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Non-alternating knots: alternation and dealternating numbers

Tesis que presenta

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Director de la Tesis:

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Nomenclature

χ_i Basic 3-tangle, page 22

$\Delta(\cdot)$ Alexander polynomial, page 11

δ_{max} The maximum δ-grading, where $Kh(\cdot)$ is non-trivial, page 15

δ_{min} The minimum δ-grading, where $Kh(\cdot)$ is non-trivial, page 15

$\Gamma(\cdot)$ Γ-polynomial, page 13

$\mathcal{E}$ A 3-braid with a half twist, page 10

$\nabla(\cdot)$ Conway polynomial, page 12

$\sharp$ Connected sum, page 7

$\sqcup$ Split union, page 13

$alt(\cdot)$ The alternation number of a link, page 17

$alt_{\mathcal{D}}(\cdot)$ The alternation number of a diagram, page 17

$b(L)$ Braid index of L, page 9

B^3 A 3-ball, page 9

$br(L)$ Bridge index of L, page 8

$cr(L)$ Crossing number of L, page 8

D_0 The number of loops obtained from D by applying 0-smoothings , page 17

D_1 The number of loops obtained from D by applying 1-smoothings , page 17

$dalt(\cdot)$ The dealternating number of a link, page 17

$dalt_{\mathcal{D}}(\cdot)$ The dealternating number of a diagram, page 17

$g_T(\cdot)$ The Turaev genus, page 18

$gcd(q,r)$ Greatest common divisor of q and r, page 7

$Kh(\cdot)$ Khovanov homology, page 14

$lk(K_1, K_2)$ Linking number of K_1 and K_2, page 9

N_i Closure of a 3-tangle, page 22

$neg(D)$ The number of negative crossings of D, page 15

$P(\cdot)$ HOMFLY-PT polynomial, page 13

$pos(D)$ The number of positive crossings of D, page 15

S^3 3-sphere, page 5

$span(P)$ The difference between the highest and lowest degrees of P, page 14

$T(p,q)$ Torus knot p, q, page 7

U The trivial knot, page 5

$V(\cdot)$ Jones polynomial, page 12

$w_{Kh}(\cdot)$ The Khovanov width, page 15

Resumen

En esta tesis proponemos fórmulas para obtener el polinomio HOMFLY-PT de ciertos 3-ovillos orientados y de enlaces formados con estos 3-ovillos. También mostramos fórmulas para los polinomios de Conway y Alexander. Proponemos una construcción para formar enlaces no alternantes y damos una condición en esta construcción para obtener nudos. Las fórmulas de Conway y Alexander son explícitas y no recursivas, por lo que son fáciles de implementar. Las familias contienen los primeros nudos no alternantes: 8_{19}, 8_{20}, 8_{21}, 9_{42}. Además, mostramos familias infinitas de nudos primos con número de alternancia igual a uno. Más específicamente estos nudos después de un cambio de cruce se vuelven nudos de dos puentes o el nudo trivial. Además, los nudos son hiperbólicos excepto por los dos únicos nudos tóricos con número de alternancia uno, que son 8_{19} y 10_{124}. Por otra parte, para cada entero positivo n probaremos, usando homología de Khovanov, que existe una familia infinita de nudos hiperbólicos primos con número de alternancia igual a uno, número de dealternancia igual a n, índice de trenza igual a $n+3$ y genero de Turaev igual a n.

Palabras clave: Nudos no alternantes, invariantes polinomiales, número de alternancia, número de dealternancia.

Abstract

In this thesis we give formulae to obtain the HOMFLY-PT polynomial of certain oriented 3-tangles and links formed by the closure of them. Also we show formulae for the Conway-Alexander polynomial.

We propose a construction to form non-alternating links and give a condition over this construction to obtain only knots. For this construction, the formulae for Conway-Alexander polynomial are explicit and non recursive. Hence, these formulae are easy to implement. The families of the constructed knots contain the first non-alternating knots: 8_{19}, 8_{20}, 8_{21}, 9_{42}.

Moreover, by using the Alexander polynomial, some infinite families of non-alternating prime knots, which have alternation number equal to one are given. More specifically, these knots with one crossing change yield 2-bridge knots or the trivial knot. Furthermore, the knots are hyperbolic except for the only two torus knots with alternation number one: 8_{19} and 10_{124}.

On the other hand, for each positive integer n we will prove, by using Khovanov homology, that a family of infinitely many hyperbolic prime knots has alternation number 1, dealternating number equal to n, braid index equal to $n+3$ and Turaev genus equal to n.

Key Words: Non-alternating knots, polynomial invariants, alternation number, dealternating number.

Chapter 1

Introduction

Links can be divided into alternating and non-alternating depending on if they possess an alternating diagram or not, respectively. After the proof of the Tait flype conjecture on alternating links, given by Menasco and Thistlethwaite in [40], it became an important question to ask how non-alternating links are “close to” alternating links [30]. Moreover, recently Greene [24] and Howie [27], independently, gave a characterization of alternating links. Such a characterization shows that being alternating is a topological property of the knot exterior and not just a property of the diagrams, answering an old question of Ralph Fox “What is an alternating knot? ”.

Furthermore, with the purpose of measure how “far”are non-alternating links from alternating links and generalize properties to non-alternating knots, new concepts were created. For example, Kawauchi in [30] introduced the invariant called alternation number. The *alternation number of a link diagram D* is the minimum number of crossing changes necessary to transform D into a diagram (possibly non-alternating) of an alternating link. The *alternation number of a link L*, denoted $alt(L)$, is the minimum alternation number of any diagram of L. Another new invariant is the *dealternating number*, which was introduced by Adams et al. [5]. The *dealternating number of a link diagram D* is the minimum number of crossing changes necessary to transform D into an alternating diagram. The *dealternating number of a link L*, denoted $dalt(L)$, is the minimum dealternating number of any diagram of L. A link L with dealternating number k is also called *k-almost alternating* and we say that a link is *almost alternating* if it is 1-almost alternating. It is immediate from their definitions that $alt(L) \leq dalt(L)$ for any link L. Another invariant, which is zero if the link is alternating, is the *Turaev genus of a knot*: Given a knot diagram D of a knot K, Turaev [48] associated a closed orientable surface embedded in S^3, called the *Turaev surface* (see also [36], [19]). From it the *Turaev genus*, denoted by $g_T(K)$, was defined as the minimal number of the genera of the Turaev surfaces of all diagrams of K [19].

In 1978, W. Thurston proved that every knot is either a torus knot, a satellite knot, or a hyperbolic knot and that these categories are mutually exclusive. Adams et al. proved that a prime almost alternating knot is either a torus knot or a hyperbolic knot [5], this generalizes Menasco’s idea of the same fact in the case of alternating links [39]. Moreover they also demonstrated that the result does not extend to almost alternating links or to 2-almost alternating knots or links. Several authors have worked with these invariants, for instance Abe and Kishimoto gave examples where the alternation number equals the dealternating number [3]. In particular, they determined dealternating numbers, alternation numbers and

Turaev genus for a family of closed positive 3-braids. They also showed that there exist infinitely many positive knots with any dealternating number (or any alternation number) and any braid index.

Besides, recently Lowrance in [35] demonstrated that there exist families of links for which the difference between certain alternating distances is arbitrarily large. In order to obtain this result he gave three families of knots; the first one denoted $F(W_n)$ consists of iterated Whitehead doubles of the figure-eight knot, the second one $F(\hat{T}(p,q))$ consists of links obtained by changing certain crossings of torus links. The last family $F(T(3,q))$ consists of the $(3,q)$-torus knots. In particular, $F(W_n)$ are satellite knots with alternation number one and dealternating number arbitrarily large, where for each positive integer n there exists a knot K such that $alt(K) = 1$ and $n \leq dalt(K)$. In addition to the results given in [3] and [35], we prove the following theorem in Section 5.2.

Theorem 1.0.1. *For all $n \in \mathbb{N}$ there exists an infinite knot family, $\mathcal{D}_n$ with $l \in \mathbb{N} \cup \{0\}$, such that if $K \in \mathcal{D}_n$ then $alt(K) = 1$ and $dalt(K) = g_T(K) = n$.*

So, for each n, we give a infinite family of hyperbolic prime knots such that $alt(K) = 1$ and $dalt(K) = n$, instead of one knot as in [26]. Moreover, in each family we have that $dalt(K) = g_T(K) = n$.

On the other hand, a way to prove that links are non-alternating is by using polynomial invariants. By using Conway's room theory [22], Giller developed a method to calculate the Conway polynomial of a link L formed with certain oriented 3-tangles. Posteriorly in [10], given another type of oriented 3-tangles, Cabrera considered five different ways for closing it to obtain knots or links and gave formulae for calculating the Conway polynomials of the closures of the composition of two such 3-tangles.

In this dissertation, we consider another different type of oriented 3-tangles and we give formulae to obtain the HOMFLY-PT polynomial [21] of the closure of it. In addition, these formulae can be reduced to obtain the Alexander and Conway polynomials [6] [15]. The formulae given in order to obtain the Alexander polynomial are non recursive. Further, by using this type of 3-tangles, we have constructed infinite families of non-alternating links. This fact is proved by using Alexander polynomial, HOMFLY-PT polynomial and the Khovanov width, which are defined in Chapter 2. The knots in these families have alternation number one, are prime and almost all are hyperbolic. The following infinite families of non-alternating knot are presented in Chapter 4. These families of knots contain the smaller non-alternating knots, namely 8_{19}, 8_{20}, 9_{21} and 9_{42}.

- For $k \geq 3$, the family of knots $\{N_1(\mathcal{T}(2l+1, 2n_1, 2m_1, ..., 2n_r, 2m_r) \cdot c \cdot \mathcal{E}^{2k})\}$, where $k, l, n_i, m_i, r \in \mathbb{N}, i = 1, ..., r$.
- For all $l \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$ the knots $N_1(\mathcal{T}(2l+1) \cdot c \cdot \mathcal{E}^{2k})$.
- For all $l, m \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$ the knots $N_1(\mathcal{T}(2l+1, 2, 2m) \cdot c \cdot \mathcal{E}^{2k})$.
- The links $N_1(\mathcal{E}^{2k})$, $N_3(\mathcal{E}^{2k})$ and $N_1(\mathcal{E}^{2k} \cdot c)$.
- For all $l \in \mathbb{N}$, the knots $N_1(\mathcal{T}(-(2l+1), 2, 2) \cdot c)$ and $N_1(\mathcal{T}(3, 2, -2l) \cdot c)$.

The content of this thesis is organized as follows. Chapter 2 contents the preliminary concepts and notation used in this dissertation. Chapter 3 presents formulae to obtain the HOMFLY-PT polynomial, the Conway polynomial, and the Alexander polynomial of certain 3-tangles and of links formed by their closures. Chapter 4 introduces some families of non-alternating knots. Chapter 5 deals with some invariants of the families introduced in Chapter 4. In particular, for each n we give an infinite family of hyperbolic prime knots such that $alt(K) = 1$ and $dalt(K) = g_T(K) = n$.

Chapter 2

Preliminaries

In this chapter we introduce some basic concepts with the purpose of having a complete background in this work. In order to fathom in these concepts we recommend to see the following books: [4], [17], and [29].

2.1 Basic definitions

Definition 1. *A **link** of m components is a subset of S^3, which consist in m disjoint curves, piecewise linear, closed and simple. A link of one component is a **knot**.*

Definition 2. *Two links L_1 and L_2 are **ambient isotopic** if there exists a continuous function $H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that*

- *For all $t \in [0,1], h_t = H(\cdot,t)$ of $\mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism.*
- *If $h_0 = H(\cdot,0)$ then $h_0(L_1) = L_1$.*
- *If $h_1 = H(\cdot,1)$ then $h_1(L_1) = L_2$.*

*Such an H is called an **ambient isotopy**.*

The ambient isotopy defines an equivalence relation of links: **Two links are equivalent** if, and only if, they are related by an ambient isotopy. Each equivalence class is called a **link type**; two equivalent links have the same link type. If a link L_1 is equivalent to a link L_2 we write $L_1 \approx L_2$.

We work with link projections, which are called **regular diagrams** if they neither have tangency points nor triple points. In the double points the segments that pass under other segment are marked with a discontinuity. Double points are called **crossings** and the connected components are called **strings**. The simplest knot of all is the **trivial knot**, which is denoted by U (see Figure 2.1 (c)).

(a) Knot diagram. (b) Link diagram. (c) Trivial knot diagram.

Figure 2.1: Regular diagrams.

Two regular diagrams can be related by local moves.

Definition 3. *The local moves shown in Figure 2.2 are called* ***Reidemeister moves****.*

Figure 2.2: Reidemeister moves.

Each Reidemeister move represents an ambient isotopy.

Definition 4. *If a regular diagram D is deformed into a regular diagram D' by means of a finite number of Reidemeister moves or their inverses, then we say that D and D' are* ***equivalent*** *denoted by $D \approx D'$.*

Theorem 2.1.1. *[42, Theo. 4.1.1] Let D and D' be a two regular diagrams of knots (or links) K and K', respectively. Then*

$$K \approx K' \Longleftrightarrow D \approx D'.$$

Definition 5. *If a link L can be separated by two disjoint spheres, then L is called* ***split link****.*

This implies that if a link L is a split link then L has a disconnected diagram, otherwise all its diagrams are connected, see Figure 2.3.

Figure 2.3: Two regular link diagrams of two components; the first one is split and the second one is connected.

Hereinafter we will only deal with regular diagrams and will simply call them diagrams instead of regular diagrams.

There exist various types of links; one of these types are the alternating links. The definition is based on their diagrams.

Definition 6. *A link diagram is called* ***alternating*** *if, as we traverse each component of the link, the crossings of the diagram alternate between over and under crossings, see Figure 2.4. A link that admits an alternating diagram is called an* ***alternating link****.*

Figure 2.4: At the left an alternating knot diagram and at the right a non-alternating knot diagram.

Other types of links are the following:

- A link is a **torus link** if it is equivalent to a link that can be drawn without any points of self-intersection on the trivial torus sitting inside S^3. We denote a torus link by $T(q,r)$ where q,r are integers and if they are co-prime then $T(q,r)$ is a knot and if they are not co-prime the number of components of $T(q,r)$ is $gcd(q,r)$.

- A **satellite link** is one that orbits a non-trivial companion knot K in the sense that it lies inside a regular neighborhood of the companion.

- A **hyperbolic link** is a link in the 3-sphere with complement that has a complete Riemannian metric of constant negative curvature, i.e. the complement has a hyperbolic geometry.

In 1978, W. Thurston proved that every knot is either a torus knot, a satellite knot, or a hyperbolic knot, and that these categories are mutually exclusive.

It is possible to construct a knot with the **connected sum** of two knots, which is described as follows: regard each knot in a copy of S^3, remove a ball in each S^3 that meets the given knot in an unknotted arc, and finally identify together the resulting boundary spheres, and theirs intersections with the knots, see Figure 2.5.

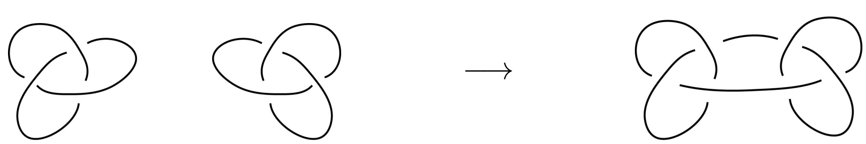


Figure 2.5: Connected sum of two knots.

Definition 7. *A knot K is a* ***prime knot*** *if it is not the trivial knot, and $K = K_1 \sharp K_2$ implies that K_1 or K_2 is the trivial knot. If a knot is non-prime then it is a* ***composite knot****.*

In 1949, Schubert proved that every knot can be uniquely decomposed, up to order, as a connected sum of prime knots. The composite knots are then easily constructed by taking connected sums. These type of knots are included in the class of satellite knots. Furthermore, if K_1 and K_2 are alternating knots then the connected sum of them will be alternating, since it is possible to obtain an alternating diagram as shown in Figure 2.6.

Figure 2.6: Two equivalent diagrams; the second one is alternating.

Definition 8. *The **bridge index** of a link L in S^3, denoted by $br(L)$, is the least integer n for which there is an S^2 separating S^3 into two balls, each meeting K in n standard (unknotted and unlinked) spanning arcs. An n-**bridge link** is a link with bridge index n.*

It is known that 2-bridge links have alternating diagrams, see for example [23] and [42]. This implies the following proposition.

Proposition 2.1.1. *2-bridge knots (or links) are alternating.*

Besides, by convention the bridge index of the trivial knot is one. Then n-bridge links for $n = 1, 2$ are alternating. For n less than or equal to 3, there are alternating and non-alternating links. In particular, for prime knots we have the following proposition.

Proposition 2.1.2. *[29] If a prime knot K has bridge index less than or equal to 3 and is not a torus knot, then K is hyperbolic.*

In 1954 Schubert proved an equality, which relates the bridge index of the connected sum of two links and the bridge index of their components.

Proposition 2.1.3. *[47] Suppose that K_1 and K_2 are two knots (or links). Then $br(K_1 \sharp K_2) = br(K_1) + br(K_2) - 1$.*

Definition 9. *The **crossing number** of a link L, denoted by $cr(L)$, is the minimal number of crossings for all diagrams of L.*

It is known that alternating links have an alternating diagram which have minimal number of crossings. However, it is complicate determinate if a link diagram have minimal number of crossing between all the diagrams of the link.

Definition 10. *An **oriented link** is a link for which each connected component has been given an orientation.*

In Figure 2.7 two oriented diagrams of the same unoriented link are shown, in which one component has different orientation.

Figure 2.7: Two Hopf link diagrams, $L2a\{0\}$ and $L2a\{1\}$, endowed with different orientations.

Definition 11. *Given an oriented diagram D of a link. At a crossing, c, of D we assign $sign(c) = +1$ and $sign(c) = -1$ as shown in Figure 2.8. The crossing, in the first case is said to be **positive** and in the second case is said **negative**.*

Figure 2.8: At the left a positive crossing and at the right a negative crossing.

Definition 12. *Let D be an oriented diagram of a link of two components, K_1 and K_2, and c_i for $i = 1,...,m$ the crossing where the two components intersect (we ignore self-intersections of each component). Then, the number*

$$\frac{1}{2}(sign(c_1) + sign(c_2) + ... + sign(c_m))$$

is called the ***linking number*** *of K_1 and K_2, which will be denoted by $lk(K_1, K_2)$.*

Example 1. *The linking numbers of the trivial knots, which form the Hopf link diagrams $L2a\{0\}$ and $L2a\{1\}$, are -1 and 1, respectively, see Figure 2.7.*

An n**-tangle** is a pair (B^3, T) where B^3 is a 3-ball and T is a one-dimensional, embedded sub manifold with non-empty boundary, which contains n arcs (i.e., n subsets homeomorphic to $[0,1]$) and satisfies $\partial T = T \cap \partial B^3$. In Figure 2.9 two 3-tangle diagrams are shown.

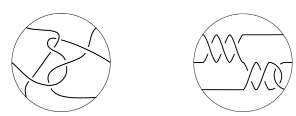


Figure 2.9: Two 3-tangle diagrams.

As in the study of general knot theory, tangles are studied via their diagrams. Given two n-tangles S and T, their **product** $S \cdot T$ is defined as the n-tangle obtained by the concatenation of S to the left of T, in Figure 2.10 is shown the case of 3-tangles. In general, we will work with 3-tangles and we will denote them by T instead of (B^3, T).

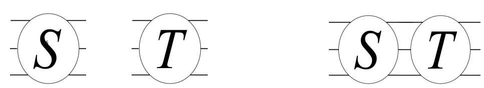


Figure 2.10: At the left 3-tangles S and T and at the right their concatenation $S \cdot T$.

A subfamily of the n-tangles is the set of the n-braids. An n**-braid** is a set of n strings attached to vertical bars at their left and right endpoints, with the property that each string heads rightwards at every point as it is traversed from left to right. The **braid index**, $b(L)$, of a link L is the minimal number of strands of any braid whose closure is equivalent to L.

The following notation is taken from [11]. Given a 3-braid B, there exists a finite sequence of integers $a_1, \dots, a_n$, such that B admits a diagram of the form $\mathcal{T}(a_1, \dots, a_n)$(see Figure 2.11), where $\mathcal{T}(a_1, \dots, a_n)$ indicates $|a_1|$ crossings of the two uppermost strands, followed by $|a_2|$ crossings of the two lowermost strands, and then $|a_3|$ crossings of the two uppermost strands, and so on, with the following sign convention. For odd i, positive values of the a_i indicate that the uppermost strand passes over the middle strand, whereas for even i, a positive value of a_i indicates that the lowermost strand passes over the middle strand. This notation is illustrated in Figure 2.12. It is clear that a diagram $\mathcal{T}(a_1,\ldots,a_n)$ equals the concatenation of diagrams $\mathcal{T}(a_1)\cdot\mathcal{T}(0,a_2)\cdot\ \ldots\ \cdot\mathcal{T}(0,a_n)$, if n is even, or $\mathcal{T}(a_1)\cdot\mathcal{T}(0,a_2)\cdot\ \ldots\ \cdot\mathcal{T}(a_n)$, if n is odd.

Figure 2.11: 3-braid $\mathcal{T}(a_1,a_2,...,a_n)$ with n odd and n even.

In Figure 2.12 are shown some examples of 3-braids; the 3-braid $\mathcal{T}(1,-1,1)$, in particular, is one half twist and we will denote it by $\mathcal{E}$.

(a) $\mathcal{T}(3,2,2)$ (b) $\mathcal{T}(1,-1,1)$ (c) $\mathcal{T}(-1,1,-1)$ (d) $\mathcal{T}(1,-1,2,-1,1)$

Figure 2.12: Examples of notation for 3-braids.

An 3-**braid is said to be alternating** if, and only if, it admits an alternating diagram, that is, a diagram $\mathcal{T}(a_1,...,a_n)$ such that $a_i \geq 0$ for all $i = 1,2,...,n$ or $a_i \leq 0$ for all $i = 1,2,...,n$. As an example, the 3-braid diagrams in Figure 2.12, except for $\mathcal{T}(3,2,2)$, are non-alternating.

By using 3-braid diagrams is possible to construct 2-bridge knots due to that a standard diagram of 2-bridge knot can be represented as the diagram showed in Figure 2.13. Now, let $\mathcal{T}(a_1,a_2,...,a_n)$ be a 3-braid, it define the **type** A **closure** of $\mathcal{T}(a_1,a_2,...,a_n)$ as the knot or link obtained of close the diagram like is showed in Figure 2.13, it will be denoted by $A(\mathcal{T}(a_1,a_2,...,a_n))$. This fact is described by the following theorem, which is a modification of Theorem 9.3.1 in [42].

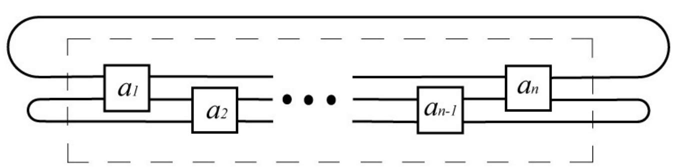


Figure 2.13: 2-bridge link generated by the type A closure of $\mathcal{T}(a_1,a_2,...,a_n)$.

Theorem 2.1.2.

1) *A 2-bridge knot (or link) is the type A closure of a 3-braid.*

2) *On the other side, the type A closure of a 3-braid is a 2-bridge knot (or link) or trivial link.*

We remark that 2-bridge knots are alternating.

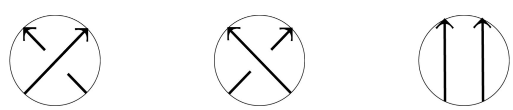
2.2 Polynomial invariants

It is possible to reduce the complexity of a link by changing crossings. Given a link diagram one changes the crossings in it, one by one, until a collection of trivial links is obtained. This process of repeatedly choosing a crossing, and then applying a skein relation to obtain simpler links, yields a tree of links called the **resolving tree**.

Resolving trees can be used with various skein relations to define several knot invariants. In 1923 was defined the first polynomial invariant for oriented links, the **Alexander polynomial** $\Delta(L;t)$ in $\mathbb{Z}[t^{\pm\frac{1}{2}}]$. The definition, which does not use skein relation, can be found in [6]; however, this polynomial can be computed by using skein relations from the following recursive relations:

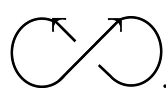
1. $\Delta(L_+;t) - \Delta(L_-;t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta(L_0;t)$
2. $\Delta(U;t) = 1$

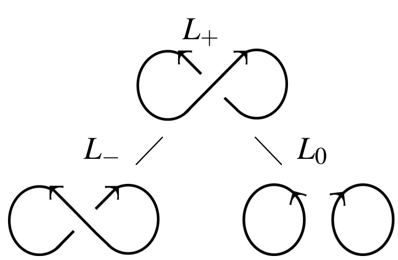
where U is the trivial knot and (L_+, L_-, L_0) is a skein triple of oriented links that are the same, except in a crossing neighbourhood where they look as shown in Figure 2.14.

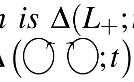


(a) L_+ (b) L_- (c) L_0

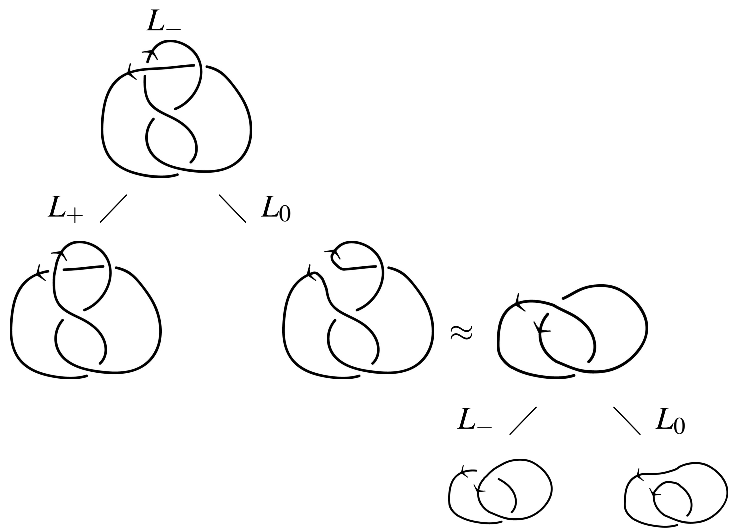
Figure 2.14: Skein triple.

Example 2. *The resolving tree of the diagram:* .



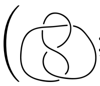
Furthermore the relation is $\Delta(L_+;t) = \Delta(L_-;t) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta(L_0;t)$*, since* $\Delta(L_+;t) = \Delta(L_-;t) = 1$ *it follows that* $\Delta($  $;t) = 0$.

Example 3. *The resolving tree of the diagram*  *is the following.*



Then the Alexander polynomial is obtained as follows:

$$
\begin{aligned}
\Delta(L_-;t) &= \Delta(L_+;t) - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta(L_0;t)\\
&= 1 - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta(L_0;t)\\
&= 1 - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(0 + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}))\\
&= 1 - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2\\
&= -t - t^{-1} + 3
\end{aligned}
$$

Hence, $\Delta\Bigg($  $;t\Bigg) = -t - t^{-1} + 3.$

In [15], around fifty years after the discovering of the Alexander polynomial, it was defined another polynomial for oriented links called the **Conway polynomial** $\nabla(L;z)$ in $\mathbb{Z}[z]$. This polynomial is computed by the following recursive relations:

1. $\nabla(L_+;z) - \nabla(L_-;z) = z\nabla(L_0;z)$

2. $\nabla(U;z) = 1$

An important result is that the Conway and the Alexander polynomials are essentially the same. The following theorem is in order.

Theorem 2.2.1. *[42, Theo. 6.2.1] Let L be an oriented link then* $\Delta(L;t) = \nabla(L;t^{\frac{1}{2}} - t^{-\frac{1}{2}})$.

Essentially, in Theorem 2.2.1, it only needs a variable change $z = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$. A property of these polynomials is the following.

Proposition 2.2.1. *[42] If L is an oriented link which is also a split link, then* $\nabla(L;z) = 0$.

In 1984, another different polynomial discovered is the **Jones polynomial** $V(L;t)$, which is a Laurent polynomial in $t^{1/2}$ with integer coefficients, which can be computed by the following recursive relations:

1. $t^{-1}V(L_+;t) - tV(L_-;t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V(L_0;t)$

2. $V(U;t) = 1$

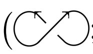
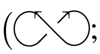
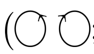
Considering the similarities of the previous polynomials it was defined, by Hoste, Ocneanu, Millett, Freyd, Lickorish, and Yetter, and independently by Przytycki and Traczyk, a new polynomial, which carries theirs initials. The **HOMFLY-PT polynomial** $P(L;v,z)$ in $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ of an oriented link L is computed by the following recursive formulae:

1. $v^{-1}P(L_+;v,z) - vP(L_-;v,z) = zP(L_0;v,z)$

2. $P(U;v,z) = 1$

Furthermore, the HOMFLY-PT polynomial of a split union of two links L_1 and L_2 is given by $P(L_1 \sqcup L_2;v,z) = \delta P(L_1;v,z)P(L_2;v,z)$, where $\delta = \frac{v^{-1}-v}{z}$.

Example 4. *$P(U \sqcup U)$ can be calculated in the following form.*

$v^{-1}P\Big($  $;v,z\Big) = vP\Big($  $;v,z\Big) + zP\Big($  $;v,z\Big)$, *hence* $P\Big($  $;v,z\Big) = \dfrac{v^{-1}-v}{z}$.

It is possible to recover from the HOMFLY-PT polynomial the Conway, the Alexander and the Jones polynomials, by using the following relations.

$$\nabla(L;z) = P(L;1,z)$$

$$\Delta(L;t) = P(L;1,t^{\frac{1}{2}} - t^{-\frac{1}{2}})$$

$$V(L;t) = P(L;t,t^{\frac{1}{2}} - t^{-\frac{1}{2}})$$

Another polynomial, which can be derived from the HOMFLY-PT polynomial, was defined by Kawauchi in [31]. The **Γ-polynomial** is the common zero-th coefficient polynomial of both; the HOMFLY-PT polynomial, $P(L;v,z) \in \mathbb{Z}[y^{\pm 1}, z^{\pm 1}]$, and the Kauffman polynomial, $F(L;a,b) \in Z[a^{\pm 1}, b^{\pm 1}]$. The Γ-polynomial $\Gamma(K) \in \mathbb{Z}[x^{\pm 1}]$ of a knot K is calculated by the following formulas:

1. $-x\Gamma(K_+) + \Gamma(K_-) = (1-x)x^{-lk(K',K'')}\Gamma(K')\Gamma(K'')$

2. $\Gamma(U) = 1$

where $(K_+;K_-;K_0)$ is a skein triple such that K_+, K_-, K', K'' are knots, K_0 is a link with components K' and K'', and $lk(K',K'')$ is the linking number of K' and K''.

Example 5. *In Figure 2.15 a diagram of $T(2,3)$, which is denoted by K_+, is shown. Since $K_- \approx U$ and $K_0 \approx$ the Hopf link, then*

$$\begin{aligned}\Gamma(T(2,3)) &= x^{-1}\Gamma(U) - x^{-1}(1-x)x^{-lk(U,U)}\Gamma(U)\Gamma(U)\\ &= x^{-1} - x^{-1}(1-x)x^{-1}\\ &= 2x^{-1} - x^{-2}.\end{aligned}$$

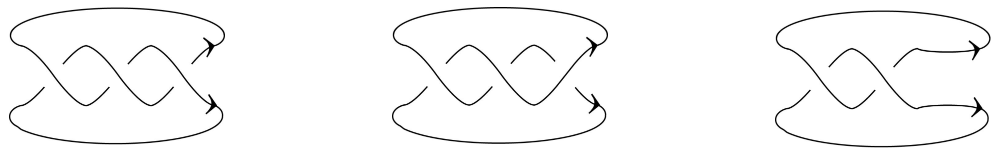


Figure 2.15: A diagram of the torus knot $T(2,3)$, K_- and K_0.

Proposition 2.2.2. *We have* $\Gamma(T(2,2r+1)) = (r+1)x^{-r} - rx^{-(r+1)}$*, where* $r \in \mathbb{N}$.

Proof. We will prove the proposition, by using induction on r. For $r=1$, due to Example 5, we have that $\Gamma(T(2,3)) = 2x^{-1} - x^{-2}$.

Let us see the case $r = n+1$; by induction hypothesis we have

$$\begin{aligned}
\Gamma(T(2,2(n+1)+1)) &= x^{-1}\Gamma(T(2,2n+1)) - x^{-1}(1-x)x^{-lk(U,U)}\Gamma(U)\Gamma(U)\\
&= x^{-1}((n+1)x^{-n} - nx^{-(n+1)}) - x^{-1}(1-x)x^{-(n+1)}\\
&= (n+2)x^{-(n+1)} - (n+1)x^{-(n+2)}.
\end{aligned}$$

□

The **span of a polynomial** P is the difference between the highest and lowest degrees of P, denoted by $span(P)$. In particular, the y-$span\,(P(L;v,z))$ is the difference between the maximum and the minimum degrees of the $P(L;v,z)$ polynomial in the variable y. The Morton-Franks-Williams inequality, which was proved in [41] and [20], relates the y-span $(P(L;v,z))$ and the braid index as follows:

$$\frac{1}{2}y\text{-}span\,(P(L;v,z)) + 1 \leq b(L) \tag{2.1}$$

In particular for the Γ-polynomial:

$$span\,(\Gamma(L)) + 1 \leq b(L). \tag{2.2}$$

On the other hand, in a similar form as the HOMFLY-PT polynomial is defined for links, it is possible to define it for oriented 3-tangles. Hereinafter, we will write $\Delta(L), \nabla(L), V(L), P(L)$ instead of $\Delta(L;t), \nabla(L;z), V(L;t), P(L;v,z)$.

2.3 Khovanov width

We will use the Khovanov width to obtain a lower bound of the dealternating number. This last invariant will be described in the following section. Khovanov in [32] introduced an invariant of links, now called the Khovanov homology, which is a bigraded $\mathbb{Z}$-module with homological grading i and polynomial (or Jones) grading j so that $Kh(L) = \bigoplus_{i,j} Kh^{i,j}(L)$ and whose graded Euler characteristic is the Jones polynomial (see the construction of the Khovanov homology in the appendix).

The support of $Kh(L)$ lies on a finite number of slope 2 lines with respect to the bigrading. Therefore, it is convenient to define the δ-grading by $\delta = j - 2i$ so that $Kh(L) = \bigoplus_{\delta} Kh^{\delta}(L)$.

Also, all the δ-gradings of $Kh(L)$ are either odd or even. Let δ_{min} and δ_{max} be the minimum and the maximum δ-grading, respectively, where $Kh(L)$ is non-trivial.

Then $Kh(L)$ is said to be $[\delta_{min}, \delta_{max}]$-thick, and the **Khovanov width of** L is defined as

$$w_{Kh}(L) = \frac{1}{2}(\delta_{max} - \delta_{min}) + 1.$$

If D is a diagram for L, then denote the Khovanov homology of L by either $Kh(L)$ or $Kh(D)$. Similarly, let $w_{Kh}(L)$ and $w_{Kh}(D)$ equivalently denote the Khovanov width of L. If $\mathbb{F}$ is a field, then let $Kh(L;\mathbb{F})$ denote $Kh(L) \otimes \mathbb{F}$ and $w_{Kh}(L;\mathbb{F})$ denote the width of $Kh(L;\mathbb{F})$.

Example 6. *Knot 9_{42} at Rolfsen's table. The rank of each group $Kh^{i,j}(9_{42})$ is shown. There are two critical diagonals $j-2i=3$ and $j-2i=1$. Non-zero entries on the critical diagonals are highlighted in yellow and non-zero entries off the critical diagonals are highlighted in red. Since $\delta_{max}=3$ and $\delta_{min}=-1$ then $w_{Kh}(9_{42})=3$.*

$j \diagdown i$	-4	-3	-2	-1	0	1	2	X
7							*1*	1
5								0
3					*1*	*1*		0
1				*1*	*1*			0
-1				*1*	*1*			0
-3		*1*	*1*					0
-5								0
-7	*1*							1

Let L be an oriented link, and let C be a component of L, denote by l the linking number of C with its complement $L-C$. Let L' be the link L with the orientation of C reversed, and let D be a diagram for L and D' be the diagram D with the component C reversed. Denote the number of negative and positive crossings in D by $neg(D)$ and $pos(D)$, respectively, where the sign of a crossing is as in Figure 2.16.
Each $Kh^{i,j}(D)$ can be obtained by suitable normalization from a homology group (see appendix) of the following form:

$$Kh^{i,j}(D) := H^{i+neg(D),j-pos(D)+2neg(D)}(D).$$

Since D' is the diagram D with the component C reversed, it follows that

$$pos(D') = pos(D) - 2l \text{ and } neg(D') = neg(D) + 2l.$$

Therefore, we have that for $i, j \in \mathbb{Z}$ there are isomorphisms of groups

$$Kh^{i,j}(D') = Kh^{i+2l,j+6l}(D). \tag{2.3}$$

Considering the δ-grading and setting $s = neg(D') - neg(D)$ it follows that:

$$Kh^{\delta}(D') = Kh^{\delta+s}(D). \tag{2.4}$$

Let D_+, D_-, D_v and D_h be diagrams of links that agree outside a neighborhood of a distinguished crossing as in Figure 2.16 and define $e = neg(D_h) - neg(D_+)$. There are long exact sequences relating the Khovanov homology of each of these links, as indicated in Theorem 2.3.1. Khovanov [32] implicitly describes these sequences. The graded versions are taken from Rasmussen [46] and Manolescu-Ozsvath [37].

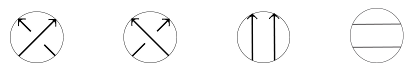


Figure 2.16: The distinguished crossings of the diagrams D_+, D_-, D_v, D_h respectively.

Theorem 2.3.1. *[32] There are long exact sequences relating the Khovanov homology of D_+, D_-, D_v and D_h as follows:*

$$\cdots Kh^{i-e-1,j-3e-2}(D_h) \to Kh^{i,j}(D_+) \to Kh^{i,j-1}(D_v) \to Kh^{i-e,j-3e-2}(D_h) \to \cdots$$

and

$$\cdots Kh^{i,j+1}(D_v) \to Kh^{i,j}(D_-) \to Kh^{i-e+1,j-3e+2}(D_h) \to Kh^{i+1,j+1}(D_v) \to \cdots.$$

When only the $\delta = j - 2i$ grading is considered, the long exact sequence become

$$\cdots Kh^{\delta-e}(D_h) \xrightarrow{f_+^{\delta-e}} Kh^{\delta}(D_+) \xrightarrow{g_+^{\delta}} Kh^{\delta-1}(D_v) \xrightarrow{h_+^{\delta-1}} Kh^{\delta-e-2}(D_h) \to \cdots$$

and

$$\cdots Kh^{\delta+1}(D_v) \xrightarrow{f_-^{\delta+1}} Kh^{\delta}(D_-) \xrightarrow{g_-^{\delta}} Kh^{\delta-e}(D_h) \xrightarrow{h_-^{\delta-e}} Kh^{\delta-1}(D_v) \to \cdots.$$

Lowrance pointed out that Theorem 2.3.1 directly implies the following corollary:

Corollary 2.3.1. *[34] Let D_+, D_-, D_v and D_h be as in Figure 2.16. Suppose $Kh(D_v)$ is $[v_{min}, v_{max}]$-thick and $Kh(D_h)$ is $[h_{min}, h_{max}]$-thick. Then $Kh(D_\pm)$ is $[\delta^{\pm}_{min}, \delta^{\pm}_{max}]$-thick, where*

$$\delta^{\pm}_{min} = \begin{cases} min\{v_{min} \pm 1, h_{min} + e\} & \text{if } v_{min} \neq h_{min} + e \pm 1 \\ v_{min} + 1 & \text{if } v_{min} = h_{min} + e \pm 1 \text{ and } h_{\pm}^{v_{min}} \text{ is surjective} \\ v_{min} - 1 & \text{if } v_{min} = h_{min} + e \pm 1 \text{ and } h_{\pm}^{v_{min}} \text{ is not surjective,} \end{cases}$$

and

$$\delta^{\pm}_{max} = \begin{cases} max\{v_{max} \pm 1, h_{max} + e\} & \text{if } v_{max} \neq h_{max} + e \pm 1 \\ v_{max} - 1 & \text{if } v_{max} = h_{max} + e \pm 1 \text{ and } h_{\pm}^{v_{max}} \text{ is injective} \\ v_{max} + 1 & \text{if } v_{max} = h_{max} + e \pm 1 \text{ and } h_{\pm}^{v_{max}} \text{ is not injective.} \end{cases}$$

We will use Khovanov homology to obtain a lower bound for the delternating number and the Turaev genus of K, which are defined in the following section.

2.4 Alternation and dealternating numbers

Some invariants are defined as the minimum of a characteristic of the link diagram; however, this class of invariants are not easy to calculate.

Definition 13. *The **alternation number of a link diagram** D, denoted $alt_{\mathcal{D}}(D)$, is the minimum number of crossing changes necessary to transform D into some diagram (possibly non-alternating) of an alternating link. The **alternation number of a link** L, denoted **alt(L)**, is the minimum alternation number of any diagram of L.*

In [25] were shown two infinite families of knots with alternation number 1.

Definition 14. *The* ***dealternating number of a link diagram*** *D, denoted $dalt_{\mathcal{D}}(D)$, is the minimum number of crossing changes necessary to transform D into an alternating diagram, see Figure 2.17. The* ***dealternating number of a link*** *L, denoted* ***dalt(L)****, is the minimum dealternating number of any diagram of L. A link with dealternating number k is also called k****-almost alternating****. We say that a link is* ***almost alternating*** *if it is 1-almost alternating.*

Figure 2.17: At the left a diagram with dealternating number one and at the right a diagram with dealternating number zero.

Both link invariants are zero if, and only if, the link is alternating; however, the difference between these invariants lies in whether the Reidemeister moves are admitted after a crossing change or not. And it is immediate from their definitions that for any link L we have that

$$alt(L) \leq dalt(L) \tag{2.5}$$

Adams et al. [5] showed a satellite knot K, see Figure 2.18, with $alt(K) = 1$ and $dalt(K) = 2$ by using that an almost alternating knot is either a torus knot or a hyperbolic knot.

Figure 2.18: Two diagrams, D_1 and D_2 for the same knot; the first one has $alt_{\mathcal{D}}(D_1) = 1$ and the second one has $dalt_{\mathcal{D}}(D_2) = 2$.

Another invariant related to both the alternation and the dealternating number is the Turaev genus, which will be described as follows. In [19], to a link diagram D, Turaev associated a closed orientable surface embedded in S^3, called the **Turaev surface** (see also [36]). Two local moves in a crossing on a link diagram are used to define a Turaev surface, we call them as 0**-smoothing** and a 1**-smoothing**, see Figure 2.19. We denote by $|D_0|$ and $|D_1|$ the number of loops, which are obtained from D by applying an 0-smoothing and a 1-smoothing at each crossing, respectively.

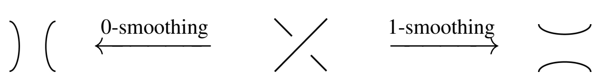


Figure 2.19: A crossing and its 0-smoothing and 1-smoothing.

Example 7. *Considering the diagram D in Figure 2.20 we have $|D_0| = 3$ and $|D_1| = 2$.*

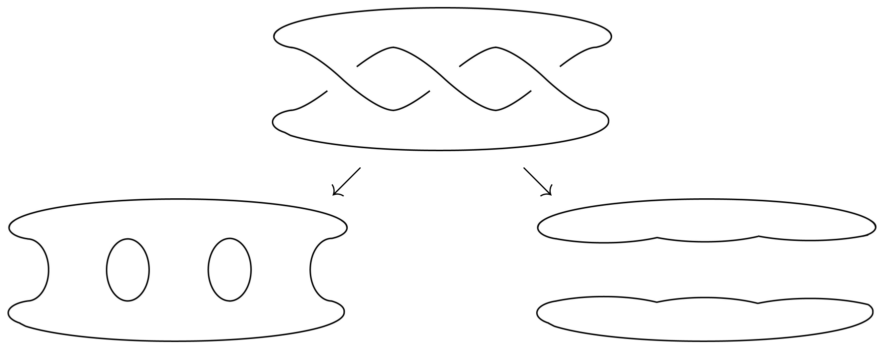


Figure 2.20: A diagram D of the knot $T(2,3)$ and all its 0-smoothings and 1-smoothings, respectively.

Let G in S^2 be the diagram D of a connected link diagram without the over and under information at each crossing and V the union of the vertices of G. Let $G \times [-1,1]$ be a surface with singularities $V \times [-1,1]$ naturally embedded in $S^2 \times [-1,1]$. Replace the neighborhoods of $V \times [-1,1]$ with saddle surfaces, as in Figure 2.21, positioned in such a way that the boundary curves in $S^2 \times \{1\}$ and $S^2 \times \{-1\}$ correspond to 0-smoothing and 1-smoothing, respectively.

Figure 2.21: A neighborhood of a singularity and its associated saddle surface.

Finally, the Turaev surface is completed by attaching disjoint discs to all the boundary circles in S^3. In Figure 2.22 a diagram and its Turaev surface is shown before attaching the discs to the boundary circles.

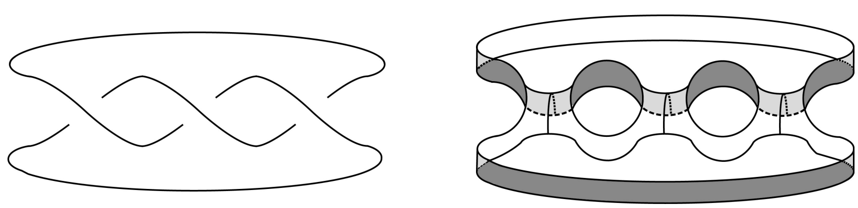


Figure 2.22: A diagram and its Turaev surface before attaching the disjoint discs.

Definition 15. *[19] The* ***Turaev genus****, $g_T(L)$, of a link L is the minimal number of the genera of the Turaev surfaces associated to all connected diagrams of L.*

Considering the Euler characteristic of the Turaev surface associated to a connected link diagram D, we can estimate the Turaev genus of D, denoted $g_T(D)$. We remark that $|D_0|$ and $|D_1|$ are the number of loops, which are obtained from a diagram D by applying a 0-smoothing and a 1-smoothing at each crossing, respectively. The following results gives the Turaev genus of a diagram D.

Proposition 2.4.1. *[19][36] Let D be a connected link diagram. Then we have*

$$g_T(D) = \frac{1}{2}(cr(D) + 2 - |D_0| - |D_1|).$$

Example 8. *Calculate the Turaev genus of the diagram D in Figure 2.20. Since $cr(D) = 3$ and by Example 7 we have $|D_0| = 3$ and $|D_1| = 2$ it follows that $g_T(D) = 0$.*

In [19], it was shown that a non-split link L is alternating if, and only if, $g_T(L) = 0$. Besides, from its definitions, $alt(L)$ and $dalt(L)$, are equal to zero if, and only if, L is an alternating link. So, for a non-split alternating link L we have that $alt(L) = g_T(L) = dalt(L) = 0$.

The following results show the relations between the Khovanov width, the Turaev genus and the dealternating number. Lemma 2.4.1 was proved by Manturov [38] and Champanerkar, Kofman and Stoltzfus [14], and Corollary 2.4.1 was proved by Abe and Kishimoto in [3].

Lemma 2.4.1. *[38][14] Let K be a knot then we have*

$$w_{Kh}(K) - 2 \leq g_T(K).$$

Corollary 2.4.1. *[3] Let L be a non-split link then we have*

$$g_T(L) \leq dalt(L).$$

The alternation number has been calculated for Torus knots, $T(p,q)$. In particular, there exist only two torus knots with alternation number one.

Theorem 2.4.1. *[1] We have*

- $alt(T(p,q)) = 0 \Leftrightarrow p = 2.$
- $alt(T(p,q)) = 1 \Leftrightarrow (p,q) = (3,4)$ *or* $(3,5).$
- $alt(T(p,q)) \geq 2 \Leftrightarrow$ *otherwise.*

The torus knots $T(3,4)$ and $T(3,5)$ are denoted by 8_{19} and 10_{124} in the Rolfsen knot table, respectively.

By using the lower bound given in Lemma 6.0.4, the alternation number also has been calculated for torus knots with three strands.

Proposition 2.4.2. *[28] For any positive integer n,*

- $alt(T(3,4)) = alt(T(3,5)) = 1,$
- $alt(T(3,6n+1)) = alt(T(3,6n+2)) = 2n,$
- $alt(T(3,6n+4)) = alt(T(3,6n+5)) = 2n$ *or* $2n+1.$

Besides, the Turaev genus and the dealternating number have been obtained for torus knots $T(3,q)$.

Proposition 2.4.3. *[3][34] Let n be a non-negative integer, and let i = 1 or 2. Then*

$$g_T(T(3,3n+i)) = dalt(T(3,3n+i)) = n.$$

If we merge Propositions 2.4.2 and 2.4.3 into one proposition we obtain the following.

Proposition 2.4.4. *Let n be a non-negative integer, and let i = 1 or 2. Then*

- *If* $n = 1$ *or even then*

$$alt(T(3,3n+i)) = g_T(T(3,3n+i)) = dalt(T(3,3n+i)),$$

- *If* $n \neq 1$ *or odd then*

$$alt(T(3,3n+i)) + 1 = g_T(T(3,3n+i)) = dalt(T(3,3n+i))$$

or

$$alt(T(3,3n+i)) = g_T(T(3,3n+i)) = dalt(T(3,3n+i)).$$

Further, Abe and Kishimoto in [3] proved, by using closed 3-braids, that there exist infinitely many knots with $alt(K) = dalt(K)$.

Theorem 2.4.2. *[3] Let K be a knot of the form* $N_1(\mathcal{E}^{2n} \cdot \mathcal{T}(0, p_1, q_1, ..., p_r, q_r))$ *such that* $p_i, q_i \in \mathbb{N}$ *for* $i = 1, 2, ..., r$ *and* $p_i, q_i \geq 2$ *(see Figure 2.23). Then we have that* $alt(K) = dalt(K) = n + r - 1$.

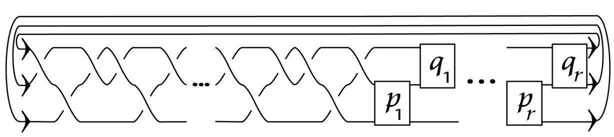


Figure 2.23: A family of closed 3-braids with $alt(L) = dalt(L)$.

Since, the torus knots $T(3,q)$ are closed 3-braids from Proposition 2.4.4 and Theorem 2.4.2 it follows that for many closed 3-braid the alternation number and the dealternating number are equal.

On the other hand, recently, Lowrance in [35] studied several alternating distances and demonstrated that there exist families of links for which the difference between certain alternating distances is arbitrarily large. In particular, he gave a family of satellite knots $\{W_n\}$, which consists of iterated Whitehead doubles of the figure-eight knot, such that for each positive integer n there exists a knot W_n such that $alt(W_n) = 1$ and $n \leq dalt(W_n)$. Each W_n is the satellite knot of the trivial knot twisted inside a torus, where the companion knot is the knot W_{n-1} and W_0 is the figure-eight knot, see Figure 2.24.

Theorem 2.4.3. *[35] For all* $n \in \mathbb{N}$ *there exist a knot* W_n *with* $alt(W_n) = 1$ *and* $dalt(W_n) \geq n$.

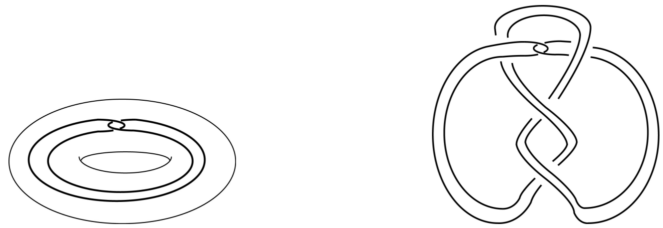


Figure 2.24: At the left the trivial knot twisted inside a torus and at the right W_1.

2.5 Knots and the 3-room R

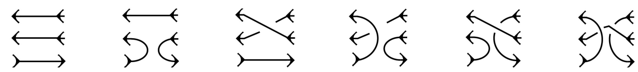
Similarly to the case of oriented links, the n-tangles can also be endowed with an orientation. An **oriented n-tangle** is an n-tangle (B^3, T) such that each connected component of T is oriented. In Figure 2.25 we show two diagrams of an unoriented 3-tangle, which are endowed with distinct orientations, one of the strands have the opposite orientation.

Figure 2.25: Diagrams of an oriented 3-tangle.

Following [22] we have that a **room** R' is a connected domain in $\mathbb{R}^2$ which possesses an equal number of oriented ingoing and outgoing strands. The **skein of a room**, $S(R')$, is the set of all collections of strands which connect ingoing to outgoing strands of the room. An **inhabitant** is an element of $S(R')$. An example of a room, which possesses three ingoing strands, and an inhabitant of this room are shown in Figure 2.26, hereinafter we will work with this room and we will denoted it only by R. In Figures 2.27, 2.29 (b) and (c) there are other inhabitants of $S(R)$. In fact, an inhabitant of the room $S(R)$ with the orientation induced by the ingoing and outgoing strands is an oriented 3-tangle.

Figure 2.26: The room R and an inhabitant.

In $S(R)$, there are six different forms to connect the ingoing strands with the outgoing strands of R. Each inhabitants of $S(R)$, denoted by χ_i for $i = 1, \ldots, 6$, in Figure 2.27 define one of these forms and has the minimal number of crossings to connect the ingoing and outgoing strands of R. This set of inhabitants, $\{\chi_i,\, i = 1, \ldots, 6\}$ will be called the **set of basic inhabitants of** $S(R)$.



(a) χ_1 (b) χ_2 (c) χ_3 (d) χ_4 (e) χ_5 (f) χ_6

Figure 2.27: Basic inhabitants of $S(R)$.

As above, the concatenation of inhabitants of $S(R)$, denoted by "$\cdot$", is a binary operation defined in $S(R)$, see Figure 2.28.

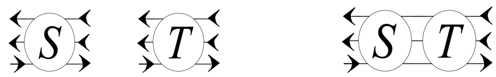
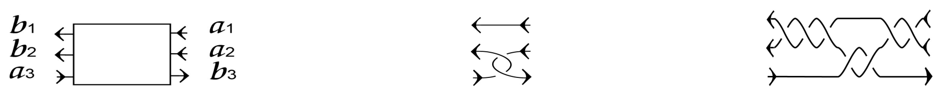


Figure 2.28: Inhabitants S and T in $S(R)$ and their concatenation $S \cdot T$.

By using the concatenation in $S(R)$, we will construct some links. In order to count the number of components of these links we will label the ingoing and outgoing strands of R. The ingoing strands of R are labelled with a_j and the outgoing strands are labelled with b_j, as shown in Figure 2.29 (a). Let be $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$.



(a) a_j and b_j. (b) 3-tangle c. (c) $\mathcal{T}(3,2,2)$.

Figure 2.29: The room R with labels and two inhabitants.

For each inhabitant T in $S(R)$ the connections, determined by the strands of T, between ingoing with outgoing strands determine a unique bijective function $\hat{T} : A \to B$. Note that there are only six different bijective functions from A to B, in particular, each basic inhabitant in Figure 2.27 determines a function $\hat{\chi}_i$. Some of the functions $\hat{\chi}_i$ are:

$$
\begin{array}{lll}
\hat{\chi}_1 : A \to B & \hat{\chi}_2 : A \to B & \hat{\chi}_3 : A \to B \\
\quad a_1 \mapsto b_1 & \quad a_1 \mapsto b_1 & \quad a_1 \mapsto b_2 \\
\quad a_2 \mapsto b_2 & \quad a_2 \mapsto b_3 & \quad a_2 \mapsto b_1 \\
\quad a_3 \mapsto b_3 & \quad a_3 \mapsto b_2 & \quad a_3 \mapsto b_3
\end{array}
$$

Then for each T in $S(R)$ there exists a unique basic elements, such that $\hat{T} = \hat{\chi}_i$. For instance the oriented 3-tangle c and the 3-braid $\mathcal{T}(3,2,2)$ shown in Figures 2.29 (b) and (c), respectively, satisfy $\hat{c} = \hat{\chi}_2$ and $\hat{\mathcal{T}}(3,2,2) = \hat{\chi}_3$. Moreover, if $T \in S(R)$ is a 3-braid then $\hat{T}(a_3) = b_3$ and hence $\hat{T} = \hat{\chi}_1$ or $\hat{T} = \hat{\chi}_3$.

Now, given T in $S(R)$ we define $N_j(T)$ as the link obtained from the closure N_j for $j = 1, \ldots, 2$, as in Figure 2.30. In particular, the closure N_2 is the type A closure endowed with the induced orientation of R.

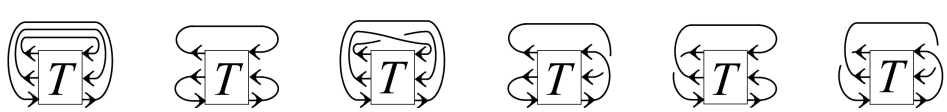


Figure 2.30: Closures N_j of a 3-tangle T in $S(R)$ for $j = 1, \ldots, 6$.

Note 2.5.1. *Note that for $j = 1, ..., 6$ we have that $N_j(T) \approx N_1(T \cdot \chi_j) \approx N_1(\chi_j \cdot T)$.*

The following results will be used to construct knot diagrams by using oriented 3-tangles T in $S(R)$ and the closure N_1.

Lemma 2.5.1. *Let B and C be oriented 3-tangles in $S(R)$ such that B is a 3-braid and $\hat{C} = \hat{\chi}_2$, then $N_1(B \cdot C)$ is a knot if, and only if, $\hat{B} = \hat{\chi}_3$.*

Proof. Suppose that $\hat{B} = \hat{\chi}_3$, since $\hat{C} = \hat{\chi}_2$, from here it is easy to see that the number of components of $N_1(B \cdot C)$ is equal to that of $N_1(\chi_3 \cdot \chi_2)$, which is one. On the other hand, if $B \in S(R)$ and $\hat{B} \neq \hat{\chi}_3$ then $\hat{B} = \hat{\chi}_1$, in this case as the number of components of $N_1(B \cdot C)$ is equal to that of $N_1(\chi_1 \cdot \chi_2)$, which is two, the conclusion follows. $\square$

Lemma 2.5.2. *Let be $B = \mathcal{T}(2a_1 + 1, 2a_2, 2a_3, \cdots, 2a_m) \cdot \mathcal{E}^{2k}$, with $a_1, a_2, \ldots, a_m, k \in \mathbb{Z}$, such that $B \in S(R)$ then $\hat{B} = \hat{\chi}_3$.*

Proof. The 3-braid B can be rewritten of the following form: if m is even $B = \mathcal{T}(2a_1 + 1) \cdot \mathcal{T}(0, 2a_2) \cdot \mathcal{T}(2a_3) \cdot \ldots \cdot \mathcal{T}(0, 2a_m) \cdot \mathcal{E}^{2k}$ and if m is odd $B = \mathcal{T}(2a_1 + 1) \cdot \mathcal{T}(0, 2a_2) \cdot \mathcal{T}(2a_3) \cdot \ldots \cdot \mathcal{T}(2a_m) \cdot \mathcal{E}^{2k}$. Then, since $\hat{\mathcal{T}}(2a_1 + 1) = \hat{\chi}_3$ and

$$\hat{\mathcal{T}}(2a_i) = \hat{\mathcal{T}}(0, 2a_i) = \hat{\mathcal{E}}^{2k} = \hat{\chi}_1,$$

it follows that $\hat{B} = \hat{\chi}_3$. $\square$

Theorem 2.5.1. *For all $a_1, a_2, \ldots, a_m, k \in \mathbb{Z}$ we have that $N_1(\mathcal{T}(2a_1 + 1, 2a_2, 2a_3, \cdots, 2a_m) \cdot \mathcal{E}^{2k} \cdot c)$ is a knot.*

Proof. Since Lemma 2.5.2 states that $\hat{\mathcal{T}}(2a_1 + 1, 2a_2, 2a_3, \cdots, 2a_m) = \chi_3$. Then by Lemma 2.5.1 we have that $N_1(\mathcal{T}(2a_1 + 1, 2a_2, 2a_3, \cdots, 2a_m) \cdot \mathcal{E}^{2k} \cdot c)$ is a knot. $\square$

Lemma 2.5.3. *Let B be an oriented 3-braid such that $B \in S(R)$ and $\hat{B} = \hat{\chi}_3$ then $N_2(B)$ is a 2-bridge knot or the trivial knot.*

Proof. Since $N_2(B) = N_1(B \cdot \chi_2)$, from Lemma 2.5.1 it follows that $N_2(B)$ is a knot. From Theorem 2.1.2 it follows that $N_2(B)$ is a 2-bridge knot or the trivial knot. $\square$

In the Chapter 3 we will give formulae to obtain the HOMFLY-PT polynomial and the Alexander polynomial of inhabitants of $S(R)$. After that, in Chapter 4 we will use these polynomials to prove some links are non-alternating.

Chapter 3

Formulae to obtain polynomials

In the present discussion we will deal with oriented 3-tangles in $S(R)$. The operation in $S(R)$ is the concatenation; it is shown in Figure 2.28. In the first section, we will give formulae to obtain the HOMFLY-PT polynomial of the concatenation of 3-tangles and of links obtained by the closure of them, in both cases the polynomial will be denoted by P. Further, if D is a diagram for L, then denote the HOMFLY-PT polynomial of L by either $P(L)$ or $P(D)$.

In the second and third section, we will specialize these formulae to the Conway polynomial and the Alexander polynomial.

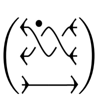
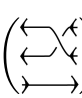
3.1 HOMFLY-PT polynomial

Let T be a 3-tangle diagram. The **HOMFLY-PT polynomial of a 3-tangle** T is obtained by applying to the diagram T the formulas which define the HOMFLY-PT polynomial repeatedly, until only the basic inhabitants are left, see Figure 2.27. Therefore,

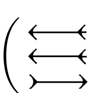
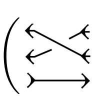
$$P(T) = \sum_{i=1}^{6} p_i \, P(\chi_i)\,, \text{ where } p_i \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}].$$

In the following examples, we will highlight the crossings where the skein relations for the HOMFLY-PT polynomial will be applied.

Example 9. *Calculate the HOMFLY-PT polynomial of $\mathcal{T}(2)$.*

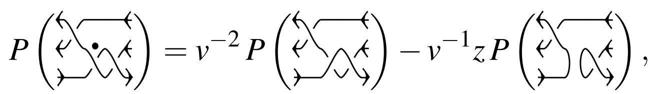
$P\Big($  $\Big) \;=\; v^2 P\Big($  $\Big) + vzP\Big($  $\Big)$,

Then, we have that

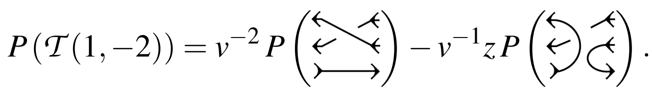
$P(\mathcal{T}(2)) \;=\; v^2 P\Big($  $\Big) + vzP\Big($  $\Big)$.

Hence, $P(\mathcal{T}(2)) = v^2P(\chi_1) + vzP(\chi_3)$. In this case $p_2 = p_4 = p_5 = p_6 = 0$, $p_1 = v^2$, and $p_3 = vz$.

Example 10. *Calculate the HOMFLY-PT polynomial of $\mathcal{T}(1,-2)$.*

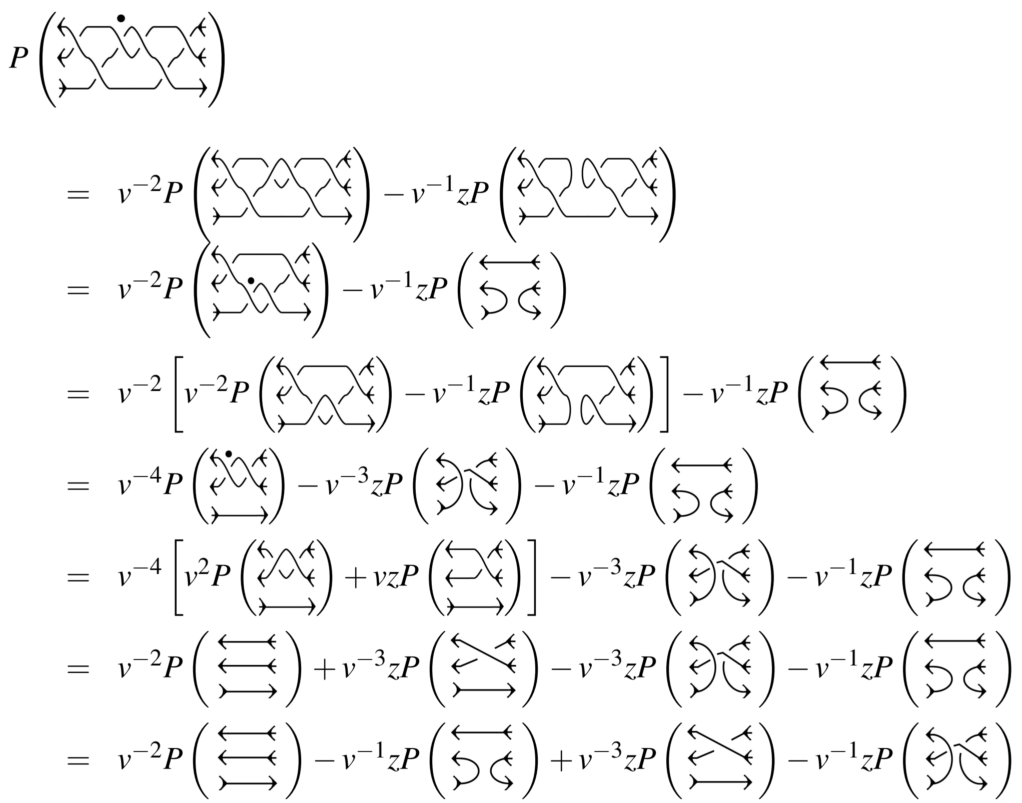


this implies that,



Then, $P(\mathcal{T}(2)) = v^{-2}P(\chi_3) - v^{-1}z(\chi_4)$.

Example 11. *Calculate the HOMFLY-PT polynomial of $\mathcal{E}^2$.*



Hence, $P(\mathcal{E}^2) = v^{-2}P(\chi_1) - v^{-1}zP(\chi_2) + v^{-3}zP(\chi_3) - v^{-1}zP(\chi_6)$.

Given two 3-tangles, we will give some formulae to calculate the HOMFLY-PT polynomial of the concatenation of them by using their HOMFLY-PT polynomials of each of them.

Theorem 3.1.1. *Let T_1 and T_2 be two 3-tangles such that*

$$P(T_1) = \sum_{i=1}^{6} p_i P(\chi_i) \ \text{ and } \ P(T_2) = \sum_{j=1}^{6} q_j P(\chi_j).$$

then, $P(T_1 \cdot T_2) =$

$$
\begin{aligned}
&[p_1q_1+p_3q_3v^2]P(\chi_1)\\
&+[p_1q_2+p_2(q_1+q_4+q_2\delta)+p_3q_4v^2+p_5(q_2+q_3v^2+(vz+v^2\delta)q_4)]P(\chi_2)\\
&+[p_1q_3+p_3q_1+vzp_3q_3]P(\chi_3)\\
&+[p_1q_4+p_3(q_2+vzq_4)+p_4(q_1+q_4+q_2\delta)+p_6(q_2+q_3v^2+(vz+v^2\delta)q_4)]P(\chi_4)\\
&+[p_1q_5+p_2(q_3+q_6+q_5\delta)+p_3q_6v^2+p_5(q_1+q_5+vzq_3+(vz+v^2\delta)q_6)]P(\chi_5)\\
&+[p_1q_6+p_3(q_5+vzq_6)+p_4(q_3+q_6+q_5\delta)+\\
&\quad p_6(q_1+q_5+vzq_3+(vz+v^2\delta)q_6)]P(\chi_6),
\end{aligned}
$$

where $\delta = \dfrac{v^{-1}-v}{z}$.

Proof. Given $T_1 \cdot T_2$, fix the 3-tangle T_2 and apply the skein relations for the HOMFLY-PT polynomial to T_1. Then

$$P(T_1 \cdot T_2) = \sum_{i=1}^{6} p_i\, P(\chi_i \cdot T_2).$$

Now, fix the 3-tangles χ_i and calculate the polynomial, then

$$\sum_{i=1}^{6} p_i\, P(\chi_i \cdot T_2) = \sum_{i=1}^{6} p_i \sum_{j=1}^{6} q_j\, P(\chi_i \cdot \chi_j).$$

After that, calculate $P(\chi_i \cdot \chi_j)$ for each pair i, j, the resultant expression can be simplified to the result. □

The HOMFLY-PT polynomial of the 3-tangle $\mathcal{E}^{2k}$, which is formed by k full twists, is given in Theorem 3.1.2. Also, Lemma 3.1.1 gives the HOMFLY-PT polynomial of some 3-braids. These results will be utilized to obtain $P(N_1(T \cdot \mathcal{E}^{2k}))$ and they are proved by induction.

Since $\delta = \dfrac{v^{-1}-v}{z}$ the following equations can be derived, and these will be used to calculate the HOMFLY-PT polynomial.

$$v^{-2} - zv^{-1}\delta = 1 \tag{3.1}$$

$$v^2 + zv\delta = 1 \tag{3.2}$$

Previously, in Example 11 we calculated $P(\mathcal{E}^2)$, now we will calculate $P(\mathcal{E}^{2k})$ for all k.

Theorem 3.1.2. *For all $k \in \mathbb{N}$ we have that $P(\mathcal{E}^{2k}) = \sum_{i=1}^{6} A_{i_k} P(\chi_i)$, where:*

$$
\begin{aligned}
A_{1_k} &= A_{1_{k-1}} v^{-2} + A_{3_{k-1}} v^{-1} z, && (3.3)\\
A_{2_k} &= [-A_{3_k} - (\delta + zv^{-1})(1 - A_{1_k})]/(1-\delta^2), && (3.4)\\
A_{3_k} &= A_{1_{k-1}} v^{-3} z + (1+z^2) v^{-2} A_{3_{k-1}}, && (3.5)\\
A_{4_k} &= [\delta A_{3_k} + v^{-2}(1 - A_{1_k})]/(1-\delta^2), && (3.6)\\
A_{5_k} &= [\delta A_{3_k} + v^{-2}(1 - A_{1_k})]/(1-\delta^2), && (3.7)\\
A_{6_k} &= -A_{3_k} - \delta[\delta A_{3_k} + v^{-2}(1 - A_{1_k})]/(1-\delta^2), && (3.8)
\end{aligned}
$$

with $A_{1_0} = 1$, $A_{3_0} = 0$, and $\delta = \dfrac{v^{-1}-v}{z}$.

Proof. By induction on k. For $k=1$, the polynomial $P\left(\mathcal{E}^2\right)$ has been obtained in Example 11:

$$P\left(\mathcal{E}^2\right) \quad = \quad v^{-2}P(\chi_1) - v^{-1}zP(\chi_2) + v^{-3}zP(\chi_3) - v^{-3}zP(\chi_6).$$

In this case $A_{1_1} = v^{-2}$, $A_{2_1} = -v^{-1}z$, $A_{3_1} = v^{-3}z$, $A_{4_1} = A_{5_1} = 0$ and $A_{6_1} = -v^{-3}z$.

These values satisfy that

$$\begin{aligned}
A_{1_1} &= v^{-2} \\
&= A_{1_0}v^{-2} + A_{3_0}v^{-1}z, \\
A_{2_1} &= -v^{-1}z \\
&= [-v^{-1}z](1-\delta^2)/(1-\delta^2) \\
&= [-v^{-1}z + v^{-1}z\delta^2]/(1-\delta^2) \\
&= [-v^{-1}z + \delta(v^{-1}z\delta)]/(1-\delta^2) \\
&= [-v^{-1}z - \delta(1-v^{-2})]/(1-\delta^2) \\
&= [-v^{-1}z - \delta(1-v^{-2}) + (v^{-3}z - v^{-3}z)]/(1-\delta^2) \\
&= [-v^{-3}z - \delta(1-v^{-2}) - v^{-1}z + v^{-3}z]/(1-\delta^2) \\
&= [-v^{-3}z - \delta(1-v^{-2}) - zv^{-1}(1-v^{-2})]/(1-\delta^2) \\
&= [-v^{-3}z - (\delta + zv^{-1})(1-v^{-2})]/(1-\delta^2) \\
&= [-A_{3_1} - (\delta + zv^{-1})(1-A_{1_1})]/(1-\delta^2), \\
A_{3_1} &= v^{-2} \\
&= A_{1_0}v^{-3}z + (1+z^2)v^{-2}A_{3_0}, \\
A_{4_1} &= 0 \\
&= [v^{-2}(1-1)]/(1-\delta^2) \\
&= [v^{-2}(1+(\delta v^{-1}z - v^{-2}))]/(1-\delta^2) \\
&= [\delta v^{-3}z + v^{-2}(1-v^{-2})]/(1-\delta^2) \\
&= [\delta A_{3_1} + v^{-2}(1-A_{1_1})]/(1-\delta^2), \\
A_{5_1} &= A_{4_1} = [\delta A_{3_1} + v^{-2}(1-A_{1_1})]/(1-\delta^2), \\
A_{6_1} &= -v^{-3}z \\
&= -v^{-3}z - [\delta v^{-2}(1-1)]/(1-\delta^2) \\
&= -v^{-3}z - [\delta v^{-2}(1+(\delta v^{-1}z - v^{-2}))]/(1-\delta^2) \\
&= -v^{-3}z - [\delta(\delta v^{-3}z + v^{-2}(1-v^{-2}))]/(1-\delta^2) \\
&= -A_{3_1} - [\delta(\delta A_{3_1} + v^{-2}(1-A_{1_1}))]/(1-\delta^2).
\end{aligned}$$

For k, by applying Theorem 3.1.1, we have that:

$P(\mathcal{E}^{2k}) = P(\mathcal{E}^{2(k-1)} \cdot \mathcal{E}^2) = \sum_{i=1}^{6} Q_{i_k}P(\chi_i)$ where

$$\begin{aligned}
Q_{1_k} &= A_{1_{k-1}}A_{1_1} + A_{3_{k-1}}A_{3_1}v^2, \\
Q_{2_k} &= A_{1_{k-1}}A_{2_1} + A_{2_{k-1}}(A_{1_1} + A_{4_1} + \delta A_{2_1}) + A_{3_{k-1}}A_{4_1}v^2 \\
&\quad + A_{5_{k-1}}(A_{2_1} + A_{3_1}v^2 + (vz + v^2\delta)A_{4_1}), \\
Q_{3_k} &= A_{1_{k-1}}A_{3_1} + A_{3_{k-1}}A_{1_1} + vzA_{3_{k-1}}A_{3_1}, \\
Q_{4_k} &= A_{1_{k-1}}A_{4_1} + A_{3_{k-1}}(A_{2_1} + vzA_{4_1}) + A_{4_{k-1}}(A_{1_1} + A_{4_1} + \delta A_{2_1}) \\
&\quad + A_{6_{k-1}}(A_{2_1} + v^2A_{3_1} + (vz + v^2\delta)A_{4_1}), \\
Q_{5_k} &= A_{1_{k-1}}A_{5_1} + A_{2_{k-1}}(A_{3_1} + A_{6_1} + \delta A_{5_1}) + A_{3_{k-1}}A_{6_1}v^2 \\
&\quad + A_{5_{k-1}}(A_{1_1} + A_{5_1} + vzA_{3_1} + (vz + v^2\delta)A_{6_1}), \\
Q_{6_k} &= A_{1_{k-1}}A_{6_1} + A_{3_{k-1}}(A_{5_1} + vzA_{6_1}) + A_{4_{k-1}}(A_{3_1} + A_{6_1} + \delta A_{5_1}) \\
&\quad + A_{6_{k-1}}(A_{1_1} + A_{5_1} + vzA_{3_1} + (vz + v^2\delta)A_{6_1}).
\end{aligned}$$

After replacing the values A_{r_i} for $i = 1,...,6$ in each Q_{i_k}, we have that

$$
\begin{aligned}
Q_{1_k} &= A_{1_{k-1}} v^{-2} + A_{3_{k-1}} v^{-3} z v^2,\\
Q_{2_k} &= -v^{-1} z A_{1_{k-1}} + A_{2_{k-1}} (v^{-2} - v^{-1} z\delta) + A_{5_{k-1}} (-v^{-1} z + v^{-3} z v^2),\\
Q_{3_k} &= A_{1_{k-1}} v^{-3} z + A_{3_{k-1}} v^{-2} + v z A_{3_{k-1}} v^{-3} z,\\
Q_{4_k} &= A_{3_{k-1}} (-v^{-1} z) + A_{4_{k-1}} (v^{-2} - v^{-1} z\delta) + A_{6_{k-1}} (-v^{-1} z + v^2 v^{-3} z),\\
Q_{5_k} &= A_{2_{k-1}} (v^{-3} z - v^{-3} z) - v^{-3} z v^2 A_{3_{k-1}} + A_{5_{k-1}} (v^{-2} + v z v^{-3} z) - v^{-3} z (v z + v^2 \delta),\\
Q_{6_k} &= -v^{-3} z A_{1_{k-1}} + A_{3_{k-1}} (-v^{-3} z v z) + A_{4_{k-1}} (v^{-3} z - v^{-3} z)\\
&\quad + A_{6_{k-1}} (v^{-2} + v z v^{-3} z - v^{-3} z (v z + v^2 \delta)).
\end{aligned}
$$

Then, by using (3.1) for $i = 2,4,5,6$, simplify Q_{i_k}.

$$
\begin{aligned}
Q_{1_k} &= A_{1_{k-1}} v^{-2} + A_{3_{k-1}} v^{-1} z, & Q_{4_k} &= -v^{-1} z A_{3_{k-1}} + A_{4_{k-1}},\\
Q_{2_k} &= -v^{-1} z A_{1_{k-1}} + A_{2_{k-1}}, & Q_{5_k} &= -v^{-1} z A_{3_{k-1}} + A_{5_{k-1}},\\
Q_{3_k} &= A_{1_{k-1}} v^{-3} z + A_{3_{k-1}} v^{-2} (1 + z^2), & Q_{6_k} &= -v^{-3} z A_{1_{k-1}} - v^{-2} z^2 A_{3_{k-1}} + A_{6_{k-1}}.
\end{aligned}
$$

Comparing Q_{i_k} and A_{i_k} it easy to see that $Q_{i_k} = A_{i_k}$ for $i = 1,3$. In order to obtain the equalities for $i = 2,4,5,6$ we use the inductive hypothesis and (3.1) as follows.

$$
\begin{aligned}
Q_{2_k} &= -v^{-1} z A_{1_{k-1}} + A_{2_{k-1}}\\
&= -v^{-1} z A_{1_{k-1}} + [-A_{3_{k-1}} - (\delta + z v^{-1})(1 - A_{1_{k-1}})]/(1 - \delta^2)\\
&= [-v^{-1} z A_{1_{k-1}} + v^{-1} z \delta^2 A_{1_{k-1}} - A_{3_{k-1}} - (\delta + z v^{-1}) + (\delta + z v^{-1}) A_{1_{k-1}}]/(1 - \delta^2)\\
&= [-v^{-1} z A_{1_{k-1}} (v^{-2} - v^{-1} z\delta) + v^{-1} z \delta A_{1_{k-1}} - A_{3_{k-1}} [(v^{-2} - v^{-1} z\delta) + t^{-2} z^2 - t^{-2} z^2]\\
&\quad - (\delta + z v^{-1}) + (\delta + z v^{-1}) A_{1_{k-1}} (v^{-2} - v^{-1} z\delta)]/(1 - \delta^2)\\
&= [-(v^{-3} z A_{1_{k-1}} + A_{3_{k-1}} v^{-2} (1 + z^2)) - (1 - (A_{1_{k-1}} v^{-2}\\
&\quad + A_{3_{k-1}} v^{-1} z))(\delta + z v^{-1})]/(1 - \delta^2)\\
&= [-(v^{-3} z A_{1_{k-1}} + A_{3_{k-1}} v^{-2} - A_{3_{k-1}} v^{-2} z^2) - ((\delta + z v^{-1}) - (A_{1_{k-1}} v^{-2} (\delta + z v^{-1})\\
&\quad + A_{3_{k-1}} v^{-1} z\delta + A_{3_{k-1}} v^{-2} z^2)]/(1 - \delta^2)\\
&= [-(A_{3_k}) - (1 - (A_{1_k}))(\delta + z v^{-1})]/(1 - \delta^2)\\
&= A_{2_k},
\end{aligned}
$$

$$
\begin{aligned}
Q_{4_k} &= -v^{-1} z A_{3_{k-1}} + A_{4_{k-1}}\\
&= -v^{-1} z A_{3_{k-1}} + [\delta A_{3_{k-1}} + v^{-2} (1 - A_{1_{k-1}})]/(1 - \delta^2)\\
&= (-v^{-1} z A_{3_{k-1}} + \delta^2 v^{-1} z A_{3_{k-1}} + (\delta A_{3_{k-1}} + v^{-2} (1 - A_{1_{k-1}}))/(1 - \delta^2)\\
&= (-v^{-1} z A_{3_{k-1}} (v^{-2} - v^{-1} z\delta) + \delta^2 v^{-1} z A_{3_{k-1}}\\
&\quad + (\delta A_{3_{k-1}} + v^{-2} (1 - A_{1_{k-1}}) (v^{-2} - v^{-1} z\delta)))/(1 - \delta^2)\\
&= (-v^{-3} z A_{3_{k-1}} + v^{-2} z^2 \delta A_{3_{k-1}} + \delta A_{3_{k-1}} v^{-2} + v^{-2} (v^{-2} - v^{-1} z\delta)\\
&\quad - A_{1_{k-1}} v^{-4} + A_{1_{k-1}} v^{-3} z\delta)/(1 - \delta^2)\\
&= (\delta (A_{1_{k-1}} v^{-3} z + A_{3_{k-1}} v^{-2} (1 + z^2)) + v^{-2} (1 - (A_{1_{k-1}} v^{-2} + A_{3_{k-1}} v^{-1} z)))/(1 - \delta^2)\\
&= (\delta A_{3_k} + v^2 (1 - A_{1_k}))/(1 - \delta^2)\\
&= A_{4_k},
\end{aligned}
$$

$$
\begin{aligned}
Q_{5_k} &= -v^{-1} z A_{3_{k-1}} + A_{5_{k-1}}\\
&= -v^{-1} z A_{3_{k-1}} + A_{5_{k-1}}\\
&= -v^{-1} z A_{3_{k-1}} + A_{4_{k-1}}\\
&= A_{4_k},
\end{aligned}
$$

$$
\begin{aligned}
Q_{6_k} &= -v^{-3}zA_{1_{k-1}} - v^{-2}z^2A_{3_{k-1}} + A_{6_{k-1}} \\
&= -v^{-3}zA_{1_{k-1}} - v^{-2}z^2A_{3_{k-1}} + (-A_{3_{k-1}} - \delta[\delta A_{3_{k-1}} + v^{-2}(1-A_{1_{k-1}})]/(1-\delta^2)) \\
&= -v^{-3}zA_{1_{k-1}} - v^{-2}z^2A_{3_{k-1}} - A_{3_{k-1}}(v^{-2} - v^{-1}z\delta) \\
&\quad -\delta[\delta A_{3_{k-1}} + v^{-2}(1-A_{1_{k-1}})]/(1-\delta^2) \\
&= ((-v^{-3}zA_{1_{k-1}} - v^{-2}z^2A_{3_{k-1}} - A_{3_{k-1}}v^{-2}) + A_{3_{k-1}}v^{-1}z\delta) \\
&\quad -\delta[\delta A_{3_{k-1}} + v^{-2}(1-A_{1_{k-1}})]/(1-\delta^2) \\
&= -A_{3_k} + A_{3_{k-1}}v^{-1}z\delta - \delta[\delta A_{3_{k-1}} + v^{-2}(1-A_{1_{k-1}})]/(1-\delta^2) \\
&= -A_{3_k} - \delta[-A_{3_{k-1}}v^{-1}z(1-\delta^2) + (\delta A_{3_{k-1}} + v^{-2}(1-A_{1_{k-1}}))]/(1-\delta^2) \\
&= -A_{3_k} - \delta[A_{3_{k-1}}v^{-1}z\delta^2 + (v^{-2} - v^{-1}z\delta)(-A_{3_{k-1}}v^{-1}z + A_{3_{k-1}}\delta \\
&\quad + v^{-2}(1-A_{1_{k-1}}))]/(1-\delta^2) \\
&= -A_{3_k} - \delta[\delta(A_{1_{k-1}}v^{-3}z + A_{3_{k-1}}v^{-2}(1+z^2)) + v^{-2}((v^{-2} - v^{-1}z\delta) \\
&\quad -(A_{1_{k-1}}v^{-2} + A_{3_{k-1}}zv^{-1}))]/(1-\delta^2) \\
&= -A_{3_k} - \delta[\delta(A_{3_k}) + v^{-2}(1-A_{1_k})]/(1-\delta^2) \\
&= A_{6_k}.
\end{aligned}
$$

□

The polynomials A_{i_k} in $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ for $k = 2,4,5,6$ are linear combinations of A_{1_k} and A_{3_k}, which are recursive. These polynomials will be used to obtain formulae to the HOMFLY-PT polynomial of others 3-braids. Further, from (3.3) and (3.5) we obtain that

$$
A_{3_{k+1}} = A_{1_{k+1}}v^{-1}z - A_{3_k}v^{-2} \tag{3.9}
$$

Lemma 3.1.1. *For all $t \in \mathbb{N}$ we have that*

i) $P(\mathcal{T}(2t)) = A_{1_t}v^{4t}P(\chi_1) + A_{3_t}v^{4t}P(\chi_3)$,

ii) $P(\mathcal{T}(2t+1)) = A_{3_t}v^{4t+2}P(\chi_1) + A_{1_{t+1}}v^{4t+2}P(\chi_3)$,

iii) $P(\mathcal{T}(-2t)) = A_{1_{t+1}}v^2P(\chi_1) - A_{3_t}P(\chi_3)$,

iv) $P(\mathcal{T}(-(2t+1))) = -A_{3_t}v^2P(\chi_1) + A_{1_{t+1}}P(\chi_3)$,

v) $P(\mathcal{T}(0,2t)) = v^{2t}P(\chi_1) + z\sum_{i=1}^{t}v^{2i-1}P(\chi_2)$,

vi) $P(\mathcal{T}(0,-2t)) = v^{-2t}P(\chi_1) - z\sum_{i=1}^{t}v^{-(2i-1)}P(\chi_2)$.

Proof.

i) By induction on t. For $t = 1$, this case has been calculate in Example 9: $P(\mathcal{T}(2)) = v^2P(\chi_1) + vzP(\chi_3)$. Since $A_{1_1} = v^{-2}$ and $A_{3_1} = v^{-3}z$, then $P(\mathcal{T}(2)) = A_{1_1}v^4P(\chi_1) + A_{3_1}v^4P(\chi_3)$.
We calculate for t by using Theorem 3.1.1.

Let be $P(\mathcal{T}(2t-2)) = p_1P(\chi_1) + p_2P(\chi_2) + p_3P(\chi_3) + p_4P(\chi_4) + p_5P(\chi_5) + p_6P(\chi_6)$, so

$$
\begin{aligned}
P(\mathcal{T}(2t-2)\cdot\mathcal{T}(2)) \quad = \quad & (p_1v^2 + v^3zp_3)P(\chi_1) + (p_2v^2 + v^3zp_5)P(\chi_2) \\
& +(p_1vz + p_3v^2 + v^2z^2p_3)P(\chi_3) + (p_4v^2 + p_6v^3z)P(\chi_4) \\
& +(p_2vz + p_5v^2 + v^2z^2p_5)P(\chi_5) \\
& +(p_4vz + p_6v^{-2} + v^2z^2p_6)P(\chi_6).
\end{aligned}
$$

By inductive hypothesis $P(\mathcal{T}(2t-2)) = A_{1_{t-1}}v^{4(t-1)}P(\chi_1)+A_{3_{t-1}}v^{4(t-1)}P(\chi_3)$, so $p_2 = p_4 = p_5 = p_6 = 0$. Replacing each value of p_i for $i = 1,...,6$ we have that

$$\begin{aligned}
P(\mathcal{T}(2t-2)\cdot\mathcal{T}(2)) &= [(A_{1_{t-1}})v^{4(t-1)}v^2+v^3z(A_{3_{t-1}}v^{4(t-1)})]P(\chi_1) \\
&\quad +[(A_{1_{t-1}}v^{4(t-1)})vz+(A_{3_{t-1}}v^{4(t-1)})v^2(1+z^2)]P(\chi_3) \\
&= v^{4t}[A_{1_{t-1}}v^{-2}+v^{-1}zA_{3_{t-1}}]P(\chi_1) \\
&\quad +v^{4t}[A_{1_{t-1}}v^{-3}z+A_{3_{t-1}}v^{-2}(1+z^2)]P(\chi_3).
\end{aligned}$$

Due to equations (3.3) and (3.5) we have that $P(\mathcal{T}(2(t-1))\cdot\mathcal{T}(2)) = v^{4t}A_{1_t}P(\chi_1)+v^{4t}A_{3_t}P(\chi_3)$, hence $P(\mathcal{T}(2t)) = A_{1_t}v^{4t}P(\chi_1)+A_{3_t}v^{4t}P(\chi_3)$.

ii) By Theorem 3.1.1, the result in *i)* and (3.3) we obtain

$$\begin{aligned}
P(\mathcal{T}(2t+1)) &= P(\mathcal{T}(2t)\cdot\mathcal{T}(1)) \\
&= A_{3_t}v^{4t}v^2P(\chi_1)+(A_{1_t}v^{4t}+vzA_{3_t}v^{4t})P(\chi_3) \\
&= v^{4t+2}A_{3_t}P(\chi_1)+v^{4t+2}(A_{1_t}v^{-2}+zA_{3_t}v^{-1})P(\chi_3) \\
&= v^{4t+2}A_{3_t}P(\chi_1)+v^{4t+2}A_{1_{t+1}}P(\chi_3).
\end{aligned}$$

iii) By induction on t. For $t = 1$, $P(\mathcal{T}(-2)) = v^{-2}(1+z^2)P(\chi_1)-v^{-3}zP(\chi_3)$. Since $A_{1_2} = v^{-4}(1+z^2)$ and $A_{3_1} = v^{-3}z$, then $P(\mathcal{T}(-2)) = A_{1_2}v^2P(\chi_1)-A_{3_1}P(\chi_3)$.
We calculate for t by using Theorem 3.1.1,inductive hypothesis $P(\mathcal{T}(-2(t-1))) = A_{1_t}v^2P(\chi_1)-A_{3_{t-1}}P(\chi_3)$, and (3.9). Then

$$\begin{aligned}
P(\mathcal{T}(-2t)) &= P(\mathcal{T}(-2(t-1))\cdot\mathcal{T}(-2)) \\
&= [v^2A_{1_t}v^{-2}(1+z^2)+(-A_{3_{t-1}})(-v^{-3}z)v^2]P(\chi_1) \\
&\quad +[v^2A_{1_t}(-v^{-3}z)-A_{3_{t-1}}v^{-2}]P(\chi_3) \\
&= v^2[A_{1_t}v^{-2}(1+z^2)+A_{3_{t-1}}v^{-3}z]P(\chi_1) \\
&\quad -[A_{1_t}v^{-1}z+A_{3_{t-1}}v^{-2}]P(\chi_3) \\
&= v^2[v^{-1}z(A_{1_t}v^{-1}z+A_{3_{t-1}}v^{-2})+A_{1_t}v^{-2}]P(\chi_1) \\
&\quad -[A_{3_t}]P(\chi_3), \\
&= v^2[v^{-1}z(A_{3_t})+A_{1_t}v^{-2}]P(\chi_1)-A_{3_t}P(\chi_3) \\
&= v^2A_{1_{t+1}}P(\chi_1)-A_{3_t}P(\chi_3).
\end{aligned}$$

iv) By Theorem 3.1.1, the result in *iii)* and (3.9) we obtain

$$\begin{aligned}
P(\mathcal{T}(-(2t+1))) &= P(\mathcal{T}(-2t)\cdot\mathcal{T}(-1)) \\
&= (v^2A_{1_{t+1}}(-v^{-1}z)+(-A_{3_t})v^{-2}v^2)P(\chi_1) \\
&\quad +(v^2A_{1_{t+1}}v^{-2}+(-A_{3_t})(-v^{-1}z)+(-A_{3_t}(-v^{-1}z)))P(\chi_3) \\
&= -v^2(A_{1_{t+1}}v^{-1}z-A_{3_t}v^{-2})P(\chi_1)+A_{1_{t+1}}P(\chi_3) \\
&= -v^2A_{3_{t+1}}P(\chi_1)+A_{1_{t+1}}P(\chi_3).
\end{aligned}$$

v) By induction on t. For $t = 1$, $P(\mathcal{T}(0,2)) = v^2P(\chi_1)+vzP(\chi_2)$, then $P(\mathcal{T}(0,2)) = v^2P(\chi_1)+z\sum_{i=1}^{1}v^{2t-1}P(\chi_2)$.
We calculate for t by using Theorem 3.1.1 and (3.2).

$$
\begin{aligned}
P(\mathcal{T}(0,2t)) &= P(\mathcal{T}(0,2(t-1))\cdot\mathcal{T}(0,2)) \\
&= v^{2(t-1)}v^2P(\chi_1)+(v^{2(t-1)}zv+z\sum_{i=1}^{t-1}v^{2i-1}v^2+z\sum_{i=1}^{t-1}v^{2i-1}zv\delta)P(\chi_2) \\
&= v^{2t}P(\chi_1)+(v^{2t-1}z+z\sum_{i=1}^{t-1}v^{2i-1}(v^2+zv\delta))P(\chi_2) \\
&= v^{2t}P(\chi_1)+(v^{2t-1}z+z\sum_{i=1}^{t-1}v^{2i-1})P(\chi_2) \\
&= v^{2t}P(\chi_1)+z\sum_{i=1}^{t}v^{2i-1}P(\chi_2).
\end{aligned}
$$

vi) By induction on t. For $t=1$, $P(\mathcal{T}(0,-2))=v^{-2}P(\chi_1)-v^{-1}zP(\chi_2)$, so, $P(\mathcal{T}(0,-2))=v^{-2}P(\chi_1)-z\sum_{i=1}^{1}v^{-(2t-1)}P(\chi_2)$.

We calculate for t by using Theorem 3.1.1 and (3.1).

$$
\begin{aligned}
P(\mathcal{T}(0,-2t)) &= P(\mathcal{T}(0,-2(t-1))\cdot\mathcal{T}(0,-2)) \\
&= v^{-2(t-1)}v^{-2}P(\chi_1)+(v^{-2(t-1)}(-zv^{-1})-z\sum_{i=1}^{t-1}v^{-(2i-1)}v^{-2} \\
&\quad -z\sum_{i=1}^{t-1}v^{-(2i-1)}(-zv^{-1}\delta))P(\chi_2) \\
&= v^{-2t}P(\chi_1)+(-v^{-(2t-1)}z-z\sum_{i=1}^{t-1}v^{-(2i-1)}(v^{-2}-zv^{-1}\delta))P(\chi_2) \\
&= v^{-2t}P(\chi_1)+(-v^{-(2t-1)}z-z\sum_{i=1}^{t-1}v^{-(2i-1)})P(\chi_2) \\
&= v^{-2t}P(\chi_1)-z\sum_{i=1}^{t}v^{-(2i-1)}P(\chi_2).
\end{aligned}
$$

□

Up to this point we have obtained the HOMFlY-PT polynomial of 3-tangles. Now, given the polynomial of a 3-tangle T, we will give formulae to obtain the HOMFLY-PT polynomial for six different closures of the 3-tangle T.

Lemma 3.1.2. *Let T be a 3-tangle, if $P(T)=\sum_{i=1}^{6}p_iP(\chi_i)$ then*

$$
\begin{aligned}
&P(N_1(T))=\delta^2p_1+\delta p_2+\delta p_3+p_4+p_5+(v^2\delta+vz)p_6,\\
&P(N_2(T))=\delta p_1+\delta^2p_2+p_3+\delta p_4+\delta p_5+p_6,\\
&P(N_3(T))=\delta p_1+p_2+(v^2\delta^2+vz\delta)p_3+(v^2\delta+vz)p_4+(v^2\delta+vz)p_5\\
&\qquad +(v^2+v^3z\delta+v^2z^2)p_6,\\
&P(N_4(T))=p_1+\delta p_2+(v^2\delta+vz)p_3+p_4+(v^2\delta^2+vz\delta)p_5+(v^2\delta+vz)p_6,\\
&P(N_5(T))=p_1+\delta p_2+(v^2\delta+vz)p_3+(v^2\delta^2+vz\delta)p_4+p_5+(v^2\delta+vz)p_6,\\
&P(N_6(T))=(v^2\delta+vz)p_1+p_2+(v^2+vz(v^2\delta+vz))p_3+(v^2\delta+vz)p_4+(v^2\delta+vz)p_5\\
&\qquad +(v^2(v^2\delta^2+vz\delta)+vz(v^2\delta+vz))p_6.
\end{aligned}
$$

Proof. Suppose that $P(T)=\sum_{i=1}^{6}p_iP(\chi_i)$. Since $P(N_j(T))=P(N_1(T\cdot\chi_j))$ for $j=1,\ldots,6$, then $P(N_j(T))=\sum_{i=1}^{6}p_iP(N_1(\chi_i\cdot\chi_j))=\sum_{i=1}^{6}p_iP(N_j(\chi_i))$.

We calculate $P(N_1(T))$, the other $P(N_j(T))$ cases are analogous.

$N_1(\chi_1) = U \sqcup U \sqcup U$, $N_1(\chi_2) = N_1(\chi_3) = U \sqcup U$, $N_1(\chi_4) = N_1(\chi_5) = U$, and $N_1(\chi_6)$ is the Hopf link (see Figure 3.1).

Then, by relations of HOMFLY-PT polynomial,

$P(N_1(\chi_1)) = \delta^2$, $P(N_1(\chi_2)) = P(N_1(\chi_3)) = \delta$, $P(N_1(\chi_4)) = P(N_1(\chi_5)) = 1$, and $P(N_1(\chi_6)) = v^2\delta + vz$.

This implies the result. $\square$

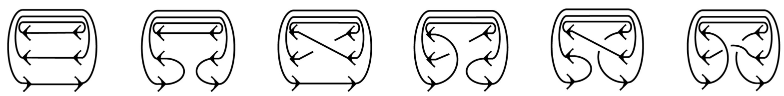


Figure 3.1: Link diagrams $N_1(\chi_i)$, for $i = 1, \ldots, 6$.

Corollary 3.1.1 states formulae to obtain $P(N_1(E^{2k}))$ and $P(N_3(E^{2k}))$, by using the polynomials A_{1_k} and A_{3_k}.

Corollary 3.1.1. *$P(N_1(\mathcal{E}^{2k})) = -(A_{1_k} + A_{1_{k+1}} v^2) + \delta^2 A_{1_k} + \delta A_{3_k} + 2$ and*
$P(N_3(\mathcal{E}^{2k})) = -(A_{3_k} + A_{3_{k+1}})v^2 + \delta^2(1+z^2)A_{1_k} + (3tz\delta + z^3\delta + \delta^2 v^2)A_{3_k} - (\delta z^2 - zv^{-1})$.

Proof. Lemma 3.1.2 and Theorem 3.1.2 implies that:

$P(N_1(\mathcal{E}^2)) = \delta^2 A_{1_k} + \delta[-A_{3_k} - (\delta + zv^{-1})(1 - A_{1_k})]/(1-\delta^2) + \delta A_{3_k}$
$+2[\delta A_{3_k} + v^{-2}(1 - A_{1_k})]/(1-\delta^2) + (v^2\delta + vz)(-A_{3_k} - \delta[\delta A_{3_k} + v^{-2}(1 - A_{1_k})]/(1-\delta^2))$,
and

$P(N_3(\mathcal{E}^2)) = \delta A_{1_k} + [-A_{3_k} - (\delta + zv^{-1})(1 - A_{1_k})]/(1-\delta^2) + (v^2\delta^2 + vz\delta)A_{3_k}$
$+2(v^2\delta + vz)[\delta A_{3_k} + v^{-2}(1 - A_{1_k})]/(1-\delta^2) + (v^2 + v^3 z\delta + v^2 z^2)\left(-A_{3_k}\right.$
$\left.-\delta[\delta A_{3_k} + v^{-2}(1 - A_{1_k})]\right)/(1-\delta^2)$.

A long but otherwise straightforward computation leads to the result. $\square$

Theorem 3.1.3. *Let T_1 and T_2 be two 3-tangles, if*

$$P(T_1) = \sum_{i=1}^{6} p_i P(\chi_i) \text{ and } P(T_2) = \sum_{i=1}^{6} q_i P(\chi_i)$$

then

$$P(N_1(T_1 \cdot T_2)) = \sum_{i=1}^{6} p_i P(N_i(T_2)) = \sum_{i=1}^{6} q_i P(N_i(T_1)).$$

Proof. Fix the 3-tangle T_2. After to apply the HOMFLY-PT formulae over the 3-tangle T_1 we obtain that

$$P(N_1(T_1 \cdot T_2)) = \sum_{i=1}^{6} p_i P(N_1(\chi_i \cdot T_2)),$$

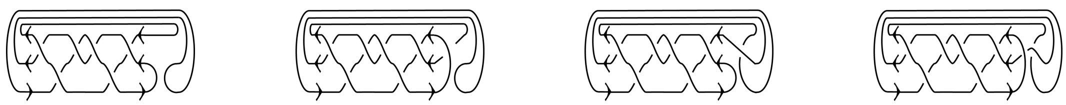
since $N_1(\chi_i \cdot T_2) = N_i(T_2)$ the first equality follows. The proof for the second equality is analogous. $\square$

Given a 3-tangle T, we will construct links of the form $N_1(T \cdot \mathcal{E}^{2k})$. Therefore it is important to have formulae to compute $P(T \cdot \mathcal{E}^{2k})$.

Corollary 3.1.2. *Let T be a 3-tangle. If $P(T) = \sum_{i=1}^{6} p_i P(\chi_i)$ and $k \in \mathbb{N}$, then*

$$P(N_1(T \cdot \mathcal{E}^{2k})) = p_1 P(N_1(\mathcal{E}^{2k})) + p_2\delta + p_3 P(N_3(\mathcal{E}^{2k})) + p_4 + p_5 + p_6(v^2\delta + vz). \quad (3.10)$$

Proof. By using diagrams, it easy to see that $N_2(\mathcal{E}^2) = U \sqcup U$, $N_4(\mathcal{E}^2)$ and $N_5(\mathcal{E}^{2k})$ are the trivial knot, and $N_6(\mathcal{E}^2)$ is the Hopf link (see Figure 3.2), so for all k in $\mathbb{N}$, $N_2(\mathcal{E}^{2k}) = U \sqcup U$, $N_4(\mathcal{E}^{2k})$ and $N_5(\mathcal{E}^{2k})$ are the trivial knot, and $N_6(\mathcal{E}^{2k})$ is the Hopf link. Then $P(N_2(\mathcal{E}^{2k})) = \delta$, $P(N_4(\mathcal{E}^{2k})) = P(N_5(\mathcal{E}^{2k})) = 1$, and $P(N_6(\mathcal{E}^{2k})) = v^2\delta + vz$, so by Theorem 3.1.2 we obtain the result. □



$N_2(\mathcal{E}^2)$ $N_4(\mathcal{E}^2)$ $N_5(\mathcal{E}^2)$ $N_6(\mathcal{E}^2)$

Figure 3.2: $N_2(\mathcal{E}^2), N_4(\mathcal{E}^2), N_5(\mathcal{E}^2)$, and $N_6(\mathcal{E}^2)$, respectively.

The previous results hold for any 3-tangle T; however, in general we will use Corollary 3.1.2 where T is the concatenation of a 3-braid and the 3-tangle c. In order to clarify this construction we show the diagrams in Figure 3.3, the first three diagrams have $k = 0$ and the last one has $k = 1$.

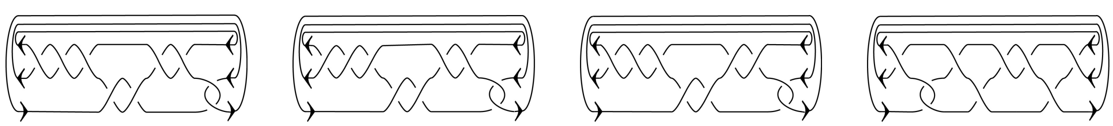


Figure 3.3: $N_1(\mathcal{T}(3,2,2)\cdot c), N_1(\mathcal{T}(-3,2,2)\cdot c), N_1(\mathcal{T}(3,2,-2)\cdot c)$ and $N_1(\mathcal{T}(1)\cdot c \cdot \mathcal{E}^2)$.

We define α_i in $\mathbb{Z}[z]$ for $i = 1,3$ as (3.11) and (3.12), respectively. These polynomials will be useful to obtain the Conway polynomial, which we will see the next section.

$$\alpha_{1_k} = z\alpha_{3_{k-1}} + \alpha_{1_{k-1}}, \quad (3.11)$$
$$\alpha_{3_k} = z\alpha_{1_{k-1}} + (1+z^2)\alpha_{3_{k-1}}. \quad (3.12)$$

Where $\alpha_{1_0} = 1$ and $\alpha_{3_0} = 0$.

The polynomials A_{1_k} and A_{3_k} can be factorized as the product of a α_{i_k} and a monomial in the variable v.

Lemma 3.1.3. *For all $k \in \mathbb{N}$ we have that $A_{1_k} = v^{-2k}\alpha_{1_k}$ and $A_{3_k} = v^{-(2k+1)}\alpha_{3_k}$, where $\alpha_i \in \mathbb{Z}[z]$ are described in the equations (3.11) and (3.12).*

Proof. By induction over k. For $k = 1$, $A_{1_1} = v^{-2}$ and $A_{3_1} = v^{-(2+1)}z$, since $\alpha_{1_1} = 1$ and $\alpha_{3_1} = 0$, it is possible to rewrite A_{1_1} and A_{3_1} as follows. $A_{1_1} = v^{-2}\alpha_{1_1}$ and $A_{3_1} = v^{-(2+1)}\alpha_{3_1}$.

For $k = n+1$, we assume $A_{1_n} = v^{-2n}\alpha_{1_n}$ and $A_{3_n} = v^{-(2n+1)}\alpha_{3_n}$.

$$\begin{aligned}
A_{1_{n+1}} &= A_{1_n}v^{-2} + A_{3_n}v^{-1}z \\
&= (v^{-2n}\alpha_{1_n})v^{-2} + (v^{-(2n+1)}\alpha_{3_n})v^{-1}z \\
&= v^{-2(n+1)}\alpha_{1_n} + v^{-2(n+1)}z\alpha_{3_n} \\
&= v^{-2(n+1)}(\alpha_{1_n} + z\alpha_{3_n}) \\
&= v^{-2(n+1)}\alpha_{1_{n+1}}. \\
A_{3_{n+1}} &= A_{1_n}v^{-3}z + A_{3_n}v^{-2}(1+z^2) \\
&= (v^{-2n}\alpha_{1_n})v^{-3}z + (v^{-(2n+1)}\alpha_{3_n})v^{-2}(1+z^2) \\
&= v^{-(2(n+1)+1)}\alpha_{1_n}z + v^{-(2(n+1)+1)}\alpha_{3_n}(1+z^2) \\
&= v^{-(2(n+1)+1)}(\alpha_{1_n}z + \alpha_{3_n}(1+z^2)) \\
&= v^{-(2(n+1)+1)}\alpha_{3_{n+1}}.
\end{aligned}$$

□

In the following section we will deal with the Conway polynomial and we will use the polynomials α_{1_k} and α_{3_k} instead of A_{1_k} and A_{3_k}.

3.2 Conway polynomial

In this section it will be computed the Conway polynomials of oriented 3-tangles, as well as of knots or links obtained from some closures of these 3-tangles and, in both cases the polynomial will be denoted by ∇.

The Conway polynomial $\nabla(L;z) \in \mathbb{Z}[z]$ can be obtained from the HOMFLY-PT polynomial via the following variable changes $\nabla(L;z) = P(L;1,z)$. In particular, (3.10) takes the following form:

$$\nabla(N_1(T \cdot \mathcal{E}^{2k});z) = p_1\nabla(N_1(\mathcal{E}^{2k});z) + p_3\nabla(N_3(\mathcal{E}^{2k});z) + \nabla(N_1(T);z). \tag{3.13}$$

Furthermore, the polynomials described at (3.3) and (3.5) are simplified to those described at (3.11) and (3.12), respectively. Therefore, the results in this section are the Conway version to the ones given in the previous section. Some of these formulae partially were obtained in [25], which are extended in this work.

Similarly to Theorem 3.1.1, in order to calculate the Conway polynomial of the product $T_1 \cdot T_2$ of two 3-tangles T_1 and T_2 we have the following theorem:

Theorem 3.2.1. *[25] Let T_1 and T_2 be two 3-tangles, such that*

$$\nabla(T_1) = \sum_{i=1}^{6} p_i\nabla(\chi_i) \text{ and } \nabla(T_2) = \sum_{i=1}^{6} q_i\nabla(\chi_i),$$

then

$$\begin{aligned}
\nabla(T_1 \cdot T_2) = (&p_1q_1 + p_3q_3)\nabla(\chi_1) \\
&+(p_1q_2 + p_2q_1 + p_2q_4 + p_3q_4 + p_5q_2 + p_5q_3 + zp_5q_4)\nabla(\chi_2) \\
&+(p_1q_3 + p_3q_1 + zp_3q_3)\nabla(\chi_3) \\
&+(p_1q_4 + p_3q_2 + p_4q_1 + p_4q_4 + p_6q_2 + zp_3q_4 + p_6q_3 + zp_6q_4)\nabla(\chi_4) \\
&+(p_1q_5 + p_2q_3 + p_2q_6 + p_3q_6 + p_5q_1 + p_5q_5 + zp_5q_3 + zp_5q_6)\nabla(\chi_5) \\
&+(p_1q_6 + p_3q_5 + p_4q_3 + p_4q_6 + p_6q_1 + p_6q_5 + zp_3q_6 + zp_6q_3 + zp_6q_6)\nabla(\chi_6).
\end{aligned}$$

In order to obtain $\nabla(N_1(\mathcal{E}^{2k}))$ and $\nabla(N_3(\mathcal{E}^{2k}))$ we have Theorem 3.2.2. We omit the proofs of Theorem 3.2.2 and Lemma 3.2.1 because are analogous to those given in Theorem 3.1.2 and Lemma 3.1.1, respectively, when we substitute $v = 1$.

Theorem 3.2.2. *[25] For all $k \in \mathbb{N}$ we have that* $\nabla(\mathcal{E}^{2k}) = \sum_{i=1}^{6} \alpha_{i_k} \nabla(\chi_i)$,

$$\begin{array}{rclcrcl} \alpha_{1_k} & = & z\alpha_{3_{k-1}} + \alpha_{1_{k-1}}, & \qquad & \alpha_{4_k} & = & 1 - \alpha_{1_k}, \\ \alpha_{2_k} & = & -\alpha_{3_k} - z(1 - \alpha_{1_k}), & & \alpha_{5_k} & = & 1 - \alpha_{1_k}, \\ \alpha_{3_k} & = & z\alpha_{1_{k-1}} + (1 + z^2)\alpha_{3_{k-1}}, & & \alpha_{6_k} & = & -\alpha_{3_k}. \end{array}$$

Where $\alpha_{1_0} = 1$ *and* $\alpha_{3_0} = 0$.

Note that α_{1_k} and α_{3_k} can be computed in a recursively manner and by the definition of $\alpha_{2_k}, \alpha_{4_k}, \alpha_{5_k}$ and α_{6_k} they can be obtained from α_{1_k} and α_{3_k}.

Some formulae useful to obtain the Conway polynomials associated to 3-braids, which are based on recursive formulae (3.11) and (3.12), are shown. Also, we rewrite (3.11) by using (3.12).

$$\alpha_{3_k} = z\alpha_{1_k} + \alpha_{3_{k-1}} \tag{3.14}$$

Lemma 3.2.1. *[25] Let be $t \in \mathbb{N} \cup \{0\}$, then*

i) $\nabla(\mathcal{T}(2t)) = \alpha_{1_t}\nabla(\chi_1) + \alpha_{3_t}\nabla(\chi_3)$.

ii) $\nabla(\mathcal{T}(2t+1)) = \alpha_{3_t}\nabla(\chi_1) + \alpha_{1_{t+1}}\nabla(\chi_3)$.

iii) $\nabla(\mathcal{T}(-2t)) = \alpha_{1_{t+1}}\nabla(\chi_1) - \alpha_{3_t}\nabla(\chi_3)$.

iv) $\nabla(\mathcal{T}(-(2t-1)) = -\alpha_{3_t}\nabla(\chi_1) + \alpha_{1_t}\nabla(\chi_3)$.

v) $\nabla(\mathcal{T}(0, 2t)) = \nabla(\chi_1) + tz\nabla(\chi_2)$.

vi) $\nabla(\mathcal{T}(0, -2t)) = \nabla(\chi_1) - tz\nabla(\chi_2)$.

Note that by using formulae of Theorem 3.2.1 and Lemma 3.2.1 it possible to calculate the Conway polynomial of 3-braids. In order to facilitate some calculations, which involves the polynomials (3.11) and (3.12), we have Propositions 3.2.1 and 3.2.2.

Proposition 3.2.1. $\alpha_{1_k} = 1 + z\sum_{i=1}^{k-1} \alpha_{3_i}$ *and* $\alpha_{3_k} = z\sum_{i=1}^{k} \alpha_{1_i}$, *where* $\alpha_{1_0} = 1$ *and* $\alpha_{3_0} = 0$.

Proof. By induction on k. For $k = 1$, we have $\alpha_{3_1} = z$ and $\alpha_{1_1} = 1$.
Suppose for $k = n$ that $\alpha_{3_n} = z\sum_{i=1}^{n} \alpha_{1_i}$ and we will calculate $\alpha_{3_{n+1}}$.

From (3.14) we have that $\alpha_{3_{n+1}} = z\alpha_{1_{(n+1)}} + \alpha_{3_n}$ and, by inductive hypothesis, $\alpha_{3_{n+1}} = z\alpha_{1_{(n+1)}} + z\sum_{i=1}^{n} \alpha_{1_i}$. It follows that $\alpha_{3_{n+1}} = z\sum_{i=1}^{n+1} \alpha_{1_i}$.

Now, suppose for $k = n$ that $\alpha_{1_k} = 1 + z\sum_{i=1}^{k-1} \alpha_{3_i}$ and we will calculate $\alpha_{1_{n+1}}$.

From, (3.11) we have that $\alpha_{1_{(n+1)}} = z\alpha_{3_n} + \alpha_{1_n}$ and by inductive hypothesis $\alpha_{1_{n+1}} = z\alpha_{3_n} + 1 + z\sum_{i=1}^{n-1}\alpha_{3_i}$. It follows that $\alpha_{1_{(n+1)}} = 1 + z\sum_{i=1}^{n}\alpha_{3_i}$. Therefore, $\alpha_{3_k} = z\sum_{i=1}^{k}\alpha_{1_i}$ and $\alpha_{1_k} = 1 + z\sum_{i=1}^{k-1}\alpha_{3_i}$. □

Proposition 3.2.2. *Let* $\alpha_{1_0} = 1$ *and* $\alpha_{3_0} = 0$ *then* $\alpha_{3_l}\alpha_{1_m} + \alpha_{1_{l+1}}\alpha_{3_m} = \alpha_{3_{m+l}}$ *and* $\alpha_{3_l}\alpha_{3_m} + \alpha_{1_{l+1}}\alpha_{1_{m+1}} = \alpha_{1_{m+l+1}}$ *for all* $m, l \in \mathbb{N} \cup \{0\}$.

Proof. By induction on m, set l. For $m = 0$, as $\alpha_{3_0} = 0$ and $\alpha_{1_0} = 1$ then $\alpha_{3_l}\alpha_{1_0} + \alpha_{1_{l+1}}\alpha_{3_0} = \alpha_{3_l}$ and $\alpha_{3_l}\alpha_{3_0} + \alpha_{1_{l+1}}\alpha_{1_1} = \alpha_{1_{l+1}}$.

For $m = 1$, due to that $\alpha_{3_1} = z$ and $\alpha_{1_1} = 1$ then $\alpha_{1_2} = 1 + z^2$ and we have that

$$\begin{aligned}
\alpha_{3_l}\alpha_{1_1} + \alpha_{1_{l+1}}\alpha_{3_1} &= \alpha_{3_l} + \alpha_{1_{l+1}}z \\
&= \alpha_{3_{l+1}}.
\end{aligned}$$

And

$$\begin{aligned}
\alpha_{3_l}\alpha_{3_1} + \alpha_{1_{l+1}}\alpha_{1_2} &= \alpha_{3_l}z + \alpha_{1_{l+1}}(1 + z^2) \\
&= \alpha_{1_{l+1}} + z(\alpha_{3_l} + z\alpha_{1_{l+1}}) \\
&= \alpha_{1_{l+1}} + z\alpha_{3_{l+1}} \\
&= \alpha_{1_{l+2}}.
\end{aligned}$$

Suppose for $m = n$ that $\alpha_{3_l}\alpha_{1_n} + \alpha_{1_{l+1}}\alpha_{3_n} = \alpha_{3_{n+l}}$ and $\alpha_{3_l}\alpha_{3_n} + \alpha_{1_{l+1}}\alpha_{1_{n+1}} = \alpha_{1_{n+l+1}}$. By Proposition 3.2.1 we have that

$$\begin{aligned}
\alpha_{3_l}\alpha_{1_{n+1}} + \alpha_{1_{l+1}}\alpha_{3_{n+1}} &= \alpha_{3_l}\left(1 + z\sum_{i=1}^{n}\alpha_{3_i}\right) + \alpha_{1_{l+1}}\left(z\sum_{i=1}^{n+1}\alpha_{1_i}\right) \\
&= \alpha_{3_l}\left(1 + z\sum_{i=1}^{n-1}\alpha_{3_i} + z\alpha_{3_n}\right) + \alpha_{1_{l+1}}\left(z\sum_{i=1}^{n}\alpha_{1_i} + \alpha_{1_{n+1}}\right) \\
&= \alpha_{3_l}\left(1 + z\sum_{i=1}^{n-1}\alpha_{3_i}\right) + \alpha_{1_{l+1}}\left(z\sum_{i=1}^{n}\alpha_{1_i}\right) + z(\alpha_{3_l}\alpha_{3_n} + \alpha_{1_{l+1}}\alpha_{1_{n+1}}) \\
&= \alpha_{3_l}\alpha_{1_n} + \alpha_{1_{l+1}}\alpha_{3_n} + z\alpha_{1_{l+n+1}} \\
&= \alpha_{3_{l+n}} + z\alpha_{1_{l+n+1}} \\
&= \alpha_{3_{l+n+1}}.
\end{aligned}$$

Also,

$$\begin{aligned}
\alpha_{3_l}\alpha_{3_{n+1}} + \alpha_{1_{l+1}}\alpha_{3_{n+2}} &= \alpha_{3_l}\left(z\sum_{i=1}^{n+1}\alpha_{1_i}\right) + \alpha_{1_{l+1}}\left(1 + z\sum_{i=1}^{n+1}\alpha_{3_i}\right) \\
&= \alpha_{3_l}\left(z\sum_{i=1}^{n}\alpha_{1_i} + z\alpha_{1_{n+1}}\right) + \alpha_{1_{l+1}}\left(1 + z\sum_{i=1}^{n}\alpha_{3_i} + \alpha_{3_{n+1}}\right) \\
&= \alpha_{3_l}\left(z\sum_{i=1}^{n}\alpha_{1_i}\right) + \alpha_{1_{l+1}}\left(1 + z\sum_{i=1}^{n}\alpha_{3_i}\right) + z(\alpha_{3_l}\alpha_{1_{n+1}} + \alpha_{1_{l+1}}\alpha_{3_{n+1}}) \\
&= \alpha_{3_l}\alpha_{3_n} + \alpha_{1_{l+1}}\alpha_{3_{n+1}} + z\alpha_{3_{l+n+1}} \\
&= \alpha_{1_{l+n+1}} + z\alpha_{3_{l+n+1}} \\
&= \alpha_{1_{(l+n+1)+1}}.
\end{aligned}$$

Now set m, by induction on l. For $l = 0$, as $\alpha_{3_0} = 0$ and $\alpha_{1_0} = 1$ then $\alpha_{3_0}\alpha_{1_m} + \alpha_{1_1}\alpha_{3_m} = \alpha_{3_m}$ and $\alpha_{3_0}\alpha_{3_m} + \alpha_{1_1}\alpha_{1_{m+1}} = \alpha_{1_{m+1}}$.

For $m = 1$, due to that $\alpha_{3_1} = z$ and $\alpha_{1_1} = 1$ $\alpha_{1_2} = 1 + z^2$ it follows that

$$\begin{aligned}\alpha_{3_1}\alpha_{1_m}+\alpha_{1_2}\alpha_{3_m} &= z\alpha_{3_m}+(1+z^2)\alpha_{3_m}\\ &= \alpha_{3_m}+z(\alpha_{1_m}+z\alpha_{3_m})\\ &= \alpha_{3_m}+z\alpha_{1_{m+1}}\\ &= \alpha_{3_{m+1}}.\end{aligned}$$

Also,

$$\begin{aligned}\alpha_{3_1}\alpha_{3_m}+\alpha_{1_2}\alpha_{1_{m+1}} &= z\alpha_{3_m}+(1+z^2)\alpha_{1_{m+1}}\\ &= \alpha_{1_{m+1}}+z(\alpha_{3_m}+z\alpha_{1_{m+1}})\\ &= \alpha_{1_{m+1}}+z\alpha_{3_{m+1}}\\ &= \alpha_{1_{m+2}}.\end{aligned}$$

Suppose for $l=n$ that $\alpha_{3_n}\alpha_{1_m}+\alpha_{1_{n+1}}\alpha_{3_m}=\alpha_{3_{n+m}}$ and $\alpha_{3_n}\alpha_{3_m}+\alpha_{1_{n+1}}\alpha_{1_{m+1}}=\alpha_{1_{n+m+1}}$. By Proposition 3.2.1 we have that

$$\begin{aligned}\alpha_{3_{n+1}}\alpha_{1_m}+\alpha_{1_{n+1}}\alpha_{3_m} &= (z\sum_{i=1}^{n+1}\alpha_{1_i})\alpha_{1_m}+(1+z\sum_{i=1}^{n+1}\alpha_{3_i})\alpha_{3_m}\\ &= (z\sum_{i=1}^{n}\alpha_{1_i})\alpha_{1_m}+(1+z\sum_{i=1}^{n}\alpha_{3_i})\alpha_{3_m}+z(\alpha_{1_{n+1}}\alpha_{1_m}+\alpha_{3_{n+1}}\alpha_{3_m})\\ &= \alpha_{3_l}\alpha_{1_m}+\alpha_{1_{l+1}}\alpha_{3_m}+z(\alpha_{1_{n+1}}\alpha_{1_m}+\alpha_{3_{n+1}}\alpha_{3_m})\\ &= \alpha_{3_{n+m}}+z(\alpha_{1_{n+1}}\alpha_{1_m}+(\alpha_{3_n}+z\alpha_{1_{n+1}})\alpha_{3_m})\\ &= \alpha_{3_{n+m}}+z(\alpha_{3_n}\alpha_{3_m}+\alpha_{1_{n+1}}(\alpha_{1_m}+z\alpha_{3_m}))\\ &= \alpha_{3_{n+m}}+z(\alpha_{3_n}\alpha_{3_m}+\alpha_{1_{n+1}}\alpha_{1_{m+1}})\\ &= \alpha_{3_{n+m}}+z\alpha_{1_{n+m+1}}\\ &= \alpha_{3_{n+m+1}}.\end{aligned}$$

Besides,

$$\begin{aligned}\alpha_{3_{n+1}}\alpha_{3_m}+\alpha_{1_{n+2}}\alpha_{3_{m+2}} &= (z\sum_{i=1}^{n+1}\alpha_{1_i})\alpha_{3_m}+(1+z\sum_{i=1}^{n+1}\alpha_{3_i})\alpha_{1_{m+1}}\\ &= (z\sum_{i=1}^{n}\alpha_{1_i})\alpha_{3_m}+(1+z\sum_{i=1}^{n}\alpha_{3_i})\alpha_{1_{m+1}}\\ &\quad +z(\alpha_{1_{n+1}}\alpha_{3_m}+\alpha_{3_{n+1}}\alpha_{1_{m+1}})\\ &= \alpha_{3_n}\alpha_{3_m}+\alpha_{1_{n+1}}\alpha_{1_{m+1}}+z(\alpha_{1_{n+1}}\alpha_{3_m}+\alpha_{3_{n+1}}\alpha_{1_{m+1}})\\ &= \alpha_{1_{n+m+1}}+z(\alpha_{1_{n+1}}\alpha_{3_m}+(\alpha_{3_n}+z\alpha_{1_{n+1}})(\alpha_{1_m}+z\alpha_{3_m}))\\ &= \alpha_{1_{n+m+1}}+z(\alpha_{1_{n+1}}\alpha_{3_m}+\alpha_{3_n}\alpha_{1_m}+z(\alpha_{1_{n+1}}\alpha_{1_m}\\ &\quad +\alpha_{3_n}\alpha_{3_m}+z\alpha_{1_{n+1}}\alpha_{3_m}))\\ &= \alpha_{1_{n+m+1}}+z(\alpha_{3_{n+m}}+z(\alpha_{3_n}\alpha_{3_m}+\alpha_{1_{n+1}}(\alpha_{1_m}+z\alpha_{3_m})))\\ &= \alpha_{1_{n+m+1}}+z(\alpha_{3_{n+m}}+z(\alpha_{3_n}\alpha_{3_m}+\alpha_{1_{n+1}}\alpha_{1_{m+1}}))\\ &= \alpha_{1_{n+m+1}}+z(\alpha_{3_{n+m}}+z\alpha_{1_{n+m+1}})\\ &= \alpha_{1_{(n+m+1)+1}}.\end{aligned}$$

□

In Lemma 3.2.2, we will obtain two of the six polynomials, which constitute the Conway polynomial of the 3-braid $\mathcal{T}(2l+1,2n_1,2m_1,...,2n_r,2m_r)$ where $l,n_i,m_i,r\in\mathbb{N}\cup\{0\}$ and $i=1,...,r$.

Lemma 3.2.2. *If* $\nabla(\mathcal{T}(2l+1,2n_1,2m_1,...,2n_r,2m_r))=\sum_{i=1}^{6}p_{i_r}\nabla(\chi_i)$ *then* $p_{1_r}=\alpha_{3_{g_r}}$ *and* $p_{3_r}=\alpha_{1_{g_r+1}}$ *such that* $g_r=l+\sum_{i=1}^{r}m_i$ *for* $r\geq 1$ *and* $g_0=l$.

Proof. By induction on r. For $r=0$, due to Lemma 3.2.1 we have that $\nabla(\mathcal{T}(2l+1))=\alpha_{3_l}\nabla(\chi_1)+(\alpha_{1_l}+z\alpha_{3_l})\nabla(\chi_3)$, then $P_{1_0}=\alpha_{3_l}$ and $P_{3_0}=\alpha_{1_l}+z\alpha_{3_l}$. By (3.14) $P_{3_0}=\alpha_{1_{l+1}}$.

By Theorem 3.2.1, if $T = T_1 \cdot T_2$ such that $\nabla(T_1) = \sum_{i=1}^{6} p_i \nabla(\chi_i)$ and $\nabla(T_2) = \sum_{i=1}^{6} q_i \nabla(\chi_i)$ then, $\nabla(T) = \sum_{i=1}^{6} Q_i \nabla(\chi_i)$ where $Q_1 = (p_1 q_1 + p_3 q_3)$ and $Q_3 = (p_1 q_3 + p_3 q_1 + z p_3 q_3)$.

Due to Lemma 3.2.1 and Theorem 3.2.1 we have that

$$\nabla(\mathcal{T}(0, 2n_s, 2m_s)) = \alpha_{1_{m_s}} \nabla(\chi_1) + \alpha_{3_{m_s}} \nabla(\chi_3).$$

For $r = 1$, $\mathcal{T}(2l+1, 2n_1, 2m_1) = \mathcal{T}(2l+1) \cdot \mathcal{T}(0, 2n_1, 2m_1)$. By Theorem 3.2.1 we have that $P_{1_1} = (\alpha_{3_l} \alpha_{1_{m_1}} + \alpha_{1_{l+1}} \alpha_{3_{m_1}})$ and $P_{3_1} = (\alpha_{3_l} \alpha_{3_{m_1}} + \alpha_{1_{l+1}} \alpha_{1_{m_1+1}})$.

By Proposition 3.2.2 it follows that $P_{1_1} = \alpha_{3_{l+m_1}}$ and $P_{3_1} = \alpha_{1_{l+m_1+1}}$.

Suppose for $r = k$ that $p_{1_k} = \alpha_{3_{g_k}}$ and $p_{3_k} = \alpha_{1_{g_k+1}}$, we will calculate for $r = k+1$. Note that

$$\mathcal{T}(2l+1, ..., 2n_{k+1}, 2m_{k+1}) = \mathcal{T}(2l+1, ..., 2n_k, 2m_k) \cdot \mathcal{T}(0, 2n_{k+1}, 2m_{k+1}),$$

and due to Lemma 3.2.1 and Theorem 3.2.1 we have that

$$\nabla(\mathcal{T}(0, 2n_{k+1}, 2m_{k+1})) = \alpha_{1_{m_{k+1}}} \nabla(\chi_1) + \alpha_{3_{m_{k+1}}} \nabla(\chi_3).$$

Therefore by Theorem 3.2.1 and (3.14) we have that

$$p_{1_{k+1}} = \alpha_{3_{g_k}} \alpha_{1_{m_{k+1}}} + \alpha_{1_{g_k+1}} \alpha_{3_{m_{k+1}}} \text{ and } p_{3_{k+1}} = \alpha_{3_{g_k}} \alpha_{3_{m_{k+1}}} + \alpha_{1_{g_k+1}} \alpha_{1_{m_{k+1}+1}}.$$

By Proposition 3.2.2 it follows that $p_{1_{k+1}} = \alpha_{3_{g_k+m_{k+1}}}$ and $p_{3_{k+1}} = \alpha_{1_{g_k+m_{k+1}+1}}$. Further

$g_k + m_{k+1} = l + \sum_{i=1}^{k} m_i + m_{k+1} = l + \sum_{i=1}^{k+1} m_i = g_{k+1}$,

hence $p_{1_{k+1}} = \alpha_{3_{g_{k+1}}}$ and $p_{3_{k+1}} = \alpha_{1_{g_{k+1}+1}}$. □

In order to obtain the Conway polynomial of the N_j-closure of a 3-tangle we have the following lemma, which is the Conway version of Lemma 3.1.2.

Lemma 3.2.3. *Let T be a 3-tangle, if $\nabla(T) = \sum_{i=1}^{6} p_i \nabla(\chi_i)$, then*

$$\begin{aligned}
\nabla(N_1(T)) &= p_4 + p_5 + z p_6, \\
\nabla(N_2(T)) &= p_3 + p_6, \\
\nabla(N_3(T)) &= p_2 + z p_4 + z p_5 + (1+z^2) p_6, \\
\nabla(N_4(T)) &= p_1 + z p_3 + p_4 + z p_6, \\
\nabla(N_5(T)) &= p_1 + z p_3 + p_5 + z p_6, \\
\nabla(N_6(T)) &= z p_1 + p_2 + (1+z^2) p_3 + z p_4 + z p_5 + z^2 p_6.
\end{aligned}$$

Proof. The proof is analogous to the proof of Theorem 3.1.2, considering that the Conway polynomial can be obtained from the HOMFLY-PT polynomial. So, when $v = 1$ we have that $\delta = 0$. □

The Conway version of Corollary 3.1.1 is the Corollary 3.2.1.

Corollary 3.2.1. *For all $n \in \mathbb{N}$ we have that $\nabla(N_1(\mathcal{E}^{2k})) = 2(1 - \alpha_{1_k}) - z\alpha_{3_k}$ and $\nabla(N_3(\mathcal{E}^{2k})) = -(2+z^2)\alpha_{3_k} + z(1 - \alpha_{1_k})$.*

Proof. The results follow from Lemma 3.2.3 and Theorem 3.2.2. □

The proof of Theorem 3.2.3 is analogous to the proof of Theorem 3.1.3.

Theorem 3.2.3. *Let T_1 and T_2 be two 3-tangles, if $\nabla(T_1) = \sum_{i=1}^{6} p_i \nabla(\chi_i)$ and $\nabla(T_2) = \sum_{i=1}^{6} q_i P(\chi_i)$ then $\nabla(N_1(T_1 \cdot T_2)) = \sum_{i=1}^{6} p_i \nabla(N_i(T_2))$.*

The Conway polynomial version of Corollary 3.1.2 involves the polynomial of $N_1(T)$ as is shown in Corollary 3.2.2.

Corollary 3.2.2. *Let T be a 3-tangle. If $\nabla(T) = \sum_{i=1}^{6} p_i \nabla(\chi_i)$ and $k \in \mathbb{N}$, then*

$$\nabla(N_1(T \cdot \mathcal{E}^{2k})) = p_1 \nabla(N_1(\mathcal{E}^{2k})) + p_3 \nabla(N_3(\mathcal{E}^{2k})) + \nabla(N_1(T)). \tag{3.15}$$

Proof. By Corollary 3.1.2 we have that

$$P(N_1(T \cdot \mathcal{E}^{2k})) = p_1 P(N_1(\mathcal{E}^{2k})) + p_2\delta + p_3 P(N_3(\mathcal{E}^{2k})) + p_4 + p_5 + p_6(v^2\delta + vz).$$

After the variable change $v = 1$ we obtain that

$$\nabla(N_1(T \cdot \mathcal{E}^{2k})) = p_1 \nabla(N_1(\mathcal{E}^{2k})) + p_3 \nabla(N_3(\mathcal{E}^{2k})) + p_4 + p_5 + p_6 z.$$

Since Lemma 3.2.3 states that $p_4 + p_5 + zp_6 = \nabla(N_1(T))$ then

$$\nabla(N_1(T \cdot \mathcal{E}^{2k})) = p_1 \nabla(N_1(\mathcal{E}^{2k})) + p_3 \nabla(N_3(\mathcal{E}^{2k})) + \nabla(N_1(T)). \tag{3.16}$$

□

Note that the Equality (3.16) involves the polynomial of the 3-tangle T and this fact is not obvious from the HOMFLY-PT polynomial. So, this version has an elegant form. In the following section we will obtain some formulae for the Alexander polynomial, in particular the polynomials α_{1_k} and α_{3_k} will be described in a non recursive form.

3.3 Alexander polynomial

Due to Theorem 2.2.1 all the results for the Conway polynomial given in the previous section can be used to obtain the Alexander polynomial version. However, after the variable change $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ some of these formulae become non recursive.

The polynomials α_{1_k} and α_{3_k}, which are described in (3.11) and (3.12) will be denoted, after the variable change $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$, by $\overline{\alpha}_{1_k}$ and $\overline{\alpha}_{3_k}$, respectively.

Proposition 3.3.1 gives non recursive formulae for $\overline{\alpha}_{1_k}$ and $\overline{\alpha}_{3_k}$ and Corollary 3.3.1 also gives a non recursive formula for the polynomials in Corollary 3.2.1.

Proposition 3.3.1. *For all $k \in \mathbb{N}$ we have*

$$\overline{\alpha}_{1_k} = \left[\sum_{i=1}^{k}(t^{-(i-1)}+t^{i-1})(-1)^{k-i}\right]+(-1)^k, \qquad (3.17)$$

$$\overline{\alpha}_{3_k} = -\sum_{i=1}^{k}(t^{-(i-\frac{1}{2})}-t^{i-\frac{1}{2}})(-1)^{k-i}. \qquad (3.18)$$

Proof. Remember that $\overline{\alpha}_{1_k}=\alpha_{1_k}$ and $\overline{\alpha}_{3_k}=\alpha_{3_k}$ after a variable change. We prove (3.17) and (3.18) by using induction on k. For $k=1$ we have

$$\begin{aligned}\overline{\alpha}_{1_1} &= \alpha_{1_1}\\ &= 1\\ &= \textstyle\sum_{i=1}^{1}(t^{-(i-1)}+t^{i-1})(-1)^{1-i}\\ &\quad +(-1)^1.\end{aligned} \qquad \begin{aligned}\overline{\alpha}_{3_1} &= \alpha_{3_1}\\ &= z\\ &= -(t^{-\frac{1}{2}}-t^{\frac{1}{2}})(-1)^{1-1}\\ &= -\textstyle\sum_{i=1}^{1}(t^{-(i-\frac{1}{2})}-t^{i-\frac{1}{2}})(-1)^{1-i}.\end{aligned}$$

We show that the lemma is true for the case k. In order to ease the calculation we take $t^{\frac{1}{2}}=x$. Since $\alpha_{3_k}=z\alpha_{1_{k-1}}+(1+z^2)\alpha_{3_{k-1}}$ then $\overline{\alpha}_{3_k}=(x-x^{-1})(\overline{\alpha}_{1_{k-1}})+(1+(x-x^{-1})^2)\overline{\alpha}_{3_{k-1}}$. By the inductive hypothesis,

$$\begin{aligned}\overline{\alpha}_{3_k} &= (x-x^{-1})\left(\left[\sum_{i=1}^{k-1}(x^{-(2i-2)}+x^{2i-2})(-1)^{k-1-i}\right]+(-1)^{k-1}\right)\\ &\quad +(1+(x-x^{-1})^2)\left(-\sum_{i=1}^{k-1}(x^{-(2i-1)}-x^{2i-1})(-1)^{(k-1)-i}\right).\end{aligned}$$

After some computations,

$$\overline{\alpha}_{3_k} = \left(-\sum_{i=1}^{k-1}(x^{-(2(i+1)-1)}-x^{2(i+1)-1})(-1)^{(k-1)-i}\right)-(-1)^{(k-1)}(x^{-1}-x).$$

We take $r=i+1$ and rewrite $\overline{\alpha}_{3_k}$,

$$\overline{\alpha}_{3_k} = -\sum_{r=1}^{k}(x^{-(2r-1)}-x^{2r-1})(-1)^{k-r}.$$

As $x=t^{\frac{1}{2}}$ we have the result. The proof for $\overline{\alpha}_{1_k}$ is analogous. □

Note that for all $k \in \mathbb{N}$ the equations (3.17) and (3.18) from Theorem 3.1.3 can be rewritten as:

$$\overline{\alpha}_{1_k} = -\overline{\alpha}_{1_{k-1}}+(t^{-(k-1)}-t^{k-1}), \qquad (3.19)$$

$$\overline{\alpha}_{3_k} = -\overline{\alpha}_{3_{k-1}}-(t^{-(k-\frac{1}{2})}-t^{k-\frac{1}{2}}). \qquad (3.20)$$

Lemma 3.3.1. *For $k \in \mathbb{N}\cup\{0\}$ we have that*

$$\Delta(N_1(\mathcal{E}^{2k})) = -(t^{-k}+t^k)+2 \quad \textit{and} \quad \Delta(N_3(\mathcal{E}^{2k})) = (t^{-(k+\frac{1}{2})}-t^{k+\frac{1}{2}})+t^{\frac{1}{2}}-t^{-\frac{1}{2}}.$$

Proof. By Corollary 3.2.1 we have $\nabla(N_1(\mathcal{E}^{2k})) = 2(1-\alpha_{1_k}) - z\alpha_{3_k}$ and $\nabla(N_3(\mathcal{E}^{2k})) = -(2+z^2)\alpha_{3_k} + z(1-\alpha_{1_k})$. Further, the variable change $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ on $\nabla(N_1(\mathcal{E}^{2k}))$ and $\nabla(N_3(\mathcal{E}^{2k}))$ generates $\Delta(N_1(\mathcal{E}^{2k}))$ and $\Delta(N_3(\mathcal{E}^{2k}))$. Therefore

$$\Delta(N_1(\mathcal{E}^{2k})) = 2(1-\overline{\alpha}_{1_k}) - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\overline{\alpha}_{3_k},$$

and

$$\Delta(N_3(\mathcal{E}^{2k})) = -(2+(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2)\overline{\alpha}_{3_k} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(1-\overline{\alpha}_{1_k}).$$

From (3.11) and (3.19) we have that

$$(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\overline{\alpha}_{3_{k-1}} + \overline{\alpha}_{1_{k-1}} = -\overline{\alpha}_{1_{k-1}} + (t^{-(k-1)} - t^{k-1}),$$

so

$$2\overline{\alpha}_{1_{k-1}} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\overline{\alpha}_{3_{k-1}} = (t^{-(k-1)} - t^{k-1}).$$

Then

$$2 - 2\overline{\alpha}_{1_{k-1}} - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\overline{\alpha}_{3_{k-1}} = -(t^{-(k-1)} - t^{k-1}) + 2.$$

Therefore, $\Delta(N_1(\mathcal{E}^{2k})) = -(t^{-k} - t^k) + 2$.

From (3.12) and (3.20) we obtain that

$$(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\overline{\alpha}_{1_{k-1}} + (1+(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2)\overline{\alpha}_{3_{k-1}} = -\overline{\alpha}_{3_{k-1}} - (t^{-(k-\frac{1}{2})} - t^{k-\frac{1}{2}}),$$

hence

$$(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\overline{\alpha}_{1_{(k-1)}} + (2+(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2)\overline{\alpha}_{3_{(k-1)}} = -(t^{-(k-\frac{1}{2})} - t^{k-\frac{1}{2}}).$$

Then

$$-(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\overline{\alpha}_{1_{(k-1)}} - (2+(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2)\overline{\alpha}_{3_{(k-1)}} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = (t^{-(k-\frac{1}{2})} - t^{k-\frac{1}{2}}) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}).$$

Therefore,

$$\Delta(N_3(\mathcal{E}^{2k})) = (t^{-(k+\frac{1}{2})} - t^{k+\frac{1}{2}}) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}).$$

□

Note that (3.15) takes the following form for the Conway polynomial:

$$\nabla(N_1(T\cdot\mathcal{E}^{2k})) = p_1(2(1-\alpha_{1_k}) - z\alpha_{3_k}) + p_3(-(2+z^2)\alpha_{3_k} + z(1-\alpha_{1_k})) + \nabla(N_1(T)). \tag{3.21}$$

And, in the Alexander polynomial version we have that

$$\Delta(N_1(T\cdot\mathcal{E}^{2k})) = p_1(-(t^{-k}+t^k)+2) + p_3(t^{-(k+\frac{1}{2})} - t^{(k+\frac{1}{2})} + t^{\frac{1}{2}} - t^{-\frac{1}{2}}) + \Delta(N_1(T)). \tag{3.22}$$

Then, for some families of knots, it will be more convenient to calculate the Alexander polynomial instead to the HOMFLY-PT polynomial. In the first section of Chapter 4 the Alexander polynomial will be used to prove that some families of knots are non-alternating. However, in some cases this polynomial will be not enough to prove that certain knots are non-alternating.

Chapter 4

Detecting families of non-alternating knots

We will prove that certain families of links are non-alternating by using different methods. The first one will use the Alexander polynomial and the second one will use the HOMFLY-PT polynomial. The purpose of to use different methods is to extend the families of knots obtained and the properties that these could have. We will focus in knots of the form $N_1(T \cdot \mathcal{E}^{2k})$ such that T is a 3-tangle in $S(R)$ and $k \in \mathbb{N} \cup \{0\}$. Further, in the case where T is the concatenation of a 3-braid $\mathcal{T}$ and the 3-tangle c (see Figure 2.29 (b)) we give $\Delta(N_1(\mathcal{T} \cdot c \cdot \mathcal{E}^{2k}))$.

4.1 By using the Alexander polynomial

In the previous chapter we have obtained formulae for the Alexander polynomial of knots formed by the closure of 3-tangles. Theorem 4.1.1 can be used to prove that some knots are non-alternating; however this theorem does not hold for links with more than one component.

Theorem 4.1.1. *[44] Suppose K is an alternating knot and*

$$\Delta(K) = a_{-m}t^{-m} + a_{-m+1}t^{-m+1} + \ldots + a_m t^m \text{ with } a_m \neq 0 \neq a_{-m}.$$

Then

(i) $a_{-m}, a_{-m+1}, \ldots, a_m$ are never equal to zero;

(ii) the sign of two consecutive coefficients alternates, i.e.,

$$a_i a_{i+1} < 0 \quad (i = -m, -m+1, \ldots, m-1).$$

Note that if K is a knot such that its Alexander polynomial does not satisfy (i) or (ii), then K is non-alternating. We will use Theorem 4.1.1 to prove that certain knots are non-alternating.

Theorem 4.1.2. *Let T be a 3-tangle such that $N_1(T)$ is a knot, then there exists $k \in \mathbb{N}$ such that for all $l \geq k$, $l \in \mathbb{N}$ the family $\{N_1(T \cdot \mathcal{E}^{2l})\}$ is non-alternating.*

Proof. Let $\nabla(T)=\sum_{i=1}^{6} p_i\nabla(\chi_i)$. By Corollary 3.2.2 we have that

$$\nabla(N_1(T\cdot\mathcal{E}^{2k}))=p_1\nabla(N_1(\mathcal{E}^{2k}))+p_3\nabla(N_3(\mathcal{E}^{2k}))+\nabla(N_1(T)).$$

After a variable change and Lemma 3.3.1 we obtain

$$\Delta(N_1(T\cdot\mathcal{E}^{2k}))=\overline{p}_1(-(t^{-k}+t^k)+2)+\overline{p}_3(t^{-(k+\frac{1}{2})}-t^{k+\frac{1}{2}}+t^{\frac{1}{2}}-t^{-\frac{1}{2}})+\Delta(N_1(T)). \quad (4.1)$$

Note that if $N_1(T)$ is knot, then $N_1(T\cdot\mathcal{E}^{2k})$ also is knot. This due to $\hat{\mathcal{E}}^{2k}=\hat{\chi}_1$. Therefore, if we choose k large enough then some coefficients are zero and by Theorem 4.1.1 the knots in $\{N_1(T\cdot\mathcal{E}^{2l})\}$ with $l\geq k$ are non-alternating. □

In the case when the 3-tangle $T=\mathcal{T}(2l+1,2n_1,2m_1,...,2n_r,2m_r)\cdot c$ where l,r,n_j,m_j are in $\mathbb{N}$ and $j=1,\dots,r$ we define $g_r=l+\sum_{i=1}^{r} m_i$. Let $\nabla(T)=\sum_{i=1}^{6} p_{i_r}\nabla(\chi_i)$ be the Conway polynomial of $T=\mathcal{T}(2l+1,2n_1,2m_1,...,2n_r,2m_r)\cdot c$.

We will use Theorem 4.1.2 to obtain a family of non-alternating knots of the form $\{N_1(\mathcal{T}(2l+1,2n_1,2m_1,...,2n_r,2m_r)\cdot c\cdot\mathcal{E}^{2k})\}$. In order to estimate a value of k in this case we will estimate p_{1_r} and p_{3_r}, respectively.

Lemma 4.1.1. *If $r\geq 1$ the span$(\Delta(N_1(\mathcal{T}(2l+1,2n_1,2m_1,...,2n_r,2m_r)\cdot c)))=2g_r+2$ and the span$(\Delta(N_1(T(2l+1)\cdot c)))=2l$.*

Proof. For $T=\mathcal{T}(2l+1)\cdot c$ we calculate $\nabla(c)=z\nabla(\chi_1)+\nabla(\chi_2)$. Lemma 3.2.1 states that

$$\nabla(\mathcal{T}(2l+1))=\alpha_{3_l}\nabla(\chi_1)+\alpha_{1_{l+1}}\nabla(\chi_3).$$

And, by Theorem 3.2.3 we have that $\nabla(N_1(\mathcal{T}(2l+1)\cdot c))=\alpha_{3_l}\nabla(N_1(c))+\alpha_{1_{l+1}}\nabla(N_3(c))$ where $\nabla(N_1(c))=0$, it follows that $\nabla(N_1(\mathcal{T}(2l+1)\cdot c))=\alpha_{1_{l+1}}$. Then by (3.17) we have that *span*$(\Delta(N_1(\mathcal{T}(2l+1)\cdot c)))=2l$.

If $r\geq 1$ and $T'=\mathcal{T}(2l+1,...,2n_r,2m_r)$ such that $\nabla(T')=\sum_{i=1}^{6} p_i\nabla(\chi_i)$ we will prove that $span(p_3)=span(p_6)=2g_r$ and $span(p_1)=span(p_2)=span(p_4)=span(p_5)=2g_r-1$. We will do this by induction on r.

For $r=1$, by Theorem 3.2.1 and Lemma 3.2.1 we have that,

$$\begin{aligned}\nabla(\mathcal{T}(2l+1,2n_1,2m_1))= \quad & \alpha_{3_{l+m_1}}\nabla(\chi_1)+\alpha_{1_{l+1}}n_1z\alpha_{1_{m_1}}\nabla(\chi_2)\\ & +\alpha_{1_{l+m_1+1}}\nabla(\chi_3)+\alpha_{1_{l+1}}n_1z\alpha_{1_{m_1}}\nabla(\chi_4)\\ & +\alpha_{3_l}n_1z\alpha_{3_{m_1}}\nabla(\chi_5)+\alpha_{1_{l+1}}n_1z\alpha_{3_{m_1}}\nabla(\chi_6). \quad (4.2)\end{aligned}$$

So, $g_1=l+m_1$ and due to equations (3.17) and (3.18) the *span* of each polynomial is $2(l+m_1)-1,2(l+m_1)-1,2(l+m_1),2(l+m_1)-1,2(l+m_1)-1,2(l+m_1)$, respectively.

Now, we will prove for $r=k+1$. Let $\nabla(\mathcal{T}(2l+1,...,n_k,m_k)=\sum_{i=1}^{r} Q_i\nabla(\chi_i)$ and assume that the hypothesis is satisfied in the case $r=k$. Then, by Lemmas 3.2.1 and 3.2.2 and Theorem 3.2.1 we have that:

$$\begin{aligned}\nabla(\mathcal{T}(2l+1,...,2n_k,2m_k,2n_{k+1},2m_{k+1})) = {} & \alpha_{3g_{r+1}}\nabla(\chi_1)+(Q_4\alpha_{1_{m_{k+1}}}+Q_6\alpha_{3_{m_{k+1}}})\nabla(\chi_2)\\ & +\alpha_{1_{g_{r+1}+1}}\nabla(\chi_3)+(Q_4\alpha_{1_{m_{k+1}}}+Q_6\alpha_{3_{m_{k+1}}})\nabla(\chi_4)\\ & +(\alpha_{1_{m_{k+1}}}+(Q_2+zQ_5)\alpha_{3_{m_{k+1}}})\nabla(\chi_5)\\ & +(Q_6n_{k+1}\alpha_{1_{m_{k+1}}}+(Q_4+zQ_6)\alpha_{3_{m_{k+1}}})\nabla(\chi_6).\end{aligned}$$

So, by inductive hypothesis and (3.17) and (3.18) we obtain the *span* of each polynomial, $2g_{r+1}-1$, $2g_{r+1}-1$, $2(g_{r+1}+1)-2$, $2g_{r+1}-1, 2g_{r+1}-1, 2(g_{r+1}+1)-2$, respectively. Therefore, $span(p_3)=span(p_6)=2g_r$ and $span(p_1)=span(p_2)=span(p_4)=span(p_5)=2g_r-1$.

On the other hand, by Theorems 3.2.1 and 3.2.3 it follows that

$$\nabla(N_1(T'\cdot c)) = z(p_4+p_5+zp_6)+p_3+p_6 = z\nabla(N_1(T'))+p_3+p_6. \tag{4.3}$$

Hence $span(\Delta(N_1(T))) = span((p_6)z^2) = 2g_r+2$. □

Theorem 4.1.3. *If $k \geq 3$, then the family of knots $\{N_1(\mathcal{T}(2l+1,2n_1,2m_1,...,2n_r,2m_r)\cdot c\cdot \mathcal{E}^{2k})\}$ is non-alternating.*

Proof. Theorem 2.5.1 states that $N_1(T\cdot\mathcal{E}^{2k})$ is a knot. Let

$$\nabla(\mathcal{T}(2l+1,2n_1,2m_1,...,2n_r,2m_r)) = \sum_{i=1}^{6} P_{i_r}\nabla(\chi_i)$$

and

$$\nabla(\mathcal{T}(2l+1,2n_1,2m_1,...,2n_r,2m_r)\cdot c) = \sum_{i=1}^{6} p_{i_r}\nabla(\chi_i).$$

By Lemma 3.2.2 we have $P_{1_r}=\alpha_{3_{g_r}}$ and $P_{3_r}=\alpha_{1_{g_{r+1}}}$, and $\nabla(c)=z\nabla(\chi_1)+\nabla(\chi_2)$ then, by Theorem 3.2.1, $p_1 = z\alpha_{3_{g_r}}$ and $p_3 = z\alpha_{1_{g_{r+1}}}$. Substituting in (4.1) and set $x=t^{\frac{1}{2}}$ we have:

$$\begin{aligned}\Delta(N_1(T\cdot\mathcal{E}^{2k})) = {} & \Delta(N_1(T))+(x-x^{-1})\overline{\alpha}_{3_{g_r}}[-(x^{-2k}+x^{2k})+2]\\ & +(x-x^{-1})\overline{\alpha}_{1_{g_{r+1}}}[(x^{-(2k+1)}-x^{(2k+1)})+(x-x^{-1})].\end{aligned}$$

Hereinafter we will use only g instead g_r. From equations (3.14) and (3.20) it follows that

$$(x-x^{-1})\overline{\alpha}_{1_{(g+1)}} = (-2\overline{\alpha}_{3_g}-(x^{-(2g+1)}-x^{(2g+1)})).$$

Thus,

$$\begin{aligned}\Delta(N_1(T\cdot\mathcal{E}^{2k})) &= \Delta(N_1(T))+(x-x^{-1})\overline{\alpha}_{3_g}[-(x^{-2k}+x^{2k})+2]\\ &\quad +(-2\overline{\alpha}_{3_g}-(x^{-(2g+1)}-x^{(2g+1)}))[(x^{-(2k+1)}-x^{(2k+1)})+(x-x^{-1})]\\ &= \Delta(N_1(T))+\overline{\alpha}_{3_g}[(x^{-(2k+1)}-x^{(2k+1)})-(x^{-(2k-1)}-x^{(2k-1)})+2(x-x^{-1})]\\ &\quad +(-2\overline{\alpha}_{3_g}-(x^{-(2g+1)}-x^{(2g+1)}))[(x^{-(2k+1)}-x^{(2k+1)})+(x-x^{-1})]\\ &= \Delta(N_1(T))+\overline{\alpha}_{3_g}[-(x^{-(2k+1)}-x^{(2k+1)})-(x^{-(2k-1)}-x^{(2k-1)})]\\ &\quad +(x^{-(2g+1)}-x^{(2g+1)})[-(x^{-(2k+1)}-x^{(2k+1)})-(x-x^{-1})].\end{aligned} \tag{4.4}$$

On the other hand,

$$\begin{array}{rl} & \overline{\alpha}_{3_g}[-(x^{-(p+2)}-x^{(p+2)})-(x^{-p}-x^{p})] \\ = & -\sum_{i=1}^{g}(x^{-(2i-1)}-x^{2i-1})(-1)^{g-i}[-(x^{-(p+2)}-x^{(p+2)})-(x^{-p}-x^{p})] \\ = & \sum_{i=1}^{g}(x^{-(2i+p-1)}+x^{2i+p-1}-x^{-(2i-p-1)}-x^{2i-p-1})(-1)^{g-i} \\ & +\sum_{i=1}^{g}(x^{-(2i+p+1)}+x^{2i+p+1}-x^{-(2i-p-3)}-x^{2i-p-3})(-1)^{g-i} \\ = & \sum_{i=1}^{g}(x^{-(2i+p-1)}+x^{2i+p-1}-x^{-(2i-p-1)}-x^{2i-p-1})(-1)^{g-i} \\ & +\sum_{i=1}^{g}(x^{-(2(i+1)+p-1)}+x^{2(i+1)+p-1})(-1)^{g-i} \\ & +\sum_{i=1}^{g}(x^{-(2(i-1)-p-1)}+x^{2(i-1)-p-1})(-1)^{g-i}. \end{array}$$

We rewrite the last equality, by variable changes $j=i+1$ and $t=i-1$,

$$\begin{array}{rl} = & \sum_{i=1}^{g}(x^{-(2i+p-1)}+x^{2i+p-1}-x^{-(2i-p-1)}-x^{2i-p-1})(-1)^{g-i} \\ & -\sum_{j=2}^{g+1}(x^{-(2j+p-1)}+x^{2j+p-1})(-1)^{g-j} \\ & -\sum_{t=0}^{g-1}(-x^{-(2t-p-1)}-x^{2t-p-1})(-1)^{g-t} \\ = & (x^{-(p+1)}+x^{p+1})(-1)^{g-1}-(x^{-(2g+p+1)}+x^{2g+p+1})(-1)^{g-(g+1)} \\ & -(x^{-(2g-p-1)}+x^{2g-p-1})(-1)^{g-g}+(x^{-(-p-1)}+x^{-p-1})(-1)^{g} \\ = & (x^{-(2g+p+1)}+x^{2g+p+1})-(x^{-(2g-p-1)}+x^{2g-p-1}). \end{array}$$

Therefore,

$$\overline{\alpha}_{3_g}[-(x^{-(p+2)}-x^{(p+2)})-(x^{-p}-x^{p})]=(x^{-(2g+p+1)}+x^{2g+p+1})-(x^{-(2g-p-1)}+x^{2g-p-1}).$$

It follows that (4.4) can be rewritten as

$$\begin{array}{rl} \Delta(N_1(T\cdot\mathcal{E}^{2k}))= & \Delta(N_1(T))+[(x^{-(2g+2k)}+x^{(2g+2k)})-(x^{-(2g-2k)}+x^{(2g-2k)})] \\ & -(x^{-(2g+2k+2)}-x^{(2g+2k+2)})+(x^{-(2g-2k)}+x^{(2g-2k)}) \\ & +(x^{-(2g+2)}-x^{(2g+2)})-(x^{-(2g)}+x^{(2g)}). \end{array}$$

Finally, simplifying and taking $x=t^{\frac{1}{2}}$ we obtain

$$\begin{array}{rl} \Delta(N_1(T\cdot\mathcal{E}^{2k}))= & \Delta(N_1(T))-(t^{-g_r}+t^{g_r})+(t^{-(g_r+1)}+t^{(g_r+1)})+(t^{-(g_r+k)}+t^{(g_r+k)}) \\ & -(t^{-(g_r+k+1)}+t^{(g_r+k+1)}). \end{array} \tag{4.5}$$

Due to Lemma 4.1.1 if $k\geq 3$ then some coefficients in (4.5) are zero and Theorem 4.1.1 implies the result. $\square$

As we can see in the following results, for certain 3-tangles T, the family $\{N_1(\mathcal{T}(2l+1,2n_1,2m_1,...,2n_r,2m_r)\cdot c\cdot\mathcal{E}^{2k})\}$ is also non-alternating when $0\leq k<3$. We will prove that for all $k,l-1\in\mathbb{N}$ the knots $N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2k})$ are non-alternating.

Theorem 4.1.4. *For all $l\in\mathbb{N}\cup\{0\}$ and $k\in\mathbb{N}$ the knots $N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2k})$ are non-alternating.*

Proof. By Theorem 2.5.1 the closure $N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2k})$ is a knot and from (4.5) it implies that

$$\Delta(N_1(T\cdot\mathcal{E}^{2k})) = \Delta(N_1(T)) - (t^{-g_r}+t^{g_r}) + (t^{-(g_r+1)}+t^{(g_r+1)})$$
$$+(t^{-(g_r+k)}+t^{(g_r+k)}) - (t^{-(g_r+k+1)}+t^{(g_r+k+1)}).$$

Besides, Lemma 3.2.1 states that $\nabla(\mathcal{T}(2l+1)) = \alpha_{3_l}\nabla(\chi_1) + (\alpha_{1_l}+z\alpha_{3_l})\nabla(\chi_3)$ and by Lemma 3.2.3 we have $\Delta(N_1(\mathcal{T}(2l+1)\cdot c)) = \overline{\alpha}_{1_{l+1}}$. Therefore, for all $l,k\in\mathbb{N}\cup\{0\}$ we have that (4.6) holds.

$$\begin{aligned}
\Delta(N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2k})) &= \left[\sum_{i=1}^{l}(t^{-i}+t^{i})(-1)^{l-i}\right]+(-1)^l\\
&\quad +(-t^{-l}-t^{l}+t^{-(l+1)}+t^{(l+1)})\\
&\quad +(t^{-(l+k)}+t^{(l+k)}-t^{-((l+k)+1)}-t^{((l+k)+1)})\\
&= \left[\sum_{i=1}^{l-1}(t^{-i}+t^{i})(-1)^{l-1-i}\right]+(-1)^l\\
&\quad +(t^{-(l+1)}+t^{(l+1)})\\
&\quad +(t^{-(l+k)}+t^{(l+k)}-t^{-((l+k)+1)}-t^{((l+k)+1)}). \qquad (4.6)
\end{aligned}$$

We will prove that for all $k,l\in\mathbb{N}$ the knots $N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2k})$ are non-alternating.

Let be $k,l\in\mathbb{N}$, from (4.6) we have that $\Delta(N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2k}))$ is a polynomial with variable t whose exponents are among $l+k+1$ and $-(l+k+1)$. Furthermore the coefficients of the terms with exponents $\pm l$ are zero. Therefore, for all $l,k\in\mathbb{N}$ the polynomial $\Delta(N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2k}))$ have zero coefficients and due to Theorem 4.1.1 the knots $N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2k})$ are non-alternating.

Now we will prove for all $k\in\mathbb{N}$ and $l=0$, the knots $N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2k})$ are non-alternating. By the equation (4.6) we have that $\Delta(N_1(\mathcal{T}(1)\cdot c\cdot\mathcal{E}^{2k})) = (-1+t^{-1}+t+t^{-(k)}+t^{(k)}-t^{-(k+1)}-t^{(k+1)})$. For $k=1$, the diagram $N_1(\mathcal{T}(1)\cdot c\cdot\mathcal{E}^{2})$ has 9 crossings and it can be proved that is a diagram of the knot 9_{42}, which is a non-alternating knot. For $k=2$, the polynomial is $\Delta(N_1(\mathcal{T}(1)\cdot c\cdot\mathcal{E}^{4})) = -1+t^{-1}+t+t^{-2}+t^{2}-(t^{-3}+t^{3})$, whose coefficients do not alternate and, by Theorem 4.1.1, this knot is non-alternating. For $k\geq 3$, the polynomial $\Delta(N_1(\mathcal{T}(1)\cdot c\cdot\mathcal{E}^{2k}))$ have coefficients zero, so and due to Theorem 4.1.1 for all $k\in\mathbb{N}$ and $l=0$ the knots $N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2k})$ are non-alternating.

Therefore for all $l\in\mathbb{N}\cup\{0\}$ and $k\in\mathbb{N}$ the knots $N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2k})$ are non-alternating.

□

Corollary 4.1.1. *For all $k\in\mathbb{N}$ and $l\in\mathbb{N}\cup\{0\}$ the knots $N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2k})$ are pairwise distinct.*

Proof. The Alexander polynomial is an invariant of oriented links and by (4.6) we have that

$$\Delta(N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2k_1})) \neq \Delta(N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2k_2})),$$

for all $k_1,k_2\in\mathbb{N}$ with $k_1\neq k_2$. Furthermore,

$$\Delta(N_1(\mathcal{T}(2l_1+1)\cdot c\cdot\mathcal{E}^{2k})) \neq \Delta(N_1(\mathcal{T}(2l_2+1)\cdot c\cdot\mathcal{E}^{2k})),$$

for all $l_1,l_2\in\mathbb{N}$ with $l_1\neq l_2$. □

The following proposition shows other family of non-alternating knots. In this case k is in $\mathbb{N}\cup\{0\}$.

Proposition 4.1.1. *For all $l \in \mathbb{N}$ and $k \in \mathbb{N}\cup\{0\}$ the knots $N_1(\mathcal{T}(2l+1,2,2)\cdot c\cdot \mathcal{E}^{2k})$ are non-alternating.*

Proof. Similarly to Theorem 4.1.4, for all $l,k \in \mathbb{N}\cup\{0\}$ we have (4.7)

$$\begin{aligned}\Delta(N_1(\mathcal{T}(2l+1,2,2)\cdot c\cdot \mathcal{E}^{2k})) \quad = \quad & \left[\sum_{i=1}^{l}(t^{-(i-1)}+t^{(i-1)})(-1)^{l-i}\right]+(-1)^l \\ & +2(-t^{-(l+1)}-t^{l+1}+t^{-(l+2)}+t^{l+2}) \\ & +(t^{-(l+k+1)}+t^{l+k+1}-t^{-(l+k+2)}-t^{l+k+2}). \quad (4.7)\end{aligned}$$

By (4.7) for all $l \in \mathbb{N}$ and $k \in \mathbb{N}\cup\{0\}$ the polynomial $\Delta(N_1(\mathcal{T}(2l+1,2,2)\cdot c\cdot \mathcal{E}^{2k}))$ have exponents between $\pm(l+k+2)$, and the coefficients of terms with exponents $\pm l$ are zero. Thus for all $l \in \mathbb{N}$ and $k \in \mathbb{N}\cup\{0\}$ the polynomials $\Delta(N_1(\mathcal{T}(2l+1,2,2)\cdot c\cdot \mathcal{E}^{2k}))$ have zero coefficients and due to Theorem 4.1.1 the knots $N_1(\mathcal{T}(2l+1,2,2)\cdot c\cdot \mathcal{E}^{2k})$ are non-alternating. □

The proof of the Corollary 4.1.2 is analogous to the one given in Corollary 4.1.1.

Corollary 4.1.2. *For all $k \in \mathbb{N}\cup\{0\}$ and $l \in \mathbb{N}$ the knots $N_1(\mathcal{T}(2l+1,2,2)\cdot c\cdot \mathcal{E}^{2k})$ are pairwise distinct.*

Theorem 4.1.5. *For all $l,m \in \mathbb{N}$ and $k \in \mathbb{N}\cup\{0\}$ the knots $N_1(\mathcal{T}(2l+1,2,2m)\cdot c\cdot \mathcal{E}^{2k})$ are non-alternating.*

By Theorem 2.5.1 we have that for all $l,m \in \mathbb{N}$ and $k \in \mathbb{N}\cup\{0\}$ the link $N_1(\mathcal{T}(2l+1,2,2m)\cdot c\cdot \mathcal{E}^{2k})$ is a knot.

We take $\nabla(\mathcal{T}(2l+1,2,2m)) = \sum_{i=1}^{6} p_i\chi_i$. By (4.2) for $n_1 = 1$ we have that

$$\begin{aligned}\nabla(\mathcal{T}(2l+1,2,2m_1)) = \quad & \alpha_{3l+m_1}\nabla(\chi_1)+\alpha_{1_{l+1}}z\alpha_{1_{m_1}}\nabla(\chi_2) \\ & +\alpha_{1_{l+m_1+1}}\nabla(\chi_3)+\alpha_{1_{l+1}}z\alpha_{1_{m_1}}\nabla(\chi_4) \\ & +\alpha_{3_l}z\alpha_{3_{m_1}}\nabla(\chi_5)+\alpha_{1_{l+1}}z\alpha_{3_{m_1}}\nabla(\chi_6)\end{aligned}$$

Furthermore, by (4.3) if follows that $\nabla(N_1(\mathcal{T}(2l+1,2,2m)\cdot c)) = z\nabla(p_4+p_5+zp_6)+p_3+p_6$. Calculate $\nabla(N_1(\mathcal{T}(2l+1,2,2m)\cdot c))$ by using (3.11) and Proposition 3.2.2.

$$\begin{aligned}&\nabla(N_1(\mathcal{T}(2l+1,2,2m)\cdot c)) \\ &= z(\alpha_{1_{l+1}}z\alpha_{1m}+\alpha_{3_l}z\alpha_{3_m}+\alpha_{1_{l+1}}z^2\alpha_{3_m})+\alpha_{1_{l+m+1}}+\alpha_{1_{l+1}}z\alpha_{3_m} \\ &= z^2\alpha_{1_{l+1}}(\alpha_{1m}+\alpha_{3_m}z)+\alpha_{1_{l+1}}\alpha_{3_m}z+\alpha_{3_l}\alpha_{3_m}z^2+\alpha_{1_{l+m+1}} \\ &= z^2\alpha_{1_{l+1}}\alpha_{1m+1}+\alpha_{1_{l+1}}\alpha_{3_m}z+\alpha_{3_l}\alpha_{3_m}z^2+\alpha_{1_{l+m+1}} \\ &= z^2(\alpha_{1_{l+1}}\alpha_{1m+1}+\alpha_{3_l}\alpha_{3_m})+\alpha_{1_{l+1}}\alpha_{3_m}z+\alpha_{1_{l+m+1}} \\ &= z^2\alpha_{1_{l+m+1}}+\alpha_{1_{l+1}}\alpha_{3_m}z+\alpha_{1_{l+m+1}} \\ &= (1+z^2)\alpha_{1_{l+m+1}}+\alpha_{1_{l+1}}\alpha_{3_m}z\end{aligned}$$

We rewrite it as before by the variable change $z = x - x^{-1}$ where $x = t^{\frac{1}{2}}$, and also by using equations (3.17) and (3.18) and the fact that $\alpha_{1_k} = 1-\alpha_{4_k}$.

$$
\begin{aligned}
& \Delta(N_1(\mathcal{T}(2l+1,2,2m)\cdot c)) \\
= \quad & (-1+x^2+x^{-2})\left(\left[-\sum_{i=1}^{l+m+1}(x^{-(2i-2)}+x^{2i-2})(-1)^{l+m+1+1-i}\right]-(-1)^{l+m+1+1}\right) \\
& +\left(\left[-\sum_{i=1}^{l+1}(x^{-(2i-2)}+x^{2i-2})(-1)^{l+1+1-i}\right]-(-1)^{l+1+1}\right) \\
& \left(-\sum_{j=1}^{m}(x^{-(2j-1)}-x^{2j-1})(-1)^{m-j}\right)(x-x^{-1}) \\
= \quad & \left[\sum_{i=1}^{l+m+1}(x^{-(2i)}+x^{2i}-x^{-(2i-2)}-x^{2i-2}+x^{-(2i-4)}+x^{2i-4})(-1)^{l+m+1-i}\right] \\
& -(-1)^{l+m}(-1+x^2+x^{-2}) \\
& +\left(\left[\sum_{i=1}^{l+1}(x^{-(2i-2)}+x^{2i-2})(-1)^{l+1-i}\right]-(-1)^{l}\right) \\
& \left(\sum_{j=1}^{m}(x^{-(2j)}+x^{2j}-x^{-(2j-2)}-x^{2j-2})(-1)^{m-j}\right) \\
= \quad & \left[\sum_{i=1}^{l+m+1}(x^{-(2i)}+x^{2i})(-1)^{l+m+1-i}-\sum_{i=1}^{l+m+1}(x^{-(2(i-1))}+x^{2(i-1)})(-1)^{l+m+1-i}\right. \\
& \left.+\sum_{i=1}^{l+m+1}(x^{-(2(i-2))}+x^{2(i-2)})(-1)^{l+m+1-i}\right] \\
& -(-1)^{l+m}(-1+x^2+x^{-2}) \\
& +\left(\left[\sum_{i=1}^{l+1}(x^{-(2i-2)}+x^{2i-2})(-1)^{l+1-i}\right]-(-1)^{l}\right) \\
& \left(\sum_{j=1}^{m}(x^{-(2j)}+x^{2j})(-1)^{m-j}-\sum_{j=1}^{m}(x^{-(2(j-1))}+x^{2(j-1)})(-1)^{m-j}\right) \\
= \quad & \left[\sum_{i=1}^{l+m+1}(x^{-(2i)}+x^{2i})(-1)^{l+m+1-i}+\sum_{i=0}^{l+m}(x^{-(2i)}+x^{2i})(-1)^{l+m+1-i}\right. \\
& \left.+\sum_{i=-1}^{l+m-1}(x^{-(2i)}+x^{2i})(-1)^{l+m+1-i}\right] \\
& -(-1)^{l+m}(-1+x^2+x^{-2}) \\
& +\left(\left[\sum_{i=1}^{l+1}(x^{-(2i-2)}+x^{2i-2})(-1)^{l+1-i}\right]-(-1)^{l}\right) \\
& \left(\sum_{j=1}^{m}(x^{-(2j)}+x^{2j})(-1)^{m-j}-\sum_{j=0}^{m}(x^{-(2j)}+x^{2j})(-1)^{m-j}(-1)^{m-j}\right) \\
= \quad & \left[3\sum_{i=1}^{l+m-1}(x^{-(2i)}+x^{2i})(-1)^{l+m+1-i}-2(x^{-2(l+m)}+x^{2(l+m)})\right. \\
& \left.+(x^{-2(l+m+1)}+x^{2(l+m+1)})-3(-1)^{l+m}\right] \\
& +\left(\left[\sum_{i=1}^{l+1}(x^{-(2i-2)}+x^{2i-2})(-1)^{l+1-i}\right]-(-1)^{l}\right) \\
& \left(2\sum_{j=1}^{m-1}(x^{-(2j)}+x^{2j})(-1)^{m-j}+(x^{-(2m)}+x^{2m})+(x^{-(2)}+x^{2})(-1)^{m}\right) \\
= \quad & \left[3\sum_{i=1}^{l+m-1}(x^{-(2i)}+x^{2i})(-1)^{l+m+1-i}-2(x^{-2(l+m)}+x^{2(l+m)})\right. \\
& \left.+(x^{-2(l+m+1)}+x^{2(l+m+1)})-3(-1)^{l+m}\right] \\
& +\left(2\sum_{i=1,j=1}^{i=l+1,j=m-1}(x^{-(2(i+j)-2)}+x^{2(i+j)-2}+x^{-(2(i-j)-2)}+x^{2(i-j)-2})(-1)^{l+m+1-(i+j)}\right) \\
& +\left(\sum_{i=1}^{l+1}(x^{-(2(i+m)-2)}+x^{2(i+m)-2}+x^{-(2(i-m)-2)}+x^{2(i-m)-2})(-1)^{l+1-i}\right) \\
& +\left(\sum_{i=1}^{l+1}(x^{-(2i-2)}-x^{2i-2})\right)(x^{-(2)}+x^{2})(-1)^{m} \\
& -(-1)^{l}\left(2\sum_{j=1}^{m-1}(x^{-(2j)}+x^{2j})(-1)^{m-j}+(x^{-(2m)}+x^{2m})+(x^{-(2)}+x^{2})(-1)^{m}\right)\Bigg)
\end{aligned}
$$

After a variable change $x=t^{\frac{1}{2}}$, due to Lemma 4.1.1 we have that $span(\Delta(N_1(\mathcal{T}(2l+1,2,2m)\cdot c)))=l+m+1$. Besides, as Alexander polynomial is symmetric then the exponents of $\Delta(N_1(\mathcal{T}(2l+1,2,2m)\cdot c))$ are between $-(l+m+1)$ and $l+m+1$. Further, it easy to see that the coefficients of t^{l+m-1} and $t^{-(l+m-1)}$ are zero in $\Delta(N_1(\mathcal{T}(2l+1,2,2m)\cdot c))$ therefore for $k=0$ the knots $N_1(\mathcal{T}(2l+1,2,2m)\cdot c\cdot\mathcal{E}^{2k})$ are non-alternating. For $k\geq 1$, (4.5) holds, thus

$$\Delta(N_1(\mathcal{T}(2l+1,2,2m)\cdot c\cdot \mathcal{E}^{2k})) = \Delta(N_1(\mathcal{T}(2l+1,2,2m)\cdot c)) - (t^{-(g+k+1)} + t^{(g+k+1)}) \\ + (t^{-(t+k)} + t^{(g+k)}) + (t^{-(g+1)} - t^{(g+1)}) - (t^{-g} + t^{g}),$$

where $g = l+m$ and therefore for all $k \geq 0$ the knots $N_1(\mathcal{T}(2l+1,2,2m)\cdot c\cdot \mathcal{E}^{2k})$ are non-alternating. □

In the following section we will use the HOMFLY-PT polynomial to prove some links are non-alternating.

4.2 By using the HOMFLY-PT polynomial

Let D be a diagram of an oriented link and let G be its Seifert graph. Each edge in G can be given a sign $+$ or $-$ depending on whether it passed through a positive or negative crossing. If the sign of all edges in each block of G is the same, then D is a **homogeneous diagram**. A link L is **homogeneous** if L has a homogeneous diagram.

It is known that the class of homogeneous links includes alternating links and positive links [16]. In this section we will use a result given by Cromwell in [16] to prove that certain links are non-homogeneous and therefore they are non-alternating links.

Theorem 4.2.1. *[16] A link L is non-homogeneous if $P(L)$ has no terms of the form*

$$\lambda(-1)^{\frac{1}{2}(r-s)} v^s z^r$$

for some $\lambda \in \mathbb{N}$, where r is the highest degree of $P(L)$ in the variable z, $s \leq r$.

Theorem 4.2.2. *For all $k \in \mathbb{N}$ and c, we have that:*

1. *The links $N_1(\mathcal{E}^{2k})$, $N_3(\mathcal{E}^{2k})$ and $N_1(\mathcal{E}^{2k}\cdot c)$ are non-homogeneous.*
2. *For all $l \in \mathbb{N}$, the knots $N_1(\mathcal{T}(-(2l+1),2,2)\cdot c)$ and $N_1(\mathcal{T}(3,2,-2l)\cdot c)$ are non-homogeneous knots.*
3. *For all $l \in \mathbb{N}\cup\{0\}$, the knots $N_1(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2k})$ are non-homogeneous knots.*

Proof. We will prove for $N_1(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2k})$; the other cases are analogous. First, we obtain $P(c) = vz\,P(\chi_1) + v^2\,P(\chi_2)$ and by Lemma 3.1.1 *ii*) we have that $P(\mathcal{T}(2l+1)) = A_{3_l} v^{4l+2}\,P(\chi_1) + A_{1_{l+1}} v^{4l+2}\,P(\chi_3)$. Hence, by Theorem 3.1.1, we have that

$$P(\mathcal{T}(2l+1)\cdot c) = A_{3_l} v^{4l+3} z\,P(\chi_1) + A_{3_l} v^{4l+4}\,P(\chi_2) + A_{1_{l+1}} v^{4l+3} z\,P(\chi_3) + A_{1_{l+1}} v^{4l+4}\,P(\chi_4).$$

Due to Corollary 3.1.2 we have that

$$P(N_1(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2k})) = \\ A_{3_l} v^{4l+3} zP(N_1(\mathcal{E}^{2k})) + A_{3_l} v^{4l+4}\delta + A_{1_{l+1}} v^{4l+3} zP(N_3(\mathcal{E}^{2k})) + A_{1_{l+1}} v^{4l+4}.$$

Then, by using and Corollary 3.1.1, we can obtain the polynomial with maximal degree in the variable z of $P(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2k})$, namely $-v^{2l-2k+2} z^{2l+2k+2}$, which is given in the polynomial $A_{1_{l+1}} v^{4l+3} z P(N_3(\mathcal{E}^{2k}))$. The polynomial $-v^{2l-2k+2} z^{2l+2k+2}$, for $\lambda \in \mathbb{N}$ and $s \leq r$, does not have terms of the form $\lambda(-1)^{\frac{1}{2}(r-s)} v^s z^r$. So, by Theorem 4.2.1 the knots $N_1(\mathcal{T}(2l+1) \cdot c \cdot \mathcal{E}^{2k})$ are non-homogeneous. $\square$

Cromwell in [16] showed that the first three non-homogeneous knots are 8_{20}, 8_{21} and 9_{42}. The diagrams $N_1(\mathcal{T}(-3,2,2) \cdot c)$, $N_1(\mathcal{T}(3,2,-2) \cdot c)$ and $N_1(\mathcal{T}(1) \cdot c \cdot \mathcal{E}^2)$, are diagrams of the knots 8_{20}, 8_{21} and, 9_{42}, respectively. In particular, they have the minimal number of crossings in each family.

In the following chapter we will estimate the alternation number and the dealternating number of some families of links given in the present chapter.

Chapter 5

Classification of families of non-alternating knots

In this chapter we will estimate the alternation number and the dealternating number of the families of knots given in the previous chapter. Besides, we will determine if they are prime and hyperbolic. We will begin with knots composed with the closure N_1 of the concatenation of a 3-braid $\mathcal{T}$, the 3-tangle c and $\mathcal{E}^{2k}$, all of them inhabitants in $S(R)$. Then the knots will be of the form $N_1(\mathcal{T} \cdot c \cdot \mathcal{E}^{2k})$.

5.1 Knots of the form $N_1(\mathcal{T} \cdot c \cdot \mathcal{E}^{2k})$

Let $\mathcal{NK}$ denote the set of non-alternating knots of the form $N_1(T \cdot \mathcal{E}^{2k})$ such that T is a 3-tangle of the form $\mathcal{T} \cdot c$ where $\mathcal{T}$ is a 3-braid in $S(R)$ and $k \in \mathbb{N} \cup \{0\}$.

Lemma 5.1.1. *If K is a knot in $\mathcal{NK}$, then $br(K) = 3$.*

Proof. The knot K has a diagram of three bridges (see right Figure 5.1), and so $br(K) \leq 3$. Suppose $br(K) \leq 2$, it follows that K is alternating. However, since $K \in \mathcal{NK}$ it implies that K is non-alternating then $br(K) = 3$. □

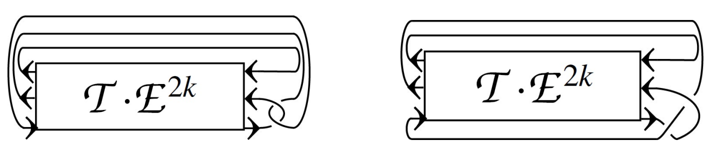


Figure 5.1: Two equivalent diagrams: $N_1(\mathcal{T} \cdot \mathcal{E}^{2k} \cdot c)$ and a 3-bridge knot diagram.

Note that Lemma 5.1.1 holds also for the link case. If we take $\mathcal{T}$ such that $\hat{\mathcal{T}} = \hat{\chi}_1$ then $N_1(\mathcal{T} \cdot c \cdot \mathcal{E}^{2k})$ will be a link.

Theorem 5.1.1. *If K is a knot in $\mathcal{NK}$, then it is a prime knot.*

Proof. Suppose that K is non-prime then K is the connected sum of non-trivial knots K_1 and K_2. Due to Proposition 2.1.3 we have that $br(K) = br(K_1) + br(K_2) - 1$. By Lemma 5.1.1 we have that $br(K) = 3$, thus as K_1 and K_2 are non-trivial then $br(K_1) = br(K_2) = 2$ and therefore they are alternating knots. Further, since the connected sum of alternating knots is also an alternating knot it implies that $K \notin \mathcal{NK}$. Hence K is a prime knot. $\square$

The previous theorem implies that the knots corresponding to the diagrams given in Figure 3.3 are prime. Furthermore, by using their Alexander polynomial we can determine that those diagrams are of the knots 8_{19}, 8_{20}, 8_{21}, and 9_{42}, respectively.

We remark that the alternation number of a link L, denoted by $alt(L)$, is the minimum number of crossing changes necessary to transform a diagram D of L into some diagram of an alternating link. For example, the knot diagram showed in the Figure 5.2 after a crossing change is a diagram of the trivial knot, then this diagram has alternation number one.

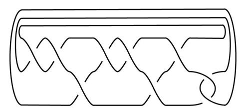
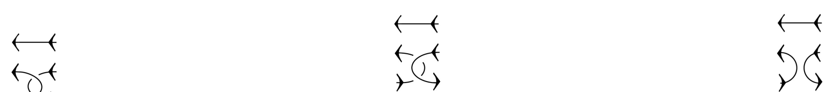


Figure 5.2: Knot diagram with alternation number one.

In order to prove that knots $N_1(T \cdot \mathcal{E}^{2k})$ have alternation number equal to one, we will realize a crossing change on the 3-tangle c in T. Note that, after a crossing change in the 3-tangle c, we obtain the 3-tangle χ_2 as shown in Figure 5.3.



The 3-tangle c | The 3-tangle c after a crossing change | Another diagram of c after a crossing change

Figure 5.3: 3-tangle c before and after a crossing change.

The following theorem shows that the knots in $\mathcal{NK}$, after one crossing change, are alternating knots. In particular they are 2-bridge knots or the trivial knot.

Theorem 5.1.2. *If K is a knot in $\mathcal{NK}$, then $alt(K) = 1$.*

Proof. If K is in $\mathcal{NK}$ then it is non-alternating and by definition $alt(K) > 0$. It is only needed to prove that $alt(K) < 2$ to prove the result. After a crossing change in the 3-tangle c and several Reidemeister moves, the k full twists vanish and we obtain the diagram $N_2(\mathcal{T})$. Since Lemma 2.2 implies that $N_2(\mathcal{T})$ is a diagram of a 2-bridge knot or the trivial knot, which are alternating, then $alt(K) = 1$. $\square$

Corollary 5.1.1. *If K is a knot in $\mathcal{NK}$ different from 8_{19} or 10_{124}, then it is a hyperbolic knot.*

Proof. Let $K \in \mathcal{NK}$. Then due to Theorems 5.1.1 and 5.1.2 and Lemma 5.1.1 we have that K is prime, have alternation number one, and is a 3-bridge knot. Proposition 2.1.2 states that the prime knots with bridge index less than or equal to 3 that are not torus knots are hyperbolic. Besides, due to Theorem 2.4.1 the only torus knots with alternation number one are 8_{19} and 10_{124}. Therefore, if K is different from 8_{19} or 10_{124} then K is a hyperbolic knot. $\square$

The knots 8_{19} and 10_{124} have diagrams $N_1(\mathcal{T}(3,2,2)\cdot c)$ and $N_1(\mathcal{T}(5,2,2)\cdot c)$, respectively. Then they are of the form $N_1(\mathcal{T}(2l+1,2,2)\cdot c)$.

5.1.1 Knots of the form $N_1(\mathcal{T}(2l+1,2n_1,2m_1,...,2n_r,2m_r)\cdot c\cdot \mathcal{E}^{2k})$

Now, we will deal with the family of knots $N_1(\mathcal{T}(2l+1,2n_1,2m_1,...,2n_r,2m_r)\cdot c\cdot \mathcal{E}^{2k})$ where $l,r,n_j,m_j\in\mathbb{N}$ and $j=1,\dots,r$. Note that Theorem 2.5.1 states that $N_1(\mathcal{T}(2l+1,2n_1,2m_1,...,2n_r,2m_r)\cdot c\cdot \mathcal{E}^{2k})$ has only one component, then it is a knot.

Corollary 5.1.2. *For all $l,r\in\mathbb{N}$, if K is a knot of the form*

$$N_1(\mathcal{T}(2l+1,2n_1,2m_1,...,2n_r,2m_r)\cdot c\cdot \mathcal{E}^{2k})$$

and $k\geq 3$, then K is a prime knot and $alt(K)=1$.

Proof. Theorem 4.1.3 implies that for all $l,r\in\mathbb{N}$ and $k\geq 3$ we have that K is a knot in $\mathcal{NK}$. Hence, from Theorems 5.1.1 and 5.1.2 it follows that K is a prime knot and has $alt(K)=1$. □

Corollary 5.1.3. *If K is either a knot of the form $N_1(\mathcal{T}(2l+1,2,2)\cdot c\cdot \mathcal{E}^{2k})$ for $l\in\mathbb{N}$ and $k\in\mathbb{N}\cup\{0\}$ or a knot of the form $N_1(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2k})$ for $l\in\mathbb{N}\cup\{0\}$ and $k\in\mathbb{N}$, then $alt(K)=1$.*

Proof. Let K be a knot with diagram $N_1(\mathcal{T}(2l+1,2,2)\cdot c\cdot \mathcal{E}^{2k})$ with $l\in\mathbb{N}$ and $k\in\mathbb{N}\cup\{0\}$. Proposition 4.1.1 states that the knots $N_1(\mathcal{T}(2l+1,2,2)\cdot c\cdot \mathcal{E}^{2k})$ are non-alternating. Therefore, K is a knot in $\mathcal{NK}$. From Theorems 5.1.1 and 5.1.2 it follows that K is a prime knot and Figure 5.4 shows that $alt(K)=1$. The other case is analogous by using Theorem 4.1.4. □

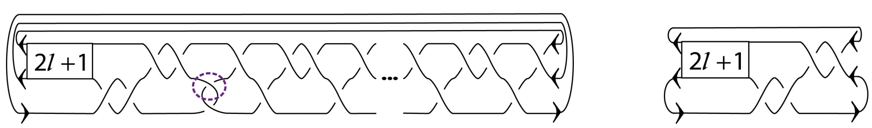


Figure 5.4: At the left $N_1(\mathcal{T}(2l+1,2,2)\cdot c\cdot \mathcal{E}^{2k})$ and at the right the resultant diagram after one crossing change in the highlight part and several Reidemeister moves.

In the following section we will describe the family of knots, $\mathcal{D}$, with dealternating number arbitrarily large while the alternation number is one. In order to do this, we will use the Khovanov width of some knots to obtain a lower bound for the dealternating number (also for the Turaev genus) of knots in $\mathcal{D}$.

5.2 Infinite family of knots with $alt(K)=1$ and $dalt(K)=n$

In Figure 5.5 oriented 3-tangles are shown, which have two different orientations. The orientation of $\mathcal{T}(2,-1,2,-1)$ and $\mathcal{T}(0,-1,2,-1)$ is the usual for 3-braids and they are not inhabitants of $S(R)$, the other 3-tangles are inhabitants of $S(R)$. We remark that the 3-braid

$\mathcal{T}(2l+1)$ has $2l+1$ crossings, $\mathcal{E}^2$ is a full twist and $\mathcal{T}(2,-1,2,-1)$ is another diagram for $\mathcal{E}^2$.

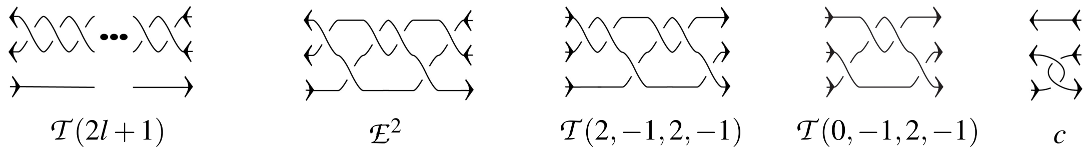


Figure 5.5: Some 3-tangles diagrams endowed with two orientations.

For all n in $\mathbb{N}$ the family $\mathcal{D}_n$ is defined as follows:

$$\mathcal{D}_n = \{N_1(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})\,|\,l+1\in\mathbb{N}\},$$

where $\mathcal{T}(2l+1)$, c and $\mathcal{E}^2$ are defined as above and are inhabitants of $S(R)$, and N_1 is the usual closure of 3-tangles (see Figures 5.5 and 5.6).

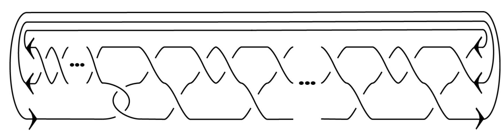


Figure 5.6: $N_1(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})$.

Now, by using the Khovanov width of some closed 3-braids it will be proved that if K is in $\mathcal{D}_n$ then $dalt(K)=n$.

In [34] Lowrance determined the Khovanov width of closed 3-braids based upon Murasugi classification of closed 3-braids up to conjugation. In particular, by using our notation but with the usual orientation of 3-braids (from left to right), we rewrite Proposition 4.8 part (1) of Lowrance as follows.

Proposition 5.2.1. *[34] If* $n>0$ *and* $m\geq 0$*, then* $Kh(N_1(\mathcal{T}(m)\cdot\mathcal{E}^{2n}))$ *is* $[4n+m-3, 6n+m-1]$*-thick and* $w_{Kh}(N_1(\mathcal{T}(m)\cdot\mathcal{E}^{2n}))=n+2$.

Let $\sigma(L)$ be the signature of a link L [43], where the right-hand trefoil knot has signature -2. In [34] Lowrance showed that the results in [37] implies that if L is alternating then $Kh(L)$ is $[-\sigma(L)-1,-\sigma(L)+1]$-thick and $w_{Kh}(L)=2$. He considered a class of links, which contains the set of alternating links, and proved that for a link L in this class we have that $Kh(L)$ is $[-\sigma(L)-1,-\sigma(L)+1]$-thick and $w_{Kh}(L)=2$. We will consider this result for alternating knots to obtain the Khovanov width of $K\in\mathcal{D}$.

Lemma 5.2.1. *If* $K\in\mathcal{D}_n$ *then* $w_{Kh}(K)=n+2$.

Proof. Let $K=N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2n})$ and D_+ the diagram $N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2n})$. Resolve the crossing in the neighborhood marked with a circle to obtain D_v and D_h (see Figure 5.7) and take $m=2l+1$. So D_v is a diagram for $N_1(\mathcal{T}(m)\cdot\mathcal{E}^{2n})$, which is a closed 3-braid without the usual orientation and D_h is a diagram for $N_1(\mathcal{T}(m)\cdot\mathcal{T}(0,-1)\cdot\mathcal{E}^{2n})$, which has the usual orientation for 3-braids. In order to obtain δ_{min} and δ_{max} of $Kh(D_+)$ we will calculate δ_{min} and δ_{max} of $Kh(D_v)$ and $Kh(D_h)$, respectively.

Let D_v^* be the diagram $N_1(\mathcal{T}(m)\cdot\mathcal{E}^{2n})$ with the usual orientation for 3-braids. Note that D_v has a component with reverse orientation to D_v^*, also note that $neg(D_v^*)=0$ and $neg(D_v)=4n$, then (A.9) implies that $Kh^{\delta}(D_v)\cong Kh^{\delta+s}(D_v^*)$ where $s=4n$. Since that by Proposition 5.2.1 we have that $Kh(N_1(D_v^*))$ is $[4n+m-3,6n+m-1]$-thick, it follows that $Kh(D_v)$ is $[m-3,2n+m-1]$-thick.

Resolve D_h at the crossing of $\mathcal{T}(0,-1)$ to obtain D_{h_v} and D_{h_h}. Note that $D_{h_v}=D_v*$ then $Kh(D_{h_v})$ is $[4n+m-3,6n+m-1]$-thick. Also note that D_{h_h} is a diagram for $T(2,m)$, which is alternating, then $Kh(D_{h_h})$ is $[-\sigma(T(2,m))-1,-\sigma(T(2,m))+1]$-thick where $\sigma(T(2,m))=-m+1$ and therefore $Kh(D_{h_h})$ is $[m-2,m]$-thick. As $neg(D_{h_h})-neg(D_h)=4n$ by Corollary 2.3.1 the group $Kh(D_h)$ is $[4n+m-2,6n+m]$-thick.

Now, $e=neg(D_h)-neg(D_+)=-4n$, since $(m-3)\neq(4n+m-2)+e+1$ and $(2n+m-1)\neq(6n+m)+e+1$ Corollary 2.3.1 implies that $Kh(D_+)$ is $[m-2,2n+m]$-thick. Hence, $w_{Kh}(N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2n}))=n+2$. □

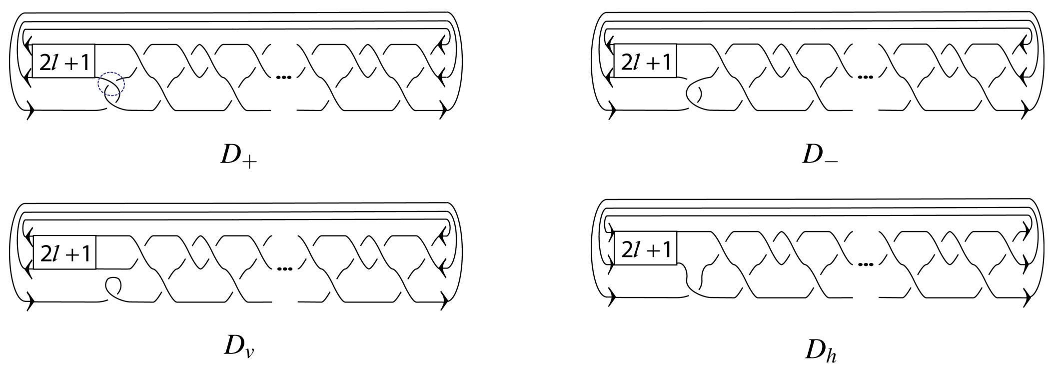


Figure 5.7: Diagrams D_+, D_-, D_v, and D_h. In D_+ it is marked the neighborhood of the crossing that differs from D_-, D_v and D_h.

Lemma 5.2.1 is an important result to prove Theorem 5.2.1.

Theorem 5.2.1. *For all $n\in\mathbb{N}$ there exists an infinite knot family, $\mathcal{D}_n$, such that if $K\in\mathcal{D}_n$ then $dalt(K)=g_T(K)=n$.*

Proof. Let $n\in\mathbb{N}$ and $K=N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2n})$ with $l\in\mathbb{N}\cup\{0\}$. Since $\mathcal{E}^2$ is a full twist then $\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2n}=\mathcal{T}(2l+1)\cdot\mathcal{E}^{2n}\cdot c$.

Note that, the following 3-braids are equivalent (see Figure 5.8):

$$\begin{aligned}\mathcal{E}^{2n} &= (\mathcal{T}(1,-1,2,-1,1))^n\\ &= (\mathcal{T}(2,-1,2,-1))^n\\ &= (\mathcal{T}(2)\cdot\mathcal{T}(0,-1,2,-1))^n\\ &= (\mathcal{T}(2))^n\cdot(\mathcal{T}(0,-1,2,-1))^n\\ &= \mathcal{T}(2n)\cdot(\mathcal{T}(0,-1,2,-1))^n.\end{aligned}$$

Therefore,

$$\begin{aligned}N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2n}) &= N_1(\mathcal{T}(2l+1)\cdot\mathcal{T}(2n)\cdot(\mathcal{T}(0,-1,2,-1))^n\cdot c)\\ &= N_1(\mathcal{T}(2l+1+2n)\cdot(\mathcal{T}(0,-1,2,-1))^n\cdot c),\end{aligned}$$

where $\mathcal{T}(2l+1+2n)$ is alternating and $(\mathcal{T}(0,-1,2,-1))^n$ is non-alternating.

The diagram $N_1(\mathcal{T}(2l+1+2n)\cdot(\mathcal{T}(0,-1,2,-1))^n\cdot c)$ can be rewritten as diagram D, see Figure 5.9. The n crossings that we will change in order to obtain an alternating diagram are marked in D. Since D is another diagram for K, it follows that

$$dalt(K) \leq n. \tag{5.1}$$

On the other hand, by Lemma 2.4.1 and Corollary 2.4.1, we have the inequalities:

$$w_{Kh}(K) - 2 \leq g_T(K) \leq dalt(K). \tag{5.2}$$

Since, $dalt(K) \leq n$ and Lemma 5.2.1 states that $w_{Kh}(K) = n+2$ we conclude that $g_T(K) = dalt(K) = n$. □

Figure 5.8: Two equivalent 3-braid diagrams: The first one is $\mathcal{E}^4$ and the second one is the diagram $\mathcal{T}(4)\cdot\mathcal{T}(0,-1,2,-1)\cdot\mathcal{T}(0,-1,2,-1)$.

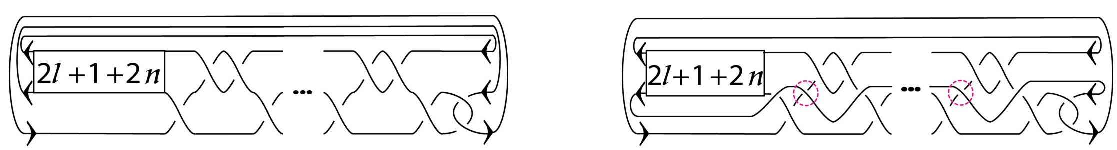


Figure 5.9: Two equivalent knot diagrams: The first one is $N_1(\mathcal{T}(2l+1+2n)\cdot(\mathcal{T}(0,-1,2,-1))^n\cdot c)$ and the second one is the diagram D.

Now for each natural n we have a knot family with dealternating number and Turaev genus equal to n. Further, these knots have alternation number equal to 1,

Theorem 5.2.2. *For all $n \in \mathbb{N}$ there exists an infinite knot family, $\mathcal{D}_n$ with $l \in \mathbb{N} \cup \{0\}$, such that if $K \in \mathcal{D}_n$ then $alt(K) = 1$ and $dalt(K) = g_T(K) = n$.*

Proof. By Theorem 5.2.1 for all $n \in \mathbb{N}$ if $K \in \mathcal{D}_n$ then $dalt(K) = g_T(K) = n$. Now, since that $dalt(K) = n$ it follows that K is not alternating and by definition $alt(K) \geq 1$. Furthermore, the knot K has the diagram D_+, which by one crossing change is transformed into D_-, see diagrams in Figure 5.7. Since D_- is a diagram of an alternating knot, it follows that $alt(K) = 1$. Besides, Corollary 4.1.1 implies that for each n the knots $N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2n})$ are pairwise distinct. □

Previously, in [25] was introduced the family of knots $\mathcal{D}_n$ as an example of knots with alternation number equal to one, also the Alexander polynomial of these knots was obtained.

In [49], Yamada gave an upper bound for the braid index.

Lemma 5.2.2. *[49] Let L be a link and D a diagram of L and let $o(D)$ be the number of Seifert circles of D. Then we have $b(L) \leq o(D)$.*

In order to obtain the braid index for the knots in $\mathcal{D}_n$ we calculate their Γ-polynomial, and we use Lemma 5.2.2.

Lemma 5.2.3. *We have that* $b(N_1(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})) = n+3$ *for all* $l,n \in \mathbb{N}$.

Proof. Let $K = N_1(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})$, by using the formulas to obtain the Γ-polynomial and Proposition 2.2.2 we have the following:

$$\begin{aligned}
&\Gamma(N_1(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n}))\\
&= x^{-1}\Gamma(T(2,2l+1)) - x^{-1}(1-x)x^{-lk(T(2,2l+1+2n),U)}\Gamma(T(2,2l+1+2n))\Gamma(U)\\
&= x^{-1}((l+1)x^{-l} - lx^{-(l+1)}) - x^{-1}(1-x)x^{-(-2n)}((l+n+1)x^{-(l+n)} - (l+n)x^{-(l+n+1)})\\
&= (l+1)x^{-l-1} - lx^{-(l+2)} - (x^{-1}+1)x^{2n}((l+n+1)x^{-(l+n)} - (l+n)x^{-(l+n+1)})\\
&= -lx^{-(l+2)} + (l+1)x^{-(l+1)} + (l+n)x^{-l+n-2} - (2l+2n+1)x^{-l+n-1} + (l+n+1)x^{-l+n}.
\end{aligned}$$

Then, if $l \neq 0$ the $span\,\Gamma(K) = n+2$, and by inequality (2.2) we have that $b(K) \geq n+3$. By Lemma 5.2.2 and diagrams in Figure 5.10 we have that $b(K) \leq n+3$, therefore $b(K) = n+3$.

□

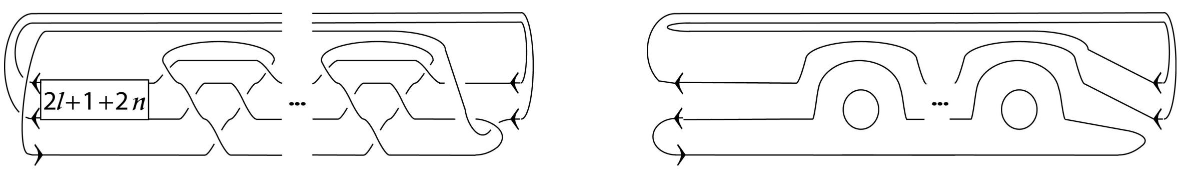


Figure 5.10: Another diagram of $N_1(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})$ and its Seifert circles.

Finally, we have the following result.

Theorem 5.2.3. *If K is a knot in $\mathcal{D}_n$ then it is hyperbolic.*

Proof. Let K be a knot in $\mathcal{D}_n$. Theorem 5.1.1 and Lemma 5.1.1 imply that K is prime and has bridge number 3. It is known that the prime knots with $br(K) \leq 3$ that are not torus knots, are hyperbolic (Proposition 2.1.2), then K is torus knot or hyperbolic knot. Furthermore, by Theorem 5.2.1 we have that K has alternation number one and by Lemma 2.4.1 the only torus knots with alternation number one are 8_{19} and 10_{124}. However, Lemma 5.2.3 implies that if $l \neq 0$ then $4 \leq b(K)$ and it is known (see [12]) that $b(8_{19}) = b(10_{124}) = 3$ this implies that K is different to both 8_{19} and 10_{124}. Therefore K is hyperbolic. In the case $l = 0$ and $n \geq 2$, by Theorem 5.2.1 the dealternating number is greater than 1 and as $dalt(8_{19}) = dalt(10_{124}) = 1$ (see [5]) then K is hyperbolic. Finally, in the case when $l = 0$ and $n = 1$ it is easy to see that K is different to both 8_{19} and 10_{124} and therefore K is hyperbolic. □

To summarize, for each integer n we have a knot family with dealternating number and Turaev genus equal to n and alternation number equal to 1. All the knot in these families are hyperbolic prime knots, non-homogeneous and they have braid index $n+3$ and bridge index 3. Furthermore we obtained the Γ-polynomial for $K \in \mathcal{D}_n$ and used it to calculate the braid index of K. We obtained that for any knot $K \in \mathcal{D}_n$ with $l \in \mathbb{N}$ the Morton-Franks-Williams inequality is sharp, in particular the Morton-Franks-Williams inequality applied to Γ-polynomial is sharp too.

Chapter 6

Conclusions and perspectives

The set of the alternating links has been extensively studied. Recently, Howie in [27] independently of Greene [24] gave a characterization of the alternating links. In order to measure how far is a link to be alternating some definitions have been given. Two of these definitions, the alternation and the dealternating numbers, are defined as the minimal number of crossing changes needed to obtain a diagram of an alternating link. However, the difference between these invariants is that for the dealternating number the diagram obtained, after crossing changes, must be alternating, while that for the alternation number the diagram can be non-alternating. So, from the definitions it follows that for all link L we have that $alt(L) \leq dalt(L)$. Therefore, the goal of this work was to obtain non-alternating links and their alternation and dealternating numbers. To this end, we used different tools; namely, the HOMFLY-PT polynomial, the Alexander polynomial and the Khovanov width. The information that we obtain when we use these tools is distinct, by using the HOMFLY-PT polynomial it is possible to prove if some links are non-homogeneous. But directly from this polynomial is not clear to prove that homogeneous knots are non-alternating. By using the Alexander polynomial, we can prove that some knots are non-alternating, but it is not possible apply it for links with more than one components. In both cases, by using the HOMFLY-PT polynomial and the Alexander polynomial, we cannot estimate the dealternating number. We use Khovanov width to obtain a lower bound of the dealternating number.

Therefore, in order to obtain non-alternating links, we gave formulae to calculate the HOMFLY-PT polynomial of 3-tangles in $S(R)$ and of links formed by the closure of them. We have used the HOMFLY-PT polynomial to prove that some links are non-homogeneous and hence non-alternating links. After that we have gave formulae to obtain the Conway polynomial and the Alexander polynomial and we proved that other families of knots were non-alternating. For some families of knots, we gave their explicit Alexander polynomial and we used it to prove that knots in these families are distinct pairwise.

The diagrams $N_1(\mathcal{T}(3,2,2)\cdot c)$, $N_1(\mathcal{T}(-3,2,2)\cdot c)$, $N_1(\mathcal{T}(3,2,-2)\cdot c)$ and $N_1(\mathcal{T}(1)\cdot c\cdot \mathcal{E}^2)$ are diagrams of 8_{19}, 8_{20}, 8_{21} and 9_{42}, respectively. And, all of them are included in the families of knots that were constructed. All the knots in this construction are 3-bridge knots and after a crossing change yields to 2-bridge knots or the trivial knot.

Furthermore, by using Khovanov width we proved that for each integer n we have a knot family $\mathcal{D}_n = \{N_1(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n}) \,|\, l+1\}$, which is an infinite family of prime hyperbolic knots with alternation number one, dealternating number n and Turaev genus n. Beside, these knots are non-homogeneous and they have braid index $n+3$ and bridge index 3.

On the other hand, we will remark some bounds for the alternation number, the dealternating number, and the Turaev genus, which are different to the presented in Section 2.4. We denote the *signature* of a link L by $\sigma(L)$ [43] and the *Rasmussen s-invariant* of a knot by $s(K)$ [45]. In [1], Abe used the behavior of the s-invariant and signature under crossing changes to give a lower bound for the alternation number.

Lemma 6.0.4. *[1] Let K be a knot, then we have* $|s(K)+\sigma(K)| \leq 2\ alt(K)$.

This lower bound has been useful for links with diagrams with positive crossings, in particular for torus knots has been used. But, for many knots the inequality does not sharp, for example for the knot 9_{42}. Also, a lower bound for the Turaev genus was given for Dasbach and Lowrance.

Lemma 6.0.5. *[18] Let K be a knot, then we have that* $|s(K)+\sigma(K)| \leq 2\ g_T(K)$.

Although, the alternation number and the Turaev genus have the same lower and upper bounds, it is not known what is the relation between the alternation number and the Turaev genus. For the knots in the family $\mathcal{D}_n$ we have that $alt(K) \leq g_T(K)$.

Previously, we use the Khovanov width to obtain a lower bound of the dealternating number of knots, however, for a non-split link L, Champanerkar and Kofman showed

Lemma 6.0.6. *[13] For a non-split link L we have that* $w_{Kh}(L)-2 \leq dalt(L)$.

It is known that the Turaev genus is closely related to algebraic invariants. For a link L, Bae and Morton [7] and separately Dasbach et al. [19] showed Lemma 6.0.7, where $cr(L)$ is the crossing number and $V(L;t)$ the Jones polynomial of the knot K.

Lemma 6.0.7. *[7][19] Let L be a link, then we have* $g_T(L) \leq cr(L) - span(V(L;t))$.

Lickorish and Thistlethwaite [33] introduced the concept of an adequate link, which is a generalization of an alternating link and Abe in [2] showed that if $g_T(K) < cr(K) - span(V(K;t))$ then L is not adequate. For some knots in $\mathcal{D}_n$ the inequality holds but, does the inequality hold for any knot $K \in \mathcal{D}_n$? Will it be true that for each integer n the knots $N_1(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2n})$ satisfy that the difference between $cr(K)-span(V(K;t))$ and $g_T(K)$ is greater than n? Finally all these invariants have interesting relations that have not been completely understood.

Chapter A

Khovanov homology

Khovanov in [32] (also see [8]) introduced an invariant of links, now called the Khovanov homology, which is a bigraded $\mathbb{Z}$-module with homological grading i and polynomial grading j so that $Kh(L) = \bigoplus_{i,j} Kh^{i,j}(L)$ and whose graded Euler characteristic is the Jones polynomial.

The Khovanov homology was defined in the following form. Let L be an oriented link and D a diagram of L. Take an ordering of the crossings of D. A **resolution** of D is a diagram where each crossing of D is changed to either its 0-smoothing or 1-smoothing. Then, D has 2^n resolutions, where n is the number of crossings of D and each resolution is a collection of disjoint circles. Taking into consideration the given ordering of the crossings of D, to any $\varepsilon = (\varepsilon_1, ..., \varepsilon_n) \in \{0,1\}^n$ we associate the smoothing D_ε where the i-th crossing of D is ε_i-smoothed. This defines a bijective function between the set of resolutions of D and the set $\{0,1\}^n$. Let V be a graded $\mathbb{Q}$-module, which is freely generated by two basic elements 1 and X with degree 1 and -1, respectively. Let k_ε be the number of the circles of the resolution D_ε. Then we assign to D_ε a graded module $M_\varepsilon = V^{\otimes k_\varepsilon}$, which is the tensor product of v's over all circles in the resolution. So, for a monomial $v = v_1 \otimes \cdots \otimes v_{k_\varepsilon} \in M_\varepsilon$, we define $deg(v) := deg(v_1) + \cdots + deg(v_k)$. Then we define the i-th chain group $C^i(D)$ by

$$C^i(D) := \bigoplus_{|\varepsilon|=i} M_\varepsilon\{i\},$$

where $|\varepsilon| = \sum_1^m \varepsilon_i$ and $M_\varepsilon\{i\}$ denotes M_ε with its gradings shifted by i, namely, for a graded module $M = \oplus_{j\in Z} M^j$ and an integer i, we define the graded module $M\{i\} = \oplus_{j\in Z} M\{i\}^j$ by $M\{i\}^j = M^{j-i}$.

Now, the differential map between two consecutive chain groups, $d_i : C^i(D) \to C^{i+1}(D)$ is given as follows. Fix an ordering of the circles for each smoothing D_ε and associate the i-th tensor factor of M_ε to the i-th circle of D_ε. Take elements ε and $\varepsilon' \in \{0,1\}^n$ such that $\varepsilon_j = 0$ and $\varepsilon'_j = 1$ for some j and that $\varepsilon_i = \varepsilon'_i$ for any $i \neq j$. For such a pair $(\varepsilon, \varepsilon')$, we will define a map $d_{\varepsilon\to\varepsilon'} : M_\varepsilon \to M_{\varepsilon'}$ depending either two circles of D_ε merge into a one circle of D'_ε or one circle of D_ε split into two circles of D'_ε, while all other circles remain the same.

In the case where two circles of D_ε merge into one circle of $D_{\varepsilon'}$, the map $d_{\varepsilon\to\varepsilon'}$ is the identity on all factors except the tensor factors corresponding to the merged circles where it is a multiplication map $m : V \otimes V \to V$ defined by:

$$m(1 \otimes 1) = 1, m(1 \otimes X) = m(X \otimes 1) = X, m(X \otimes X) = 0.$$

In the case where one circle of D_ε splits into two circles of $D_{\varepsilon'}$, the map $d_\varepsilon \to \varepsilon'$ is the identity on all factors except the tensor factor corresponding to the split circle where it is a comultiplication map $\triangle : V \to V \otimes V$ defined by:

$$\triangle\,(1) = 1 \otimes X + X \otimes 1, \triangle\,(X) = X \otimes X.$$

Besides, if there exist distinct integers i and j such that $\varepsilon_i \neq \varepsilon'_i$ and that $\varepsilon_j \neq \varepsilon'_j$, then define $d_{\varepsilon \to \varepsilon'} = 0$. To obtain the differential d^i of the chain complex $C(D)$ we define a map $d^i : C^i(D) \to C^{i+1}(D)$ by $\sum_{|\varepsilon|=i} d^i_\varepsilon$, where $d^i_\varepsilon : M_\varepsilon \to C^{i+1}(D)$ is as follows

$$d^i_\varepsilon(v) := \sum_{|\varepsilon'|=i+1} (-1)^{r(\varepsilon,\varepsilon')} d_{\varepsilon \to \varepsilon'}(v).$$

Where $v \in M_\varepsilon \subset C^i(D)$ and $r(\varepsilon, \varepsilon')$ is equal to the number of 1's ordered before of the factor of ε which is different from ε'. We can check that $(C^i(D), d^i)$ is a cochain complex and we denote its i-th homology group by $H^i(D)$, which is called the **unnormalized Khovanov homology** of D. The i-th chain group induces a graded structure $H^i(D) = \otimes_{j \in Z} H^{i,j}(D)$ due to the map d^i preserves the grading of $C^i(D)$. For any link diagram D, we define its Khovanov homology $Kh^{i,j}(D)$ by

$$Kh^{i,j}(D) := H^{i+neg(D), j-pos(D)+2neg(D)}(D). \tag{A.1}$$

where $pos(D)$ and $neg(D)$ are the number of the positive and negative crossings of D according Figure 2.8, respectively. The grading i is called the homological degree and j is called the q-grading. Also, if we set $C^{i,j}_{Kh}(D) := C^{i+neg(D), j-pos(D)+2neg(D)}(D)$ and $d^i_{Kh} := d^{i+neg(D)}$, the Khovanov homology $Kh^{i,j}(D)$ is the homology group of the chain complex $(C^{i,j}_{Kh}(D), d^i_{Kh})$.

Theorem A.0.4. *[9][8][32] Let L be an oriented link and D a diagram of L. Then $Kh(L) := Kh(D)$ is a link invariant. Moreover, the graded Euler characteristic of the homology $Kh(L)$ equals the Jones polynomial of L, that is,*

$$(q + q^{-1}) V_L(q^2) = \sum_{i,j} (-1)^i rank\, Kh^{i,j}(L) q^j \Big|_{q=-t^{\frac{1}{2}}},$$

where $V_L(t)$ is the Jones polynomial of L.

Let L be an oriented link, and let C be a component of L, denote by l the linking number of C with its complement $L - C$. Fixing a diagram D of L we count the linking number with weights $+1$ or -1 according to Figure 2.8. Let L' be the link L with the orientation of C reversed. Let D be a diagram for L and D' be the diagram D with the component C reversed. Denote the number of negative and positive crossings in D where C is not involved by n_{1_-} and n_{1_+}, respectively; the number of negative and positive crossings where C is involved by n_{2_-} and n_{2_+}; the number of negative and positive crossings in D' where C is not involved by n'_{1_-} and n'_{1_+}, respectively; and the number of negative and positive crossings where C is involved by n'_{2_-} and n'_{2_+}. Then,

$$pos(D) = n_{1_+} + n_{2_+} \text{ and } neg(D) = n_{1_-} + n_{2_-}. \tag{A.2}$$

Similarly,

$$pos(D') = n'_{1_+} + n'_{2_+} \text{ and } neg(D') = n'_{1_-} + n'_{2_-}. \tag{A.3}$$

However, since D' is the diagram D with the component C reversed then

$$n'_{1_+} = n_{1_+}, \quad n'_{1_-} = n_{1_-}, \quad n'_{2_+} = n_{2_-}, \text{ and } n'_{2_-} = n_{2_+}. \tag{A.4}$$

Besides, by definition of linking number, $2l = n_{2_+} - n_{2_-}$, then (A.2), (A.3) and (A.4) imply that

$$pos(D') = pos(D) - 2l \text{ and } neg(D') = neg(D) + 2l. \tag{A.5}$$

Furthermore, by A.1 for a diagram D' we have that

$$Kh^{i,j}(D') := H^{i+neg(D'),j-pos(D')+2neg(D')}(D'). \tag{A.6}$$

Substituting (A.5) in (A.6) we obtain that

$$Kh^{i,j}(D') := H^{i+neg(D)+2l,j-pos(D)+2neg(D)+6l}(D). \tag{A.7}$$

Therefore, for $i, j \in \mathbb{Z}$ there are isomorphisms of groups

$$Kh^{i,j}(D') \cong Kh^{i+2l,j+6l}(D). \tag{A.8}$$

Now, take $\delta = j - 2i$. Then (A.8) implies that:

$$Kh^{\delta}(D') \cong Kh^{\delta+s}(D). \tag{A.9}$$

where $s = 2l$, which is equivalent to $s = neg(D') - neg(D)$.

Example 12. *To exemplify (A.8), we will use the Hopf link diagrams $L2a\{0\}$ and $L2a\{1\}$ as D' and D, respectively. In Figure 2.7 diagrams $L2a\{0\}$ and $L2a\{1\}$, which are endowed with different orientations are shown. The rank of each group $Kh^{i,j}(L2a\{0\})$ and $Kh^{i,j}(L2a\{1\})$ are given, respectively, in the following arrays.*

$j \diagdown i$	-2	-1	0
0			*1*
-2			*1*
-4	*1*		
-6	*1*		

$j \diagdown i$	0	1	2
6			*1*
4			*1*
2	*1*		
0	*1*		

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