Spectral parameter power series representation for Hill's discriminant

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Annals of Physics 325 (Nov. 2010) pp. 2512-2521

Abstract

We establish a series representation of the Hill discriminant based on the spectral parameter power series (SPPS) recently introduced by V. Kravchenko. We also show the invariance of the Hill discriminant under a Darboux transformation and employing the Mathieu case the feasibility of this type of series for numerical calculations of the eigenspectrum.

Keywords: Hill's discriminant, spectral parameter power series, supersymmetric partner equation, Mathieu equation.

1 Introduction

There is recent strong interest in differential equations with periodic coefficients due to their numerous applications in modern material sciences and engineering. The main goal in their theoretical framework is the description of the spectrum and getting the periodic and quasiperiodic solutions. In the case of linear second-order ordinary differential equation, a function of the spectral parameter known as Hill's discriminant is the basic quantity containing important information both about the spectrum of the differential operator and also about the construction of the (quasi)periodic solutions.

This paper focuses on a representation of the Hill discriminant which is given in the form of a power series in the spectral parameter. This representation is obtained by using recent results of Kravchenko [1, 2, 3]. It is worth mentioning that since the Hill discriminant for each value of the spectral parameter can be obtained from a couple of corresponding linearly independent solutions, any available representation for these solutions leads to a representation of Hill's discriminant. Nevertheless, in general no easy and practically treatable representation for Hill's discriminant is known unless for the case of some well-studied equations such as the Mathieu and the Lamé equations.

Long ago, Jagerman [4] introduced and studied in detail the so-called cardinal series representation of Hill's discriminant. In the famous book of Magnus and Winkler [5] the Hill discriminant for Schroedinger type equations is expressed as an infinite determinant involving the Fourier coefficients of the potential as well as the spectral parameter. The phase-integral method is used by Fröman [6] to obtain a representation involving a matrix whose entries are complicated phase integrals, while Boumenir [7] wrote it in terms of integrals derived from the inverse spectral theory.

In all the aforementioned series representations the spectral parameter enters in a quite sophisticated way with all the terms being functions of the spectral parameter. Instead, our result herein gives the Hill discriminant in the form of a power series in the spectral parameter with the series coefficients independent of it and calculated only once, i.e., for a single value of the spectral parameter. Moreover, its practical implementation is easy and as an illustration we show in the present work the numerical results obtained for the Mathieu equation.

We also show that the SPPS representation gives additional information not only concerning the original equation but also on its Darboux-related partners. As a corollary, we prove the invariance of the Hill discriminant under the Darboux transformation under some additional conditions.

2 Hill's type equations

The Sturm-Liouville differential equation

$$L\left[f(x,\lambda)\right]=-(p(x)f'(x,\lambda))'+q(x)f(x,\lambda)=\lambda f(x,\lambda) \tag{1}$$

with T-periodic coefficients $p(x)$ and $q(x)$ and real parameter λ is known as of Hill type. We first recall some necessary definitions and basic properties associated with the Eq. (1) from the Floquet (Bloch) theory. For more details see, e.g., [5, 8]. In what follows we assume that $p(x)>0$, $p'(x)$ and $q(x)$ are continuous bounded functions.

For each λ there exists a fundamental system of solutions, i.e., two linearly independent solutions of (1) $f_1(x,\lambda)$ and $f_2(x,\lambda)$ which satisfy the initial conditions

$$f_1(0,\lambda)=1,\quad f_1'(0,\lambda)=0,\quad f_2(0,\lambda)=0,\quad f_2'(0,\lambda)=1. \tag{2}$$

Then the Hill discriminant associated with Eq. (1) is defined as a function of λ as follows

$$D(\lambda)=f_1(T,\lambda)+f_2'(T,\lambda).$$

Employing $D(\lambda)$ one can easily describe the spectrum of the corresponding equation. Namely, the values of λ for which $|D(\lambda)|\leq 2$ form the allowed bands or stability intervals meanwhile the values of λ such that $|D(\lambda)|>2$ belong to forbidden bands or instability intervals [5]. The band edges (values of λ such that $|D(\lambda)|=2$) represent the discrete spectrum of the operator, i.e., they are the eigenvalues of the operator with periodic ($D(\lambda)=2$) or antiperiodic

$(D(\lambda) = -2)$ boundary conditions. The eigenvalues λ_n, $n = 0, 1, 2, ...$ form an infinite sequence $\lambda_0 < \lambda_1 \leqslant \lambda_2 < \lambda_3 ...$, and an important property of the minimal eigenvalue λ_0 is the existence of a corresponding periodic nodeless solution $f_0(x, \lambda_0)$ [5]. In general solutions of (1) are not of course periodic, and one of the important tasks related to Sturm-Liouville equations with periodic coefficients is the construction of quasiperiodic solutions. In this paper, we use the matching procedure from [9] for which the main ingredient is the pair of solutions $f_1(x, \lambda)$ and $f_2(x, \lambda)$ of (1) satisfying conditions (2). Namely, using $f_1(x, \lambda)$ and $f_2(x, \lambda)$ one obtains the quasiperiodic solutions $f_\pm(x+T) = \beta_\pm f_\pm(x)$ as follows

$$f_\pm(x, \lambda) = \beta_\pm^n F_\pm(x - nT, \lambda), \quad \begin{cases} nT \leq x < (n+1)T \\ n = 0, \pm 1, \pm 2, ... \end{cases} . \tag{3}$$

where $F_\pm(x, \lambda)$ are the so-called self-matching solutions, which are the following linear combinations $F_\pm(x, \lambda) = f_1(x, \lambda) + \alpha_\pm f_2(x, \lambda)$ with $\alpha_\pm$ being roots of the algebraic equation $f_2(T, \lambda)\alpha^2 + (f_1(T, \lambda) - f_2'(T, \lambda))\alpha - f_1'(T, \lambda) = 0$. The Bloch factors $\beta_\pm$ are a measure of the rate of increase (or decrease) in magnitude of the self-matching solutions $F_\pm(x, \lambda)$ when one goes from the left end of the cell to the right end, i.e., $\beta_\pm(\lambda) = \frac{F_\pm(T,\lambda)}{F_\pm(0,\lambda)}$. The values of $\beta_\pm$ are directly related to the Hill discriminant, $\beta_\pm(\lambda) = \frac{1}{2}(D(\lambda) \mp \sqrt{D^2(\lambda) - 4})$, and obviously at the band edges $\beta_+ = \beta_- = \pm 1$ for $D(\lambda) = \pm 2$, correspondingly.

3 The SPPS series representation of Hill's discriminant

In this section, we will give an efficient representation for the Hill discriminant using the method of spectral parameter power series (SPPS) [1, 3], which was used in [10, 11] for studying quantum mechanical models related to one-dimensional Dirac systems, also for solving electromagnetic scattering problems [12] and for solving inverse problems [13]. The SPPS method offers a procedure for constructing the solutions $f_1(x, \lambda)$ and $f_2(x, \lambda)$ of (1) which satisfy the initial conditions (2) . This construction is based on the knowledge of one non-vanishing particular solution $f_0(x, \lambda_0)$ of (1) being bounded on $[0, T]$ together with $\frac{1}{f_0(x,\lambda_0)}$. In the case of Hill's equation the first eigenvalue λ_0 of the Eq. (1) generates nodeless periodic eigenfunction $f_0(x, \lambda_0)$. In what follows we initially suppose that the value of λ_0 is known. Note that, it can be obtained by different methods including the same SPPS method [3] as we explain in subsection 3.4.

Given λ_0, we proceed in three steps in order to obtain the representation of Hill's discriminant:

i) the first one is the construction of a particular nodeless solution $f_0(x, \lambda_0)$ of (1) which is periodic, i.e., $f_0(x + T, \lambda_0) = f_0(x, \lambda_0)$,

ii) the second one is the construction of the fundamental system of solutions $f_1(x, \lambda)$ and $f_2(x, \lambda)$ of (1) for all values of the parameter λ,

iii) the final step is getting the representation of Hill's discriminant.

We detail each of the steps in the following subsections.

3.1 The nodeless periodic solution

In order to obtain the nodeless periodic solution $f_0(x,\lambda_0)$ of (1) for $\lambda=\lambda_0$, i.e.,

$$-\left(p(x)(f_0(x))'\right)'+q(x)f_0(x)=\lambda_0 f_0(x) \tag{4}$$

we first construct two linearly independent SPPS solutions of (4), $f_{0,1}(x,\lambda_0)$ and $f_{0,2}(x,\lambda_0)$, which satisfy the initial conditions $f_{0,1}(0,\lambda_0)=f'_{0,2}(0,\lambda_0)=1$ and $f'_{0,1}(0,\lambda_0)=f_{0,2}(0,\lambda_0)=0$ [1]. These solutions are not necessarily periodic. The procedure of James [9] allows one to obtain from $f_{0,1}(x,\lambda_0)$ and $f_{0,2}(x,\lambda_0)$ the Floquet type solutions which degenerate to a single periodic solution $f_0(x,\lambda_0)$ since λ_0 represents a band edge.

The functions $f_{0,1}(x,\lambda_0)$ and $f_{0,2}(x,\lambda_0)$ can be calculated as follows [1]

$$f_{0,1}(x,\lambda_0)=\sum_{even\ n=0}^{\infty}\widetilde{X}_0^{(n)} \qquad \text{and} \qquad f_{0,2}(x,\lambda_0)=p(0)\sum_{odd\ n=1}^{\infty}X_0^{(n)}, \tag{5}$$

where

$$\widetilde{X}_0^{(0)}\equiv 1, \qquad X_0^{(0)}\equiv 1,$$

$$\widetilde{X}_0^{(n)}(x)=\begin{cases}\int_0^x \widetilde{X}_0^{(n-1)}(\xi)(q(\xi)-\lambda_0)d\xi & \text{for an odd } n\\[2mm] \int_0^x \widetilde{X}_0^{(n-1)}(\xi)\frac{1}{p(\xi)}d\xi & \text{for an even } n\end{cases}$$

$$X_0^{(n)}(x)=\begin{cases}\int_0^x X_0^{(n-1)}(\xi)\frac{1}{p(\xi)}d\xi & \text{for an odd } n\\[2mm] \int_0^x X_0^{(n-1)}(\xi)(q(\xi)-\lambda_0)d\xi & \text{for an even } n\end{cases}$$

Now following [9] we obtain the periodic nodeless solution of (4)

$$f_0(x,\lambda_0)=f_{0,1}(x-nT,\lambda_0)+\alpha f_{0,2}(x-nT,\lambda_0), \tag{6}$$
$$\begin{cases} nT\le x<(n+1)T\\ n=0,1,2,\ldots\,. \end{cases}$$

where $\alpha=\frac{f'_{0,2}(T,\lambda_0)-f_{0,1}(T,\lambda_0)}{2f_{0,2}(T,\lambda_0)}$ since λ_0 is the band edge eigenvalue [9].

3.2 Fundamental system of solutions

Once having the function $f_0(x,\lambda_0)$ the solutions $f_1(x,\lambda)$ and $f_2(x,\lambda)$ of (1) and (2) for all values of the parameter λ can be given using the SPPS method once again [1]

$$
\begin{aligned}
f_1(x,\lambda) &= \frac{f_0(x)}{f_0(0)}\widetilde{\Sigma}_0(x,\lambda,\lambda_0) + p(0)f_0'(0)f_0(x)\Sigma_1(x,\lambda,\lambda_0), \\
f_2(x,\lambda) &= -p(0)f_0(0)f_0(x)\Sigma_1(x,\lambda,\lambda_0).
\end{aligned} \tag{7}
$$

The summations $\widetilde{\Sigma}_0$ and Σ_1 are the spectral parameter power series

$$
\widetilde{\Sigma}_0(x,\lambda,\lambda_0) = \sum_{n=0}^{\infty} \widetilde{X}^{(2n)}(x)(\Delta\lambda)^n, \quad \Sigma_1(x,\lambda,\lambda_0) = \sum_{n=1}^{\infty} X^{(2n-1)}(x)(\Delta\lambda)^{n-1} \ ,
$$

where $\Delta\lambda = \lambda - \lambda_0$ and the coefficients $\widetilde{X}^{(n)}(x)$, $X^{(n)}(x)$ are given by the following recursive relations

$$
\widetilde{X}^{(0)} \equiv 1, \qquad X^{(0)} \equiv 1,
$$

$$
\tilde{X}^{(n)}(x) = \begin{cases} \int_0^x \tilde{X}^{(n-1)}(\xi) f_0^2(\xi) d\xi & \text{for an odd } n \\ - \int_0^x \tilde{X}^{(n-1)}(\xi) \frac{d\xi}{p(\xi) f_0^2(\xi)} & \text{for an even } n \end{cases} \tag{8}
$$

$$
X^{(n)}(x) = \begin{cases} - \int_0^x X^{(n-1)}(\xi) \frac{d\xi}{p(\xi) f_0^2(\xi)} & \text{for an odd } n \\ \int_0^x X^{(n-1)}(\xi) f_0^2(\xi) d\xi & \text{for an even } n \ . \end{cases} \tag{9}
$$

One can check by a straightforward calculation that the solutions f_1 and f_2 fulfill the initial conditions (2), for this the following relations are useful

$$
\left(\widetilde{\Sigma}_0(x,\lambda,\lambda_0)\right)_x' = -\frac{\widetilde{\Sigma}_1(x,\lambda,\lambda_0)}{p(x)f_0^2(x)}, \text{ where } \widetilde{\Sigma}_1(x,\lambda,\lambda_0) = \sum_{n=1}^{\infty} \widetilde{X}^{(2n-1)}(x)(\Delta\lambda)^n \tag{10}
$$

and

$$
\left(\Sigma_1(x,\lambda,\lambda_0)\right)_x' = -\frac{\Sigma_0(x,\lambda,\lambda_0)}{p(x)f_0^2(x)}, \text{ where } \Sigma_0(x,\lambda,\lambda_0) = \sum_{n=0}^{\infty} X^{(2n)}(x)(\Delta\lambda)^n. \tag{11}
$$

Having obtained the fundamental system of solutions for any value of λ, one can apply the construction (3) in order to obtain the Bloch solutions which become eigenfunctions for λ being eigenvalues.

3.3 Hill's discriminant in SPPS form

Now the Hill discriminant $D(\lambda) = f_1(T,\lambda) + f_2'(T,\lambda)$ can be written in a simple explicit form. For this we write $f_1(T,\lambda)$ and $f_2'(T,\lambda)$ in a form of a spectral parameter power series using (7) and taking into account (11):

$$D(\lambda) = \frac{f_0(T)}{f_0(0)}\widetilde{\Sigma}_0(T,\lambda,\lambda_0) + \frac{f_0(0)}{f_0(T)}\Sigma_0(T,\lambda,\lambda_0) \\ + \left(f_0'(0)f_0(T) - f_0(0)f_0'(T)\right)p(0)\Sigma_1(T,\lambda,\lambda_0)\ . \tag{12}$$

Finally, taking into account that $f_0(x)$ is a T-periodic function $f_0(0) = f_0(T)$ and writing the explicit expressions for $\widetilde{\Sigma}_0(T,\lambda,\lambda_0)$ and $\Sigma_0(T,\lambda,\lambda_0)$ we obtain a representation for Hill's discriminant associated with (1)

$$D(\lambda) \equiv \sum_{n=0}^{\infty}\left(\tilde{X}^{(2n)}(T) + X^{(2n)}(T)\right)(\Delta\lambda)^n. \tag{13}$$

Thus, only one particular nodeless and periodic solution $f_0(x,\lambda_0)$ of (1) is needed for constructing the associated Hill discriminant. We formulate the result (13) as the following theorem:

Theorem 1 *Let λ_0 be the lowest eigenvalue of the Sturm-Liouville problem for the operator L on the segment $[0,T]$ with periodic boundary conditions and $f_0(x,\lambda_0)$ be the corresponding eigenfunction. Then the Hill discriminant for (1) has the form (13) where $\tilde{X}^{(2n)}$ and $X^{(2n)}$ are calculated according to (8) and (9), and the series converges uniformly on any compact set of values of λ.*

To illustrate the formula (13) we consider a simple example. Let in equation (1) $q(x) = 0$, $p(x) = 1$. It is easy to see that the associated discriminant is $D(\lambda) = 2\cos\sqrt{\lambda}T$, from where we obtain $\lambda_0 = 0$ and a corresponding non-trivial periodic solution is $f_0(x) = 1$. Now making use of this solution we construct the discriminant by means of the formula (13). The coefficients $\tilde{X}^{(2n)}(T)$ and $X^{(2n)}(T)$ given by (8) and (9) take the form

$$\tilde{X}^{(2n)}(T) = X^{(2n)}(T) = (-1)^n\frac{T^{2n}}{(2n)!}, \quad n = 0,1,2,\dots\ .$$

The substitution in (13) gives $D(\lambda) = 2\cos\sqrt{\lambda}T$.

3.4 Construction of the first eigenvalue λ_0 by the SPPS method

Notice that in the expression (12) for $D(\lambda)$ and in all reasonings previous to it we do not use the periodicity of the solution $f_0(x,\lambda_0)$, therefore (12) and the whole procedure for obtaining it are valid for any λ_* such that there exists a corresponding solution $f_*(x,\lambda_*)$ which is bounded on $[0,T]$ together with $1/(pf_*^2)$. Such a solution $f_*(x,\lambda_*)$ can be obtained in the following way

$$f_*(x,\lambda_*) = f_{*,1}(x,\lambda_*) + if_{*,2}(x,\lambda_*) \tag{14}$$

where $f_{*,1}(x,\lambda_*)$ and $f_{*,2}(x,\lambda_*)$ are given by (5) with λ_* instead of λ_0. For more details see [2]. The pair of the independent solutions $f_1(x,\lambda)$ and $f_2(x,\lambda)$ of (1) given by (7) of course are independent of the choice of the solution $f_0(x,\lambda_0)$, hence instead of $f_0(x,\lambda_0)$ in (12) one can take $f_*(x,\lambda_*)$ given by (14). Thus, in terms of $f_*(x,\lambda_*)$ where λ_* is essentially arbitrary, $D(\lambda)$ can be represented as a series in powers of $(\lambda-\lambda_*)$

$$D(\lambda) = \sum_{n=0}^{\infty}\Bigg(\frac{f_*(T)}{f_*(0)}\tilde{X}^{(2n)}(T) + \frac{f_*(0)}{f_*(T)}X^{(2n)}(T) + \tag{15}$$
$$+ \left(f_*'(0)f_*(T) - f_*(0)f_*'(T)\right)p(0)X^{(2n+1)}(T)\Bigg)(\lambda-\lambda_*)^n.$$

Now the band edge λ_0 required for the formula (13) can be calculated as a first zero of the expression $D(\lambda)-2$ where $D(\lambda)$ is given by (15). For the numerical purpose it can be useful to know the interval containing λ_0. Since q is a bounded periodic function, there is a number Λ which satisfies the inequality $q(x) > \Lambda\ \forall x \in \mathbf{R}$. It is known [8] that $D(\lambda) > 2$ for all $\lambda \leqslant \Lambda$, therefore the lower estimate for λ_0 is the following

$$\lambda_0 \geqslant \min q(x).$$

The upper bound can be obtained considering the Rayleigh quotient for periodic problem [14]

$$\lambda_0 \leqslant \frac{\int_0^T \left(p\left(x\right)\left(u'\left(x\right)\right)^2 + q\left(x\right)\left(u(x)\right)^2\right)dx}{\int_0^T \left(u(x)\right)^2 dx},$$

where $u(x) \in \mathbf{C}^2\left[0,T\right]$ is periodic with period T. The equality occurs if and only if $u(x)$ is an eigenfunction corresponding to λ_0.

4 Hill's discriminant of the supersymmetric (SUSY)-related equation

In this section, we consider the SUSY-related equation of Eq. (1) and obtain its SPPS solutions. These solutions allow us to prove the equality between the Hill discriminants of equation (1) and its SUSY-related Eq. (18). For various aspects of SUSY periodic problems, see [15, 16, 17].

The left-hand side of the equation (1) can be factorized in the following way [18]

$$L\left[f(x,\lambda)\right] = \left(-d_x p^{\frac{1}{2}}(x) + \Phi(x)\right)\left(p^{\frac{1}{2}}(x)d_x + \Phi(x)\right)f(x), \tag{16}$$

where d_x means the x-derivative, the superpotential $\Phi(x)$ is defined as follows $\Phi(x) = -p^{\frac{1}{2}}(x)\frac{f_0'(x,\lambda_0)}{f_0(x,\lambda_0)}$. Using this factorization the coefficient $q(x)$ can be expressed as

$$q(x) = \Phi^2(x) - \left(p^{\frac{1}{2}}(x)\Phi(x)\right)' + \lambda_0.$$

Introducing the following Darboux transformation

$$\left(p^{\frac{1}{2}}(x)d_x + \Phi(x)\right) f(x,\lambda) = \tilde{f}(x,\lambda), \tag{17}$$

one obtains the equation supersymmetrically related to equation (1)

$$\tilde{L}\left[\tilde{f}(x,\lambda)\right] = \left(p^{\frac{1}{2}}(x)d_x + \Phi(x)\right)\left(-d_x p^{\frac{1}{2}}(x) + \Phi(x)\right)\tilde{f}(x,\lambda) = \lambda\tilde{f}(x,\lambda),$$

which can be written as follows

$$-d_x(p(x)d_x\tilde{f}(x,\lambda)) + \tilde{q}(x)\tilde{f}(x,\lambda) = \lambda\tilde{f}(x,\lambda), \tag{18}$$

where $\tilde{q}(x)$ is the SUSY partner of the potential $q(x)$ given by

$$\tilde{q}(x) = q(x) + 2p^{\frac{1}{2}}(x)\Phi'(x) - p^{\frac{1}{2}}(x)(p^{\frac{1}{2}}(x))''. \tag{19}$$

It is worth noting that as $\Phi(x)$ is a T-periodic function, the Darboux transformation assures the T-periodicity of $\tilde{q}(x)$.

The pair of linearly independent solutions $\tilde{f}_1(x,\lambda)$ and $\tilde{f}_2(x,\lambda)$ of (18) can be obtained directly from the solutions (7) by means of the Darboux transformation (17). We additionally take the linear combinations in order that the solutions $\tilde{f}_1(x,\lambda)$ and $\tilde{f}_2(x,\lambda)$ satisfy the initial conditions $\tilde{f}_1(0,\lambda) = \tilde{f}_2'(0,\lambda) = 1$ and $\tilde{f}_1'(0,\lambda) = \tilde{f}_2(0,\lambda) = 0$

$$\tilde{f}_1(x,\lambda) = \frac{p^{\frac{1}{2}}(0)f_0(0)}{p^{\frac{1}{2}}(x)f_0(x)}\Sigma_0(x,\lambda) + \frac{[p^{\frac{1}{2}}(x)]'|_{x=0} - \Phi(0)}{(\Delta\lambda)\, f_0(0)p^{\frac{1}{2}}(x)f_0(x)}\widetilde{\Sigma}_1(x,\lambda), \tag{20}$$

$$\tilde{f}_2(x,\lambda) = \frac{p^{\frac{1}{2}}(0)}{(\Delta\lambda)\, f_0(0)p^{\frac{1}{2}}(x)f_0(x)}\widetilde{\Sigma}_1(x,\lambda). \tag{21}$$

These two solutions allow us to write the expression for Hill's discriminant associated to the equation (18), that is $\widetilde{D}(\lambda) = \tilde{f}_1(T,\lambda) + \tilde{f}_2'(T,\lambda)$. We consider first the derivative of $\tilde{f}_2(x,\lambda)$ and evaluate it for $x = T$

$$\tilde{f}_2'(x,\lambda)|_{x=T} = -p^{\frac{1}{2}}(0)\frac{[p^{\frac{1}{2}}(x)]'|_{x=T}f_0(T) + p^{\frac{1}{2}}(T)f_0'(T)}{(\Delta\lambda)\, f_0(0)p(T)f_0^2(T)}\widetilde{\Sigma}_1(T,\lambda) + \frac{p^{\frac{1}{2}}(0)f_0(T)}{p^{\frac{1}{2}}(T)f_0(0)}\widetilde{\Sigma}_0(T,\lambda).$$

Notice that as the functions $f_0(x,\lambda_0)$ and $p(x)$ are T-periodic, i.e., $f_0(0,\lambda_0) = f_0(T,\lambda_0)$ and $p(0) = p(T)$, then obviously, the functions $f_0'(x,\lambda_0), p^{\frac{1}{2}}(x)$ and $[p^{\frac{1}{2}}(x)]'$ possess the same properties. Therefore we have

$$\widetilde{D}(\lambda) = \Sigma_0(T,\lambda) + \widetilde{\Sigma}_0(T,\lambda) + \left(\frac{[p^{\frac{1}{2}}(x)]'|_{x=0} - \Phi(0)}{(\Delta\lambda)\, f_0(0,\lambda_0)p^{\frac{1}{2}}(T)f_0(T,\lambda_0)} - \right.$$
$$\left. - \frac{[p^{\frac{1}{2}}(x)]'|_{x=T}f_0(T,\lambda_0) + p^{\frac{1}{2}}(T)f_0'(T,\lambda_0)}{(\Delta\lambda)\, f_0(0,\lambda_0)p^{\frac{1}{2}}(T)f_0^2(T,\lambda_0)}\right)\widetilde{\Sigma}_1(T,\lambda).$$

The substitution $\Phi(0) = -p^{\frac{1}{2}}(0)\frac{f_0'(0,\lambda_0)}{f_0(0,\lambda_0)}$ clearly shows that the expression in brackets vanishes. Therefore, we obtain

$$\widetilde{D}(\lambda) = \Sigma_0(T,\lambda) + \widetilde{\Sigma}_0(T,\lambda) = \sum_{n=0}^{\infty}\left(\tilde{X}^{(2n)}(T) + X^{(2n)}(T)\right)(\Delta\lambda)^n.$$

Comparing with (13) we have the identity

$$D(\lambda) \equiv \widetilde{D}(\lambda). \tag{22}$$

Thus, we have proven the following statement:

Theorem 2 *Let λ_0 be the first eigenvalue of (1)and $f_0(x,\lambda_0)$ the corresponding T-periodic nodeless eigenfunction. Then the Darboux transformation (17) with $\Phi(x) = -p^{\frac{1}{2}}(x)\frac{f_0'(x,\lambda_0)}{f_0(x,\lambda_0)}$ leads to a SUSY-related Eq. (18) with the preservation of the Hill discriminant, i.e., Eq. (22) holds.*

From the identity of discriminants (22) it is clear that λ_0 gives rise to a nodeless periodic solution $\tilde{f}_0(x,\lambda_0)$ of Eq. (18). Taking $\lambda = \lambda_0$ in (20) and (21) we get this eigenfunction in the form $\tilde{f}_0(x,\lambda_0) = \frac{1}{p^{\frac{1}{2}}(x)f_0(x,\lambda_0)}$.

Notice that, the factorization method can be applied to Eq. (18) with the superpotential $\Phi_1(x) = -p^{\frac{1}{2}}(x)\frac{\tilde{f}_0'(x,\lambda_0)}{\tilde{f}_0(x,\lambda_0)}$. In this case, we obtain the representation

$$\tilde{q} = \Phi_1^2(x) - \left(p^{\frac{1}{2}}(x)\Phi_1(x)\right)' + \lambda_0,$$

which reduces to the equality (19) if one notices the relationship $\Phi_1(x) = (p^{\frac{1}{2}}(x))' - \Phi(x)$. It can be also shown that $\tilde{\tilde{q}} \equiv q$, where $\tilde{\tilde{q}} = \tilde{q}(x) + 2p^{\frac{1}{2}}(x)\Phi_1'(x) - p^{\frac{1}{2}}(x)(p^{\frac{1}{2}}(x))''$ is the superpartner potential of $\tilde{q}(x)$. Thus, the Darboux transformation (17) with the superpotential $\Phi_1(x)$ applied to Eq. (18) does not produce a different potential.

5 Numerical calculation of eigenvalues based on the SPPS form of Hill's discriminant

As is well known, see e.g., [5], the zeros of the functions $D(\lambda) \pm 2$ represent eigenvalues of the corresponding operator. In this section, we show that besides other possible applications the representation (13) gives us an efficient tool for the calculation of the discrete spectrum of a periodic Sturm-Liouville operator.

The first step of the numerical realization of the method consists in calculation of the minimal eigenvalue λ_0 by means of the procedure given in subsection 3.4 and subsequently in construction of the corresponding nodeless periodic solution $f_0(x,\lambda_0)$ using formula (6). The next step of the algorithm is to compute the functions $\tilde{X}^{(n)}$ and $X^{(n)}$ given by (8) and (9), respectively. This construction is based on the eigenfunction $f_0(x, \lambda_0)$. Finally, by truncating the infinite series for $D(\lambda)$ (13) we obtain a polynomial in $\Delta\lambda$

$$D_N(\lambda) = \sum_{n=0}^{N} \left(\tilde{X}^{(2n)}(T) + X^{(2n)}(T)\right) (\Delta\lambda)^n \tag{23}$$
$$= 2 + \sum_{n=1}^{N} \left(\tilde{X}^{(2n)}(T) + X^{(2n)}(T)\right) (\Delta\lambda)^n.$$

The roots of the polynomials $D_N(\lambda) \pm 2$ give us eigenvalues corresponding to Eq. (1) with periodic and antiperiodic boundary conditions.

As an example, we consider the Mathieu equation with the following coefficients

$$p(x) = 1, \quad q(x) = 2r\cos 2x.$$

The algorithm was implemented in Matlab 2006. The recursive integration required for the construction of $\tilde{X}_0^{(n)}$, $X_0^{(n)}$, $\tilde{X}^{(n)}$ and $X^{(n)}$ was done by representing the integrand through a cubic spline using the *spapi* routine with a division of the interval $[0, T]$ into 7000 subintervals and integrating using the *fnint* routine. Next, the zeros of $D_N(\lambda) \pm 2$ were calculated by means of the *fnzeros* routine.

In the following tables, the Mathieu eigenvalues were calculated employing the SPPS representation (13) for two values of the parameter r. For comparison the same eigenvalues from the National Bureau of Standards (NBS) tables are also displayed [19].

	$r = 1$	$r = 1$
n	λ_n (SPPS)	λ_n (NBS)
0	−0.455139055973837	−0.45513860
1	−0.110248420387377	−0.11024882
2	1.859107160521687	1.85910807
3	3.917024962694820	3.91702477
4	4.371299312651704	4.37130098
5	9.047736927007582	9.04773926
6	9.078369587941564	9.07836885
7	16.033018848985410	16.03297008
8	16.033785039658117	16.03383234
9	25.020598536509114	25.02084082
10	25.021087773318282	25.02085434

	$r = 5$	$r = 5$
n	λ_n (SPPS)	λ_n (NBS)
0	-5.800045777242780	-5.80004602
1	-5.790080596840196	-5.79008060
2	1.858191484309548	1.85818754
3	2.099460384254221	2.09946045
4	7.449142541577460	7.44910974
5	9.236327731534002	
6	11.548906947651728	
7	16.648219815375526	
8	17.096668282587867	
9	25.510753265631860	25.51081605
10	25.551677357240167	25.54997175

Figures 1 and 2 display the plots of the calculated Hill discriminants for two values of the Mathieu parameter.

6 Conclusions

In summary, in the present work we obtained a representation of the Hill discriminant in the form of a series in powers of the spectral distance with respect to the first eigenvalue and also tested its efficiency for numerical calculations of the spectral bands using the Mathieu case. Moreover, we demonstrate that the Hill discriminant is invariant under the Darboux transformation generated by the first eigenfunction.

Acknowledgment. The first author thanks CONACyT for a postdoctoral fellowship allowing her to work in IPICyT.

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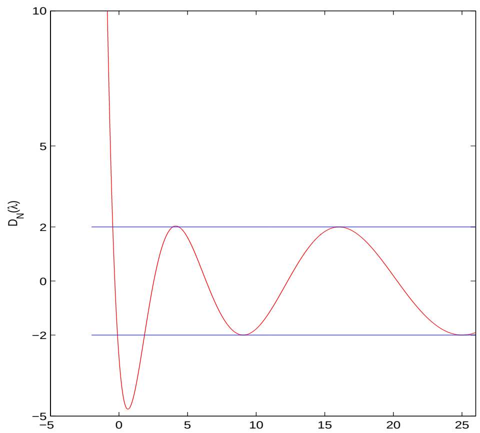

Figure 1: *The polynomial $D_N(\lambda)$ for the Mathieu equation with the parameter $r = 1$ calculated by means of formula (23) for $N = 100$.*

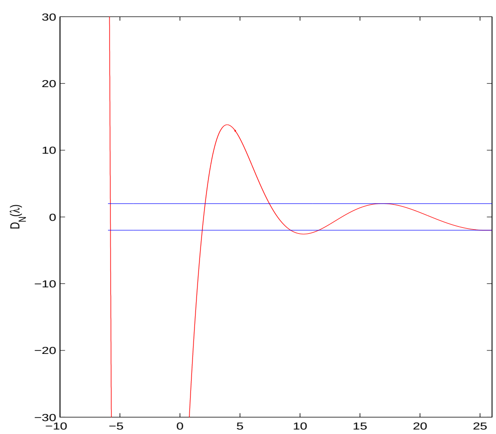

Figure 2: *Same as in the previous figure but for $r = 5$. The first minimum goes down to -292.0066.*