over K.

A set M with a map $M \times M \longrightarrow M : (x, y) \longmapsto xy$ is called a **magma**. Let $B = \{1, \dots, m\}$ and let us define inductively a family of sets B_n, for $n \geq 1$, as follows:

1. $B_1 = B$.
2. $B_n = \bigcup_{(p,q) \in C_n} X_p \times X_q$, where $C_n = \{(p, q) \in \mathbb{N} \times \mathbb{N} : p + q = n\}$. $(n \geq 2)$

Let us set $M_B := \bigcup_{n=1}^{\infty} B_n$, and define a multiplication $\bullet : M_B \times M_B \longrightarrow M_B$ on M_B as follows: For all $x, y \in M_B$, there exists $p, q \in \mathbb{N}$ such that $x \in B_p$ and $y \in B_q$; one sets $x \bullet y = (x, y) \in B_{p+q}$. The magma M_B with the multiplication thus defined is called the ***free magma*** on B. An element w of M_B is called a non-associative word on B; its ***length***, denoted $\ell(w)$, is the unique $n \in \mathbb{N}$ such that $w \in B_n$.

Example 6. *(Free magma)* Let $B = \{1,2\}$, following the construction of the free magma M_B on B, the first four sets in the sequence $(B_n)_{n \in \mathbb{N}}$ are:

$$\begin{aligned}
B_1 &= B = \{1,2\} \\
B_2 &= B_1 \times B_1 = \{(1,1),(1,2),(2,1),(2,2)\} \\
B_3 &= (B_1 \times, B_2) \bigcup (B_2 \times B_1) = \{(1,(1,1)),(1,(1,2)),(1,(2,1)),(1,(2,2)),(2,(1,1)), \\
&\quad (2,(1,2)),(2,(2,1)),(2,(2,2)),((1,1),1),((1,2),1),((2,1),1),((2,2),1), \\
&\quad ((1,1),2),((1,2),1),((2,1),2),((2,2),2)\} \\
B_4 &= (B_1 \times B_3) \bigcup (B_3 \times B_1) \bigcup (B_2 \times B_2).
\end{aligned}$$

Let N be a magma, B be a set and $f : B \longrightarrow N$ be any map. Then there exists a unique magma homomorphism $F : M_B \longrightarrow N$ which extends f. This magma homomorphism is defined inductively by $F(u,v) = F(u) \bullet F(v)$ if $u,v \in B_p \times B_q$. This means that F is the unique function that maps M_B into N, preserving the magma structure and making the following diagram commute:

$$\begin{tikzcd}
B \arrow[r, hook, "\mathrm{id}"] \arrow[dr, "f"'] & M_B \arrow[d, dashed, "F"] \\
& N
\end{tikzcd}$$

where id (identity map) is the natural inclusion of B into M_B.

Let A_B be the K-algebra constructed from the free magma M_B as follows. Let $A_B = \{f : M_B \longrightarrow K\}$, i.e., A_B is the set of all functions of M_B into K. Addition "+" and multiplication by an scalar "$*$" in A_B are defined from the sum and multiplication of functions as follows:

$$\begin{aligned}
+ : A_B \times A_B \longrightarrow A_B &: (x,y) \longmapsto x + y \\
* : K \times A_B \longrightarrow A_B &: (k,x) \longmapsto kx
\end{aligned}$$

where $x + y : M_B \longrightarrow K$ is the function that maps each $x \in M_B$ to $x(b) + y(b)$ and $kx : M_B \longrightarrow K$ is the function that maps each $x \in M_B$ to $kx(b)$.

Every $b \in B$ is naturally included in A_B as the function f_b defined by

$$f_b(\bar{b}) = \begin{cases} 1, & \text{if } \bar{b} = b \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the multiplication $\bar{\bullet}$ on A_B extends the multiplication $\bullet$ of elements in M_B. Hence every element $a \in A_B$ may be uniquely written as a finite sum $a = \sum_{m \in M_B} c_m m$, with $c_m \in k$.

The algebra A_B is called the ***free algebra generated by*** **B** and it satisfies the following "universal" property: Let D be a K-algebra and let $f : B \longrightarrow D$ be a map. There exists a unique k–algebra homomorphism $F : A_B \longrightarrow D$ which extends f. To see a proof of this property the reader may refer to [Serre, 1992b]. The fact that A_B satisfies the above property means that the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, hook, "\overline{\mathrm{id}}"] \arrow[dr, "f"'] & A_B \arrow[d, dashed, "F"] \\ & D \end{tikzcd}$$

where, for every $b \in B$, one has $\overline{\mathrm{id}}(b) = f_b$. Intuitively, the only relations that hold among elements of the free algebra A_B are the ones imposed by the definition of algebra, i.e., those that derive from the properties of the sum and the multiplication by scalars.

Example 7. Since $\mathbb{Z}$ is a commutative ring, $\mathbb{Z}$ is an algebra over itself with the usual addition and multiplication of integers. Each property that operations in $\mathbb{Z}$ must satisfy ($+$, $\bullet$ and multiplication by scalars) generates a relation in $\mathbb{Z}$; for example, $3 \bullet (4+5)$ is related with $3 \bullet 4 + 3 \bullet 5$ by the distributivity property of $\bullet$ and $+$, since $3 \bullet (4+5) = 3 \bullet 4 + 3 \bullet 5$. In addition, there are other relations given by the nature of $\mathbb{Z}$ itself, for example $(5 \bullet 3)$ is related with $10 + 5$. On the other hand, in the free algebra A_B generated by $B = \{1\}$, none of the relations among elements of A_B, other than these strictly imposed by the algebra operation, hold valid.

By construction, the free Lie algebra A_B is a graded algebra, with homogeneous elements of degree n being those equal to linear combinations of words $m \in M_B$ of length n.

Let I be the two-sided ideal of A_B generated by the elements of the forms $a \bullet a$, and $J(a,b,c) = (a \bullet b) \bullet c + (b \bullet c) \bullet a + (c \bullet a) \bullet b$, with $a, b, c \in A_B$. The quotient algebra A_B/I is called the ***free Lie algebra on*** B. This algebra is denoted by $\mathcal{L}_B(K)$ and, when the ring K is clear from the context, $\mathcal{L}_B(K)$ is simply denoted $\mathcal{L}_B$. Since A_B is a graded algebra and I is a sub-algebra of A_B, it follows that I is a graded ideal of A_B, which implies that $\mathcal{L}_B$ has a natural structure of graded algebra.

Note that by the definition of the ideal I, and by the definition of quotient algebra, elements of the form $a \bullet a$ and $J(a,b,c) = (a \bullet b) \bullet c + (b \bullet c) \bullet a + (c \bullet a) \bullet b$, with $a, b, c \in A_B$, belong to the "zero" equivalence class in A_B/I. Let $a, b \in A_B$, then $[a], [b] \in \mathcal{L}_B$, and the sum $\bar{+}$ and multiplication $\bar{\bullet}$ in $\mathcal{L}_B$ are defined in terms of the sum $+$ and multiplication $\bullet$ in A_B by:

$$[a] \bar{+} [b] = [a + b]$$

$$[a]\bar{\bullet}[b] = [a \bullet b].$$

Hence, for $[a],[b],[c] \in \mathcal{L}_B$ one has $[a]\bar{\bullet}[a] = [a\bullet a] = 0$ and $([a]\bar{\bullet}[b])\bar{\bullet}[c]+([b]\bar{\bullet}[c])\bar{\bullet}[a]+([c]\bar{\bullet}[a])\bar{\bullet}[b] = [(a\bullet b)\bullet c+(b\bullet c)\bullet a+(c\bullet a)\bullet b] = 0$. Thus, the multiplication $\bar{\bullet}$ of elements of the free Lie algebra $\mathcal{L}_B$ is skew symmetric and satisfies the Jacobi identity. Since $\bullet$ is bilinear, it is easy to prove that $\bar{\bullet}$ is bilinear too, so $\bar{\bullet}$ is a Lie bracket operation.

From here on $[\cdot,\cdot]$ will be used interchangeably to denote both the operation $\bar{\bullet}$ in $\mathcal{L}_B$ and the Lie bracket of vector fields defined in Section 2.1.

Remark 2. As in free algebras and algebras, the difference between Free Lie Algebras and Lie algebras is in the relationships between the elements of each one. In addition to the relations generated in a free algebra by the properties of the sum, multiplication and multiplication by an scalar, in a free Lie algebra there exist another two relations: one arising from the skew-symmetry, the other generated by the Jacobi identity of the Lie Bracket operation.

By contrast, in a Lie algebra there may also be other relations arising from the nature of the set upon which operations are defined. Consider for example the Lie algebra generated by the vector fields X_1 and X_2, defined on a differential manifold M by $X_1(x) = \frac{\partial}{\partial x_1}\big|_x + x_2\frac{\partial}{\partial x_3}\big|_x$, $X_2(x) = \frac{\partial}{\partial x_2}\big|_x$. Lie brackets $[X_1,[X_1,X_2]]$ and $[X_1,[X_1,[X_1,X_2]]]$ are "related" by an equation that must hold true, since $[X_1,[X_1,X_2]] = [X_1,[X_1,[X_1,X_2]]] = 0$. Therefore the concept of "length of an element" does not make sense in a Lie algebra, unlike the case of a free Lie algebra.

4.2.1 P. Hall Basis of Free Lie Algebras

Let B be a totally ordered set. A ***P. Hall Family*** in M_B, the free magma on B, is a totally strictly ordered subset H of M_B, such that:

1. $B \subseteq H$.
2. If $u,v \in H$ with $\ell(u) \prec \ell(v)$ and $\prec$ the order in $\mathbb{N}$, then $u < v$.
3. An element $u = v \bullet w$ with $v,w \in M_B$ belongs to H if and only if one of the following conditions is satisfied:
   (a) $v \in H$, $w \in H$ and $v < w$;
   (b) either $w \in B$, or there exists $w', w'' \in H$ such that $w = w' \bullet w''$ and $w' \leq v$.

As shown, for example in [Serre, 1992a], there exists a P. Hall family for any ordered set B, constructed, by induction, by defining $H^1 = B$ and $H^n = H \cap B_n$ for $n \geq 2$. For instance, let $B = \{1,2\}$; then the sets $H^1, \dots, H^5$ of the P. Hall basis for B are:

$$
\begin{aligned}
H^1 &= \{1,2\} \\
H^2 &= \{1 \bullet 2\} \\
H^3 &= \{1 \bullet (1 \bullet 2),\ 2 \bullet (1 \bullet 2)\} \\
H^4 &= \{1 \bullet (1 \bullet (1 \bullet 2)),\ 2 \bullet (1 \bullet (1 \bullet 2)),\ 2 \bullet (2 \bullet (1 \bullet 2))\} \\
H^5 &= \{1 \bullet (1 \bullet (1 \bullet (1 \bullet 2))),\ 2 \bullet (1 \bullet (1 \bullet (1 \bullet 2))),\ 2 \bullet (2 \bullet (1 \bullet (1 \bullet 2))),\ 2 \bullet (2 \bullet (2 \bullet (1 \bullet 2))), \\
&\quad (1 \bullet 2) \bullet (1 \bullet (1 \bullet 2)),\ (1 \bullet 2) \bullet (2 \bullet (1 \bullet 2))\}
\end{aligned}
$$

If H is a P. Hall family in M_B, then the natural inclusion (identity map) of the elements $h \in H$ in $\mathcal{L}_B$ form a basis of $\mathcal{L}_B$, called a ***P. Hall basis*** of $\mathcal{L}_B$. Elements in H are called ***Lie monomials***.

4.3 Relationship Between Lie Algebras and free Lie Algebras

In the following subsections the reader will find a description of concepts that relate the free Lie algebra generated by m elements with the Lie algebra generated by m vector fields, which will be important to describe the desingularization algorithm in the next chapter.

4.3.1 Evaluation map $E_{\mathcal{X}}$

Suppose that Q is a differentiable manifold and (Ω, φ) is a coordinate chart of Q. Let $\mathcal{X} = \{X_1, \dots, X_m\}$, with $X_1, \dots, X_m \in \Gamma(T\Omega)$ and set $\mathcal{I} = \{1, \dots, m\}$. Recall that $\mathcal{L}_{\mathcal{I}}$ denotes the free Lie algebra generated by $\mathcal{I}$ and $L_{\mathcal{X}}$ denotes the Lie algebra generated by $\mathcal{X}$.

Let us define the function $e : \mathcal{I} \longrightarrow L_{\mathcal{X}}$ by $e(i) = X_i$ for $i \in \mathcal{I}$. By the "universal property"of free Lie algebras, there exists an algebra homomorphism $E_{\mathcal{X}} : \mathcal{L}_{\mathcal{I}} \longrightarrow L_{\mathcal{X}}$ that extends e, called the ***evaluation map of*** $\mathcal{L}_{\mathcal{I}}$ in $L_{\mathcal{X}}$. Thus, for instance, $E_{\mathcal{X}}(1) = X_1$, $E_{\mathcal{X}}([1,[2,3]]) = [X_1,[X_2,X_3]]$ and $E_{\mathcal{X}}([[1,2],[2,3]]) = [[X_1,X_2],[X_2,X_3]]$.

4.3.2 Descendants

The definitions in this section were taken from [Chitour et al., 2013] and [Jean, 2014]. By definition, the P. Hall Basis H of $\mathcal{L}_{\mathcal{X}}$ has a strict and total order, which allows one to define a surjective sequence $(I_i)_{i \in \mathbb{N}}$ on H. In the sequel, we will use I_j to denote the j-th element of H. Let $\mathcal{H}^1 := H^1$ and $\mathcal{H}^t := \bigcup_{i=1}^t H^i$, for $t \geq 2$. Every Lie monomial

$I_j \in H$ can be expanded as

$$I_j = [I_{k_1}, [I_{k_2}, \dots, [I_{k_{\ell-1}}, I_{k_\ell}] \dots]],$$

with $\ell \in \mathbb{N}$, $I_{k_1}, \dots, I_{k_{\ell-2}} \in \mathcal{H}^{|I_j|-2}$, and $I_{k_{\ell-1}}, I_{k_\ell} \in \mathcal{H}^1$ such that $I_{k_\ell-1} < I_{k_\ell}$. One says that I_j is a ***direct descendant*** of I_{k_ℓ}. Let us define the mapping $\phi : H \to \mathcal{H}^1$ that maps an element of H to the element in H^1 from which it descends, i.e., $\phi(I_j) = I_{k_\ell}$.

Example 8. Let $A = \{1, 2\}$, and let $u = \big[[1,2],[1,[1,2]]\big]$. By construction of the P. Hall basis for the free Lie algebra generated by A, one has $u \in H$. Therefore $u = \big[I_{k_1}, [I_{k_2}, [I_{k_3}, I_{k_4}]]\big]$, with $I_{k_1} = [1, 2]$, $I_{k_2} = 1$, $I_{k_3} = 1$ and $I_{k_4} = 2$. Thus, u is a direct descendant of 2, i.e., $\phi(u) = 2$.

4.3.3 Monomial $P_j(x)$ associated to I_j

One will associate to I_j the $(j-1)$-tuple $\alpha_j = (\alpha_j^1, \dots, \alpha_j^{j-1})$, where α_j^i is the number of occurrences of I_i (the i-th element in H) among $I_{k_1}, \dots, I_{k_\ell-1}$. The authors of [Chitour et al., 2013] define the monomial $P_j(x)$ associated with I_j by:

$$P_j(x) = \frac{x_1^{\alpha_j^1} \dots x_{j-1}^{\alpha_j^{j-1}}}{\alpha_j^1! \dots \alpha_j^{j-1}!}$$

Example 9. Let us consider again $[[1,2],[1,[1,2]] = [I_{k_1}, [I_{k_2}, [I_{k_3}, I_{k_4}]]] \in H$. Since u is the 13^{th} element in H (according to the total order in H), it has associated the 12-tuple $\alpha_{13} = (\alpha_{13}^1, \dots, \alpha_{13}^{12})$, where $\alpha_{13}^1 = 2$ is the number of occurrences of 1 among $I_{k_1}, \dots, I_{k_3}$, $\alpha_{13}^2 = 0$ is the number of occurrences of 2 among $I_{k_1}, \dots, I_{k_3}$, $\alpha_{13}^3 = 1$ is the number of occurrences of $[1, 2]$ among $I_{k_1}, \dots, I_{k_3}$, and $\alpha_{13}^4 = \dots = \alpha_{13}^{12} = 0$. Thus, one has the monomial

$$P_{13}(x) = \frac{x_1^{\alpha_{13}^1} \dots x_{12}^{\alpha_{13}^{12}}}{\alpha_{13}^1! \dots \alpha_{13}^{12}!} = \frac{x_1^2 x_3}{2}$$

Chapter 5

The desingularization algorithm

One of the objectives of this thesis was the study of the "desingularization algorithm" proposed in [Chitour et al., 2013]. This chapter presents the importance and the steps of the algorithm and its application to a simple system.

5.1 Regular and singular systems

Let us consider the driftless control-affine system

$$\dot{x} = \sum_{i=1}^{m} X_i(x)\, u_i \tag{5.1}$$

where $\varphi = (x_1, \ldots, x_n)$ are coordinates for the configuration manifold M, $X_1, \ldots, X_m$ are local representations of some vector fields $\hat{X}_1, \ldots, \hat{X}_m$ in $\Gamma(TM)$, and $u_1, \ldots, u_m$ are control inputs that take values in the real numbers. Let $U \subseteq M$ be the domain of the coordinate chart φ. Hereafter, Ω will denote $\varphi(U)$ so that $X_1, \ldots, X_m$ are elements of $\Gamma(T\Omega)$.

As the reader may read, e.g. in [Nijmeijer and van der Schaft, 1991], if one assumes that the control inputs $u_1, \ldots, u_m$ belong to a set of "admissible inputs" $\mathcal{U}$, then there exists a unique solution of (5.1) at time t, for these control inputs $u_1, \ldots, u_m \in \mathcal{U}$, with x_0 and t_0 as initial conditions, denoted by $x(t, t_0, x_0, u_1, \ldots, u_m)$ or, more simply as $x(t)$, when the rest of the arguments $(t_0, x_0, u_1, \ldots, u_m)$ are clear from the context.

The authors in [Nijmeijer and van der Schaft, 1991] justify to consider $\mathcal{U}$ as the set of admissible control inputs by the following reasoning that follows from standard results on the continuity of solutions of differential equations: If one has an approximation of a more general control input $\bar{u}(\cdot) : [0, \infty] \longrightarrow \mathbb{R}^n$ by piecewise constant functions in some suitable sense, then the solutions of (5.1) for these piecewise constant functions will be an approximation of the solution of (3.1) for $\bar{u}(\cdot)$.

Let $\mathcal{X} = \{X_1, \ldots, X_m\}$ and let $A = \{1, \ldots, m\}$. Recall that $\mathcal{L}_A$ denotes the *free Lie Algebra* generated by A, $L_{\mathcal{X}}$ denotes the Lie algebra generated by $\mathcal{X}$ and one has the evaluation map $E_{\mathcal{X}}$ of $\mathcal{L}_A$ into $L_{\mathcal{X}}$. As in Chapter 3, H denotes the P. Hall basis of $\mathcal{L}_A$. Let Δ be the distribution generated by the elements in $\mathcal{X}$ and let $s \in \mathbb{N}$; in the following, Δ^s will denote the distribution spanned by the image by $E_{\mathcal{X}}$ of all the elements in H the lengths of which are less than or equal to s, i.e., the elements in $\mathcal{H}^s$.

The ***involutive closure*** of the distribution Δ is the intersection of all the involutive distributions that contain Δ, i.e., the "smallest" involutive distribution in which Δ is contained. From here on, $\overline{\Delta}$ will denote the involutive closure of Δ. The distribution Δ is said to satisfy the ***Lie Algebra Rank Condition*** at $x \in \Omega$ (for short ***LARC***(x)) if and only if $\overline{\Delta}(x) = T_x\Omega$, i.e., if and only if $\dim(\overline{\Delta}(x)) = n$. System (5.1) is said to satisfy the LARC if the LARC is satisfied at every $x \in \Omega$.

Let us suppose that (5.1) satisfies the LARC and let $x \in \Omega$. Therefore there exists a smaller integer r such that $\dim(\Delta^r)(x) = n$. This integer r is called the ***degree of nonholonomy*** of $\mathcal{X}$ at x. The degree of nonholonomy of $\mathcal{X}$ on a set $B \subseteq \Omega$ is defined by $\max\{r \in \mathbb{Z} : (\exists x \in B)(r \text{ is the degree of nonholonomy of } \mathcal{X} \text{ at } x)\}$.

Let $x \in \Omega$ and let r be the degree of nonholonomy of $\mathcal{X}$ at x. Define $n_s(x) := \dim(\Delta^s)(x))$, for $s = 1, \ldots, r$. The r-tuple $(n_1(x), \ldots, n_r(x))$ is called the ***growth vector*** of $\mathcal{X}$ at x.

A point $x \in \Omega$ is said to be a ***regular point*** of (5.1) if there exists a neighborhood V of x such that the growth vector of $\mathcal{X}$ is the same at every $y \in V$, otherwise x is said to be a ***singular point*** of (5.1). System (5.1) is said to be a ***regular system*** if every point $x \in \Omega$ is regular, otherwise (5.1) is said to be a ***singular system***.

Example 10. Let us consider the system

$$\dot{x} = \begin{pmatrix} 1 \\ 0 \\ x_2 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_2 = X_1(x)u_1 + X_2(x)u_2, \tag{5.2}$$

defined on $\mathbb{R}^3$. Since $[X_1, X_2] = \frac{\partial}{\partial x_3}$, it is easy to prove that at every $x \in \mathbb{R}^3$, $\overline{\Delta}_x = \text{span}\{X_1(x), X_2(x), [X_1, X_2](x)\}$ is the involutive closure of the distribution $\Delta_x = \text{span}\{X_1(x), X_2(x)\}$. Therefore one has $\dim(\overline{\Delta}(x)) = 3$ for every $x \in \mathbb{R}^3$, i.e., $\overline{\Delta}_x$ spans $T_x\mathbb{R}^3$. It follows that the degree of nonholonomy r of $\{X_1, X_2\}$ at $\mathbb{R}^3$ is equal to 2. By definition one has $n_1(x) = 2$ and $n_2(x) = 3$ for every $x \in \mathbb{R}^3$, i.e., the growth vector of $\{X_1, X_2\}$ at $x \in \Omega$ is (2,3). Thus, (5.2) is a regular system. An example of singular system is presented in Section 5.4.

Let us define, for $s \in \mathbb{N}$, $\tilde{n}_s = \#(\mathcal{H}^s)$. A family of vector fields $\{V_1, \ldots, V_m\}$ defined on a differential manifold M is said to be ***free up to step*** s if for every $m \in M$, the growth vector $(n_1(x), \ldots, n_s(x))$ is equal to $(\tilde{n}_1, \ldots, \tilde{n}_s)$.

5.2 Interest of the algorithm

Driftless control affine systems have been under study in control theory partly because they constitute mathematical representations of many real systems including kinematic models of mechanical systems with nonholonomic constraints. Among the problems often addressed for this type of systems is the motion planing problem (for short MPP), also called "state steering problem". Solving the MPP for a system consists in associating to every pair of points $(x, y) \in \Omega \times \Omega$, an admissible control input $u(\cdot)$, defined on some interval $[0, T]$, such that the solution starting from x at a $t = 0$ reaches y at $t = T$. In other words, solving the MPP boils down to driving a system from an initial position x to an ending position y, using an admissible control input $u(\cdot)$ in a "finite" time T.

In control references, for example [Nijmeijer and van der Schaft, 1991], the distribution Δ is called the *accessibility distribution* generated by the *accessibility algebra* $L_{\mathcal{X}}$, and the LARC is called the *accessibility rank condition*. In that sense, (5.1) is said to be locally accessible on a set $B \subseteq \Omega$ if the LARC is satisfied at every $b \in B$.

System (5.1) is said to be controllable on a set $B \subseteq \Omega$ if, for any two points $x, y \in B$, there exists a finite time $T \in \mathbb{R}$ and an admissible control input $u \in \mathcal{U}$ such that $x(T, 0, x_1, u) = y$. The reader may wish to refer to [Nijmeijer and van der Schaft, 1991] to check that in the case of systems like (5.1), which has the drift term absent, when Ω is connected one has that if the LARC is satisfied on a set B then (5.1) is controllable on B.

Over the years, various methodologies have been developed in order to solve the MPP in driftless control-affine systems, many of which are applicable only in cases when one has more information about th system that the mere satisfaction of the LARC. For instance, the authors of [Lafferriere and Sussmann, 1991] and the authors of [Lafferriere and Sussman, 1992] propose a method for nilpotentizable systems, based on Lie brackets taken within the Lie algebra generated by the system's vector fields; [Murray and Sastry, 1993] propose sinusoidal controls for systems in chained form; and in [Bullo et al., 2000] one finds techniques applicable for left invariant systems defined on Lie groups.

The aforementioned methods have proved efficient in some applications; however, it is important to keep in mind that they are focused on solving the MPP for specific types of systems, which makes them rather restrictive. A clear example of this includes chained form systems: car-like wheeled mobile robots with more that one trailer cannot be transformed into chained-form unless each trailer is hooked to the midpoint of the previous wheel axle (ref. [Chitour et al., 2013]).

For this reason, many steering techniques have been developed in order to solve the MPP in general driftles systems. To mention only a few of these steering techniques, one has the *iterated Lie brackets method* ([Lafferriere and Sussmann, 1991, Sontag, 1995]), the *generic loop method* ([Sontag, 1995, Alouges et al., 2010]) and the *continuation method* ([Chitour and Sussmann, 1993, Sontag, 1995]).

The authors of [Chitour et al., 2013] propose an algorithm to solve the MPP in systems like (5.1) whose novelty, compared to the existing procedures, is that for the development of this algorithm the authors do not rule out the existence of singular points in (5.1). Let us suppose that x is a singular point of some system Σ. Therefore there does not exist any neighborhood of x on which the growth vector is constant, which implies that the degree of nonholonomy is not constant either. From a control-theoretic point of view, the above entails that the control inputs necessary to steer the system from one point to another may be rather intricate in the sense that their expressions are somewhat involved and their nature is highly oscillatory.

The previously mentioned algorithm is based on the assumption that the system for which one wants to solve the MPP is regular. For this reason the authors of [Chitour et al., 2013] have proposed a "desingularization algorithm" for singular systems in the form of (5.1). The construction of this algorithm ensures that the suitable signals used in order to control the "desingularized system" will also be suitable control inputs to solve the MPP in the original singular system.

As in Chapter 1, among the goals of the present work is to acquire a thorough understanding of the desingularization algorithm proposed by [Chitour et al., 2013], followed by an application of this method to a particular system.

It should be mentioned that the algorithm in [Chitour et al., 2013] ensures that the system obtained via desingularization (the "desingularized system") is expressed in special coordinates called *privileged coordinates*. However, as mentioned in that reference, it is possible to apply the desingularization algorithm in a way that does not necessarily yield a system in privileged coordinates.

5.3 Desingularization Algorithm

Let us suppose that (5.1) is a driftless control-affine system with a nonempty set of singular points. Also, suppose that $r \in \mathbb{N}$ is the degree of nonholonomy of $\mathcal{X} =$

$\{X_1, \ldots, X_m\}$ on Ω. The main idea of the desingularization procedure is to construct a manifold $\widetilde{\Omega} = \Omega \times \mathbb{R}^{\tilde{n}_r - n}$ and "lift" the control vector fields $X_1, \ldots, X_m$ to vector fields $\xi_1, \ldots, \xi_m$ on $\widetilde{\Omega}$ such that:

- For $i = 1, \ldots, m$, the vector fields X_i and ξ_i are π-related by the canonical projection $\pi : \tilde{\Omega} \longrightarrow \Omega$ that maps $(x_1, \ldots, x_{\tilde{n}_r}) \in \widetilde{\Omega}$ onto $(x_1, \ldots, x_n) \in \Omega$, that is, $T\pi \circ \xi_i = X_i \circ \pi$. This property guarantees that one retrieves $X_1, \ldots, X_m$ by projecting $\xi_1, \ldots, \xi_m$ on $T\Omega$, and that the following diagram commutes:

$$\begin{tikzcd} T\widetilde{\Omega} \arrow[r, "T\pi"] & T\Omega \\ \widetilde{\Omega} \arrow[u, "\xi_i"] \arrow[r, "\pi"] & \Omega \arrow[u, "X_i"'] \end{tikzcd}$$

- The family of vector fields $\{\xi_1, \ldots, \xi_m\}$ is free up to step r. This fact guarantees that the nonholonomic system defined by $\xi_1, \ldots, \xi_m$ is regular, since its growth vector is constant on $\tilde{\Omega}$.

Suppose that the following is a "lifted"version of (5.1) obtained as result of its desingularization:

$$\dot{\tilde{x}} = \sum_{i=1}^{m} \xi_i(\tilde{x}) u_i. \tag{5.3}$$

Let us consider a trajectory $\tilde{x}(\cdot, \tilde{x}_0, u(\cdot))$ of (5.3). Since X_i and ξ_i are π-related, $\pi(\tilde{x}(\,\cdot\,, \tilde{x}_0, u(\cdot))) = x(\,\cdot\,, \pi(\tilde{x}_0), u(\cdot))$ is a trajectory of (5.1) associated to the same control input function. Therefore, any control function $u(\cdot)$ that steers (5.3) from a point $\tilde{x}_0 := (x_0, 0)$ to a point $\tilde{x}_1 := (x_1, 0)$ also steers (5.1) form x_0 to x_1.

Starting with vector fields $X_1, \ldots, X_m$ expressed in some coordinates $x = (x_1, \ldots, x_n)$, the algorithm below yields a regular system with vector fields $\xi_1, \ldots, \xi_m$ expressed in the extended coordinates $\tilde{x} = (x_1, \ldots, x_{\tilde{n}_r})$ of x.

Consider again System (5.1) and let C be a compact subset of Ω. Let us assume that the LARC is satisfied at every point of C. Let r denote the degree of nonholonomy of $\mathcal{X}$ on C, note that r exists since C is compact.

Let $B = \{1, \ldots, m\}$ and let H be the the P. Hall basis for $\mathcal{L}_B$. For every n-tuple $\mathcal{I} = (I_1, \ldots, I_n)$ of elements of $\mathcal{H}^r$, define the set

$$V_{\mathcal{I}} := \{p \in \Omega : \dim(\operatorname{span}\{X_{I_1}(p), \ldots, X_{I_n}(p)\}) = n\} \tag{5.4}$$

where $X_{I_j} = E_{\mathcal{X}}(I_j)$, i.e., X_{I_j} is the image of I_j by the evaluation map $E_{\mathcal{X}}$.

Lemma 1. For every $\mathcal{I} = (I_1, \dots, I_n) \in (\mathcal{H}^r)^n$, the set $V_{\mathcal{I}}$ given by (5.4) is open.

Proof: Let $p \in V_{\mathcal{I}}$ and let $I_p \in \mathbb{R}^{n\times n}$ denote the matrix whose columns are given by the components of $X_{I_i}(p)$, for $i = 1, \dots, n$. The set $\{0\}$ is closed in $\mathbb{R}$, so $\mathbb{R}\backslash\{0\}$ is open. The function $\det : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$ being polynomial in the entries of its argument, is continuous, therefore the set $\det^{-1}(\mathbb{R}\backslash\{0\})$ is open. Let $f : \Omega \longrightarrow \mathbb{R}^{n\times n}$ be given by $f(p) = (X_{I_1}(p), \dots, X_{I_n}(p))$. Since $X_{I_i}(p), i = 1, \dots, n$ is continuous, so is f. Therefore, $f^{-1}(\det^{-1}(\mathbb{R}\backslash\{0\}))$ is open. Because $p \in V_{\mathcal{I}}$, $\det(I_p)$ is nonzero and one has $p \in f^{-1}(\det^{-1}(\mathbb{R}\backslash\{0\}))$. Let $x \in f^{-1}(\det^{-1}(\mathbb{R}\backslash\{0\}))$, therefore there exists $y \in \mathbb{R}\backslash\{0\}$ such that $\det(f(x)) = y$, thus $\dim(\mathrm{span}\{X_{I_1}(x), \dots, X_{I_n}(p)\}) = n$ and $x \in V_{\mathcal{I}}$. It is easy to prove that $V_{\mathcal{I}} = f^{-1}(\det^{-1}(\mathbb{R}\backslash\{0\}))$, therefore $V_{\mathcal{I}}$ is open. $\square$

Let $x \in C$. Since the LARC is satisfied at x, there exists $\mathcal{I}_x = (I_1, \dots, I_n)$ such that $\dim(\mathrm{span}\ \{X_{I_1}(p), \dots, X_{I_n}(p)\}) = n$, therefore, there exists $V_{\mathcal{I}_x}$ such that $x \in V_{\mathcal{I}_x}$. It follows that $\bigcup_{x\in C} V_{\mathcal{I}_x}$ is an open cover of C. Since C is compact, that open cover admits a finite subcover of $\bigcup_{x\in C} V_{\mathcal{I}_x}$, i.e., there exists a finite family of n-tuples $\mathcal{I}_1, \dots, \mathcal{I}_M$ of elements of $\mathcal{H}^r$ such that $C \subseteq \bigcup_{i=1}^M V_{\mathcal{I}_i}$.

Lemma 2. For every compact set K and every finite cover $\bigcup_{i=1}^N B_i$ of K, there exists a compact subcover $\bigcup_{i=1}^N B_i^c$ of $\bigcup_{i=1}^N B_i$, such that, for $i = 1, \dots, N$, $B_i^c \subseteq B_i$.

Proof: Let $y \in K$ and $\bigcup_{i=1}^N B_i$ be a finite cover of K. There exists B_j, $i \in \{1, \dots, N\}$ such that y belongs to B_j. Let ∂B_j be the boundary of B_j and $x \in \partial B_j \bigcap K$. Since $x \in K$, there exists $B_k \in \bigcup_{i=1}^N B_i$ such that $x \in B_k$ and, since B_k is open, there exists an open set $W_{x,j,k} \subseteq B_k$ such that x belongs to $W_{x,j,k}$. Let $B_i^c = \overline{K \bigcap B_j \backslash \bigcup_{x\in\partial B_j \bigcap K} W_{x,j,k}}$. It follows that B_i^c is a closed subset of K and, since closed subsets of compact sets are compact, B_i^c is compact. Furthermore, $K \subseteq \bigcup_{i=1}^N B_i^c$, and it is clear that $B_i^c \subseteq B_i$. Therefore, $\bigcup_{i=1}^N B_i^c$ is a compact subcover of $\bigcup_{i=1}^N B_i$. $\square$

From Lemma 2, there exists a compact cover of C in the form $\bigcup_{i=1}^M V_{\mathcal{I}i}^c$ where, for $i = 1, \dots, M$, the set $V_{\mathcal{I}i}^c \subseteq V_{\mathcal{I}i}$ is compact.

Let $\mathcal{I} = (I_1, \dots, I_n)$ with $I_i \in \{\mathcal{I}_1, \dots, \mathcal{I}_M\}$. The authors of [Chitour et al., 2013] construct, by induction on the length of elements in a free Lie algebra, a family of m vector fields $\xi = \{\xi_1, \dots, \xi_m\}$ defined on $V_{\mathcal{I}} \times \mathbb{R}^{\tilde{n}_r - n}$ (recall that $\tilde{n}_r = \#(H^r)$), which is free up to step r and has its projection onto $V_{\mathcal{I}}$ equal to $\mathcal{X}$. In the following, an alternative explanation to the aforementioned desingularization algorithm is presented.

$\boxed{\textbf{Desingularization Algorithm steps:}}$

- Define $I^s := \{I_j \in \mathcal{I} : |I_j| = s\}$, for $s \geq 1$ and $k_s := \#(H^s \backslash I^s)$. Let $x \in V_{\mathcal{I}}$ and let $a \in V_{\mathcal{I}}$.

- $\boxed{s = 1}$ Initialization step:

  1. Set $V^1 := V_{\mathcal{I}} \times \mathbb{R}^{k_1}$. Define $\tilde{k}_1 := n$. Let $\bar{x}^1 = (x_{\tilde{k}_1+1}, \dots, x_{\tilde{k}_1+k_1})$ be coordinates on $\mathbb{R}^{k_1}$. Then a point $x^1 \in V^1$ is in the form $x^1 = (x, \bar{x}^1) = (x_1, \dots, x_{n+k_1})$.
  2. Define $\{\xi_1^1, \dots, \xi_m^1\}$ on V^1 as follows:

$$\forall x^1 \in V^1, \qquad \xi_i^1(x^1) := X_i(x) + \begin{cases} 0, & \text{if } i \in I^1 \\ \partial_{x_{i+\tilde{k}_1}}, & \text{if } i \in H^1 \backslash I^1 \end{cases} \tag{5.5}$$

     Note that $H^1 = B = \{1, \dots, m\}$ and $I^1 \subseteq H^1$, therefore $i \in I^1$ or $i \in H^1 \backslash I^1$.
  3. Define $\mathcal{K}^1 := H^1 \bigcup (\mathcal{I} \backslash I^1)$ and $a^1 := (a, 0) \in V^1$. Compute coordinates y^1 on V^1 such that $\partial_{y_j^1} = \xi_{I_k}^1(a^1)$ and $y^1(a^1) = 0$, where I_k is the j-th element in $\mathcal{K}^1$ and the k-th element in H, according to the order in H, and $\partial_{y_j^1} := \frac{\partial}{\partial y_j^1}$.

- $\boxed{s = 2, \dots, r}$ Iteration steps:

  1. Set $V^s := V^{s-1} \times \mathbb{R}^{k_s}$. Let $v^s = (v_1^s, \dots, v_{k_s}^s)$ be coordinates on $\mathbb{R}^{k_s}$. Then $x^s \in V^s$ is in the form $x^s := (y^{s-1}, v^s)$.
  2. Define $\{\xi_1^s, \dots, \xi_m^s\}$ as the vector fields on V^s which, written in coordinates (y^{s-1}, v^s), are viewed as:

$$\xi_i^s(y^{s-1}, v^s) = \xi_i^{s-1}(y^{s-1}) + \sum_{I_k \in E_i^s} P_{\mathrm{ord}(k)}(y^{s-1}) \partial_{v_k^s} \tag{5.6}$$

     where $E_i^s = \{I_j \in H^s \backslash I^s : \phi(I_j) = i\}$, I_k is the k-th element in $H^s \backslash I^s$ and the j-th element in H, $\mathrm{ord} : \{1, \dots, \#(H^s \backslash I^s)\} \to \mathbb{N}$ is the mapping defined by $\mathrm{ord}(k) = j$ and $P_j(v)$ is the multi-monomial defined in Chapter 4:

$$P_j(v) = \frac{v_1^{\alpha^1} \dots v_{j-1}^{\alpha^{j-1}}}{\alpha^1! \dots \alpha^{j-1}!} \tag{5.7}$$

     with $(\alpha^1, \dots, \alpha^{j-1})$ the first $j-1$ elements of the $(\tilde{n}_s - 1)$-tuple $\alpha = (\alpha^1, \dots, \alpha^{\tilde{n}_s - 1})$ associated with I_j.
  3. Define $\mathcal{K}^s := \mathcal{K}^{s-1} \bigcup (\mathcal{H}^s \backslash I^s)$ and $a^1 := (a, 0) \in V^s$. Compute coordinates y^s on V^s such that $\partial_{y_j^s} = \xi_{I_k}^s(a^s)$ and $y^s(a^s) = 0$, where I_k is the j-th element in $\mathcal{K}^s$ and the k-th element in H, according to the order in H, and $\partial_{y_j^s} := \frac{\partial}{\partial y_j^s}$.

- $\boxed{\text{Final step}}$
  Define $\xi_i := \xi_i^s$, for $i = 1, \dots, m$, and $y^r = y$. The vector fields $\{\xi_1, \dots, \xi_m\}$ are the "lifted" vector fields whose projection on Ω is $\mathcal{X}$, and the system given by $\dot{y} = \sum_{i=1}^m \xi_i(y) u_i$ is regular.

Remark 3. It is important to remark that, since vector fields $\{\xi_1, \ldots, \xi_m\}$ of System (4.3) are "lifted" vector fields of $\{X_1, \ldots, X_m\}$ of System (4.1), if the LARC is not satisfied at a point $x = (x_1, \ldots, x_n) \in \Omega$, then the LARC is not satisfied at any point in $\widetilde{\Omega}$ of the form $(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{\tilde{n}_r})$. This, together with the fact that X_i is π-related with ξ_i, ensures that both the singular system and the desingularized system have the same controllability properties.

5.4 Application example

Let us consider the system

$$\dot{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ \frac{x_1^2}{2} \end{pmatrix} u_2 = X_1(x)u_1 + X_2(x)u_2. \tag{5.8}$$

For System (4.8), one has $n = 3$ and $m = 2$, and the vector fields X_1 and X_2 are defined for all $\mathbb{R}^3$, i.e., $\Omega = \mathbb{R}^3$. By direct computation one gets:

$$[X_1, X_2](x) = \begin{pmatrix} 0 \\ 0 \\ x_1 \end{pmatrix} \qquad [X_1, [X_1, X_2]](x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

As mentioned in the previous section, to apply the desingularization algorithm, one must select a compact set of Ω on which the LARC is satisfied. Let $C = \{x \in \mathbb{R}^3 : \|x\| \leq 1\}$. Since C is closed and bounded, C is compact. Consider the sets $C_1 = \{x \in C : x_1 \neq 0\}$ and $C_2 = \{x \in C : x_1 = 0\}$. Clearly $C_1 \cup C_2 = C$.

Proposition 1. The LARC is satisfied at every $x \in C$.

Proof: Let $x \in C$. Let us suppose that $x \in C_1$. Then, $X_1(x)$, $X_2(x)$ and $[X_1, X_2](x)$ are linearly independent, and $X_1(x)$, $X_2(x)$, $[X_1, X_2](x) \in \bar{D}$, hence $\dim(\bar{D}) = n$. It follows that LARC is satisfied at x. Suppose that $x \in C_2$. Then, $X_1(x)$, $X_2(x)$ and $[X_1, [X_1, X_2]](x)$ are linearly independent. Therefore, by a similar reasoning, the LARC is satisfied at x. □

For $x \in C_1$ one has $\dim(D^1) = 2$ and $\dim(D^2) = 3$; therefore, the degree of nonholonomy of $\mathcal{X} = \{X_1, X_2\}$ on C_1 is $r_1 = 2$, and the growth vector at $x \in C_1$ is $(n_1(x), n_2(x)) = (2, 3)$. For $x \in C_2$ we have $\dim(D^1) = 2$, $\dim(D^2) = 2$, and $\dim(D^3) = 3$; therefore, the degree of nonholonomy of $\mathcal{X}$ on C_2 is $r_2 = 3$, and the growth vector at $x \in C_2$ is $(n_1(x), n_2(x), n_3(x)) = (2, 2, 3)$. Therefore, the degree of nonholonomy of $\mathcal{X}$ at C is $r = \max\{r_1, r_2\} = 3$. Let $\varepsilon \in (0, 1) \subseteq \mathbb{R}_{>0}$, and let U be a ball of radius ε centered at $x \in C_2$. It is easy to prove that theere exists $y \in C_1$ such that

$y \in U$, and since the growth vector of $\mathcal{X}$ at x is different from the growth vector of $\mathcal{X}$ at y, it follows that x is a singular point. Thus, C_2 is a set of singular points of System (4.8).

Let $B = \{1, 2\}$ and set $1 < 2$. Recall that $\mathcal{L}_B$ denotes the free lie algebra generated by B, H denotes the P. Hall basis of $\mathcal{L}_B$, and $\mathcal{H}^r$ is the set of elements of $\mathcal{L}_B$ whose length is smaller than or equal to r. Let $\mathcal{I} = (1, 2, [1, [1, 2]])$; it is clear that $\mathcal{I}$ is a triple of elements of $\mathcal{H}^r$. Define $V_{\mathcal{I}} = \{p \in \mathcal{R}^3 : \dim(\mathrm{span}\{E_{\mathcal{X}}(1), E_{\mathcal{X}}(2), E_{\mathcal{X}}(3)\}) = 3\}$, so that $V_{\mathcal{I}} = \mathbb{R}^3$. It is clear that $C \subseteq V_{\mathcal{I}}$ and, since $V_{\mathcal{I}}$ is open, it is an open cover of C.

By definitions from Chapters 4 and 5, one has the following sets and scalars:

- For $i = 1, 2, 3$, the sets H^i of the P. Hall basis of H are:

$$\begin{aligned} H^1 &= \{1, 2\} \\ H^2 &= \{[1, 2]\} \\ H^3 &= \{[1, [1, 2]], [2, [1, 2]]\} \end{aligned}$$

- For $i = 1, 2, 3$, the sets $\mathcal{H}^i$ are given by:

$$\begin{aligned} \mathcal{H}^1 &= \{1, 2\} \\ \mathcal{H}^2 &= \{1, 2, [1, 2]\} \\ \mathcal{H}^3 &= \{1, 2, [1, 2], [1, [1, 2]], [2, [1, 2]]\} \end{aligned}$$

- For $i = 1, 2, 3$, one has:

$$\begin{aligned} I^1 &= \{1, 2\} \\ I^2 &= \varnothing \\ I^3 &= \{[1, [1, 2]]\} \end{aligned}$$

- By simple calculations, for $i = 1, 2, 3$, one obtains:

$$\begin{aligned} k_1 &= 0 \\ k_2 &= 1 \\ k_3 &= 1 \end{aligned}$$

Let $x = (x_1, x_2, x_3)$ be coordinates on $V_{\mathcal{I}}$ and let $a = (0, 0, 0)$.

- $\boxed{\text{Step } s = 1}$

  1. $V^1 := V_{\mathcal{I}} \times \mathbb{R}^0 = V_{\mathcal{I}}$, $\quad \tilde{k}_1 := 3$, $\quad x^1 := x = (x_1, x_2, x_3)$.

2. Define, for $x^1 \in V^1$:

$$\xi_1^1(x^1) := X_1(x)$$
$$\xi_2^1(x^1) := X_2(x)$$

3. Define $a^1 := a$ and $\mathcal{K}^1 := \{1, 2, [1,[1,2]]\} = \{I_1, I_2, I_4\}$. Let $(x_1, \ldots, x_3) := (y_1^1, y_2^1, y_3^1) := y^1$, therefore $y^1(a^1) = 0$ and after straightforward computations, one obtains $\xi_1^1(a^1) = \partial_{y_1}$, $\xi_2^1(a^1) = \partial_{y_2}$ and $\xi_4^1(a^1) = \partial_{y_3}$.

- $\boxed{\text{Step } s = 2}$

1. $V^2 := V^1 \times \mathbb{R}^1, \qquad v^2 := (v_1^2)$
2. Since $H^2 \backslash I^2 = \{[1,2]\}$, and $\phi([1,2]) = 2$, one has $E_2^1 = \varnothing$, therefore, for $(y^1, v^2) \in V^2$:

$$\xi_1^2(y^1, v^2) := \xi_1^1(y^1)$$

By definition, $E_2^2 = \{[1,2]\}$. Since $[1,2]$ is the first element of E_2^2 and the third element of H, let us define, for $(y^1, v^2) \in V^2$:

$$\xi_2^2(y^1, v^2) := \xi_2^1(y^1) + P_3(y^1)\partial_{v_1^2}$$

The Lie monomial $[1,2]$ is associated with the pair $\alpha = (\alpha^1, \alpha^2) = (1,0)$; therefore, $P_3(x) = x_1$. It follows from (5.9) that:

$$\xi_2^2(y^1, v^2) = \xi_2^1(y^1) + x_1\partial_{v_1^2}$$

3. Define $a^2 := (a^1, 0) = (0,0,0,0)$ and $\mathcal{K}^2 := \{1, 2, [1,2], [1,[1,2]]\} = \{I_1, I_2, I_3, I_4\}$. Let $(y^1, v^2) := (y_1^2, \ldots, y_4^2) := y^2$. It follows that $y^2(a^2) = 0$, $\xi_1^2(a^2) = \partial_{y_1}$, $\xi_2^2(a^2) = \partial_{y_2}$. $\xi_3^2(a^2) = \partial_{y_3}$ and $\xi_4^2(a^2) = \partial_{y^4}$.

- $\boxed{s = 3}$

1. $V^3 := V^2 \times \mathbb{R}^1, \qquad v^3 := (v_1^3)$.
2. Since $H^3 \backslash I^3 = \{[2,[1,2]]\}$, and $\phi([2,[1,2]]) = 2$, $E_3^1 = \varnothing$, therefore, for $(y^2, v_1^3) \in V^3$:

$$\xi_1^3(y^2, v_1^3) := \xi_1^2(y^2)$$

By definition, $E_2^3 = \{[2,[1,2]]\}$, therefore $[2,[1,2]]$ is the first element of E_2^3 and the fifth element of H. Define, for $x^3 \in V^3$:

$$\xi_2^3(y^2, v_1^3) := \xi_2^2(y^2) + P_5(y^2)\partial_{v_1^3}$$

The 4-tuple $\beta = (\beta^1, \beta^2, \beta^3, \beta^4) = (1,1,0,0)$ is associated with the Lie monomial $[2,[1,2]]$, thus $P_5(x^2) = x_1 x_2$. Therefore one has:

$$\xi_2^3(x^3) := \xi_2^3(x^2) + x_1 x_2 \partial_{v_1^3}$$

3. By definition one has $\mathcal{K}^3 := \{1, 2, [1,2], [1,[1,2]], [2,[1,2]]\} = \{I_1, I_2, I_3, I_4, I_5\}$, and $a^3 = (0,0,0,0,0)$. Let $(y_1^2, \ldots, y_4^2, v^3) := (y_1^3, \ldots, y_5^3) := y^3$. Simple calculations yield $y^3(a^3) = 0$, $\xi_1^3(a^3) = \partial_{y_1}$, $\xi_2^2(a^3) = \partial_{y_2}$, $\xi_3^2(a^3) = \partial_{y_3}$, $\xi_4^2(a^4) = \partial_{y_4}$ and $\xi_5^2(a^4) = \partial_{y_5}$.

- Define $y^3 := y$, $\xi_1 := \xi_1^3$ and $\xi_2 := \xi_2^3$. Then the domain of ξ_i is $\tilde{\Omega} = \Omega \times \mathbb{R}^2$.

Let $y \in \mathbb{R}^5$; in coordinates y one has:

$$\xi_1(y) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \xi_2(y) = \begin{pmatrix} 0 \\ 1 \\ \frac{y_1^2}{2} \\ y_1 \\ y_1 y_2 \end{pmatrix}$$

and

$$\xi_3(y) := [\xi_1, \xi_2](y) = \begin{pmatrix} 0 \\ 0 \\ y_1 \\ 1 \\ y_2 \end{pmatrix}$$

$$\xi_4(y) := [\xi_1, [\xi_1, \xi_2]](y) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \xi_5(y) := [\xi_2, [\xi_1, \xi_2]](y) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Let s be a positive integer such that $1 \le s \le r = 3$ and $y \in \mathbb{R}^5$. Since $(n_1(y), n_2(y), n_3(y)) = (2,3,5) = (\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$, the family of vector fields $\{\xi_1, \xi_2\}$ is free up to step $r = 3$. Since for all $y \in \mathbb{R}^5$ the growth vector is the same, no point is singular for the system defined by ξ_1, ξ_2. Let π denote the canonical projection of $\mathbb{R}^5$ onto $\mathbb{R}^3$ given by $\pi((y_1, y_2, y_3, y_4, y_5)) = (y_1, y_2, y_3)$ and let $y \in \widetilde{\Omega}$; one has $T\pi \circ \xi_1(y) = \frac{\partial}{\partial_{y_1}}$, $X_1 \circ \pi(y) = \frac{\partial}{\partial_{y_1}}$. Therefore, $T\pi \circ \xi_1 = X_1 \circ \pi$ and, by a similar reasoning, one has $T\pi \circ \xi_2 = X_2 \circ \pi$.

5.5 An elementary procedure to desingularize a simple system

The nature of System (5.8) allows one to apply a "rudimentary" desingularization procedure. This procedure is helpful to shed light on the underlying mechanism of

[Chitour et al., 2013]'s algorithm. Additionally, this elementary procedure shows that there is no "unique" desingularization for System (5.8).

The desingularization algorithm mentioned in the previous section involves the extension of Ω to $\widetilde{\Omega}$ and the definition of extended vector fields on $\widetilde{\Omega}$. Let $\mathcal{E} = \{\xi_1, \dots, \xi_m\}$ be the set of vector fields obtained by the desingularization algorithm; since $\mathcal{E}$ is free up to step 3, $\xi_1, \dots, \xi_m$ are defined such that, for any $e_1, e_2 \in H^r$, the image of e_1 by $E_{\mathcal{E}}$ is linearly independent from the image of e_2 by $E_{\mathcal{E}}$.

In order to obtain a desingularization of (5.8) and may be of other simple systems, one can apply the following desingularization procedure. Let us define ξ_1, ξ_2 for $x \in \mathbb{R}^5$ as lifts of X_1, X_2, as follows:

$$\xi_1(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \xi_2(x) = \begin{pmatrix} 0 \\ 1 \\ \frac{x_1^2}{2} \\ a(x) \\ b(x) \end{pmatrix} \tag{5.9}$$

where $a, b \in C(\mathbb{R}^5, \mathbb{R})$. Recall, from Chapter 4, that the triple $(\operatorname{span}_{\mathbb{R}}\{\xi_1, \xi_2\}, +, \star)$, with $+$ the sum of vector fields and $\star$ the multiplication of vector field by scalars, is a vector spacer over $\mathbb{R}$; in that sense, it is easy to prove that ξ_1 and ξ_2 are linearly independent. One has:

$$\xi_3(x) = [\xi_1, \xi_2](x) = \begin{pmatrix} 0 \\ 0 \\ x_1 \\ \frac{\partial a(x)}{\partial x_1} \\ \frac{\partial b(x)}{\partial x_1} \end{pmatrix}$$

Note that in order for ξ_3 to be linearly independent form ξ_1 and ξ_2, it suffices to have $\frac{\partial a(x)}{\partial x_1} = 1$ or $\frac{\partial b(x)}{\partial x_1} = 1$. Let us propose $a(x) = x_1$, so that $\frac{\partial a}{\partial x_1} = 1$, and by computation:

$$\xi_4(x) = [\xi_1, [\xi_1, \xi_2]](x) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \frac{\partial^2 b(x)}{\partial x_1^2} \end{pmatrix}$$

In this case, ξ_4 is already linearly independent form ξ_1, ξ_2 and ξ_3 independently of the value of $\frac{\partial^2 b(x)}{\partial_{x_1^2}}$. Finally:

$$\xi_5(x) = [\xi_2, [\xi_1, \xi_2]](x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{\partial b(x)}{\partial x_1 \partial x_2} + \frac{x_1^2}{2}\frac{\partial b(x)}{\partial x_1 \partial x_3} + x_1 \frac{\partial b(x)}{\partial x_1 \partial x_4} + b \frac{\partial b(x)}{\partial x_1 \partial x_5} - x_1 \frac{\partial b(x)}{\partial x_3} - \frac{\partial b(x)}{\partial x_4} - \frac{\partial b(x)}{\partial x_1}\frac{\partial b(x)}{\partial x_5} \end{pmatrix}.$$

To ensure that ξ_5 is linearly independent form ξ_1, ξ_2, ξ_3 and ξ_4, it is enough to require that $\frac{\partial b}{\partial x_1 \partial x_2} + \frac{x_1^2}{2}\frac{\partial b}{\partial x_1 \partial x_3} + x_1 \frac{\partial b}{\partial x_1 \partial x_4} + b \frac{\partial b}{\partial x_1 \partial x_5} - x_1 \frac{\partial b}{\partial x_3} - \frac{\partial b}{\partial x_4} - \frac{\partial b}{\partial x_1}\frac{\partial b}{\partial x_5} = 1$. It is easy to prove that either $b(x) = x_1 x_2$, or $b(x) = -x_4$, satisfies that condition.

As one may anticipate, if one selects $a(x) = x_2$ instead of $a(x) = x_1$, the system obtained by desingularization may be different. Moreover it is reported in [Jean, 2014] that system

$$\dot{\tilde{x}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ \frac{x_2^2}{2} \\ x_1 \end{pmatrix} = \xi_1 u_1 + \xi_2 u_2 \tag{5.10}$$

results form a desingularization of (5.8).

However this intuitive and elementary desingularization of (5.8) was possible thanks to the relatively simple nature of the system. However, this need not be the case for more general driftless control-affine systems. Hence the importance of the algorithm proposed in [Chitour et al., 2013].

Chapter 6

Application example

In this chapter we present the application of the desingularization algorithm proposed in [Chitour et al., 2013] to the kinematic model of the tricycle with one trailer; the existence of singular points in the configuration manifold of this system will also be discussed here.

6.1 Singular points of the tricycle with one trailer

One of the objectives in this work was to apply the desingularization algorithm on the kinematic model of a car-like robot with nonholonomic constraints. It is worth mentioning that there exist reports in the literature on the existence or not of singular points in kinematic models of some car-like robots. For example, in [Jacquard, 1993] there is a classification for the singular points of the cart with 2, 3 and 4 trailers; the authors of [Jean, 1996] proved that a point p belonging to the configurations manifold of a cart with N trailers that satisfies $\theta_k - \theta_{k-1} = \pm\frac{\pi}{2}$, for $k = 2, \dots, N$, is a singular point of this system.

Before studying the singular points of the tricycle with 1 trailer, it is important to stress that there exist other singularities that do not necessarily have to do with the existence of singular points, nevertheless, in this work we refer to a singular point in the sense of the definition given in Chapter.

Let us consider the tricycle with 1 trailer, whose graphic representation is shown in Figure 3.6, and recall that $Q = \mathbb{R}^2 \times (S^1)^3$ is a valid configuration manifold for this system. Let us define $K_1 := \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, $K_2 := \left[0, \frac{9\pi}{10}\right]$, $K_3 := \left[0, \frac{9\pi}{10}, \pi\right] \setminus \left\{\pm \arctan\left(\frac{C_2}{\sqrt{L_1^2 - C_2^2}}\right)\right\}$, $K_4 := \left[0, \frac{9\pi}{10}\right]$ and $K := K_1 \times K_2 \times K_3 \times K_4$. Since $K_1,\ K_2,\ K_3$ and K_4 are closed and bounded, it follows that K is closed and bounded, therefore K is compact.

Let

$$f(x) = \begin{pmatrix} \frac{1}{L_2}\cos(x_3)\left(L_2\cos(x_4)\cos(x_5) + C_2\sin(x_4)\sin(x_5)\right) \\ \frac{1}{L_2}\sin(x_3)\left(L_2\cos(x_4)\cos(x_5) + C_2\sin(x_4)\sin(x_5)\right) \\ \frac{1}{L_1L_2}\left(L_2\sin(x_4)\cos(x_5) - C_2\cos(x_4)\sin(x_5)\right) \\ \frac{1}{L_1L_2}\left(L_1\sin(x_5) - L_2\sin(x_4)\cos(x_5) + C_2\cos(x_4)\sin(x_5)\right) \\ 0 \end{pmatrix} \qquad g(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Let $\mathcal{X} = \{f, g\}$ and let us define the matrices:

$$\begin{aligned} M_1(x) &= \begin{pmatrix} f(x) & g(x) & [f,g](x) & [f,[f,g]](x) & [f,[f,[f,[f,g]]]](x) \end{pmatrix} \\ M_2(x) &= \begin{pmatrix} f(x) & g(x) & [f,g](x) & [f,[f,g]](x) & [f,[f,[f,g]]](x) \end{pmatrix} \end{aligned}$$

By simple computation one checks that $A = \left\{ y \in \Omega : y_4 = \pm\arctan\left(\frac{C_2}{\sqrt{L_1^2 - C_2^2}}\right)\right\}$ and $B = \left\{ y \in \Omega : y_5 = -\arctan\left(\frac{L_2(C_2\cos y_4 + L_1)}{C_2^2 \sin y_4}\right)\right\}$ are the zero sets of $\det(M_1(x))$ and of $\det(M_2(x))$ respectively.

Lemma 3. The LARC is satisfied at every $x \in K$.

Proof: Let $x \in K$. By definition of K, $x_4 \in [0, \frac{9\pi}{10}] \setminus \pm\arctan\left(\frac{C_2}{\sqrt{L_1^2 - C_2^2}}\right)$. Therefore, $x_4 \neq \pm\arctan\left(\frac{C_2}{\sqrt{L_1^2 - C_2^2}}\right)$, which implies that $f(x)$, $g(x)$, $[f,g](x)$, $[f,[f,g]](x)$, $[f,[f,[f,[f,g]]]](x) \in \bar{D}(x)$ are linearly independent. Thus, $\dim(\bar{D}(x)) = 5$ and the LARC is satisfied at x. □

Lemma 4. The set $K \cap B$ consists of singular points for System (3.6).

Proof: Let $x \in K \cap B$. As was proved in Lemma 3, for every $x \in K$, $f(x)$, $g(x)$, $[f,g](x)$, and $[f[f,g]](x)$ are linearly independent; since $[g,[f,g]](x) = f(x)$ one has $\dim(\bar{D}_x^3) = 4$, i.e., $n_3(x) = 4$. Given that $[g[f,[f,g]]](x) = 0$, $[g,[g,[f,g]]](x) = -[f,g](x)$, and $x \in B$, one has $\dim(\bar{D}_x^4) = 4$, i.e., $n_4(x) = 4$. As was proved in Lemma 3, since $x \in K$, $\dim(\bar{D}_x^5) = 5$, therefore, the growth vector for x is $n(x) = (2,3,4,4,5)$ and the degreee of nonholonomy of $\mathcal{X}$ at x is 5. Let U be a neighborhood of x. By definition of product topology, there exists $U' \subseteq S^1$ such that U' is a neighborhood of x_4, and by definition of neighborhood, there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $(x_4 - \varepsilon, x_4 + \varepsilon) \subseteq U'$. Let $y = \frac{x_4+\varepsilon}{2}$, it is easy to see that y belongs to $(x_4 - \varepsilon, x_4 + \varepsilon)$. Therefore $y \notin B$ and $n_4(y) = 5 \neq n_4(x)$. It follows that the growth vector is not constant on any neighborhood of x, which means that x is a singular point of System (5.4). □

6.2 Desingularization of the tricycle with one trailer

As proved in Lemma 4, the degree of nonholonomy at $x \in K \cap B$ is 5. Let $y \in K \backslash (K \cap B)$. Since $[g,[f,g]](y) = f(y)$, one has $\dim(\bar{D}_x^3) \leq 4$, therefore $n_3(y) \leq 4$. Since $y \notin B$, the vectors $f(y)$, $g(y)$, $[f,g](y)$, $[f,[f,g]](y)$ and $[f,[f,[f,g]]](y)$ are linearly independent, which implies that $\dim(\bar{D}_y^4) = 5$, therefore $n(y) = (n_1(y), n_2(y), n_3(y), 5)$, i.e., the degree of nonholonomy of X at y is 4. Since $K = (K \cap B) \cup (K \backslash (K \cap B))$, the maximum degree of non holonomy of $\mathcal{X}$ in K is $r = 5$.

Given that System (3.6) is defined by two vector fields, the free Lie algebra generated by the set of two elements $A = \{1, 2\}$ and its P. Hall basis H will be used in the application of the desingularization algorithm to this system.

Recall that I_i denotes the i−th element in H according to the order in H. Let $\mathcal{I}_1 = (I_1, I_2, I_3, I_4, I_6)$ and $\mathcal{I}_2 = (I_1, I_2, I_3, I_4, I_9)$. Define $V_{\mathcal{I}_1} = \{p \in \Omega : dim(\text{span}\{X_{(I_1)}(p), X_{(I_2)}(p), X_{(I_3}(p), X_{(I_4)}(p), X_{(I_6)}(p)\}) = 5\}$ and $V_{\mathcal{I}_2} = \{p \in \Omega : \dim(\text{span}\{X_{(I_1)}(p), X_{(I_2)}(p), X_{(I_3)}(p), X_{(I_4)}(p), X_{(I_9)}(p)\}) = 5\}$. It follows from lemma 4 that $K \subseteq V_{\mathcal{I}_1} \bigcup V_{\mathcal{I}_2}$. Let $\mathcal{I} = \mathcal{I}_2$. From chapter 5 one has, for $i = 1, \ldots, 5$, the following:

- The sets H^i of the P. Hall basis H of the free Lie algebra generated by $\{1, 2\}$ are given by:

$$
\begin{aligned}
H^1 &= \{1,2\} \\
H^2 &= \{[1,2]\} \\
H^3 &= \{[1,[1,2]],[2,[1,2]]\} \\
H^4 &= \{[1,[1,[1,2]]],[2,[1,[1,2]]],[2,[2,[1,2]]]\} \\
H^5 &= \{[1,[1,[1,[1,2]]]],[2,[1,[1,[1,2]]]],[2,[2,[1,[1,2]]]],[2,[2,[2,[1,2]]]], \\
&\quad [[1,2],[1,[1,2]],[[1,2],[2,[1,2]]\}
\end{aligned}
$$

- The sets $\mathcal{H}$, which are unions of elements H^j of H with $j \leq i$, are given by:

$$
\begin{aligned}
\mathcal{H}^1 &= \{1,2\} \\
\mathcal{H}^2 &= \{1,2,[1,2]\} \\
\mathcal{H}^3 &= \{1,2,[1,2],[1,[1,2]],[2,[1,2]]\} \\
\mathcal{H}^4 &= \{1,2,[1,2],[1,[1,2]],[2,[1,2]],[1,[1,[1,2]]],[2,[1,[1,2]]],[2,[2,[1,2]]]\} \\
\mathcal{H}^5 &= \{1,2,[1,2],[1,[1,2]],[2,[1,2]],[1,[1,[1,2]]],[2,[1,[1,2]]],[2,[2,[1,2]]] \\
&\quad [1,[1,[1,[1,2]]]],[2,[1,[1,[1,2]]]],[2,[2,[1,[1,2]]]],[2,[2,[2,[1,2]]]], \\
&\quad [[1,2],[2,[1,2]],[[1,2],[2,[1,2]]\}
\end{aligned}
$$

- The sets I^i, which contain the elements of $\mathcal{I}$ whose length is smaller than or equal to i, are given by:

$$
\begin{aligned}
I^1 &= \{1,2\} \\
I^2 &= \{[1,2]\} \\
I^3 &= \{[1,[1,2]]\} \\
I^4 &= \varnothing \\
I^5 &= \{[1,[1,[1,[1,2]]]]\}
\end{aligned}
$$

- The scalars k_i, which denote the cardinalities of $H^i \backslash I^i$, $i = 1, \dots, r$, are given by:

$$
\begin{aligned}
k_1 &= 0 \\
k_2 &= 0 \\
k_3 &= 1 \\
k_4 &= 3 \\
k_5 &= 5
\end{aligned}
$$

For simplicity in the computations, let us suppose $C_1 = C_2 = 1$ and $L_1 = L_2 = 2$. Let $x = (x_1, x_2, x_3, x_4, x_5) \in V_{\mathcal{I}} = K$ be coordinates on $V_{\mathcal{I}}$, and let $a = (0, 0, 0, \frac{\pi}{2}, -\arctan(4))$; thus a is a singular point of the tricycle with 1 trailer. The following are the steps of the desingularization algorithm applied to this system:

- $\boxed{\text{Step } s{=}1}$

  1. $V^1 := V_{\mathcal{I}} \times \mathbb{R}^0 = V_{\mathcal{I}}$, $\tilde{k}_1 := 5$.
  2. For $x \in V^1$:

  $$
  \begin{aligned}
  \xi_1^1(x) &:= f(x) \\
  \xi_2^1(x) &:= g(x)
  \end{aligned}
  $$

  3. By definition $a^1 = a$ and $\mathcal{K}^1 = \{I_1, I_2, I_3, I_4, I_9\}$. Let $y^1 = (y_1^1, \dots, y^5)$ coordinates on V^1 given, for every $x \in V^1$, by:

  $$
  \begin{aligned}
  y_1^1(x) &= x_1 \\
  y_2^1(x) &= x_5 + \arctan(4) \\
  y_3^1(x) &= -2x_3 - 2x_4 + \pi \\
  y_4^1(x) &= 2x_2 + 6x_3 + 2x_4 - \pi \\
  y_5^1(x) &= -8x_2 - 8x_3 - \frac{8}{3}x_4 + \frac{4}{3}\pi
  \end{aligned}
  \tag{6.1}
  $$

  Simple computing yields $y^1(a^1) = 0$, $\xi_1^1(a^1) = \partial_{y_1^1}$, $\xi_2^1(a^1) = \partial_{y_2^1}$, $\xi_3^1(a^1) = \partial_{y_3^1}$,$\xi_4^1(a^1) = \partial_{y_4^1}$ and $\xi_9^1(a^1) = \partial_{y_5^1}$.

- $\boxed{\text{Step } s{=}2}$

    1. $V^2 := V^1 \times \mathbb{R}^0 = V_{\mathcal{I}}$.
    2. Since $H^2 \backslash I^2 = \varnothing$, one has $E_2^1 = E_2^2 = \varnothing$. Therefore, for $y^1 \in V^2$:

    $$\begin{aligned}\xi_1^2(y^1) &:= \xi_1^1(y^1)\\ \xi_2^2(y^1) &:= \xi_2^1(y^1)\end{aligned}$$

    3. Define $a^2 = a^1$ and $\mathcal{K}^2 = \{I_1, I_2, I_3, I_4, I_9\}$. Since $\mathcal{K}^2 = \mathcal{K}^1$, there no change of coordinates is required in this step.

- $\boxed{\text{Step } s{=}3}$

    1. $V^3 := V^2 \times \mathbb{R}^1 = V_{\mathcal{I}} \times \mathbb{R}$, $v^3 = (v_1^3)$.
    2. Since $H^3 \backslash I^3 = \{[2,[1,2]]\}$ and $\phi([2,[1,2]]) = 2$, $E_1^3 = \varnothing$, therefore, for $(y^1, v^3) \in V^3$:

    $$\xi_1^3(y^1, v^3) := \xi_1^2(y^1)$$

    By definition, $E_2^3 = \{[2,[1,2]]\}$. Since $[2,[1,2]]$ is the first element of E_2^3 and the fifth element of H, for $(y^1, v^3) \in V^3$:

    $$\xi_2^3(y^1, v^3) := \xi_2^2(y^1) + P_5(y^1)\partial_{v_1^3}$$

    The Lie monomial $[2,[1,2]]$ is associated with the 4-tuple $\beta = (\beta^1, \beta^2, \beta^3, \beta^4) = (1,1,0,0)$; hence $P_5(x^2) = x_1 x_2$ and, for $(y^1, v^3) \in V^3$:

    $$\xi_2^3(y^1, v^3) := \xi_2^3(y^1) + y_1^1 y_2^1 \partial_{v_1^3}$$

    3. By definition $a^3 = (a, 0)$ and $\mathcal{K}^3 = \{I_1, I_2, I_3, I_4, I_5, I_9\}$. Let $y^3 = (y_1^3, \ldots, y_6^3)$ be coordinates on V^3 given, for every $(y^1, v^3) \in V^3$, by:

    $$\begin{aligned}y_1^3(y^1, v^3) &= y_1^1 - v_1^3\\ y_2^3(y^1, v^3) &= y_2^1\\ y_3^3(y^1, v^3) &= y_3^1\\ y_4^3(y^1, v^3) &= y_4^1 - \frac{\pi}{2}\\ y_5^3(y^1, v^3) &= v_1^3\\ y_6^3(y^1, v^3) &= y_5^1 + \arctan(4)\end{aligned} \tag{6.2}$$

    By computing one obtains $y^3(a^3) = 0$, $\xi_1^1(a^3) = \partial_{y_1^3}$, $\xi_2^1(a^3) = \partial_{y_2^3}$, $\xi_3^1(a^3) = \partial_{y_3^3}$, $\xi_4^1(a^3) = \partial_{y_4^3}$, $\xi_5^1(a^3) = \partial_{y_5^3}$ and $\xi_9^1(a^3) = \partial_{y_6^3}$.

- $\boxed{\text{Step } s{=}4}$

1. $V^4 := V^3 \times \mathbb{R}^3 = V_{\mathcal{I}} \times \mathbb{R}^3$, $v^4 := (v_1^4, v_2^4, v_3^4)$.
2. Since $H^4 \backslash I^4 = H^4 = \{[1,[1,[1,2]]],[2,[1,[1,2]]],[2,[2,[1,2]]]\}$, $E_1^4 = \varnothing$, and for $(y^3, v^4) \in V^4$:

$$\xi_1^4(y^3, v^4) := \xi_1^3(y^3)$$

By definition, $E_2^4 = \{[1,[1,[1,2]]],[2,[1,[1,2]]],[2,[2,[1,2]]]\}$. Therefore, $\sum_{I_k \in E_2^4} P_k(x^3)\partial_{x_{6+k}} = P_6(x^3)\partial_{x_7} + P_7(x^3)\partial_{x_8} + P_8(x^3)\partial_{x_9} = x_1^3\partial_{x_7} + x_1^2 x_2 \partial_{x_8} + x_1 x_2^2 \partial_{x_9}$, and for $(y^3, v^4) \in V^4$:

$$\xi_2^4(y^3, v^4) := \xi_2^3(y^3) + {y_1^3}^3 \partial_{v_1^4} + {y_1^3}^2 y_2^3 \partial_{v_2^4} + y_1^3 {y_2^3}^2 \partial_{v_3^4}$$

3. Define $a^4 = (a^3, 0, 0, 0)$ and $\mathcal{K}^4 = \{I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9\}$. Let $y^4 = (y_1^4, \ldots, y_9^4)$ coordinates on V^4 given, for every $(y^1, v^3) \in V^4$, by:

$$\begin{aligned} y_1^4(y^3, v^4) &= y_1^3 \\ y_2^4(y^3, v^4) &= y_2^3 \\ y_3^4(y^3, v^4) &= y_3^3 + \frac{1}{2}v_3^4 \\ y_4^3(y^3, v^4) &= y_4^3 - \frac{\pi}{2} \\ y_5^3(y^3, v^4) &= y_5^3 + \arctan(4) \\ y_6^3(y^3, v^4) &= \frac{1}{6}v_1^4 \\ y_7^3(y^3, v^4) &= \frac{1}{2}v_2^4 \\ y_8^3(y^3, v^4) &= \frac{1}{2}v_3^4 \\ y_9^3(y^3, v^4) &= y_6^3 - \frac{1}{3}v_1^4 \end{aligned} \tag{6.3}$$

- $\boxed{\text{Step } s=5}$

1. $V^5 := V^4 \times \mathbb{R}^5$, $v^5 = (v_1^5, \ldots, v_5^5)$
2. Since $H^5 \backslash I^5 = \{[2,[1,[1,[1,2]]]],[2,[2,[1,[1,2]]]],[2,[2,[2,[1,2]]]],$ $[[1,2],[1,[1,2]]],\ [[1,2],[2,[1,2]]]\}$, $E_1^5 = \varnothing$, and for $(y^4, v^5) \in V^5$:

$$\xi_1^5(y^4, v^5) := \xi_1^4(y^4)$$

By definition, $E_2^4 = H^5 \backslash I^5$. Therefore, for $(y^4, v^5) \in V^5$:

$$\xi_2^5(y^4, v^5) := \xi_2^4(y^4) + {y_1^4}^3 y_2^4 \partial_{v_1^5} + {y_1^4}^2 {y_2^4}^2 \partial_{v_2^5} + y_1^4 {y_2^4}^3 \partial_{v_3^5} + {y_1^4}^2 y_3^4 \partial_{v_4^5} + y_1^4 y_2^4 y_3^4 \partial_{v_5^5}$$

3. By definition one has $a^5 = (a^4, 0, 0, 0, 0, 0)$ and $\mathcal{K}^5 = \{I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, I_{10}, I_{11}, I_{12}, I_{13}, I_{14}\}$. Let $y^4 = (y_1^4, \dots, y_9^4)$ coordinates on V^4 given, for every $(y^1, v^3) \in V^4$, by:

$$
\begin{aligned}
y_1^5(y^4, v^5) &= y_1^4 - \frac{1}{12} v_1^5 - \frac{5}{6} v_3^5 - \frac{1}{8} v_4^5 \\
y_2^5(y^4, v^5) &= y_2^4 \\
y_3^5(y^3, v^4) &= y_3^4 \\
y_4^5(y^4, v^5) &= y_4^4 + v_2^5 + v_5^5 - \frac{\pi}{2} \\
y_5^5(y^4, v^5) &= y_5^4 + v_3^5 + \arctan(4) \\
y_6^5(y^4, v^5) &= y_6^4 \\
y_7^5(y^4, v^5) &= y_7^4 \\
y_8^5(y^4, v^5) &= y_8^4 \\
y_9^3(y^4, v^5) &= y_9^4 \\
y_{10}^3(y^4, v^5) &= \frac{1}{6} v_1^5 \\
y_{11}^3(y^4, v^5) &= \frac{1}{4} v_2^5 \\
y_{12}^3(y^4, v^5) &= \frac{1}{6} v_3^5 \\
y_{13}^3(y^4, v^5) &= \frac{1}{2} v_1^5 + \frac{1}{-} 2 v_4^5 \\
y_{14}^3(y^4, v^5) &= v_2^5 + v_5^5
\end{aligned}
$$

- $\boxed{\text{Final Step}}$
  Define $y := y^5$, $\xi_1 := \xi_1^5$ and $\xi_2 := \xi_2^5$. Since the domain of X_1, X_2 is Q, the domain of ξ_i is $\tilde{\Omega} = Q \times \mathbb{R}^9$.

Let $y \in \tilde{\Omega}$; in coordinates y one has:

$$\xi_1(y) = \begin{pmatrix} \frac{1}{2}\cos(A_y)\,(2E_y - F_y) \\ 0 \\ -\sin(y_2) \\ \sin(A_y)(2E_y - F_y) - G_y - H_y + \sin(y_2) \\ 0 \\ 0 \\ 0 \\ 0 \\ -4\sin(A_y)(2E_y - F_y) + \frac{8}{3}G_y + \frac{4}{3}H_y - \frac{4}{3}\sin(y_2) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\xi_2(y) = \begin{pmatrix} -C_y y_2 - \frac{1}{12}D_y^3 y_2 - \frac{5}{6}D_y {y_2}^3 - \frac{1}{8}D_y^2 y_3 \\ 1 \\ \frac{1}{2}D_y {y_2}^2 \\ D_y^2 {y_2}^2 + D_y y_2 y_3 \\ -C_y y_2 + D_y {y_2}^3 \\ \frac{1}{6}D_y^3 \\ \frac{1}{2}D_y^2 y_2 \\ \frac{1}{2}D_y {y_2}^2 \\ -\frac{1}{3}D_y^3 \\ \frac{1}{6}D_y^3 y_2 \\ \frac{1}{4}D_y^2 {y_2}^2 \\ \frac{1}{6}D_y {y_2}^3 \\ \frac{1}{2}D_y^3 y_2 + \frac{1}{2}D_y^2 y_3 \\ D_y^2 {y_2}^2 + D_y y_2 y_3 \end{pmatrix},$$

where

$$
\begin{aligned}
A_y &= \frac{3}{16}y_6 + \frac{3}{32}y_9 - \frac{1}{4}y_8 + \frac{1}{4}y_3 - \frac{3}{8}y_{14} + \frac{3}{8}y_4 \\
B_y &= \frac{3}{16}y_6 + \frac{3}{32}y_9 - \frac{3}{4}y_8 + \frac{3}{4}y_3 - \frac{3}{8}y_{14} + \frac{3}{8}y_4 \\
C_y &= -y_{12} + y_5 + y_1 - \frac{1}{4}y_{10} + \frac{1}{4}y_{13} \\
D_y &= y_1 - \frac{1}{4}y_{10} + 5y_{12} + \frac{1}{4}y_{13} \\
E_y &= \cos(B)\cos(y_2) \\
F_y &= \sin(B)\sin(y_2) \\
G_y &= \sin(B)\cos(y_2) \\
H_y &= \cos(B)\sin(y_2)
\end{aligned}
$$

Thus, the system obtained by applying the desingularization procedure to System (3.6) is

$$\dot{y} = \xi_1(y)u_1 + \xi_2(y)u_2 \tag{6.4}$$

Having obtained System (6.5), the next natural step is to check that indeed the family of vector fields $\xi = \{\xi_1, \xi_2\}$ is free up to step 5, which would ensure that System (6.5) is regular. To achieve this, it is necessary to calculate the growth vector at an arbitrary $y \in \tilde{\Omega}$ and verify that the growth vector at this point is constant and equal to $(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{n}_4, \tilde{n}_5) = (2, 3, 5, 8, 14)$.

To calculate the growth vector at y one may consider all the elements in the P. Hall basis H of the free Lie algebra generated by the set $\{1, 2\}$, whose length is less that or equal to 5, i.e., consider the set $\mathcal{H}^5$. Let $\mathcal{E} = E_\xi(\mathcal{H}^5)$. If the elements in $\mathcal{E}$, evaluated at y, are all linearly independent, then the growth vector at y will be $(2, 3, 5, 8, 14)$, i.e., if the matrix $\mathfrak{M}(y)$, whose columns are the elements in $\mathcal{E}$ evaluated at y, has complete rank, then the family ξ is free up to step 5.

In general, the computations required to calculate $\det(\mathfrak{M}(y))$ for an arbitrary point $y \in \tilde{\Omega}$ are overly complicated given the complexity of the matrix $\mathfrak{M}(y)$. Nevertheless, we numerically compute the rank of $\mathfrak{M}(a^5)$ to be equal to 14 and moreover, the growth vector of ξ at a^5 is $(2, 3, 5, 8, 14)$. We obtained the same result for other points in a neighborhood of a^5.

In order to support the desingularization algorithm, the authors of [Chitour et al., 2013] have proven that if ξ_i, $i = 1, \ldots, m$, are the vector fields given by the desingularization procedure, then the family $\{\xi_1, \ldots, \xi_m\}$, defined on $\tilde{\Omega}$, is free up to step r. The conclusions about the application of this algorithm to the tricycle with one trailer and about the results obtained will be discussed on chapter 8.

Chapter 7

A solution to the motion planning problem for a desingularized system

The objective of this chapter is to address the construction of a control law to solve the motion planing problem for System (5.8). This control law is constructed from the method of sinusoidal controls for regular nilpotent systems given in [Chitour et al., 2013], whose extended explanation the reader may find in [Jean, 2014].

7.1 Approach to the problem

As mentioned in Chapter 5, the vector fields X_1 and X_2 associated to System (5.8) are defined on $\Omega = \mathbb{R}^3$ and the LARC is satisfied at every $x \in C = \{x \in \mathbb{R}^3 : \|x\| \leq 1\}$, i.e., (5.8) is controllable on C.

Solving the MPP for (5.8) on C consists in finding a control input $u(\cdot) : [0, T] \longrightarrow \mathbb{R}^2$, with $T \in \mathbb{R}_{>0}$, such that for each pair of points $(x_i, x_f) \in C \times C$, the corresponding solution of (5.8) starting from x_i at $t = 0$ reaches x_f at $t = T$, i.e., $x(T) = x_f$.

A system in the form of (5.1) is said to be ***nilpotent system of degree*** $k \in \mathbb{N}$ if the vector fields $X_1, \ldots, X_m$ generate a nilpotent algebra of degree k, i.e., if for every $I \in H$, with length equal to or larger than k, the image by $E_{\mathcal{X}}$ of I vanishes.

From the order in the P. Hall basis of the free Lie algebra generated by $\{1, \ldots, m\}$, there is a natural way to associate $I_1, \ldots, I_n \in H$ with the coordinates $(x_1, \ldots, x_n)$ of System (5.1) as follows:

$$\begin{aligned} x_1 &:= x_{I_1} \\ x_2 &:= x_{I_2} \\ &\vdots \\ x_n &:= x_{I_n}, \end{aligned}$$

with $I_1 = 1,\ I_2 = 2,\ I_3 = [1,2]$, etc.

System (5.1) is said to be in ***canonical form*** in coordinates x if (5.1) is written in the form

$$
\begin{aligned}
\dot{x}_i &= u_i, \qquad \text{if } \ i = 1, \dots, m; \\
\dot{x}_{I_j} &= \frac{1}{k!} x_{I_{j_1}} \dot{x}_{I_{j_2}}, \ \text{if } \ j = m+1, \dots, n, \ \text{if } \ I_j = \mathfrak{L}^k_{I_{j_1}} I_{j_2}, \ \text{with } \ I_{j_1}, I_{j_2} \in H,
\end{aligned}
$$

where $\mathfrak{L}^k_{I_{j_1}} I_{j_2}$ is the k-th Lie derivative of I_{j_1} in the direction of I_{j_2}, i.e., the Lie bracket $[I^1_{j_1}, [I^2_{j_1}, [\dots, [I^k_{j_1}, I_{j_2}]]]]$.

One of the contributions in [Chitour et al., 2013] is a methodology to solve the MPP for systems in the form of (5.1) satisfying the following assumptions:

1. The family $\{X_1, \dots, X_m\}$ is free up to step r.
2. The Lie algebra generated by $X_1, \dots, X_m$ is nilpotent of degree $k \in \mathbb{N}$, i.e., the system is nilpotent of degree k.
3. The vector fields $X_1, \dots, X_m$ are given in the canonical form in some coordinates x.

System (5.8) does not satisfy the previous assumptions since, as mentioned in Chapter 5, the Lie algebra generated by the vector fields associated to this system is not free up to step r. However, by applying the desingularization algorithm to (5.8) one obtains the following system:

$$
\dot{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ \frac{x_1^2}{2} \\ x_1 \\ x_1 x_2 \end{pmatrix} u_2 = \xi_1(x) u_1 + \xi_2(x) u_2, \tag{7.1}
$$

where $x = (x_1, x_2, x_3, x_4, x_5)$ are coordinates for $\tilde{\Omega} = \mathbb{R}^3 \times \mathbb{R}^2$. By construction of $\tilde{\Omega}$, the family $\{\xi_1, \xi_2\}$ is free up to step r and, since ξ_1 and ξ_2 are liftings of X_1 and X_2, respectively, the control inputs u_1, u_2 that solve the MPP for (7.1) will also solve the MPP for System (5.8).

It is easy to see that System (7.1) is not in canonical form, nevertheless by a the change of coordinates $y = \varphi(x)$, with φ given by $\varphi(x) = (x_1, x_2, x_4, x_3, x_5)$, one obtains that

$$\dot{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ y_1 \\ \frac{y_1^2}{2} \\ y_1 y_2 \end{pmatrix} u_2 = Y_1(y)u_1 + Y_2(y)u_2 \tag{7.2}$$

is the canonical form in coordinates y of (7.1). By direct computations it is easy to check that the Lie algebra generated by $\{Y_1, Y_2\}$ is nilpotent of degree 4. Thus, (7.2) satisfies the conditions previously mentioned to solve the MPP by applying the methodology in [Chitour et al., 2013].

Let $y_{in} = (y_{in_1}, \dots, y_{in_5})$ and $y_f = (y_{f_1}, \dots, y_{f_5})$ be, respectively, the initial and final conditions in the MPP of (7.2). The control functions $u_1(t), u_2(t)$ obtained by this method will be linear combinations of sinusoids with integer frequencies. To achieve the control objective one uses auxiliary control inputs $u^j = (u_1^j, u_2^j)$, for $i \in \{1, \dots, 4\}$, such that the following conditions are satisfied:

- (C1) By the action of $u^1(t)$, y_{in_1} and y_{in_2} reach respectively y_{f_1} and y_{f_2} at $t = 2\pi$.
- (C2) For $j = 2,3,4$, the action of $u^j(t)$ during an interval of length 2π makes $y_{in_{j+1}}$ reach $y_{f_{j+1}}$.
- (C3) For $j = 2,3,4$, every y_{I_k} such that $k < j$ returns at the end of the action of $u^j(t)$ to its value taken at the end of the action of u^{j-1}, i.e., u_j does not modify y_{I_k}.

Thus, $u = (u_1(t), u_2(t))$ is given by the concatenation of all the control signals u^j, defined by

$$u(t) = u^1 * \cdots * u^4(t) = u^j(t - 2(j-1)\pi), \tag{7.3}$$

for $t \in [2(j-1)\pi, 2j\pi]$ and $j \in \{1,2,3,4\}$.

7.2 Definition of u^j

As mentioned previously, $u^1 = (u_1^1, u_2^1)$ steers (7.2) from y_{in_1} to y_{f_1} and from y_{in_2} to y_{f_2} at a time $T = 2\pi$. Since there are not previous components to y_2 and y_1, is not necessary to check that u^1 satisfies (C3). Let us define

$$u_1^1(t) = \frac{y_{f_1} - y_{in_1}}{2\pi} \quad \text{and} \quad u_2^1(t) = \frac{y_{f_2} - y_{in_2}}{2\pi}. \tag{7.4}$$

Since $\dot{y}_1 = u_1(t)$ and $u_1(t) = u_1^1(t)$, for $t \in [0, 2\pi]$, it follows by integration that $y_1(t) = y_{in_1} + \frac{y_{f_1} - y_{in_1}}{2\pi}$, therefore $y_1(2\pi) = y_{f_1}$. By a similar reasoning one has $y_2(2\pi) = y_{f_2}$.

the control functions u^j, for $j = 2,3,4$, will be defined by

$$u_1^j(t) = \cos(\omega_1 t) \quad \text{and} \quad u_2^j(t) = \cos(\omega_2 t) + a_{j+1}\cos\left(\omega_3 t - \varepsilon\frac{\pi}{2}\right), \tag{7.5}$$

where $\varepsilon \in \{1,0\}$, $a \in \mathbb{R}$ is the coefficient ensuring Condition (C2), and $\omega_1, \omega_2, \omega_3 \in \mathbb{N}$ are the frequencies that guarantee Condition (C3).

7.2.1 Choice of ω_1, ω_2, ω_3 and ε

Let us denote by $|I|_i$, the number of times i occurs in $I \in H$, for $i = 1,2$. For example, for $I = [1,[1,2]]$, one has $|I|_1 = 2$ and $|I|_2 = 1$. Define, for y_{I_j}, $m_1^j = |I_j|_1$ and $m_2^j = |I_j|_2$.

Let $j \in \{1,2,3,4\}$. In [Chitour et al., 2013] it is proved, by induction, that for every $i \le j$, the dynamics $\dot{x}_i$ is a linear combination of cosine functions in the form

$$\cos\left((\ell_1\omega_1 + \ell_2\omega_2 + \ell_3\omega_3)t - (\ell_3\varepsilon + \ell_1 + \ell_2 + \ell_3 - 1)\frac{\pi}{2}\right), \tag{7.6}$$

where $\ell_1, \ell_2, \ell_3 \in \mathbb{Z}$ satisfy $|\ell_1| \le m_1^j$ and $|\ell_2| + |\ell_3| \le m_2^j$.

Note that the function that results from integration of a function in the form $\cos(\gamma t + \epsilon\frac{\pi}{2})$, with $\gamma \in \mathbb{Z}$ and $\epsilon \in \mathbb{N}$ equals zero except if $\gamma = 0$ and $\epsilon = 0 \mod 2$. Thus, in order to obtain a nontrivial contribution in the component x_{I_j}, its derivative $\dot{x}_{I_j}$ should contain at least one cosine function in the form (7.6) verifying the following conditions:

$$\begin{aligned} \ell_1\omega_1 + \ell_2\omega_2 + \ell_3\omega_3 &= 0 \\ \ell_3\varepsilon + \ell_1 + \ell_2 + \ell_3 - 1 &\equiv 0 \mod 2. \end{aligned} \tag{7.7}$$

Moreover, for every $i < j$, this condition shall not be satisfied by any cosine function appearing in $\dot{y}_{I_i}$, in order to ensure that contribution in the component y_{I_i} during the action of u^j is trivial.

Among all the cosine functions in the form of (7.6) that appear in $\dot{y}_{I_j}$, the one with $\ell_1 = m_1$, $\ell_2 = m_2^j - 1$, and $\ell_3 = -1$ is the only one that satisfies (7.7). Therefore, ω_1, ω_2, ω_3 and ε are chosen so that they satisfy the following:

$$\begin{aligned} \omega_3 &= m_1^j\omega_1 + (m_2^j - 1)\omega_2 \\ \varepsilon &= m_1^j + m_2^j - 1 \mod 2 \\ \omega_2 &> (m_1^j + m_2^j)m_1^j\omega_1 \end{aligned} \tag{7.8}$$

Thus the values shown in Table 7.1 are acceptable values for ω_1, ω_2, ω_3 and ε that ensure (7.8) is satified for every u^j with $j = 2,3,4$, which implies in turn that (7.7) is satisfied for u^j.

Table 7.1: Proposed values for ω_1, ω_2, ω_3 and ε, for every u^j.

j	m_1^j	m_2^j	ω_1	ω_2	ω_3	ε
2	1	1	1	3	1	1
3	2	1	1	7	2	0
4	1	2	1	4	5	0

Therefore control functions u_1^j and u_2^j are given by

$$\begin{aligned} u_1^2(t) &= \cos(t) \\ u_2^2(t) &= \cos(3t) + a_3\cos(t - \frac{\pi}{2}) \\ u_1^3(t) &= \cos(t) \\ u_2^3(t) &= \cos(7t) + a_4\cos(2t) \\ u_1^4(t) &= \cos(t) \\ u_2^4(t) &= \cos(4t) + a_5\cos(5t) \end{aligned} \tag{7.9}$$

7.2.2 Computing the coefficients a_j

The coefficients a_{j+1} that guarantee that $y_{in_{j+1}}$ reaches $y_{f_{j+1}}$ at a time $2j\pi$ are obtained by solving the equation $y_{j+1}(2\pi) = y_{f_{j+1}}$. Thus one obtains:

$$\begin{aligned} a_3 &= \frac{y_{f_3} - y_3(2\pi)}{\pi} \\ a_4 &= \frac{2(y_4(4\pi) - y_{f_4})}{\pi} \\ a_5 &= \frac{40(y_5(6\pi) - y_{f_5})}{\pi} \end{aligned}$$

It is noteworthy that the authors of this procedure assume that one wants to steer a system in the form of (7.2) from any point $y_{in} \in C$, with C a set in which (7.2) is controllable, to the origin $y_f = (0,0,0,0,0)$. This last assumption is not restrictive since, for a different $\tilde{y}_f$, one may employ a linear change of coordinates z such that $z(\tilde{y}_f) = y_f$.

7.3 Simulation results

In this section we present the simulation of System (7.2) with u_1 and u_2 given by concatenation of $u_1^1, \ldots, u_1^4$ and $u_2^1, \ldots, u_2^4$, respectively. Initial conditions y_{in} and final conditions y_f for this simulation were defined as follows:

$$\begin{aligned} y_{in} &= \left(0, 3, \frac{\pi}{3}, \frac{\pi}{8}, \frac{6\pi}{7}\right) \\ y_f &= (0,0,0,0,0). \end{aligned}$$

Define for $8\pi < t$, $u_1(t) = u_2(t) = 0$. Figures 7.1-7.5 show the trajectories of $y_1, \ldots, y_5$ respectively, in a numerical simulation of system (7.2) during on the interval $[0, 9\pi]$.

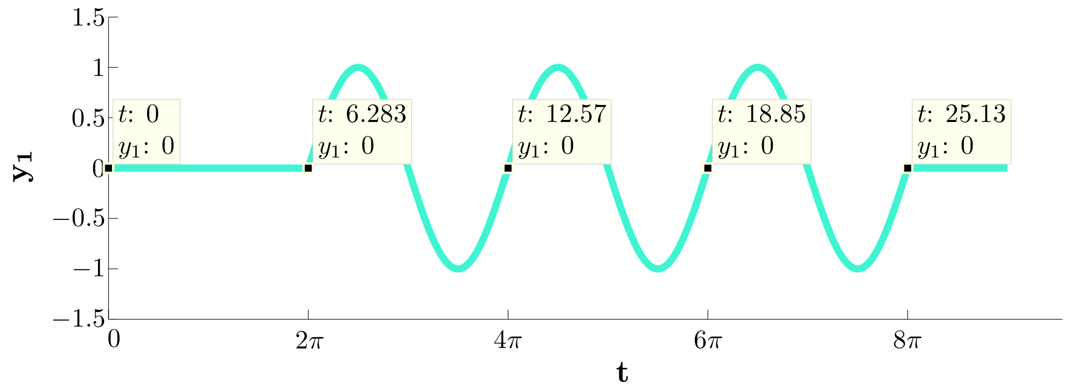

Figure 7.1: Plot of y_1 with respect to the time t.

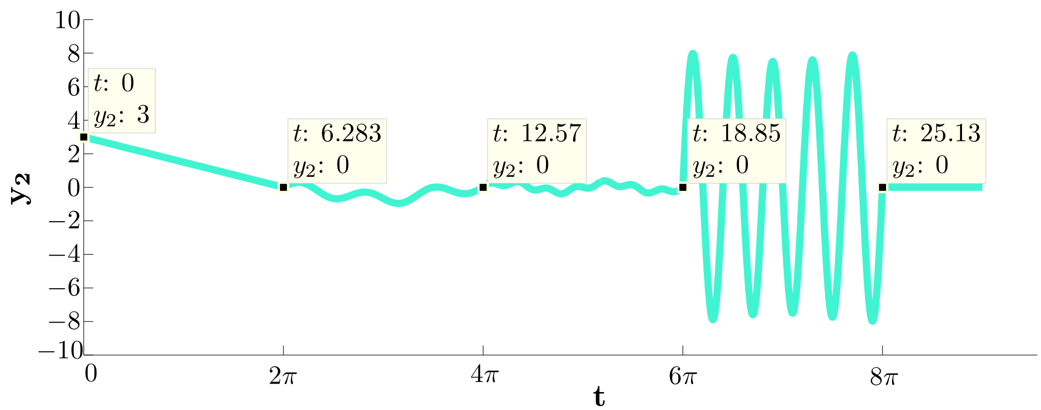

Figure 7.2: Plot of y_2 with respect to the time t.

Let $x_{in} = (x_{in_1}, \ldots, x_{in_5})$ and $x_f = (x_{f_1}, \ldots, x_{f_5})$ be the initial and final conditions for the MPP for System (7.1), respectively. The control functions u_1 and u_2 can be used to steer (7.1) by replacing y_{in} by $\varphi(x_{in})$ and y_f by $\varphi(x_f)$ in the expressions for the scalars a_3, a_4, a_5, i.e., by redefining:

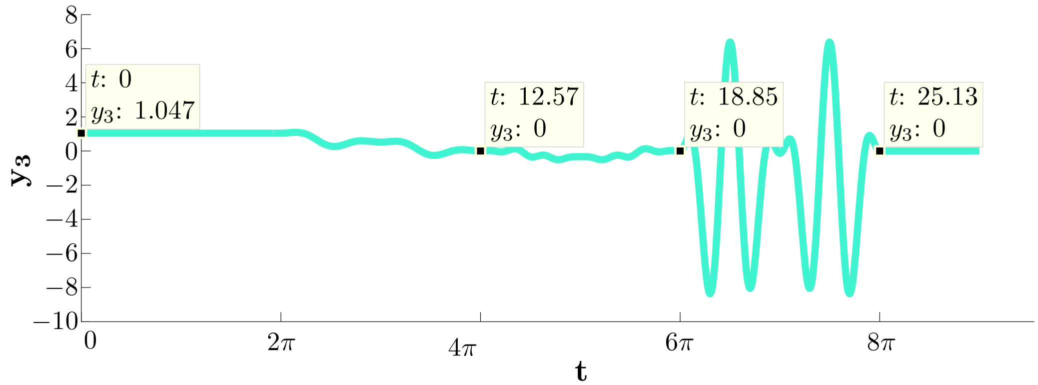

Figure 7.3: Plot of y_3 with respect to the time t.

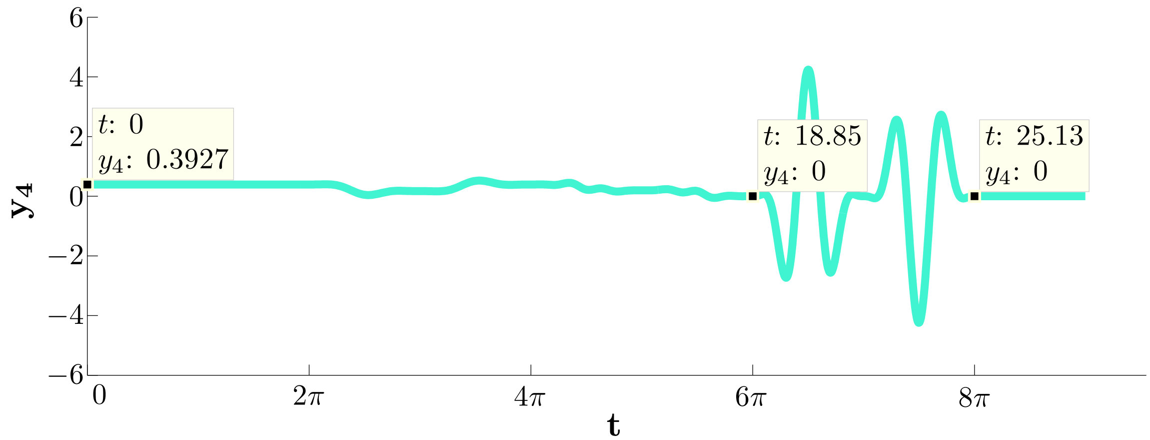

Figure 7.4: Plot of y_4 with respect to the time t.

$$
\begin{aligned}
a_3 &= \frac{x_{f_4} - x_4(2\pi)}{\pi} \\
a_4 &= \frac{2(x_3(4\pi) - x_{f_3})}{\pi} \\
a_5 &= \frac{40(x_5(6\pi) - x_{f_5})}{\pi}
\end{aligned}
$$

Figures 7.6-7.10 show the trajectories of $x_1, \ldots, x_5$ in a numerical simulation of system (7.1), during on the interval $[0, 9\pi]$, with control inputs u_1 and u_2 as defined in the previous paragraph, and initial and final states given by $x_{in} = \left(0, 3, \frac{\pi}{8}, \frac{\pi}{3}, \frac{6\pi}{7}\right)$ and $x_f = (0,0,0,0,0)$, respectively.

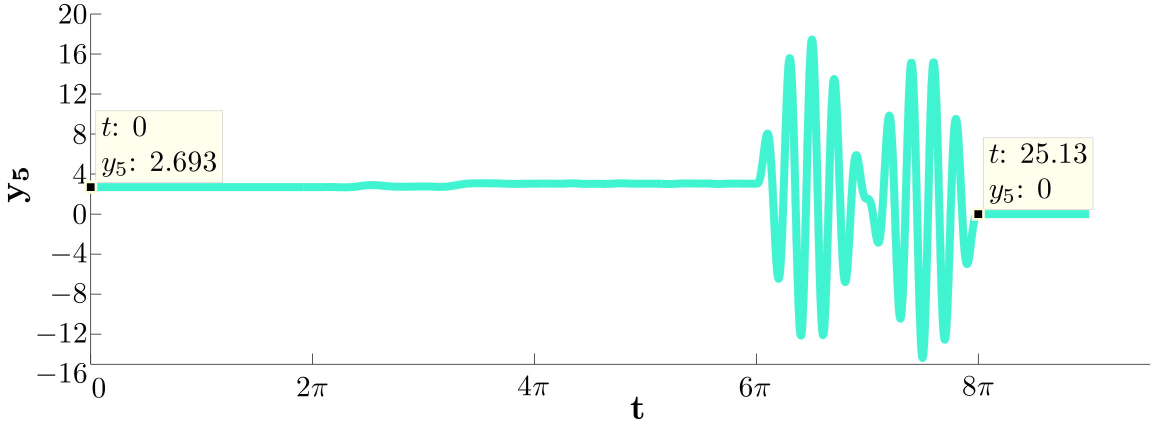

Figure 7.5: Plot of y_5 with respect to the time t.

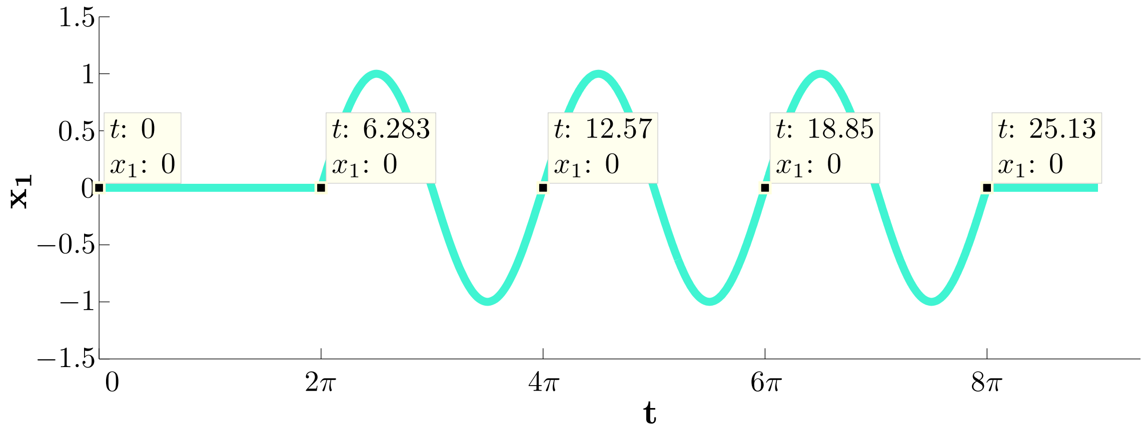

Figure 7.6: Plot of x_1 with respect to the time t.

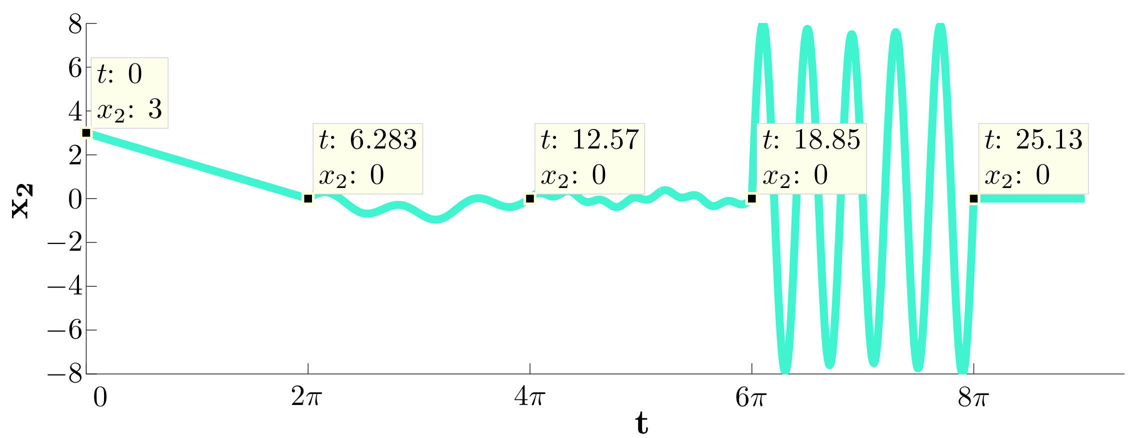

Figure 7.7: Plot of x_2 with respect to the time t.

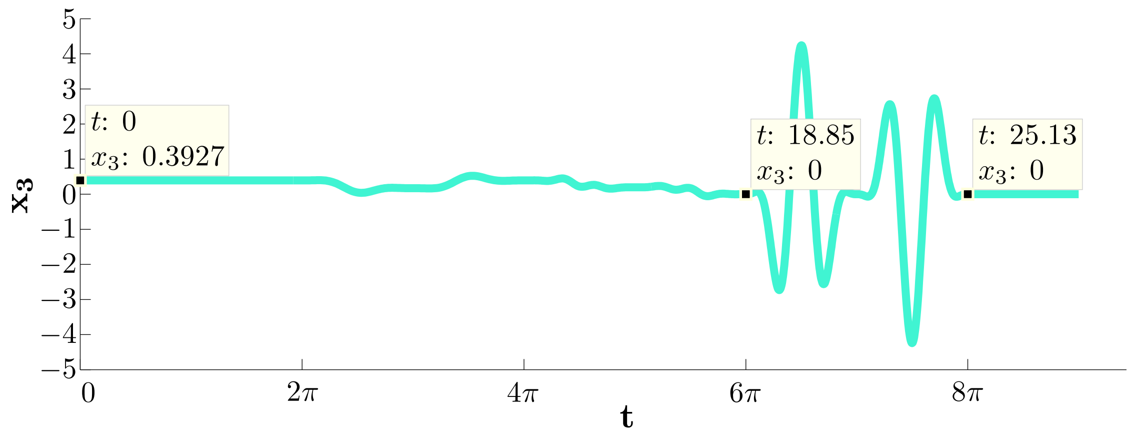

Figure 7.8: Plot of x_3 with respect to the time t.

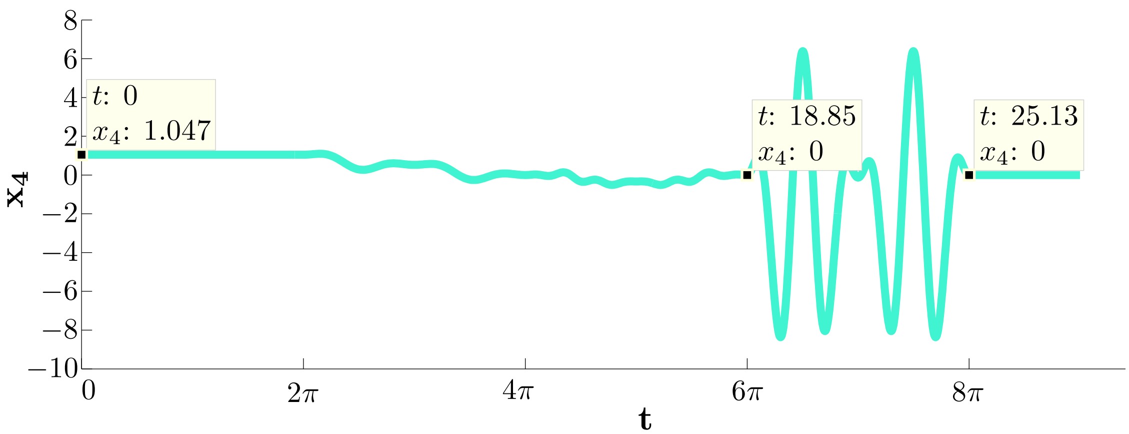

Figure 7.9: Plot of x_4 with respect to the time t.

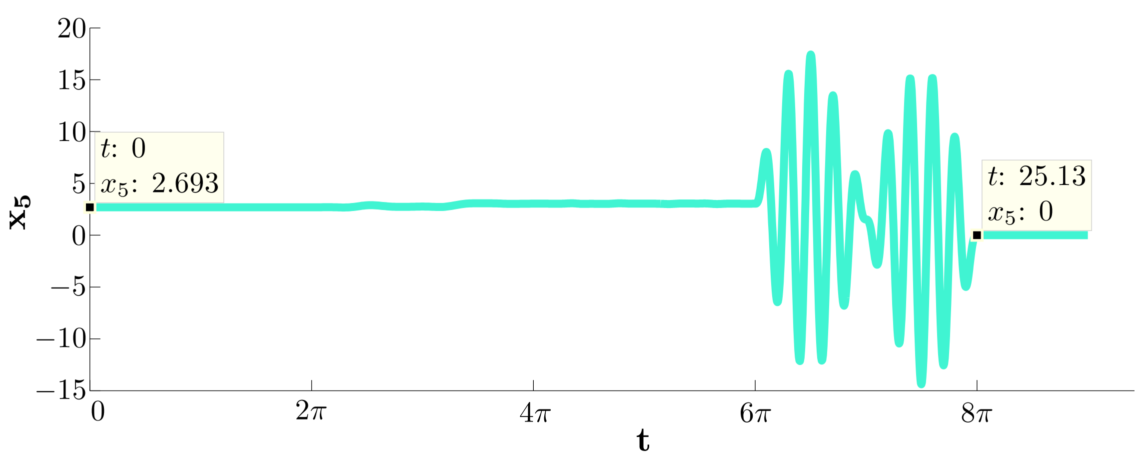

Figure 7.10: Plot of x_5 with respect to the time t.

Chapter 8

Conclusions and future work

The main interest in this work was the study of the desingularization algorithm proposed in [Chitour et al., 2013], which roughly speaking consists on the lifting of the vector fields $X_1, \ldots, X_m$ of a driftless system, defined on Ω, to vector fields $\xi_1, \ldots, \xi_m$, defined on an extended configuration manifold $\tilde{\Omega} = \Omega \times \mathbb{R}^{\tilde{n}_r}$, with $\tilde{n}_r \in \mathbb{N}$. This algorithm guarantees that the control inputs that solve the MPP for the "lifted system" will also solve it for the original system.

The desingularization procedure studied in this paper may be applied to driftless systems in general, even if they do not have singular points, and ensures that the family $\{\xi_1, \ldots, \xi_m\}$ is free U up to step r. Nevertheless, the generality as to the type of systems for which this algorithm can be applied entails, as trade-off, that the "desingularized" system is not necessarily the "smallest" regular system whose vector fields are liftings of $X_1, \ldots, X_m$. For example, in [Jean, 2014] it is reported that System (5.10), defined on a manifold of dimension 4, is regular and its vector fields are liftings of the vector fields X_1 and X_2 of the singular system (5.8); however, by applying the desingularization algorithm to (5.8), one obtains System (7.1), defined on a manifold of dimension 5.

The above trade-off may result in an increased difficulty when designing control laws for some systems obtained via the desingularization algorithm. A clear example of this can be seen on System(3.6) (a kinematic model of the tricycle with one trailer), in this case the difference between the dimensions of Ω and $\tilde{\Omega}$ is 9, which implies an increase in the difficulty of designing u_1 and u_2. For instance, if one tries to design these inputs as a concatenation, in a similar way as performed in Chapter 7 for the system (7.1), it would be necessary to concatenate at least 13 auxiliary control inputs to steer all of the state variables, which would entail a large number of computations, otherwise unnecessary if one could find a "smaller dimensional" desingularization of (3.6).

An alternative procedure was proposed in Chapter 5, which could be further developed to deal with the significant difference in dimensions mentioned in the preceding paragraphs, for some particular systems: When the system to be controlled is relatively simple, one can use a straightforward desingularization approach, such as the elementary desingularization mentioned in Chapter 5. This approach would guarantee at least the linear independence of as many Lie brackets as dimensions in the original system, i.e., it would provide an elementary desingularization ensuring that the system obtained is the "smallest-dimensional" desingularization of the original system.

It should be emphasized that this alternative is not intended as a replacement of the algorithm proposed in [Chitour et al., 2013], since this alternative would lack the generality and systematic nature of that algorithm.

We propose, as future work, the in-depth study of the proposed alternative, which would include, for example, the definition of the set of systems for which the elementary desingularization can be applied, i.e., to define under what circumstances a system could be considered "relatively simple" to apply this procedure. Once is this set defined, we might endeavor to give a formal definition of the steps of such alternative desingularization procedure.

A fringe benefit of this work is that it may be considered as a starting point to solve the local asymptotic point-stabilization problem, via continuous time-varying feedback, for singular systems. Nevertheless, given the additional difficulties in the control design that the application of the desingularization algorithm would imply, a research direction to be explored before addressing feedback stabilization would be the assessment, in terms of actual computational complexity, of the benefits of applying the desingularization algorithm to solve the steering problem for more general driftless, singular systems.

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