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Classical Harmonic Oscillator with Dirac-like Parameters and Possible Applications

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Abstract. We obtain a class of parametric oscillation modes that we call K-modes with damping and absorption that are connected to the classical harmonic oscillator modes through the “supersymmetric” one-dimensional matrix procedure similar to relationships of the same type between Dirac and Schrödinger equations in particle physics. When a single coupling parameter, denoted by K, is used, it characterizes both the damping and the dissipative features of these modes. Generalizations to several K parameters are also possible and lead to analytical results. If the problem is passed to the physical optics (and/or acoustics) context by switching from the oscillator equation to the corresponding Helmholtz equation, one may hope to detect the K-modes as waveguide modes of specially designed waveguides and/or cavities.

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1. Introduction.

Factorizations of differential operators describing simple mechanical motion have been only occasionally used in the past, although in quantum mechanics the procedure led to a vast literature under the name of supersymmetric quantum mechanics initiated by a paper of Witten [1]. However, as shown by Rosu and Reyes [2], for the damped Newtonian free oscillator the factorization method could generate interesting results even in an area settled more than three centuries ago. In the following, we apply some of the supersymmetric schemes to the basic classical harmonic oscillator. In particular, we show how a known connection in particle physics between Dirac and Schrödinger equations could lead in the case of harmonic motion to chirped (i.e., time-dependent) frequency oscillator equations whose solutions are a class of oscillatory modes depending on one more parameter, denoted by K in this work, besides the natural circular frequency ω_0 . The parameter K characterizes both the damping and the losses of these “supersymmetric” partner modes. Moreover, we do not limit this study to one K parameter extending it to several such parameters still getting analytic results. Guided by mathematical equivalence, possible applications in several areas of physics are identified.

2. Classical harmonic oscillator: The Riccati approach.

The harmonic oscillator can be described by one of the simplest Riccati equation

$$u' + u^2 + \kappa\omega_0^2 = 0, \quad \kappa = \pm 1, \quad (1)$$

where the plus sign is for the normal case whereas the minus sign is for the up side down case. Indeed, employing $u = \frac{w'}{w}$ one gets the harmonic oscillator differential equation

$$w'' + \kappa\omega_0^2 w = 0, \quad (2)$$

with the solutions

$$w_b = \begin{cases} W_+ \cos(\omega_0 t + \varphi_+) & \text{if } \kappa = 1 \\ W_- \sinh(\omega_0 t + \varphi_-) & \text{if } \kappa = -1, \end{cases}$$

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where W_{\pm} and φ_{\pm} are amplitude and phase parameters, respectively, which can be ignored in the following.

The particular Riccati solution of Eq. (1) are

$$u_p = \begin{cases} -\omega_0 \tan(\omega_0 t) & \text{if } \kappa = 1 \\ \omega_0 \coth(\omega_0 t) & \text{if } \kappa = -1. \end{cases}$$

It is well known that the particular Riccati solutions enter as nonoperatorial part in the common factorizations of the second-order linear differential equations that are directly related to the Darboux isospectral transformations [3].

Thus, for Eq. (2) one gets ($D_t = \frac{d}{dt}$)

$$(D_t + u_p)(D_t - u_p)w = w'' + (-u_p' - u_p^2)w = 0. \quad (3)$$

To fix the ideas, we shall use the terminology of Witten's supersymmetric quantum mechanics and call Eq. (3) the bosonic equation. We stress here that the supersymmetric terminology is used in this paper only for convenience and should not be taken literally. Thus, the supersymmetric partner (or fermionic) equation of Eq. (3) is obtained by reversing the factorization brackets

$$(D_t - u_p)(D_t + u_p)w_f = w'' + (u_p' - u_p^2)w = w'' + \omega_f^2(t)w = 0, \quad (4)$$

which is related to the fermionic Riccati equation

$$u' - u^2 - \omega_f^2(t) = 0, \quad (5)$$

where the free term ω_f^2 is the following function of time

$$\omega_f^2(t) = u_p' - u_p^2 = \begin{cases} \omega_0^2(-1 - 2\tan^2\omega_0 t) & \text{if } \kappa = 1 \\ \omega_0^2(1 - 2\coth^2\omega_0 t) & \text{if } \kappa = -1. \end{cases}$$

The solutions (fermionic zero modes) of Eq. (4) are given by

$$w_f = \begin{cases} \frac{-\omega_0}{\cos(\omega_0 t)} & \text{if } \kappa = 1 \\ \frac{\omega_0}{\sinh(\omega_0 t)} & \text{if } \kappa = -1, \end{cases}$$

and thus present strong periodic singularities in the first case and just one singularity at the origin in the second case. These 'partner' oscillators, as well as those to be discussed in the following, are parametric oscillators, i.e., of time-dependent frequency. Moreover, their frequencies can become infinite (periodically). In general, signals of this type are known as chirps. 'Infinite' chirps could be produced, in principle, in very special astrophysical circumstances, e.g., close to black hole horizons [4].

3. Matrix formulation.

Using the Pauli matrices $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we write the matrix equation

$$\hat{D}_0 W \equiv [\sigma_y D_t + \sigma_x (iu_p)]W = 0, \quad (6)$$

where $W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is a two component spinor. Eq. (6) is equivalent to the following decoupled equations

$$(iD_t + iu_p)w_1 = 0 \quad (7)$$

$$(-iD_t + iu_p)w_2 = 0. \quad (8)$$

Solving these equations one gets $w_1 \propto \omega_0 / \cos(\omega_0 t)$ and $w_2 \propto \omega_0 \cos(\omega_0 t)$ for the $\kappa = 1$ case and $w_1 \propto \omega_0 / \sinh(\omega_0 t)$ and $w_2 \propto \omega_0 \sinh(\omega_0 t)$ for the $\kappa = -1$ case. Thus, we obtain

$$W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_f \\ w_b \end{pmatrix}. \quad (9)$$

This shows that the matrix equation is equivalent to the two second-order linear differential equations of bosonic and fermionic type, Eq. (2) and Eq. (4), respectively, a result quite well known in particle physics. Indeed, a comparison with the true Dirac equation with a Lorentz scalar potential $S(x)$

$$[-i\sigma_y D_x + \sigma_x(m + S(x))]W = EW, \quad (10)$$

shows that Eq. (6) corresponds to a Dirac spinor of ‘zero mass’ and ‘zero energy’ in an *imaginary* scalar ‘potential’ $i u_p(t)$. We remind that a detailed discussion of the Dirac equation in the supersymmetric approach has been provided by Cooper *et al* [5] in 1988. They showed that the Dirac equation with a Lorentz scalar potential is associated with a susy pair of Schrodinger Hamiltonians. This result has been used later by many authors in the particle physics context [6].

4. Extension through parameter K.

We now come to the main issue of this work. Consider the slightly more general Dirac-like equation

$$\hat{D}_K W \equiv [\sigma_y D_t + \sigma_x(iu_p + K)]W = KW, \quad (11)$$

where K is a (not necessarily positive) real constant. On the left hand side of the equation, K stands as an (imaginary) mass parameter of the Dirac spinor, whereas on the right hand side it corresponds to the energy parameter. Thus, we have an equation equivalent to a Dirac equation for a spinor of mass iK at the fixed energy $E = iK$. This equation can be written as the following system of coupled equations

$$iD_t w_1 + (iu_p + K)w_1 = Kw_2 \quad (12)$$

$$-iD_t w_2 + (iu_p + K)w_2 = Kw_1. \quad (13)$$

The decoupling can be achieved by applying the operator in Eq. (13) to Eq. (12). For the fermionic spinor component one gets

$$D_t^2 w_1^+ - \omega_0^2 \left[(1 + 2 \tan^2 \omega_0 t) + i \frac{2K}{\omega_0} \tan \omega_0 t \right] w_1^+ = 0 \quad \text{for } \kappa = 1 \quad (14)$$

$$D_t^2 w_1^- + \omega_0^2 \left[(1 - 2 \coth^2 \omega_0 t) + i \frac{2K}{\omega_0} \coth \omega_0 t \right] w_1^- = 0 \quad \text{for } \kappa = -1, \quad (15)$$

whereas the bosonic component fulfills

$$D_t^2 w_2^+ + \omega_0^2 \left[1 - i \frac{2K}{\omega_0} \tan \omega_0 t \right] w_2^+ = 0 \quad \text{for } \kappa = 1 \quad (16)$$

$$D_t^2 w_2^- - \omega_0^2 \left[1 - i \frac{2K}{\omega_0} \coth \omega_0 t \right] w_2^- = 0 \quad \text{for } \kappa = -1. \quad (17)$$

The solutions of the bosonic equations are expressed in terms of the Gauss hypergeometric functions ${}_2F_1$

$$\begin{aligned} w_2^+(t; \alpha_+, \beta_+) &= \alpha_+ z_1^{(p-\frac{1}{2})} z_2^{(q-\frac{1}{2})} {}_2F_1 \left[p+q, p+q-1, 2p; -\frac{1}{2} z_1 \right] \\ -\beta_+ e^{-2ip\pi} 4^{(p-\frac{1}{2})} z_1^{-(p-\frac{1}{2})} z_2^{(q-\frac{1}{2})} &{}_2F_1 \left[q-p, q-p+1, 2-2p; -\frac{1}{2} z_1 \right] \end{aligned} \quad (18)$$

and

$$\begin{aligned} w_2^-(t; \alpha_-, \beta_-) &= \alpha_- z_3^r z_4^s {}_2F_1 \left[r + s, r + s + 1, 1 + 2r; \frac{1}{2} z_3 \right] \\ &+ \beta_- 4^r z_3^{-r} z_4^s {}_2F_1 \left[s - r + 1, s - r, 1 - 2r; \frac{1}{2} z_3 \right], \end{aligned} \quad (19)$$

where the variables z_i ($i = 1, \dots, 4$) are given in the following form:

$$z_1 = i \tan(\omega_0 t) - 1, \quad z_2 = i \tan(\omega_0 t) + 1, \quad z_3 = \coth(\omega_0 t) + 1, \quad z_4 = \coth(\omega_0 t) - 1,$$

respectively. The parameters are the following:

$$p = \frac{1}{2} \left(1 + \sqrt{1 - \frac{2K}{\omega_0}} \right), \quad q = \frac{1}{2} \left(1 + \sqrt{1 + \frac{2K}{\omega_0}} \right), \quad r = \frac{1}{2} \sqrt{1 + i \frac{2K}{\omega_0}}, \quad s = \frac{1}{2} \sqrt{1 - i \frac{2K}{\omega_0}}.$$

The fermionic zero modes can be obtained as the inverse of the bosonic ones. Thus

$$w_1^+ = \frac{1}{w_2^+(t; \alpha_+, \beta_+)}, \quad w_1^- = \frac{1}{w_2^-(t; \alpha_-, \beta_-)}. \quad (20)$$

A comparison of w_1^+ with the common $1/\cos t$ fermionic mode is displayed in Figures (3) and (4).

In the small K regime, $K \ll \omega_0$, one gets

$$\begin{aligned} w_2^+(t; \alpha_+, \beta_+) &\approx \alpha_+ z_1^{(p-\frac{1}{2})} z_2^{(q-\frac{1}{2})} {}_2F_1 \left[2, 1, 2 - \frac{K}{\omega_0}; -\frac{1}{2} z_1(t) \right] - \\ &\beta_+ e^{-2ip\pi} 4^{(p-\frac{1}{2})} z_1^{-(p-\frac{1}{2})} z_2^{(q-\frac{1}{2})} {}_2F_1 \left[\frac{K}{\omega_0}, 1 + \frac{K}{\omega_0}, \frac{K}{\omega_0}; -\frac{1}{2} z_1(t) \right] \end{aligned} \quad (21)$$

and

$$\begin{aligned} w_2^-(t; \alpha_-, \beta_-) &\approx \alpha_- z_3^r z_4^s {}_2F_1 \left[1, 2, 2 + i \frac{K}{\omega_0}; \frac{1}{2} z_3(t) \right] + \\ &\beta_- 4^r z_3^{-r} z_4^s {}_2F_1 \left[1 - i \frac{K}{\omega_0}, -i \frac{K}{\omega_0}, -i \frac{K}{\omega_0}; \frac{1}{2} z_3(t) \right]. \end{aligned} \quad (22)$$

Examining the bosonic equations, one can immediately see that the resonant frequencies acquired resistive time-dependent losses whose relative strength is given by the parameter K . The fermionic equations having time-dependent real parts of the frequency can be interpreted as parametric oscillators which are also affected by losses through the imaginary part.

5. More K parameters.

A more general case in this scheme is to consider the following matrix Dirac-like equation

$$\begin{aligned} &\left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} D_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} iu_p + K_1 & 0 \\ 0 & iu_p + K_2 \end{pmatrix} \right] \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \\ &\begin{pmatrix} K'_1 & 0 \\ 0 & K'_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \end{aligned} \quad (23)$$

The system of coupled first-order differential equations will be now

$$\left[-iD_t + iu_p + K_2 \right] w_2 = K'_1 w_1 \quad (24)$$

$$\left[iD_t + iu_p + K_1 \right] w_1 = K'_2 w_2 \quad (25)$$

and the equivalent second-order differential equations

$$D_t^2 w_i + [-i\Delta K] D_t w_i + [\pm D_t u_p + i(K_1 + K_2)u_p + (K_1 K_2 - K'_1 K'_2) - u_p^2] w_i = 0, \quad (26)$$

where the subindex $i = 1, 2$ and $\Delta K = K_1 - K_2$. Under the gauge transformation

$$w_i = Z_i \exp\left(-\frac{1}{2} \int^t [-i\Delta K] d\tau\right) = Z_i(t) e^{\frac{1}{2} i t \Delta K}, \quad (27)$$

one gets

$$D_t^2 Z_i + Q_i(t) Z_i = 0, \quad (28)$$

where the 'potentials' have the form

$$Q_i(t) = [\pm D_t u_p + i(K_1 + K_2)u_p + (K_1 K_2 - K'_1 K'_2) - u_p^2] - \frac{1}{4} [-i\Delta K]^2 \quad (29)$$

$Q_{1,2}$ are functions that differ from the nonoperatorial parts in Eqs. (30 - 33) only by constant terms. Indeed, one can obtain easily the following equations:

For the fermionic spinor component one gets

$$D_t^2 Z_1^+ - \omega_0^2 \left[1 + 2 \tan^2 \omega_0 t - \frac{(K_2 - K_1)^2}{4\omega_0^2} - \frac{K_1 K_2 - K'_1 K'_2}{\omega_0^2} + i \frac{K_1 + K_2}{\omega_0} \tan \omega_0 t \right] Z_1^+ = 0 \quad (30)$$

for $\kappa = 1$, and

$$D_t^2 Z_1^- + \omega_0^2 \left[1 - 2 \coth^2 \omega_0 t + \frac{(K_2 - K_1)^2}{4\omega_0^2} + \frac{K_1 K_2 - K'_1 K'_2}{\omega_0^2} + i \frac{K_1 + K_2}{\omega_0} \coth \omega_0 t \right] Z_1^- = 0 \quad (31)$$

for $\kappa = -1$.

The bosonic component fulfills

$$D_t^2 Z_2^+ + \omega_0^2 \left[1 + \frac{(K_2 - K_1)^2}{4\omega_0^2} + \frac{K_1 K_2 - K'_1 K'_2}{\omega_0^2} - i \frac{K_1 + K_2}{\omega_0} \tan \omega_0 t \right] Z_2^+ = 0, \quad (32)$$

for $\kappa = 1$, and

$$D_t^2 Z_2^- - \omega_0^2 \left[1 - \frac{(K_2 - K_1)^2}{4\omega_0^2} - \frac{K_1 K_2 - K'_1 K'_2}{\omega_0^2} - i \frac{K_1 + K_2}{\omega_0} \coth \omega_0 t \right] Z_2^- = 0, \quad (33)$$

for $\kappa = -1$. When $K_1 = K_2 = K$ one gets the particular case studied in full detail above.

The more general 'bosonic' modes have the form:

$$\begin{aligned} Z_2^+(t; \alpha_+, \beta_+) &= \alpha_+ [\tan(\omega_0 t) - i]^{\frac{\Omega_1}{4\omega_0}} [\tan(\omega_0 t) + i]^{\frac{\Omega_2}{4\omega_0}} \\ & {}_2F_1 \left[\frac{\Omega_1 + \Omega_2}{4\omega_0}, \frac{\Omega_1 + \Omega_2}{4\omega_0} + 1, 1 + \frac{\Omega_1}{2\omega_0}; \frac{1}{2} (\tan(\omega_0 t) - i) \right] \\ & + \beta_+ (-1)^{-\frac{\Omega_1}{2\omega_0}} [\tan(\omega_0 t) - i]^{-\frac{\Omega_1}{4\omega_0}} [\tan(\omega_0 t) + i]^{\frac{\Omega_2}{4\omega_0}} \\ & {}_2F_1 \left[\frac{\Omega_2 - \Omega_1}{4\omega_0}, \frac{\Omega_2 - \Omega_1}{4\omega_0} + 1, 1 - \frac{\Omega_1}{2\omega_0}; \frac{1}{2} (\tan(\omega_0 t) - i) \right] \end{aligned} \quad (34)$$

and

$$\begin{aligned} Z_2^-(t; \alpha_-, \beta_-) &= \alpha_- [\coth(\omega_0 t) - 1]^{\frac{\Omega_3}{4\omega_0}} [\coth(\omega_0 t) + 1]^{\frac{\Omega_4}{4\omega_0}} \\ & {}_2F_1 \left[\frac{\Omega_3 + \Omega_4}{4\omega_0} + 1, \frac{\Omega_3 + \Omega_4}{4\omega_0}, 1 + \frac{\Omega_3}{2\omega_0}; -\frac{1}{2} (\coth(\omega_0 t) - 1) \right] \\ & + \beta_- (-1)^{-\frac{\Omega_3}{2\omega_0}} 4^{\frac{\Omega_3}{2\omega_0}} [\coth(\omega_0 t) - 1]^{-\frac{\Omega_3}{4\omega_0}} [\coth(\omega_0 t) + 1]^{\frac{\Omega_4}{4\omega_0}} \\ & {}_2F_1 \left[\frac{\Omega_3 - \Omega_4}{4\omega_0}, \frac{\Omega_4 - \Omega_3}{4\omega_0} + 1, 1 - \frac{\Omega_3}{2\omega_0}; -\frac{1}{2} (\coth(\omega_0 t) - 1) \right], \end{aligned} \quad (35)$$

where

$$\begin{aligned}\Omega_1 &= \left(4\omega_0^2 + (K_1 + K_2)^2 + 4[(K_1 + K_2)\omega_0 - K'_1 K'_2]\right)^{1/2}, \\ \Omega_2 &= \left(4\omega_0^2 + (K_1 + K_2)^2 - 4[(K_1 + K_2)\omega_0 + K'_1 K'_2]\right)^{1/2}, \\ \Omega_3 &= \left(4\omega_0^2 - (K_1 + K_2)^2 - 4[i(K_1 + K_2)\omega_0 - K'_1 K'_2]\right)^{1/2}, \\ \Omega_4 &= \left(4\omega_0^2 - (K_1 + K_2)^2 + 4[i(K_1 + K_2)\omega_0 + K'_1 K'_2]\right)^{1/2}.\end{aligned}$$

6. Applications.

6.1 Waveguides.

In view of the correspondence between mechanics and optics, one can also provide an interpretation in terms of the Helmholtz optics for light propagation in waveguides of special profiles. The supersymmetry of the Helmholtz equation has been studied by Wolf and collaborators [7]. To get the waveguide application, one should switch from the temporal independent variable to a spatial variable $t \rightarrow x$ along which we consider the inhomogeneity of the fiber whereas the propagation of beams is along another supplementary spatial coordinate z . Thus, we turn the equations (14-17) into Helmholtz waveguide equations of the type (we take $c = 1$)

$$[\partial_z^2 + \partial_x^2 + \omega_0^2 n^2(x)]\varphi(x, z) = 0, \quad (36)$$

where the modes $\varphi(x, z)$ can be written in the form $w_{1,2}(x)e^{-ik_z z}$ for a fixed wavenumber k_z in the propagating coordinate that is common to both wavefunctions and the index profiles correspond to two pairs of bosonic-fermionic waveguides and are given by

$$n_b^2(x) \sim 1 - i\frac{2K}{k_0} \tan(k_0 x), \quad n_f^2(x) \sim -(1 + 2 \tan^2(\omega_0 x)) - i\frac{2K}{k_0} \tan(k_0 x), \quad (37)$$

and

$$n_b^2(x) \sim -1 - i\frac{2K}{k_0} \coth(k_0 x), \quad n_f^2(x) \sim 1 - 2\coth^2(k_0 x) + i\frac{2K}{k_0} \coth(k_0 x), \quad (38)$$

respectively. In our units $k_0 = \omega_0$. Eqs. (37, 38) can be obtained from Riccati equations of the type ($c \neq 1$)

$$\omega_0^2 n_{f,b}^2(x)/c^2 = k^2 \mp R_x - R^2, \quad (39)$$

where $R(x)$ are Riccati solutions directly related to the Riccati solutions discussed in the previous sections.

According to Chumakov and Wolf [7] a second waveguide interpretation is possible describing two different Gaussian beams, bosonic and fermionic, whose small difference in frequency is given in terms of a small parameter ϵ (wavelength/beam width), propagating in the *same* waveguide. In this interpretation, the index profile is the same for both beams. For illustration, let us take the normal oscillator Riccati solution in the space variable x , i.e., $\tan k_0 x$ that we approximate to first order linear Taylor term $k_0 x$. Then, the two beam interpretation leads to the following Riccati equation (for details, see the paper of Chumakov and Wolf)

$$\omega_{1,2}^2 n^2(x) - \omega_0^2 n^2(0) = \mp k_0 - k_0^2 x^2 (1 \mp \epsilon). \quad (40)$$

An almost exact, up to nonlinear corrections of order ϵ^2 and higher, supersymmetric pairing of the z wavenumbers (propagating constants) occurs, except for the 'ground state' one. As

noted by Chumakov and Wolf, supersymmetry connects in this case light beams of different frequencies but having the same wavelength in the propagation direction z . This approach is valid only in the paraxial approximation. Therefore, one should know the small x behaviour of the K-modes in order to hope to detect them through stable interference patterns along the waveguide axis.

6.2 Cavity physics.

Another very interesting application of the K-modes in a radial variable could be Schumann's resonances, i.e., the resonant frequencies of the spherical cavity provided by the Earth's surface and the ionosphere plasma layer [8]. The Schumann problem can be approached as a spherical Helmholtz equation $[\nabla_r^2 + k^2]\phi = 0$ with Robin type (mixed) boundary condition $\frac{\partial\phi}{\partial n}|_S = C(\omega)\phi_S$, where $C(\omega)$ is expressed in terms of the skin depth $\delta = \sqrt{2/(\mu_c\sigma\omega)}$ of the conducting wall, μ_c is its permeability and σ is its conductivity. The eigenfrequencies fulfilling such boundary conditions can be written as follows

$$\omega^2 \approx \omega_0^2[(1 - I) + iI], \quad (41)$$

where I is a complicated expression in terms of skin depths and surface and volume integrals of Helmholtz solutions with Neumann boundary conditions $\frac{\partial\phi}{\partial n}|_S = 0$. It is worth noting the similarity between these improved values of Schumann's eigenfrequencies and the K-eigenfrequencies. Moreover, using the Q parameter of the cavity, one can write Eq. (41) in the form

$$\omega^2 \approx \omega_0^2 \left[\left(1 - \frac{1}{Q}\right) + i\frac{1}{Q} \right]. \quad (42)$$

This form shows that the modification of the real part of ω leads to a downward shift of the resonant frequencies, while the contribution to the imaginary component changes the rate of decay of the modes.

We point out that Jackson mentions in his textbook that the near equality of the real and imaginary parts of the change in ω^2 is a consequence of the employed boundary condition, which is appropriate for relatively good conductors. Thus, by changing the form of $C(\omega)$ that could result from different surface impedances, the relative magnitude of the real and imaginary parts of the change in ω^2 can be made different. It is this latter case that corresponds better to the K-modes.

6.3 Crystal models.

There is also a strong mathematical similarity between the K-modes and the solutions of Scarf's crystal model [9] based on the singular potential $V(x) = -V_0 \text{cosec}^2(\pi x/a)$, where a is an arbitrary lattice parameter. For this model the one-dimensional Schrödinger equation has the form

$$\psi'' + (a/\pi)^2 \left[\lambda^2 + \left(\frac{1}{4} - s^2 \right) \text{cosec}^2(\pi x/a) \right] \psi = 0. \quad (43)$$

For $0 < x \leq a/2$, the general solution is

$$\begin{aligned} \psi = & [f(x)]^{\frac{1}{2}+s} {}_2F_1 \left[\frac{1}{4} + \frac{1}{2}(s + \lambda), \frac{1}{4} + \frac{1}{2}(s - \lambda); 1 + s; f^2(x) \right] + \\ & [f(x)]^{\frac{1}{2}-s} {}_2F_1 \left[\frac{1}{4} - \frac{1}{2}(s - \lambda), \frac{1}{4} - \frac{1}{2}(s + \lambda); 1 - s; f^2(x) \right], \end{aligned} \quad (44)$$

where $f(x) = \sin(\pi x/a)$ corresponds to the $z_i(t)$ functions, and s and λ corresponding to $-p$ and $-q$, respectively, are related to the potential amplitude and energy spectral parameter. Thus, by turning the K-oscillator equations into corresponding Schrodinger equations, one

could introduce another analytical crystal model with possible applications in photonics crystals.

6.4 Cosmology.

Two of the authors applied the K-mode approach to barotropic FRW cosmologies [10]. K- Hubble cosmological parameters have been introduced and expressed as logarithmic derivatives of the K-modes with respect to the conformal time. For $K \rightarrow 0$ the ordinary solutions of the common FRW barotropic fluids have been obtained.

It is also worth noticing the analogy of the nonzero K oscillator case with the phenomenon of diffraction of atomic waves in imaginary crystals of light (crossed laser beams) [11]. In fact, the K parameter is a counterpart of the modulation parameter Q introduced by Berry and O'Dell in their study of imaginary optical gratings. Roughly speaking, the nonzero K modes could occur in an *imaginary crystal of time* that could occur in some exotic astrophysical conditions.

7. Conclusion.

By a procedure involving the factorization connection between the Dirac-like equations and the simple second-order linear differential equations of harmonic oscillator type, a class of classical modes with a Dirac-like parameters describing their damping and absorption (dissipation) has been introduced in this work. While for zero values of the Dirac parameters the highly singular fermionic modes are decoupled from their normal bosonic harmonic modes, at nonzero values a coupling between the two types of modes is introduced at the level of the matrix equation. These interesting modes are given by the solutions of the Eqs. (30) - (33) and in a more general way by Eqs. (27, 34-35) of this work and are expressed in terms of hypergeometric functions. Several possible applications in different fields of physics are mentioned as well. Finally, similar to the fact that the PT quantum mechanics can be considered as a complex extension of standard quantum mechanics, we notice that what we have done here is a particular type of complex extension of the classical harmonic oscillator.

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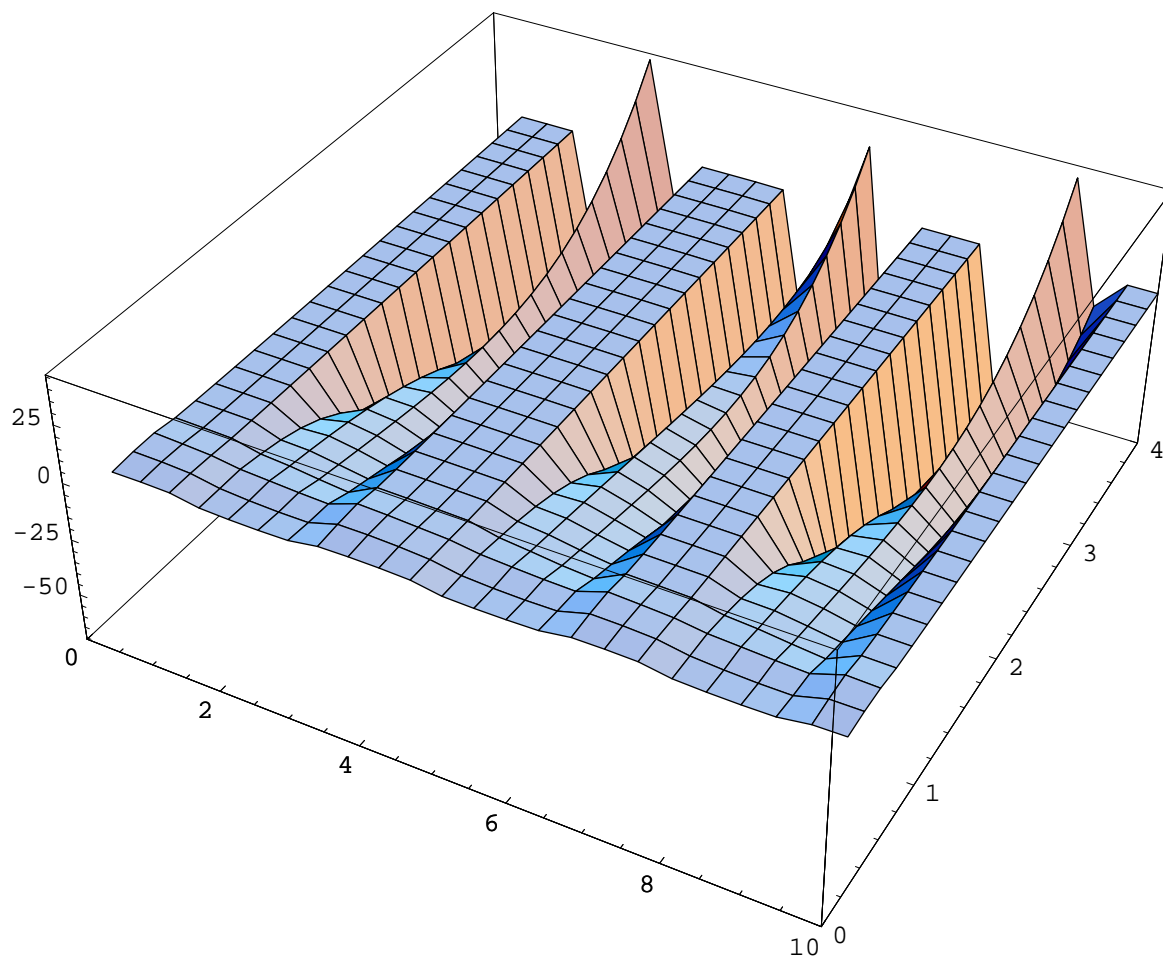


Figure 1. The real part of the bosonic mode $w_2^+(y; \frac{1}{2}, \frac{1}{2})$ for $t \in [0, 10]$ and $K \in [0, 4]$.

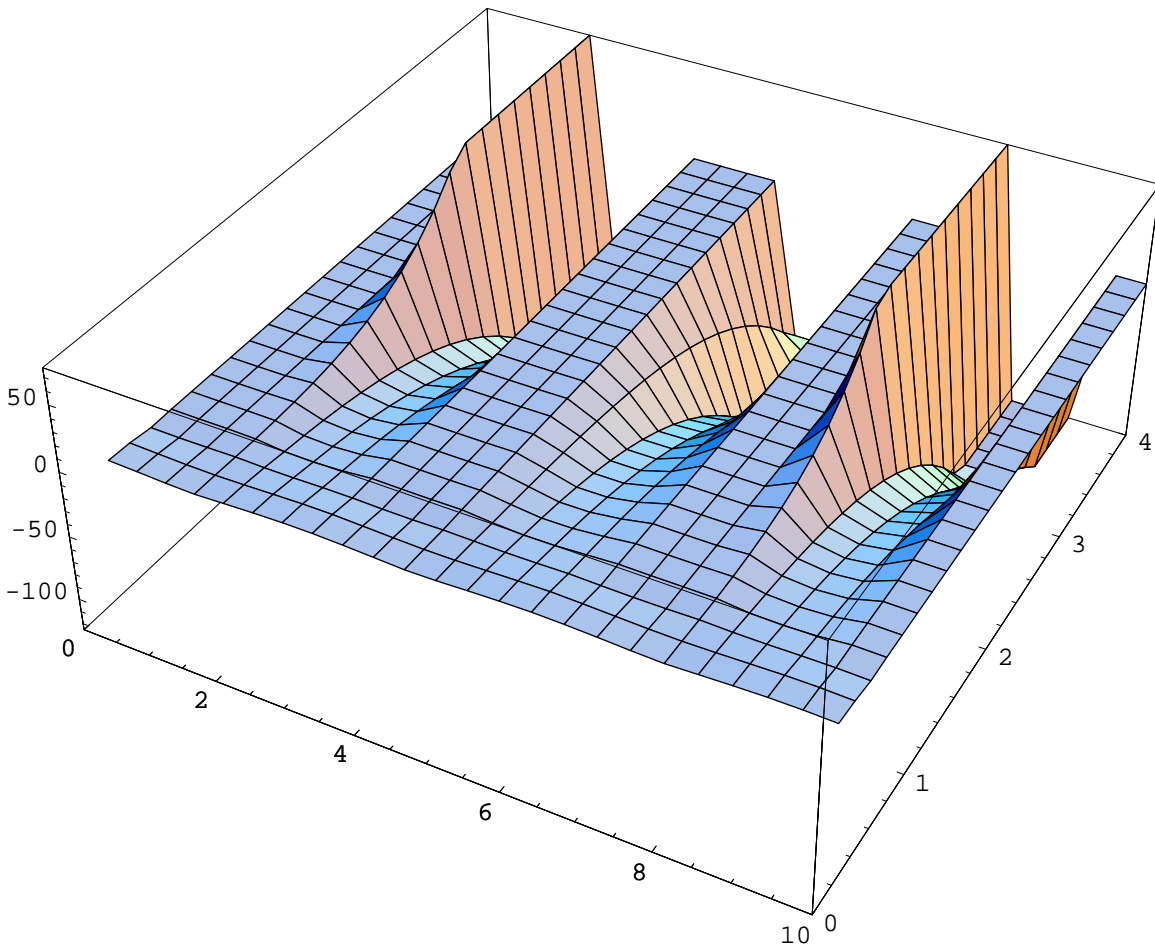


Figure 2. The imaginary part of the bosonic mode $w_2^+(y; \frac{1}{2}, \frac{1}{2})$ for $t \in [0, 10]$ and $K \in [0, 4]$.

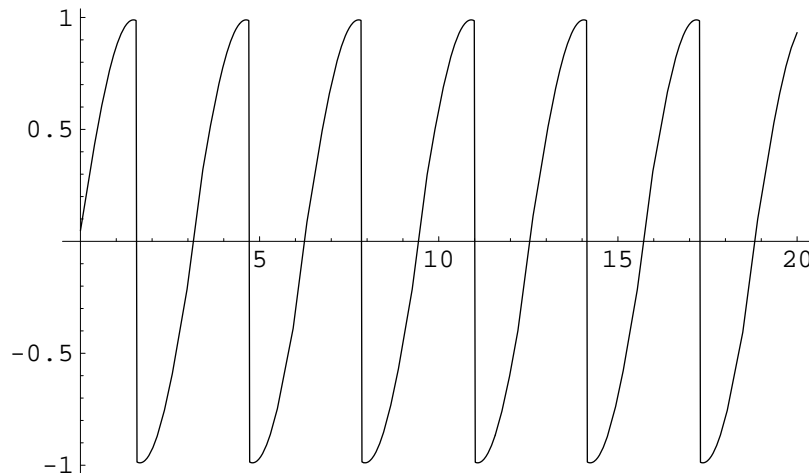


Figure 3. The real part of the bosonic mode $w_2^+(y; \frac{1}{2}, \frac{1}{2})$ for $t \in [0, 20]$ and $K = 0.01$.

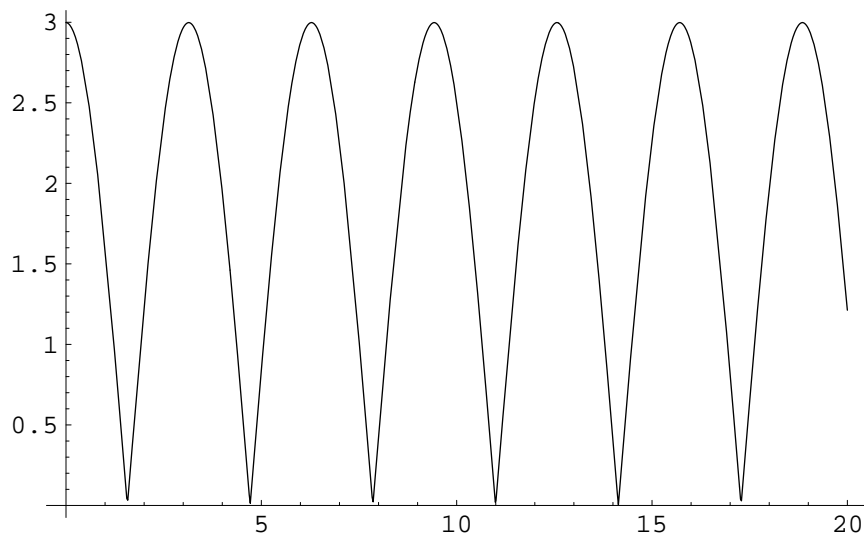


Figure 4. The imaginary part of the bosonic mode $w_2^+(y; \frac{1}{2}, \frac{1}{2})$ for $t \in [0, 20]$ and $K = 0.01$.

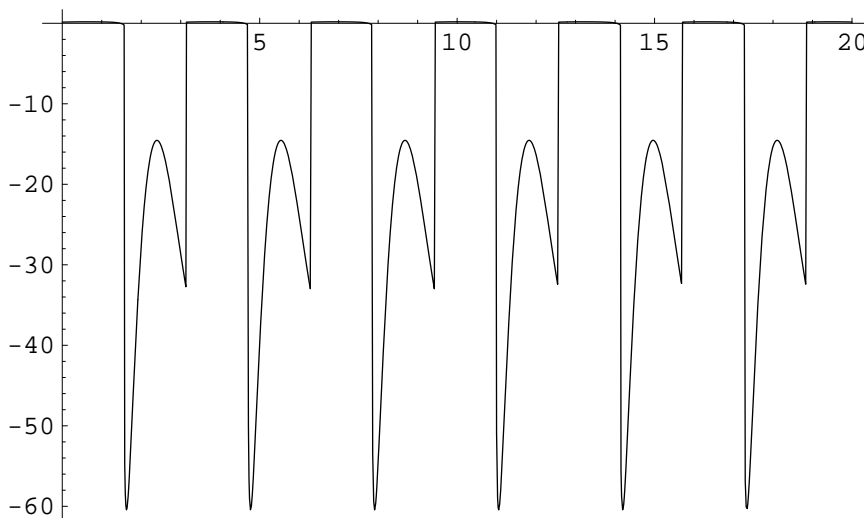


Figure 5. The real part of the bosonic mode $w_2^+(y; \frac{1}{2}, \frac{1}{2})$ for $t \in [0, 20]$ and $K = 2$.

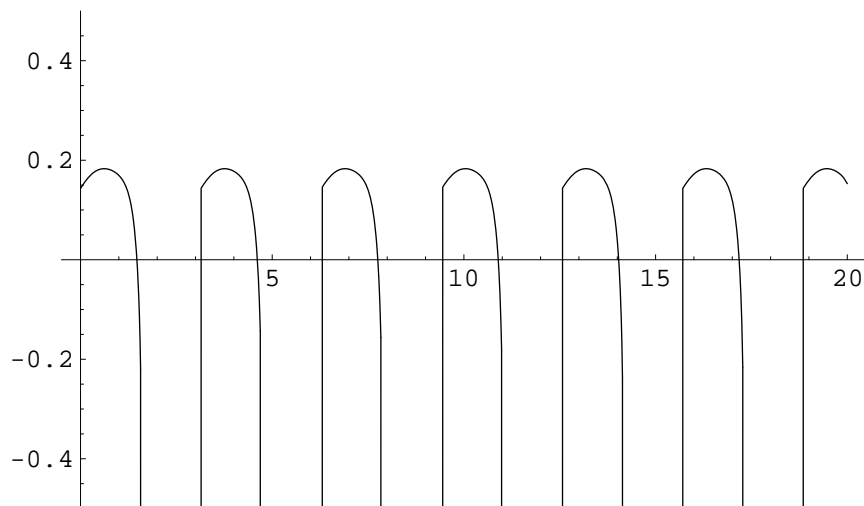


Figure 6. The real part of the bosonic mode $w_2^+(y; \frac{1}{2}, \frac{1}{2})$ for $t \in [0, 20]$ and $K = 2$ in the vertical strip $[-0.5, 0.5]$.

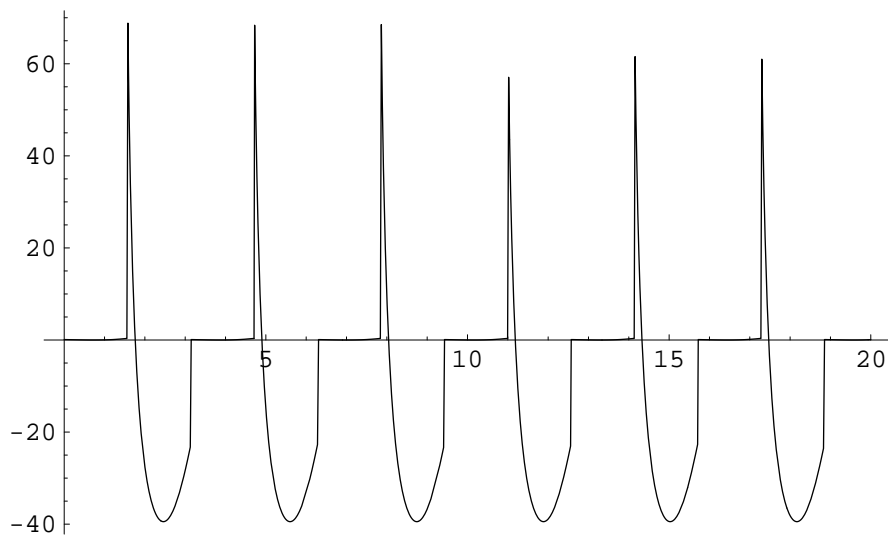


Figure 7. The imaginary part of the bosonic mode $w_2^+(y; \frac{1}{2}, \frac{1}{2})$ for $t \in [0, 20]$ and $K = 2$.

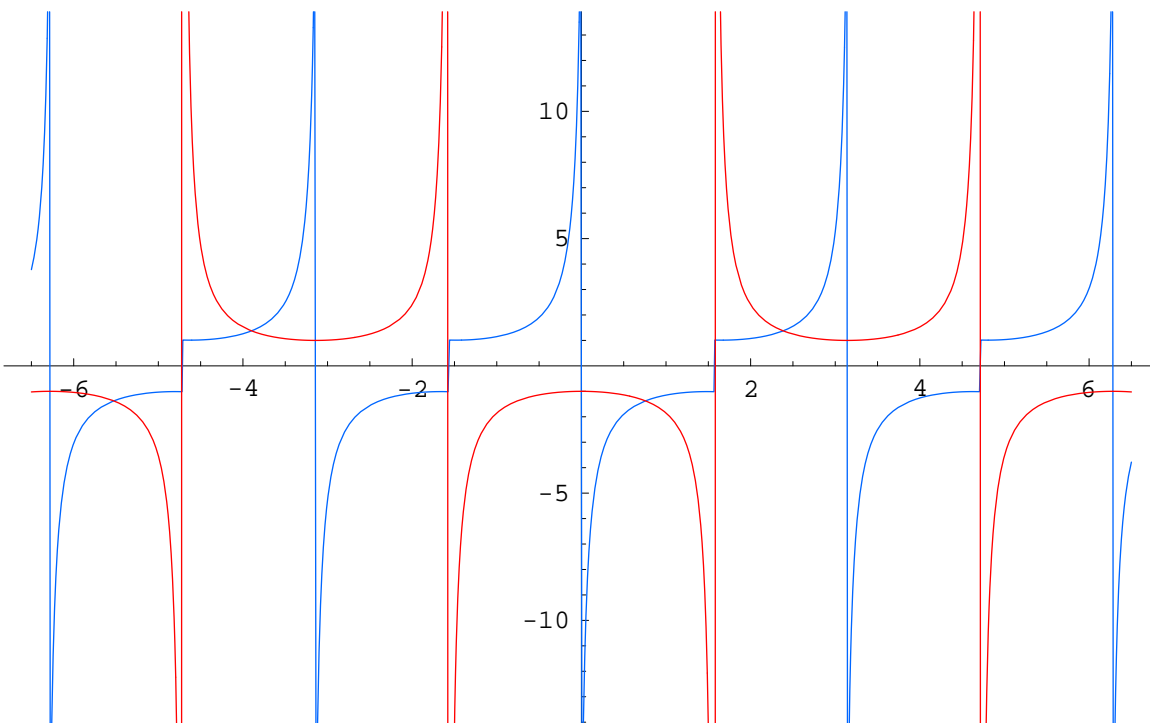


Figure 8. The fermionic zero mode $-1/\cos t$, (red curve), and the real part of $-1/w_2^+$, (blue curve), for $K = 0.01$.

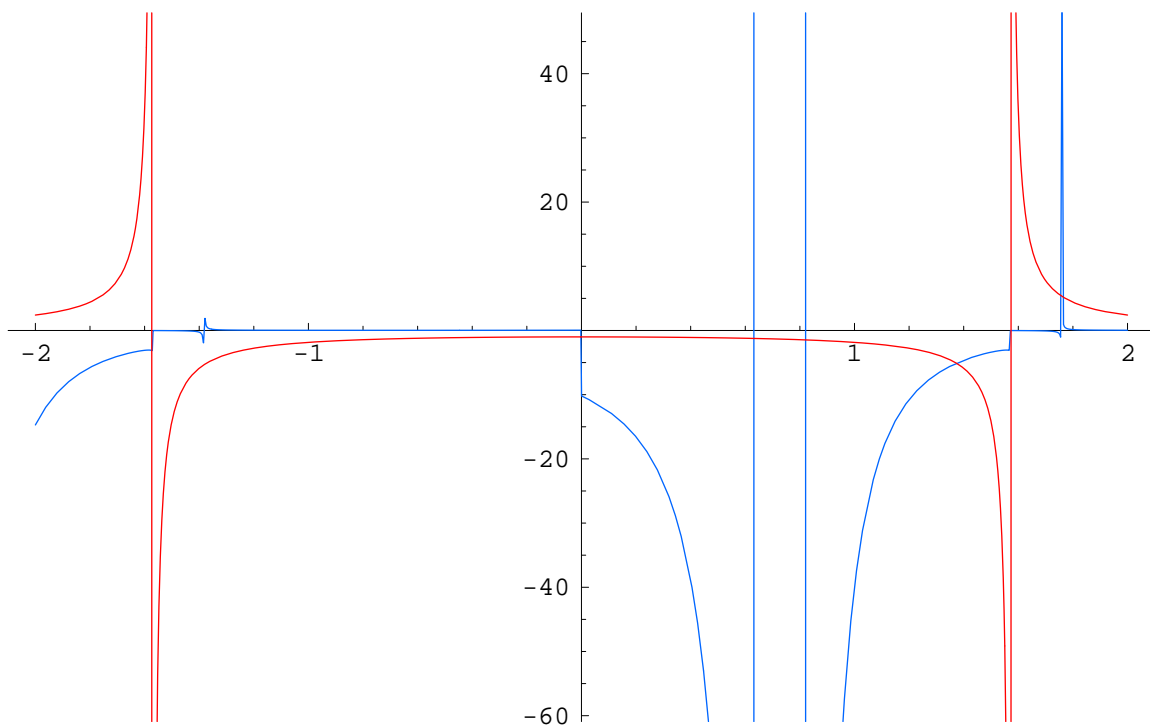


Figure 9. The fermionic zero mode $-1/\cos t$, (red curve), and the imaginary part of $-1/w_2^+$, (blue curve), for $K = 2$.